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Metric Properties of the Fuzzy Sphere

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Abstract

The fuzzy sphere, as a quantum metric space, carries a sequence of metrics which we describe in detail. We show that the Bloch coherent states, with these spectral distances, form a sequence of metric spaces that converge to the round sphere in the high-spin limit.

1 Introduction

It is common practice in several fields to "approximate" a manifold with a finite or countable subset of its points. A typical example in particle physics is the study of quantum field theories on a lattice: although they can be as difficult as continuous theories, they are of great interest since they can be studied with computer simulations. One drawback of lattice field theory is that it lacks some of the symmetries of the continuous theory it purports to approximate (say, e.g., Poincaré symmetries in flat Minkowski space).

Take the simple example of a unit two-sphere $\mathbb{S}^2$. If one replaces $\mathbb{S}^2$ with a subset of N points, the rotational symmetry is lost. In algebraic language: the algebra $\mathbb{C}^N$ of functions on N points is not an $\mathcal{U}(\mathfrak{su}(2))$-module $*$-algebra. There are no nontrivial $SU(2)$-orbits with finitely many points; thus, in order to preserve the symmetries and keep the algebra finite dimensional, one is forced to replace the algebra of functions $\mathbb{C}^N$

with a noncommutative one [26], provided that the noncommutativity be suppressed as $N \to \infty$. This is the rough idea behind the fuzzy sphere (and more general fuzzy spaces), as introduced in [26].

To be more precise, let x_1, x_2, x_3 be Cartesian coordinate functions on $\mathbb{S}^2$, and let $\mathcal{A}(\mathbb{S}^2)$ be the $*$-algebra of polynomials in these x_i. As an abstract $*$-algebra, this is the complex unital commutative $*$-algebra with three self-adjoint generators x_1, x_2, x_3 subject only to the relation

$$x_1^2 + x_2^2 + x_3^2 = 1.$$

It is well known that $\mathcal{A}(\mathbb{S}^2)$ is an $\mathcal{U}(\mathfrak{su}(2))$-module $*$-algebra, and it decomposes into a direct sum of irreducible representations $\mathcal{A}(\mathbb{S}^2) \simeq \bigoplus_{\ell=0}^{\infty} V_\ell$. Here V_ℓ is the vector space underlying the irreducible representation of $\mathcal{U}(\mathfrak{su}(2))$ with highest weight $\ell \in \mathbb{N}$, and is spanned by Laplace spherical harmonics $Y_{\ell,m}$.

In the spirit of [3, 4], suppose we want to introduce a cut-off in the energy spectrum, i.e., to neglect all but a finite number, say the first $N+1$, of representations V_ℓ in the decomposition of $\mathcal{A}(\mathbb{S}^2)$. One cannot simply take the linear span of $Y_{\ell,m}$ for $\ell = 0, 1, \ldots, N$, since this is not a subalgebra of $\mathcal{A}(\mathbb{S}^2)$. One way to proceed is to notice that, if we write $N = 2j$ and denote by $\pi_j \colon \mathcal{U}(\mathfrak{su}(2)) \to M_{2j+1}(\mathbb{C})$ the spin j representation of $\mathcal{U}(\mathfrak{su}(2))$, then the action

$$h \triangleright a := \pi_j(h_{(1)})\, a\, \pi_j(S(h_{(2)})), \qquad h \in \mathcal{U}(\mathfrak{su}(2)),\ a \in M_{2j+1}(\mathbb{C})$$

makes the matrix algebra $\mathcal{A}_N := M_{N+1}(\mathbb{C})$ an $\mathcal{U}(\mathfrak{su}(2))$-module $*$-algebra. (Here we use Sweedler notation for the coproduct; notations about $\mathfrak{su}(2)$ are recalled in the Appendix). There is a decomposition into irreducible representations:

$$\mathcal{A}_N \simeq V_j \otimes V_j^* \simeq \bigoplus_{\ell=0}^{2j} V_\ell$$

and a surjective homomorphism $\mathcal{A}(\mathbb{S}^2) \to \mathcal{A}_N$ of $\mathcal{U}(\mathfrak{su}(2))$-modules (but not of module algebras), given on generators by

$$x_k \mapsto \hat{x}_k := \frac{1}{\sqrt{j(j+1)}}\, \pi_j(J_k),$$

where the J_k are the standard real generators of $\mathcal{U}(\mathfrak{su}(2))$. We stress that the map $x_k \mapsto \hat{x}_k$ does not extend to an algebra morphism, but can be extended in a unique way, using coherent-state quantization, to an isometry between $*$-representations of $\mathcal{U}(\mathfrak{su}(2))$ sending the spherical harmonic $Y_{\ell,m}$, for $\ell \leq 2j$, into a matrix $\widehat{Y}^{(j)}_{\ell,m}$ sometimes called a "fuzzy spherical harmonic" (we give the details at the end of subsection 3.3).

Since an infinite-dimensional vector space is mapped onto a finite-dimensional one, the naïve idea is that information is lost and the space becomes "fuzzy".

The matrices $\hat{x}_k$ are normalized in such a way that the spherical relation still holds:

$$\hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2 = 1,$$

but their commutators are clearly not zero [35]:

$$[\hat{x}_k, \hat{x}_l] = \frac{1}{\sqrt{j(j+1)}}\, \mathrm{i}\, \varepsilon_{klm} \hat{x}_m,$$

where ε_{klm} is the completely skewsymmetric tensor.

Since the coefficient in the commutator vanishes for $N = 2j \to \infty$, the naïve idea is that the fuzzy sphere "converges", as $N \to \infty$, to a unit sphere. It is clear that the notion of convergence must involve the Riemannian metric of $\mathbb{S}^2$.

The correct mathematical framework for the convergence of matrix algebras to algebras of functions on Riemannian manifolds (or more generally, on metric spaces) was developed by Rieffel in a series of seminal papers, where he introduced the notion of (compact) quantum metric spaces and quantum Gromov–Hausdorff convergence [29–31]. The convergence of the fuzzy sphere to $\mathbb{S}^2$ was established in [32].

We recall briefly the notion of convergence of ordinary metric spaces (see [31, 33] for the quantum version). Given a metric space (M, d), the Hausdorff distance between two subsets X, Y of M is defined as follows. One defines

$$d(X, Y) := \sup_{p \in X} d(p, Y) := \sup_{p \in X} \inf_{q \in Y} d(p, q).$$

This expression is not symmetric in its arguments, but one can symmetrize and get an expression obeying the triangle inequality (namely, the Hausdorff distance) by taking

$$d_H(X, Y) := \max\{d(X, Y), d(Y, X)\}.$$

This is finite if both X and Y are bounded. The Gromov–Hausdorff distance $d_{GH}(X, Y)$ between two compact metric spaces (X, d_X) and (Y, d_Y) is then [19] simply the infimum of the Hausdorff distance $d_H(f(X), g(Y))$ over all isometric embeddings $f\colon X \hookrightarrow M$ and $g\colon Y \hookrightarrow M$ into a common metric space M. Thus the set of all isometry classes of compact metric spaces becomes itself a metric space.

One can show that a sequence $\{X_N\}$ of compact subspaces of a metric space (M, d) is convergent to M with respect to the Gromov–Hausdorff distance if and only if for any $p \in M$, there exists a sequence of points $p_N \in X_N$ such that $\lim_{N\to\infty} d(p, p_N) = 0$. Of course, this is an abstract notion and in principle it gives no clue as to how to choose the sequence of elements "approximating" a given point p.

In addition to the rigorous notion of convergence between spaces there are usually more "physical" ways to see in which sense a noncommutative space "looks like" a commutative one. The physical rationale behind this is that experiments are always done with some errors, and if the precision of the measuring apparatus is not sufficient to detect the differences between the two spaces then they are operationally the same. Since noncommutative spaces have no points, the notion of distance must be defined using some states which approximate them, and which in the limit converge to the usual states of the commutative algebra. What we deal with in this paper is how to "approximate" points of $\mathbb{S}^2$ using states of the fuzzy sphere. A natural candidate to replace a point of $\mathbb{S}^2$ is the corresponding coherent state of $\mathcal{A}_N$. A distance d_N on the state space of $\mathcal{A}_N$ can be defined using a generalized Dirac operator. Since the set of coherent states can be identified, for any N, with $\mathbb{S}^2$, this gives us a distance on $\mathbb{S}^2$ depending on the deformation parameter N. What we prove here is that

$$\lim_{N\to\infty} d_N(p,q) = d_{\mathrm{geo}}(p,q), \quad \text{for all} \quad p,q \in \mathbb{S}^2,$$

where d_{geo} is the geodesic distance of the round sphere.

Another example of a noncommutative space where the distance between coherent states has already been studied is the Moyal plane [16, 36]. In contrast with the Moyal example, where the distance is independent of the deformation parameter, here the distance does depend on N.

In Section 2, we recall some basic properties of noncommutative spaces. In Section 3, we introduce the spectral triples used in this paper and compare them with several other proposals in the literature. In Section 4, we recall the Bloch coherent states [5] and compute some particular distances. For instance, in subsection 4.1, we determine the distance for $N = 1$ and prove that it equals half of the chordal distance on the sphere; in subsection 4.2, we find the diameter for any N; we also compute an exact lower bound for the spectral distance. In Section 5, we prove that the spectral distance is $SU(2)$-invariant and is nondecreasing with N; we also show that it is bounded above by the geodesic distance of $\mathbb{S}^2$, and that the distance between two coherent states converges to the geodesic distance between the corresponding points of $\mathbb{S}^2$ when $N \to \infty$. Notations and conventions about $\mathcal{U}(\mathfrak{su}(2))$ are recalled in an Appendix.

2 Preliminaries on noncommutative manifolds

Material in this section is mainly taken from [14, 18]. In the spirit of Connes' noncommutative geometry, manifolds are replaced by spectral triples.

A *unital spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ has the following data: (i) a separable complex Hilbert space $\mathcal{H}$; (ii) a complex associative involutive unital algebra $\mathcal{A}$ with a faithful unital $*$-representation $\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$. The representation symbol is usually omitted, thereby identifying $\mathcal{A}$ with $\pi(\mathcal{A})$; (iii) a self-adjoint operator D on $\mathcal{H}$ with compact resolvent such that $[D, a]$ is a bounded operator, for all $a \in \mathcal{A}$.

A spectral triple is called *even* if there is a *grading* γ on $\mathcal{H}$, i.e., a bounded operator satisfying $\gamma = \gamma^*$ and $\gamma^2 = 1$, commuting with any $a \in \mathcal{A}$ and anticommuting with D.

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is called *real* if there is an antilinear isometry $J\colon \mathcal{H} \to \mathcal{H}$, called its *real structure*, such that

$$J^2 = \varepsilon 1, \qquad JD = \varepsilon' DJ, \tag{2.1a}$$

plus $J\gamma = \varepsilon'' \gamma J$ in the even case, and

$$[a, JbJ^{-1}] = 0, \qquad [[D,a], JbJ^{-1}] = 0, \qquad \text{for all} \quad a, b \in \mathcal{A}. \tag{2.1b}$$

The signs $\varepsilon, \varepsilon', \varepsilon'' \in \{\pm 1\}$ determine the KO-dimension (an integer modulo 8) of the triple [15]. The first condition of (2.1b) shows that $b \mapsto Jb^* J^{-1}$ is an injective homomorphism of $\mathcal{A}_N$ into its commutant (acting on $\mathcal{H}$).

For the notion of *equivariant* spectral triple, we refer to [34]. A group, or more generally a Hopf algebra, acts on $\mathcal{A}$ and on $\mathcal{H}$, intertwining the operator D with itself.

Remark 2.1. Note that if $(\mathcal{A}, \mathcal{H}, D, \gamma)$ is an even spectral triple and v an eigenvector of D with eigenvalue λ, then γv is an eigenvector of D with eigenvalue $-\lambda$. Thus, the eigenvalues λ and $-\lambda$ have the same multiplicity.

We use the following notations and conventions. $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded linear operators on $\mathcal{H}$. By *states* of $\mathcal{A}$ we always mean states of the C^*-algebra $\overline{\mathcal{A}}$, the norm-completion of $\mathcal{A}$; the set of all states is denoted by $\mathcal{S}(\mathcal{A})$. We denote by $\|\cdot\|_{\mathcal{B}(\mathcal{H})}$ the operator norm of $\mathcal{B}(\mathcal{H})$ (usually omitting the subscript); by $\|v\|^2_{\mathcal{H}} = \langle v \mid v \rangle$ the norm-squared of a vector $v \in \mathcal{H}$, and use the notation $\langle \cdot \mid \cdot \rangle$ for the scalar product, regardless of which Hilbert space we use. By $\mathrm{C}\ell(\mathfrak{g})$ we mean the Clifford algebra over a semisimple Lie algebra with its Killing form.

Recall that $\mathcal{S}(\mathcal{A})$ is a compact (in the weak* topology) convex set, whose extremal points are the *pure states* of $\mathcal{A}$. The set $\mathcal{S}(\mathcal{A})$ is an extended metric space,* with distance given by

$$d_{\mathcal{A},D}(\omega, \omega') := \sup_{a=a^* \in \mathcal{A}} \{\, |\omega(a) - \omega'(a)| : \|[D, a]\|_{\mathcal{B}(\mathcal{H})} \leq 1 \,\} \tag{2.2}$$

for all $\omega, \omega' \in \mathcal{S}(\mathcal{A})$. This is usually called a *Connes' metric* or *spectral distance* [13]. The supremum is usually taken over all $a \in \mathcal{A}$ obeying the side condition; but it was noted in [23] that the supremum is always attained on self-adjoint elements.

More generally, one can replace $\|[D, a]\|$ by $L(a)$ where L is a Leibniz seminorm on $\mathcal{A}$, when defining a metric. The structure $(\mathcal{A}, d_{\mathcal{A},L})$ thereby obtained is called a "compact quantum metric space" [30, 33].

3 Dirac operators for the fuzzy sphere

The classical Dirac operator $D\!\!\!/$ on a compact semisimple Lie group G with Lie algebra $\mathfrak{g}$ can be seen as a purely algebraic object $\mathfrak{D}$ living in the noncommutative Weil algebra $U(\mathfrak{g}) \otimes \mathrm{C}\ell(\mathfrak{g})$, see [22, 25]. It is equivariant in the sense that there exists a Lie algebra homomorphism $\mathfrak{g} \to U(\mathfrak{g}) \otimes \mathrm{C}\ell(\mathfrak{g})$ with whose image $\mathfrak{D}$ commutes. The spinor bundle of G is parallelizable: $L^2(G, S) \simeq L^2(G) \otimes \Sigma$, where Σ is an irreducible $\mathrm{C}\ell(\mathfrak{g})$-module. The algebra $U(\mathfrak{g}) \otimes \mathrm{C}\ell(\mathfrak{g})$ acts on the Hilbert space $L^2(G) \otimes \Sigma$ making $\mathfrak{D}$ into the "concrete" Dirac operator $D\!\!\!/$ of G, an unbounded first-order elliptic operator. Using the injection $\mathfrak{g} \hookrightarrow \mathrm{C}\ell(\mathfrak{g})$ we can also think of $\mathfrak{D}$ as an element of $U(\mathfrak{g}) \otimes U(\mathfrak{g})$, equivariant in the sense that it commutes with the image of the coproduct Δ in $U(\mathfrak{g}) \otimes U(\mathfrak{g})$.

*An extended metric space is a pair (X, d) where X is a set and $d\colon X \times X \to [0, \infty]$ a symmetric map satisfying the triangle inequality and such that $d(x, y) = 0$ if and only if $x = y$. Such a d differs from an ordinary metric only in that the value $+\infty$ is allowed.

On a compact Riemannian symmetric space G/U, this construction also applies (indeed, it works on G as a symmetric space of $G \times G$), except that the spinor bundle may not always be parallelizable. This is the point of view that we shall adopt for the fuzzy sphere.

3.1 An abstract Dirac operator

We begin with the two-sphere $\mathbb{S}^2$. Notations concerning $\mathfrak{su}(2)$ and its representations are explained in Appendix A.

The abstract *Dirac element* $\mathfrak{D} \in \mathcal{U}(\mathfrak{su}(2)) \otimes \mathcal{U}(\mathfrak{su}(2))$ is defined as

$$\mathfrak{D} := 1 \otimes 1 + 2 \sum_k J_k \otimes J_k \,. \tag{3.1}$$

Since $\sum_k [J_k \otimes J_k, J_l \otimes 1 + 1 \otimes J_l] = 0$, this element commutes with the image of the coproduct $\Delta\colon \mathcal{U}(\mathfrak{su}(2)) \to \mathcal{U}(\mathfrak{su}(2)) \otimes \mathcal{U}(\mathfrak{su}(2))$. That is an equivariance property.

The corresponding element of $\mathcal{U}(\mathfrak{su}(2)) \otimes \mathbb{C}\ell_{20}$ is

$$\mathfrak{D}_S := (\mathsf{id} \otimes \pi_{\frac{1}{2}})(\mathfrak{D}) = 1 \otimes 1 + \sum_k J_k \otimes \sigma_k = \begin{pmatrix} 1+H & F \\ E & 1-H \end{pmatrix}. \tag{3.2}$$

Its square is $\mathfrak{D}_S^2 = C_{SU(2)} + \frac{1}{4}(1 \otimes 1)$, where

$$C_{SU(2)} := \sum_k (J_k \otimes 1 + 1 \otimes \tfrac{1}{2}\sigma_k)^2$$

is the Casimir operator and $1/4 = R/8$ is the scalar curvature term (recall that $R = 2$ is the scalar curvature of $\mathbb{S}^2$) is the symmetric space version, $D^2 = C_G + R/8$, of the Schrödinger–Lichnerowicz formula for equivariant Dirac operators [17, p. 87].

Lemma 3.1. *For any $\ell \neq 0$ in $\frac{1}{2}\mathbb{N}$, the operator $(\pi_\ell \otimes \pi_{\frac{1}{2}})(\mathfrak{D}^2)$ has eigenvalues ℓ^2 with multiplicity 2ℓ and $(\ell+1)^2$ with multiplicity $2\ell+2$. For $\ell = 0$, $(\pi_0 \otimes \pi_{\frac{1}{2}})(\mathfrak{D}^2)$ has eigenvalue 1 with multiplicity 2.*

Proof. It follows from (A.1) of the Appendix that

$$C_{SU(2)} = (\mathsf{id} \otimes \pi_{\frac{1}{2}}) \sum_k (J_k \otimes 1 + 1 \otimes J_k)^2 = (\mathsf{id} \otimes \pi_{\frac{1}{2}})\Delta(J^2).$$

Since $\Delta(1) = 1 \otimes 1$, this yields $\mathfrak{D}_S^2 = (\mathsf{id} \otimes \pi_{\frac{1}{2}})\Delta(J^2 + \frac{1}{4})$. Therefore,

$$(\pi_\ell \otimes \pi_{\frac{1}{2}})(\mathfrak{D}^2) = (\pi_\ell \otimes \mathsf{id})(\mathfrak{D}_S^2) = (\pi_\ell \otimes \pi_{\frac{1}{2}})\Delta(J^2 + \tfrac{1}{4}).$$

Now $(\pi_\ell \otimes \pi_{\frac{1}{2}})\Delta$ is the Hopf tensor product of the representations π_ℓ and $\pi_{\frac{1}{2}}$. From the decomposition

$$V_\ell \otimes V_{\frac{1}{2}} \simeq V_{\ell+\frac{1}{2}} \oplus V_{\ell-\frac{1}{2}} \tag{3.3}$$

it follows that $(\pi_\ell \otimes \pi_{\frac{1}{2}})(\mathfrak{D}^2)$ is unitarily equivalent to $\pi_{\ell+\frac{1}{2}}(J^2 + \frac{1}{4}) \oplus \pi_{\ell-\frac{1}{2}}(J^2 + \frac{1}{4})$, and hence has eigenvalues

$$(\ell \pm \tfrac{1}{2})(\ell \pm \tfrac{1}{2} + 1) + \tfrac{1}{4} = \begin{cases} (\ell+1)^2 & \text{on } V_{\ell+\frac{1}{2}}\,, \\ \ell^2 & \text{on } V_{\ell-\frac{1}{2}}\,. \end{cases}$$

If $\ell = 0$, the summand $V_{\ell-\frac{1}{2}}$ in (3.3) is missing, so the only eigenvalue is 1 on the 2-dimensional $V_{\frac{1}{2}}$. ■

3.2 The Dirac operator of $\mathbb{S}^2$

The natural representation of $\mathcal{U}(\mathfrak{su}(2))$ on $\mathbb{S}^2$ as vector fields yields the Dirac operator $D\!\!\!\!/$ of the unit sphere (with round metric). The spinor bundle $S \to \mathbb{S}^2$ is trivial of rank 2, so the spinor space is $L^2(\mathbb{S}^2, S) \simeq L^2(\mathbb{S}^2) \otimes \mathbb{C}^2$.

Modulo the identification $L^2(\mathbb{S}^2) \simeq \bigoplus_{\ell\in\mathbb{N}} V_\ell$, the operator $D\!\!\!\!/$ is given by

$$D\!\!\!\!/ = \bigoplus_{\ell\in\mathbb{N}} (\pi_\ell \otimes \pi_{\frac{1}{2}})(\mathfrak{D}),$$

with $\mathfrak{D}$ as in (3.1). It follows from Lemma 3.1 that $D\!\!\!\!/^2$ has eigenvalues $\lambda_\ell = \ell^2$ with multiplicity $m_\ell = 4\ell$, for every integer $\ell \geq 1$.

The spectral triple of $\mathbb{S}^2$ is even, using the grading that exchanges the two half-spinor line bundles [18]. From Remark 2.1 it follows that $D\!\!\!\!/$ has eigenvalues $\pm\ell$ with multiplicities $\frac{1}{2}m_\ell = 2\ell$.

3.3 Dirac operators on the fuzzy sphere

On the fuzzy sphere, we require an equivariant Dirac operator whose spectrum is that of $D\!\!\!\!/$, truncated at the level $\ell = N + 1$.

Let $N \geq 1$ be a fixed integer, and write $N = 2j$ as before. The fuzzy sphere (labelled by this N) is the "noncommutative $SU(2)$ coset space" described by the algebra $\mathcal{A}_N = M_{N+1}(\mathbb{C})$ with the $SU(2)$ left action:

$$(g, a) \mapsto a^g := \pi_j(g)\, a\, \pi_j(g)^*, \quad \text{for} \quad g \in SU(2),\ a \in \mathcal{A}_N .$$

Definition 3.2. The *irreducible* spectral triple on $\mathcal{A}_N$, that we denote by $(\mathcal{A}_N, \mathcal{H}_N, D_N)$, is given by $\mathcal{H}_N := V_j \otimes \mathbb{C}^2$, with the natural representation of $\mathcal{A}_N$ via row-by-column multiplication on the factor $V_j \simeq \mathbb{C}^{N+1}$, and

$$D_N := (\pi_j \otimes \pi_{\frac{1}{2}})(\mathfrak{D}),$$

where $\mathfrak{D}$ is the abstract Dirac element in (3.1).

Proposition 3.3. *The irreducible spectral triple on $\mathcal{A}_N = \mathcal{A}_{2j}$ has the following properties:*

(i) *It is equivariant with respect to the $SU(2)$ representation $\pi_j \otimes \pi_{\frac{1}{2}}$.*

(ii) *D_N has eigenvalues $j + 1$ and $(-j)$ with respective multiplicities $2j + 2$ and $2j$.*

(iii) *No grading or real structure is compatible with this spectral triple.*

(iv) *Its metric dimension is 0.*

Proof. Equivariance follows from the commuting of the abstract Dirac operator with the image of the coproduct, so that D_N commutes with the representation $\pi_j \otimes \pi_{\frac{1}{2}}$ of $\mathcal{U}(\mathfrak{su}(2))$ —or the corresponding representation of $SU(2)$— and from the intertwining relation:

$$(\pi_j \otimes \pi_{\frac{1}{2}})(g)\,(a \otimes 1)\,(\pi_j \otimes \pi_{\frac{1}{2}})(g)^* = \pi_j(g)\,a\,\pi_j(g)^* \otimes \pi_{\frac{1}{2}}(g)\pi_{\frac{1}{2}}(g)^* = a^g \otimes 1 \quad \text{for } a \in \mathcal{A}_N\,.$$

From Lemma 3.1 it follows that D_N^2 has eigenvalues j^2 and $(j+1)^2$. But without a grading we cannot conclude that the spectrum of D_N is symmetric. In fact, an explicit computation shows that it is *not*. Indeed, $\mathcal{H}_N$ has an orthonormal basis of eigenvectors for D_N, given by

$$\begin{aligned} |j,m\rangle\!\rangle_+ &:= \sqrt{\tfrac{j+m+1}{2j+1}}\,|j,m\rangle \otimes \tbinom{1}{0} + \sqrt{\tfrac{j-m}{2j+1}}\,|j,m+1\rangle \otimes \tbinom{0}{1}, \qquad m = -j-1,\dots,j\,; \\ |j,m\rangle\!\rangle_- &:= -\sqrt{\tfrac{j-m}{2j+1}}\,|j,m\rangle \otimes \tbinom{1}{0} + \sqrt{\tfrac{j+m+1}{2j+1}}\,|j,m+1\rangle \otimes \tbinom{0}{1}, \quad m = -j,\dots,j-1\,. \end{aligned} \tag{3.4}$$

One easily checks that

$$D_N|j,m\rangle_+ = (j+1)|j,m\rangle_+\,, \qquad D_N|j,m\rangle_- = -j|j,m\rangle_-\,.$$

Therefore, D_N has eigenvalue $j+1$ with multiplicity $2j+2$, and eigenvalue $-j$ with multiplicity $2j$, as claimed.

This asymmetry of the spectrum of D_N and Remark 2.1 rule out the existence of a grading for this spectral triple.

If there were a real structure, the commutant of $\mathcal{A}_N$ would contain $J\mathcal{A}_N J^{-1}$, whose dimension is $(N+1)^2 \geq 4$. But the commutant of $\mathcal{A}_N$ has only dimension 2; hence, no real structure can exist.

The metric dimension is clearly 0, since $\mathcal{H}_N$ is finite-dimensional. ■

Definition 3.4. The *full* spectral triple on $\mathcal{A}_N$, that we denote by $(\mathcal{A}_N, \widetilde{\mathcal{H}}_N, \widetilde{\mathfrak{D}}_N, \widetilde{\mathfrak{J}}_N)$, is given by $\widetilde{\mathcal{H}}_N \simeq \mathcal{A}_N \otimes \mathbb{C}^2$, where the first factor carries the left regular representation of $\mathcal{A}_N$, i.e., the GNS representation associated to the matrix trace; and the Dirac operator and real structure are defined by:

$$\begin{aligned} \widetilde{\mathfrak{D}}_N(a \otimes v) &:= a \otimes v + \textstyle\sum_k [\pi_j(J_k), a] \otimes \sigma_k v, \\ \widetilde{\mathfrak{J}}_N(a \otimes v) &:= a^* \otimes \sigma_2 \bar{v}, \end{aligned}$$

for any $a \in \mathcal{A}_N$ and $v \in \mathbb{C}^2$ (a column vector). For $v = (v_1, v_2)^t \in \mathbb{C}^2$, $\bar{v} := (v_1^*, v_2^*)^t$ is again a column vector.

The distinction between D_N and $\widetilde{\mathfrak{D}}_N$ is that π_j is replaced by its adjoint action on the space $\mathcal{A}_N = \mathrm{End}(V_j) \simeq V_j \otimes V_j^*$.

Proposition 3.5. *The full spectral triple on $\mathcal{A}_N$ has the following properties:*

(i) *It is a real spectral triple.*

(ii) *It is equivariant with respect to the $SU(2)$ representation given by the product of the action $a \mapsto a^g$ on $\mathcal{A}_N$ and the spin-$\frac{1}{2}$ representation.*

(iii) $\widetilde{\mathfrak{D}}_N$ *has integer eigenvalues $\pm\ell$ with multiplicity 2ℓ, for every $\ell = 1, \dots, N$; and eigenvalue $N+1$ with multiplicity $2N+2$.*

(iv) *This spectral triple carries no grading.*

(v) *Its metric dimension is* 0.

Proof. Ad (i): Clearly $\widetilde{\mathfrak{J}}_N$ is antilinear, and indeed is antiunitary, since

$$\begin{aligned}\langle \widetilde{\mathfrak{J}}_N(a \otimes v) \mid \widetilde{\mathfrak{J}}_N(b \otimes w)\rangle &= \operatorname{Tr}(a^*b)\langle \sigma_2\bar{v} \mid \sigma_2\bar{w}\rangle = \operatorname{Tr}(a^*b)\langle \bar{v} \mid \bar{w}\rangle \\ &= \overline{\operatorname{Tr}(b^*a)}\;\overline{\langle w \mid v\rangle} = \overline{\langle b \otimes w \mid a \otimes v\rangle}\,.\end{aligned}$$

We need to check the conditions (2.1). The equality $\bar{\sigma}_2 = -\sigma_2$ shows that $(\widetilde{\mathfrak{J}}_N)^2 = -1$. Using $\widetilde{\mathfrak{J}}_N^{-1} = -\widetilde{\mathfrak{J}}_N$, we find that

$$\widetilde{\mathfrak{J}}_N\, b\, \widetilde{\mathfrak{J}}_N^{-1}(a \otimes v) = -\widetilde{\mathfrak{J}}_N(ba^* \otimes \sigma_2\bar{v}) = ab^* \otimes v, \quad \text{for all} \quad a, b \in \mathcal{A}_N,\ v \in \mathbb{C}^2.$$

Since left and right multiplication on $\mathcal{A}_N$ commute, $\widetilde{\mathfrak{J}}_N\, b\, \widetilde{\mathfrak{J}}_N^{-1}$ lies in the commutant of $\mathcal{A}_N \otimes M_2(\mathbb{C})$, and both conditions in (2.1) are satisfied.

Since $\sigma_2\bar{\sigma}_k = -\sigma_k\sigma_2$ for $k = 1, 2, 3$, and $[\pi_j(J_k), a]^* = -[\pi_j(J_k), a^*]$, we obtain

$$\widetilde{\mathfrak{J}}_N\widetilde{\mathfrak{D}}_N(a \otimes v) = a^* \otimes \sigma_2 v - \textstyle\sum_k [\pi_j(J_k), a^*] \otimes \sigma_2\bar{\sigma}_k v = \widetilde{\mathfrak{D}}_N\widetilde{\mathfrak{J}}_N(a \otimes v),$$

for any $a \in \mathcal{A}_N$ and $v \in \mathbb{C}^2$. Hence $\widetilde{\mathfrak{J}}_N\widetilde{\mathfrak{D}}_N = \widetilde{\mathfrak{D}}_N\widetilde{\mathfrak{J}}_N$.

Ad (ii): Equivariance follows again from the commuting of the abstract Dirac element with the image of the coproduct, since the representation $J_k \mapsto [\pi_j(J_k), \cdot]$ is the derivative of the adjoint action $a \mapsto a^g = \pi_j(g)\, a\, \pi_j(g)^*$ of $SU(2)$.

Ad (iii): Note that

$$\widetilde{\mathfrak{D}}_N = (\operatorname{ad}\pi_j \otimes \pi_{\frac{1}{2}})(\mathfrak{D}),$$

with the $\mathfrak{D}$ of (3.1) and $\operatorname{ad}\pi_j$ is given by $\operatorname{ad}\pi_j(h)\colon a \mapsto \pi_j(h_{(1)})\, a\, \pi_j(S(h_{(2)}))$ for any $h \in \mathcal{U}(\mathfrak{su}(2))$ and $a \in \mathcal{A}_N$. Using the unitary $\mathcal{U}(\mathfrak{su}(2))$-module isomorphism

$$\mathcal{A}_N \simeq V_j \otimes V_j^* \simeq \bigoplus_{\ell=0}^{2j} V_\ell$$

we obtain that $\widetilde{\mathfrak{D}}_N$ is unitarily equivalent to the operator $\bigoplus_{\ell=0}^{2j}(\pi_\ell \otimes \pi_{\frac{1}{2}})(\mathfrak{D})$.

Replacing $N = 2j$ by 2ℓ in Prop. 3.3(ii), we see that $(\pi_\ell \otimes \pi_{\frac{1}{2}})(\mathfrak{D})$ has eigenvalues $\ell+1$ and $(-\ell)$, with respective multiplicities $2\ell+2$ and 2ℓ (but if $\ell = 0$ the eigenvalue $-\ell$ is missing). Hence $\widetilde{\mathfrak{D}}_N$ has the eigenvalues $\pm\ell$, each with multiplicity 2ℓ for $\ell = 1, \dots, N$; and $N+1$ with multiplicity $2N+2$.

Ad (iv): Since the spectrum of $\widetilde{\mathfrak{D}}_N$ is not symmetric about 0, Remark 2.1 again shows that no grading can exist for this spectral triple.

Ad (v): The metric dimension is 0 since $\widetilde{\mathcal{H}}_N$ is finite-dimensional. ∎

The signs in (2.1a) are $\varepsilon = -1$ and $\varepsilon' = 1$. In the absence of a grading such that $\varepsilon'' = -1$, this would suggest that the KO-dimension be 3 rather than 2; see, e.g., [18]. This "grading anomaly" could however be rectified by adding an extra subspace to the Hilbert space $\widetilde{\mathcal{H}}_N$ on which $\mathcal{A}$ acts trivially, along the lines of [14, p. 332]. We leave this issue aside.

Proposition 3.6. *The irreducible and full spectral triples induce the same metric on the state space $\mathcal{S}(\mathcal{A}_N)$ of the fuzzy sphere.*

Proof. This follows from the calculation:

$$\begin{aligned}[\widetilde{\mathcal{D}}_N, a](b \otimes v) &= \textstyle\sum_k \big([\pi_j(J_k), ab] - a\,[\pi_j(J_k), b]\big) \otimes \sigma_k v \\ &= \textstyle\sum_k [\pi_j(J_k), a]\, b \otimes \sigma_k v = [D_N, a](b \otimes v).\end{aligned}$$

Hence $[\widetilde{\mathcal{D}}_N, a]$ is the operator of left multiplication by the matrix $[D_N, a] \in \mathcal{A}_N \otimes M_2(\mathbb{C})$, so its operator norm coincides with the norm of the matrix. Therefore, since $\big\|[\widetilde{\mathcal{D}}_N, a]\big\| = \big\|[D_N, a]\big\|$ for each $a \in \mathcal{A}_N$, it follows that the two spectral triples induce the same metric (2.2) on the state space of $\mathcal{A}_N$. ∎

It is useful to give a more explicit presentation of the full spectral triple by exhibiting its eigenspinors. Recall that the polynomial algebra $\mathcal{A}(\mathbb{S}^2)$ is linearly spanned by the spherical harmonics $Y_{\ell,m}$, each of which is a homogeneous polynomial in Cartesian coordinates of degree ℓ, with the multiplication rule

$$Y_{\ell',m'} Y_{\ell'',m''} = \sum_{\ell=|\ell'-\ell''|}^{\ell=\ell'+\ell''} \sum_{m=-\ell}^{\ell} \sqrt{\frac{(2\ell'+1)(2\ell''+1)}{4\pi(2\ell+1)}}\, C^{\ell 0}_{\ell' 0, \ell'' 0} C^{\ell m}_{\ell' m', \ell'' m''} Y_{\ell,m}\,,$$

involving $SU(2)$ Clebsch–Gordan coefficients. Form there it is clear that the subspace spanned by the $Y_{\ell,m}$ for $\ell = 0, 1, \dots, N$ does not close under multiplication. To replace them, while keeping $SU(2)$ symmetry, one can make use of the irreducible tensor operators at level $N = 2j$ [1, 8]. These are elements $\widehat{T}^{(j)}_{\ell,m} \in M_{N+1}(\mathbb{C})$ whose matrix elements are given by

$$\langle jm'' \mid \widehat{T}^{(j)}_{\ell,m} \mid jm' \rangle := \sqrt{\frac{2\ell+1}{2j+1}}\, C^{jm''}_{jm',\ell m}\,.$$

They transform like the $Y_{\ell,m}$ under $SU(2)$, but still require an appropriate normalization. For any $-1 \leq s \leq 1$, one can define an operator $\widehat{Y}^{(j)}_{\ell,m} \in M_{N+1}(\mathbb{C})$ as follows [2, 9, 12, 24]:

$$\widehat{Y}^{(j,s)}_{\ell,m} := \sqrt{\frac{4\pi}{2j+1}}\, \big(C^{jj}_{jj,\ell 0}\big)^s\, \widehat{T}^{(j)}_{\ell,m}. \tag{3.5}$$

We omit the precise multiplication rules for these operators, see [24]; but in any case it is clear, by working backwards, that the ordinary spherical harmonic $Y_{\ell,m}$ can be regarded as a "symbol" of the operator $\widehat{Y}^{(j,s)}_{\ell,m}$ for fixed j and s. The cases $s = 1$, $s = 0$ and $s = -1$ correspond respectively to the Husimi Q-function, the Moyal–Wigner W-function and the

Glauber P-function [9]. Here we put $s = 1$ in (3.5), omit the superscripts, and call these operators the *fuzzy harmonics* $\widehat{Y}_{\ell,m} \in \mathcal{A}_N$.

The commutation rules for the irreducible tensor operators and the fuzzy harmonics come directly from their symmetries [1, 8, 35]:

$$[\pi_j(J_3), \widehat{Y}_{\ell,m}] = [\widehat{Y}_{1,0}, \widehat{Y}_{\ell,m}] = m\,\widehat{Y}_{\ell,m}\,,$$
$$[\pi_j(J_1 \pm \mathrm{i}J_2), \widehat{Y}_{\ell,m}] = [\widehat{Y}_{1,\pm1}, \widehat{Y}_{\ell,m}] = \sqrt{(\ell \mp m)(\ell \pm m + 1)}\,\widehat{Y}_{\ell,m\pm1}\,.$$

Adopting a 2×2 matrix notation, as in (3.2), we can write

$$\widetilde{\mathcal{D}}_N = \begin{pmatrix} 1 + \mathcal{L}_3 & \mathcal{L}_- \\ \mathcal{L}_+ & 1 - \mathcal{L}_3 \end{pmatrix}, \quad \text{where} \quad \mathcal{L}_3 = \operatorname{ad}\pi_j(J_3),\ \mathcal{L}_\pm = \operatorname{ad}\pi_j(J_1 \pm \mathrm{i}J_2). \tag{3.6}$$

Then the normalized eigenspinors for the operators $\widetilde{\mathcal{D}}_N$ are

$$|\ell, m\rangle\!\rangle_+ := \frac{1}{\sqrt{2\ell+1}} \begin{pmatrix} \sqrt{\ell+m+1}\,\widehat{Y}_{\ell,m} \\ \sqrt{\ell-m}\,\widehat{Y}_{\ell,m+1} \end{pmatrix}, \quad |\ell, m\rangle\!\rangle_- := \frac{1}{\sqrt{2\ell+1}} \begin{pmatrix} -\sqrt{\ell-m}\,\widehat{Y}_{\ell,m} \\ \sqrt{\ell+m+1}\,\widehat{Y}_{\ell,m+1} \end{pmatrix}$$

for $\ell = 0, 1, \dots, N$; whereby

$$\widetilde{\mathcal{D}}_N|\ell, m\rangle\!\rangle_+ = (\ell+1)\,|\ell, m\rangle\!\rangle_+ \quad \text{for} \quad m = -\ell-1, \dots, \ell\,;$$
$$\widetilde{\mathcal{D}}_N|\ell, m\rangle\!\rangle_- = \quad (-\ell)\,|\ell, m\rangle\!\rangle_- \quad \text{for} \quad m = -\ell, \dots, \ell-1.$$

The full spectral triple on $\mathcal{A}_N$ is thus a truncation of the standard spectral triple over $\mathbb{S}^2$, in the following sense. The Hilbert space of spinors $L^2(\mathbb{S}^2) \otimes \mathbb{C}^2$, generated by pairs of spherical harmonics $Y_{\ell,m}$, is truncated at $l \leq N$. On replacing these by pairs of fuzzy harmonics $\widehat{Y}_{\ell,m}$, the resulting spectrum of $\widetilde{\mathcal{D}}_N$ is a truncation of the spectrum of $D\!\!\!\!/$ to the range $\{-N, \dots, N+1\}$, unavoidably breaking the parity symmetry.

3.4 Comparison with the literature

Two spectral triples on the fuzzy sphere algebra $\mathcal{A}_n$ have been introduced, one constructed with the irreducible $\mathfrak{su}(2)$-module V_j and the other with the left regular or GNS representation. Neither one is even (there exists no grading); although this could be remedied by allowing $\mathcal{A}_N$ to act trivially on a supplementary vector space. The first carries no real structure but the second one does, because the reducible action of the algebra on the Hilbert space allows for a large enough commutant. The crucial point here, however, is Prop. 3.6, showing that both spectral triples give the same metric.

Other Dirac operators for the fuzzy sphere have been proposed in [6, 7, 10, 11, 20] and are recalled below.

In [20] $\mathcal{A}_N$ is obtained as the even part of a truncated supersphere, and the Dirac operator is defined as the odd part of a truncated superfield. Reformulating the result of Section 4.3 of [20] in our language, the Hilbert space is taken to be

$$\mathcal{H}'_N := \bigoplus_{\ell=\frac{1}{2},\dots,N-\frac{1}{2}} V_\ell \oplus V_\ell\,.$$

Note that to get our $\mathcal{A}_N \otimes V_{\frac{1}{2}}$ one must add an extra $V_{N+\frac{1}{2}}$ subspace. The algebra $\mathcal{A}_N$ is generated by the three matrices $\hat{x}_k$, proportional to $\pi_j(J_k)$, which can be represented on $\mathcal{H}'_N$ using a suitable direct sum of irreducible representations of $\mathfrak{su}(2)$. The Dirac operator can be defined by representing the abstract Dirac element (3.1) on $\mathcal{H}'$ using the same representation of $\mathfrak{su}(2)$; it is proportional to the identity on each subspace V_ℓ and its spectrum is given by the eigenvalues $\pm\ell$, for $\ell = 1, \dots, N$ (restricted to $V_\ell \oplus V_\ell$ their Dirac operator is the operator $\ell \oplus -\ell$). Compared to our full spectral triple, the eigenvalue $N+1$ is missing. Since the two copies of V_ℓ carry the same representation of $\mathcal{A}_N$, the operator γ_N that exchanges these copies commutes with $\mathcal{A}_N$ (and anticommutes with the Dirac operator): therefore, one obtains an even spectral triple.

This construct is still metrically equivalent to the spectral triples of subsection 3.3. Here $\widetilde{\mathcal{H}}_N \simeq \mathcal{H}'_N \oplus V_{N+\frac{1}{2}}$; but the additional term $V_{N+\frac{1}{2}}$ carries a nontrivial subrepresentation of $\mathcal{A}_N$, and the Dirac operator $\widetilde{D}_N$ is proportional to the identity on such a subspace: hence $[\widetilde{D}_N, a]$ vanishes on the subspace $V_{N+\frac{1}{2}}$ for any $a \in \mathcal{A}_N$. Therefore the two spectral triples induce the same seminorm on $\mathcal{A}_N$, and hence the same distance.

The authors of [6,7] take another approach. Given any finite-dimensional $\mathfrak{su}(2)$-module Σ, one can construct a Dirac-like operator on $L^2(\mathbb{S}^2) \otimes \Sigma$ by using the appropriate representation of the abstract Dirac element (3.1). If Σ is the spin j representation, this can be called a "spin-j" Dirac operator. For $j = \frac{1}{2}$ we recover the ordinary Dirac operator acting on 2-spinors.

A spin-$\frac{1}{2}$ Dirac operator for the fuzzy sphere is discussed in [6], and is generalized to arbitrary spin j in [7]. These are constructed using the Ginsparg–Wilson algebra, namely, the free algebra generated by two grading operators Γ, Γ'. The linear combinations

$$\Gamma_1 = \tfrac{1}{2}(\Gamma + \Gamma'), \qquad \Gamma_2 = \tfrac{1}{2}(\Gamma - \Gamma'),$$

anticommute, and the proposal is to realize them as operators on a suitable Hilbert space, interpreting the former as the Dirac operator and the latter as the chirality operator. In the spin-$\frac{1}{2}$ case, the Hilbert space is taken to be $\mathcal{A}_N \otimes V_{\frac{1}{2}}$. From equation (2.20) of [7], or equivalently from (8.29) of [6], we see that the Dirac operator is the same as the operator (3.6) of our full spectral triple. The chirality operator, (2.21) of [7], in contrast with the Dirac operator, is constructed using the anticommutator with $\pi_j(J_k)$, i.e., $L_k^L + L_k^R$ in the notation of [7].

The asymmetry of the Dirac operator spectrum was already noticed in [6]. At the end of subsection 8.3.2 we read:

> *For* $j = 2L+1$ [$\ell = N+1$ in our notations here] *we get the positive eigenvalue correctly, but the negative one is missing. That is an edge effect caused by cutting off the angular momentum at* $2L$.

And in the same subsection, after equation (8.30):

As mentioned earlier, use of Γ_2 *as chirality resolves a difficulty addressed elsewhere [80], where* $\operatorname{sign}(\Gamma_2)$ *was used as chirality. That necessitates projecting out* V_{+1} *and creates a very inelegant situation.*

In other words, Γ_2 is not a true grading operator. Since Γ_2 anticommutes with the Dirac operator, it must vanish on $V_{N+\frac{1}{2}}$ (otherwise, the Dirac operator would have an eigenvector $\Gamma_2 v$ for $v \in V_{N+\frac{1}{2}}$, with eigenvalue $-N-1$); which entails $(\Gamma_2)^2 \neq 1$.

A third proposal is that of [10, 11]. There, the authors start by constructing, on the Hilbert space $\mathcal{A}_N$, a chirality operator that is a genuine $\mathbb{Z}_2$-grading (its square is 1), and then find a Dirac-like operator $\mathbf{D}$ by imposing anticommutation with the grading, arriving at an even spectral triple. It follows that this operator cannot be isospectral to our $\widetilde{\mathcal{D}}_N$. The earlier paper uses a chirality operator γ_χ, see (5) of [11], that does not commute with the algebra $\mathcal{A}_N$. Later, in (6) of [11], this is corrected to γ_χ° by replacing left with right multiplication. On imposing anticommutation of $\mathbf{D}$ with that grading, one arrives at a "second order" operator, (8) of [11], that in our notations is:

$$\mathbf{D}(a \otimes v) := c\, \gamma_\chi^\circ \textstyle\sum_{klm} \varepsilon_{klm}\, \pi_j(J_k)\, a\, \pi_j(J_l) \otimes \sigma_m v,$$

where c is a normalization constant.

From (17) of [11], relabelling with $\ell = j + \frac{1}{2}$, we see that the spectrum of $\mathbf{D}$ is given by the eigenvalues $\pm\lambda_\ell$, for $\ell = 1, \dots, N+1$, with

$$\lambda_\ell^2 := \frac{\ell^2((N+1)^2 - \ell^2)}{N(N+2)}\,.$$

Note that λ_ℓ is non-linear in ℓ, and that $\lambda_{N+1} = 0$, i.e., this operator has a kernel $V_{N+\frac{1}{2}}$.

The mentioned proposals, and other variants such as [21], begin with a chirality operator and then find an anticommuting self-adjoint Dirac-like operator with a plausible spectrum. Our approach, in contrast, starts from $SU(2)$-equivariance and arrives at a neater truncation of the classical spectrum, paying the price of spectral asymmetry.

4 Some distance calculations

Having reduced the problem of computing distances on the fuzzy sphere, via Prop. 3.6, to the use of the irreducible spectral triple $(\mathcal{A}_N, \mathcal{H}_N, D_N)$, we can now compute the distance between particular pairs of pure states in $\mathcal{S}(\mathcal{A}_n)$. Using (3.2), we know that

$$D_N = \begin{pmatrix} 1 + \pi_j(H) & \pi_j(F) \\ \pi_j(E) & 1 - \pi_j(H) \end{pmatrix} \tag{4.1}$$

where once again $2j = N$. From now on we omit the representation symbol π_j and use the matrix of (3.2) instead, by a small abuse of notation. The spectral distance will be denoted by d_N.

Lemma 4.1. *For any $a \in \mathcal{A}_N$, the following inequalities hold:*

$$\big\|[H,a]\big\| \leq \big\|[D_N,a]\big\|, \qquad \big\|[E,a]\big\| \leq \big\|[D_N,a]\big\|, \qquad \big\|[F,a]\big\| \leq \big\|[D_N,a]\big\|.$$

moreover, if a is a diagonal hermitian matrix, then $\|[D_N,a]\| = \|[E,a]\|$.

Proof. Using the expression

$$[D_N,a]^*[D_N,a] = \begin{pmatrix} [H,a]^*[H,a] + [E,a]^*[E,a] & \cdots \\ \cdots & \cdots \end{pmatrix},$$

we find a lower bound for $\|[D_N,a]\|$ by taking the supremum over unit vectors of the form $(x,0)^t$, with $x \in V_j$:

$$\begin{aligned} \big\|[D_N,a]\big\|^2 &\geq \sup_{\|x\|=1} \big\langle x \mid \big([H,a]^*[H,a] + [E,a]^*[E,a]\big)x\big\rangle \\ &= \sup_{\|x\|=1} \big(\|[H,a]\,x\|^2 + \|[E,a]\,x\|^2\big). \end{aligned}$$

The right hand side is a sum of two positive terms, and is greater than or equal to both $\sup_{\|x\|=1} \|[H,a]\,x\|^2 = \|[H,a]\|^2$ and $\sup_{\|x\|=1} \|[E,a]\,x\|^2 = \|[E,a]\|^2$.

Since $[F,a] = -[E,a^*]^*$, we also get $\|[F,a]\| \leq \|[D_N,a^*]\| = \|[D_N,a]^*\| = \|[D_N,a]\|$.

If $a \in \mathcal{A}_N$ is a diagonal matrix, then $[H,a] = 0$, so that

$$[D_N,a]^*[D_N,a] = \begin{pmatrix} [E,a]^*[E,a] & 0 \\ 0 & [F,a]^*[F,a] \end{pmatrix},$$

thus $\|[D_N,a]\|$ is the greater of $\|[E,a]\|$ and $\|[F,a]\|$. Furthermore, if $a = a^*$, then $[F,a] = -[E,a]^*$ and $\|[E,a]\| = \|[F,a]\|$, so that $\|[D_N,a]\| = \|[E,a]\| = \|[F,a]\|$. ∎

The $SU(2)$-coherent states on $\mathcal{A}_N$ were introduced in [5], under the names "Bloch states" or "atomic coherent states", by applying rotation operators $R_{(\varphi,\theta)}$ to the "ground" state $|j,-j\rangle \in V_j$. (Of course, one applies the whole group $SU(2)$ via π_j, but the stabilizer of $|j,-j\rangle$ is a circle, so the orbit is a copy of $\mathbb{S}^2$, labelled by the usual angular coordinates.) The coherent-state vectors are [5]:

$$|\varphi,\theta)_N := \sum_{m=-j}^{j} \binom{2j}{j+m}^{\frac{1}{2}} e^{-im\varphi} (\sin\tfrac{\theta}{2})^{j+m} (\cos\tfrac{\theta}{2})^{j-m}\, |j,m\rangle. \tag{4.2}$$

The corresponding vector states are denoted by

$$\psi^N_{(\varphi,\theta)}(a) := (\varphi,\theta \mid a \mid \varphi,\theta)_N\,.$$

These Bloch coherent states are for the group $SU(2)$ what the usual harmonic oscillator coherent states are for the Heisenberg group [28]. In particular, they are minimum uncertainty states.

By construction, the map $\mathbb{S}^2 \to V_j$ sending the point $(\varphi,\theta) \in \mathbb{S}^2$ to the vector $|\varphi,\theta)$ intertwines the rotation action of $SU(2)$ on $\mathbb{S}^2$ with the irrep π_j on V_j. At the infinitesimal level, this is expressed by the next lemma.

Lemma 4.2. *Regarding $\psi^N_{(\varphi,\theta)}$ as a vector state on $\mathcal{B}(V_j)$, we find that*

$$\psi^N_{(\varphi,\theta)}([H,a]) = -\mathrm{i}\,\frac{\partial}{\partial\varphi}\,\psi^N_{(\varphi,\theta)}(a), \tag{4.3a}$$

$$\psi^N_{(\varphi,\theta)}([E,a]) = e^{\mathrm{i}\varphi}\biggl(\frac{\partial}{\partial\theta} + \mathrm{i}\cot\theta\frac{\partial}{\partial\varphi}\biggr)\psi^N_{(\varphi,\theta)}(a), \tag{4.3b}$$

$$\psi^N_{(\varphi,\theta)}([F,a]) = -e^{-\mathrm{i}\varphi}\biggl(\frac{\partial}{\partial\theta} - \mathrm{i}\cot\theta\frac{\partial}{\partial\varphi}\biggr)\psi^N_{(\varphi,\theta)}(a). \tag{4.3c}$$

Proof. A simple direct computation. ■

4.1 The $N = 1$ case

We write the general hermitian element $a = a^* \in M_2(\mathbb{C})$ as

$$a = \begin{pmatrix} a_0 + a_3 & a_1 + \mathrm{i}a_2 \\ a_1 - \mathrm{i}a_2 & a_0 - a_3 \end{pmatrix}$$

with all a_i real. Arbitrary (not necessarily pure) states on $M_2(\mathbb{C})$ are given by

$$\omega_{\vec{x}}(a) := a_0 + \vec{x}\cdot\vec{a},$$

with $\vec{x}$ in the closed unit ball $B^3 \subset \mathbb{R}^3$ and $\vec{a} = (a_1, a_2, a_3)$. This state is pure if and only if $\vec{x} = (x_1, x_2, x_3) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ lies on the boundary $\mathbb{S}^2$ of the ball, in which case it coincides with the coherent state $\psi^1_{(\varphi,\theta)}$. Note that *for $N = 1$, all pure states are coherent states.*

We now compute the distance between arbitrary states: this will be (half of) the Euclidean distance in the ball; which is (half of) the chordal distance on the sphere, when restricted to coherent states.

Proposition 4.3. *For all $\vec{x}, \vec{y} \in B^3$, the distance between the corresponding states is*

$$d_1(\omega_{\vec{x}}, \omega_{\vec{y}}) = \frac{1}{2}\,|\vec{x} - \vec{y}|. \tag{4.4}$$

In particular, $d_1(\psi^1_{(0,\theta)}, \psi^1_{(0,0)}) = \sin(\theta/2)$.

Proof. Writing $a_\pm = a_1 \pm \mathrm{i}a_2$ and $\sigma_\pm = \sigma_1 \pm \mathrm{i}\sigma_2$, we get, for $a = a^*$:

$$[D_1, a] = \begin{pmatrix} \frac{1}{2}[\sigma_3, a] & [\sigma_-, a] \\ [\sigma_+, a] & -\frac{1}{2}[\sigma_3, a] \end{pmatrix} = \begin{pmatrix} 0 & a_+ & -a_+ & 0 \\ -a_- & 0 & 2a_3 & a_+ \\ a_- & -2a_3 & 0 & -a_+ \\ 0 & -a_- & a_- & 0 \end{pmatrix}.$$

The matrix $\mathrm{i}[D_1, a]$ is hermitian, and its characteristic polynomial is easily seen to be $\det(\lambda - \mathrm{i}[D_1, a]) = \lambda^2(\lambda^2 - 4|\vec{a}|^2)$, showing that its norm is $\|[D_1, a]\| = 2|\vec{a}|$.

The Cauchy–Schwarz inequality

$$\bigl|\omega_{\vec{x}}(a) - \omega_{\vec{y}}(a)\bigr| = \bigl|(\vec{x} - \vec{y})\cdot\vec{a}\bigr| \le |\vec{x} - \vec{y}|\,|\vec{a}|$$

is saturated when $\vec{a}$ is parallel to $\vec{x} - \vec{y}$. Thus $d_1(\omega_{\vec{x}}, \omega_{\vec{y}})$ is the supremum of $|\vec{x} - \vec{y}|\,|\vec{a}|$ over hermitian a with $\|[D_1, a]\| = 2|\vec{a}| \leq 1$. This establishes (4.4).

If $\vec{x} = (\sin\theta, 0, \cos\theta)$ and $\vec{y} = (0, 0, 1)$, then $|\vec{x} - \vec{y}|^2 = 2(1 - \cos\theta) = 4\,\sin^2(\theta/2)$, and thus $d_1(\omega_{\vec{x}}, \omega_{\vec{y}}) = \sin(\theta/2)$. ∎

4.2 Distance between basis vectors and diameter

Similarly to Prop. 3.6 of [16], the distance between basis vector can be exactly computed. For arbitrary $N = 2j$, the distance between the basic vector states

$$\omega_m(a) := \langle j, m \mid a \mid j, m \rangle$$

can be computed explicitly.

Proposition 4.4. *For any $m < n$ in $\{-j, \dots, j\}$, the following distance formula holds:*

$$d_N(\omega_m, \omega_n) = \sum_{k=m+1}^{n} \frac{1}{\sqrt{(j+k)(j-k+1)}}\,. \tag{4.5}$$

Proof. If $a \in \mathcal{A}_N$, then

$$\begin{aligned} \omega_m(a) - \omega_n(a) &= \sum_{k=m+1}^{n} \langle j, k-1 \mid a \mid j, k-1 \rangle - \langle j, k \mid a \mid j, k \rangle \\ &= \sum_{k=m+1}^{n} \frac{1}{\sqrt{(j+k)(j-k+1)}}\,\langle j, k \mid [E, a] \mid j, k-1 \rangle. \end{aligned}$$

Using Lemma 4.1, we get the estimate

$$\bigl|\langle j, k \mid [E, a] \mid j, k-1 \rangle\bigr| \leq \|[E, a]\| \leq \|[D_N, a]\|$$

which shows that the left hand side of (4.5) is no greater than the right hand side.

On the other hand, let $\hat{a}$ be the self-adjoint diagonal operator:

$$\hat{a}\,|j, m\rangle := -\biggl(\sum_{k=-j+1}^{m} \frac{1}{\sqrt{(j+k)(j-k+1)}}\biggr)|j, m\rangle. \tag{4.6}$$

The coefficients are chosen so that $[E, \hat{a}]\,|j, m\rangle = |j, m+1\rangle$ for $m = -j, \dots, j-1$. Notice that $\hat{a}\,|j, -j\rangle = 0$ and $[E, \hat{a}]\,|j, -j\rangle = 0$. Since $\hat{a} = \hat{a}^*$, Lemma 4.1 then shows that $\|[D_N, \hat{a}]\| = \|[E, \hat{a}]\| = 1$. Therefore,

$$d_N(\omega_m, \omega_n) \geq \omega_m(\hat{a}) - \omega_n(\hat{a}) = \sum_{k=m+1}^{n} \frac{1}{\sqrt{(j+k)(j-k+1)}}\,. \qquad ∎$$

Note that the distance is additive on the chain of vector states $\{\omega_m\}$:

$$d_N(\omega_m, \omega_n) = \sum_{k=m+1}^{n} d_N(\omega_{k-1}, \omega_k).$$

As a corollary, we get the distance between the north and south poles of the sphere.

Proposition 4.5. *For any N, the diameter of the fuzzy sphere is:*

$$d_N(\psi^N_{(0,0)}, \psi^N_{(0,\pi)}) = \sum_{k=1}^{N} \frac{1}{\sqrt{k(N-k+1)}} \,. \tag{4.7}$$

Proof. By construction, the Bloch state vectors at the poles are basis vectors: $|0,0)_N = |j,-j\rangle$ and $|0,\pi)_N = |j,j\rangle$. Therefore, $\psi^N_{(0,0)} = \omega_{-j}$ and $\psi^N_{(0,\pi)} = \omega_j$. From (4.5) we get (4.7), under the guise:

$$d_N(\omega_{-j}, \omega_j) = \sum_{m=-j+1}^{j} \frac{1}{\sqrt{(j+m)(j-m+1)}} \,. \qquad \blacksquare$$

4.3 An auxiliary distance

Let $\mathcal{B}_N \subset \mathcal{A}_N$ be the subalgebra of *diagonal* matrices. Note that if a is diagonal, then $\psi^N_{(\varphi,\theta)}(a) = \psi^N_{(0,\theta)}(a)$ for any φ. Define

$$\rho_N(\theta) := \sup_{a=a^* \in \mathcal{B}_N} \big\{ \big|\psi^N_{(0,\theta)}(a) - \psi^N_{(0,0)}(a)\big| : \|[D_N, a]\| \leq 1 \big\}. \tag{4.8}$$

The distance ρ_N can be easily computed.

Proposition 4.6. *For any $0 \leq \theta \leq \pi$, $\rho_N(\theta)$ is given by:*

$$\rho_N(\theta) = \sum_{n=1}^{N} \binom{N}{n} (\sin\tfrac{\theta}{2})^{2n} (\cos\tfrac{\theta}{2})^{2(N-n)} \sum_{k=1}^{n} \frac{1}{\sqrt{k(N-k+1)}} \,. \tag{4.9}$$

Proof. Let $a = (\delta_{mn} c_m) \in \mathcal{B}_N$, with $c_m \in \mathbb{R}$. Then

$$\begin{aligned} \psi^N_{(0,0)}(a) - \psi^N_{(0,\theta)}(a) &= \sum_{m=-j}^{j} \binom{2j}{j+m} (\sin\tfrac{\theta}{2})^{2(j+m)} (\cos\tfrac{\theta}{2})^{2(j-m)} \, (c_{-j} - c_m) \\ &= \sum_{m=-j}^{j} \binom{2j}{j+m} (\sin\tfrac{\theta}{2})^{2(j+m)} (\cos\tfrac{\theta}{2})^{2(j-m)} (\omega_{-j}(a) - \omega_m(a)), \end{aligned}$$

where the ω_m are the vector states in Prop. 4.4. We know that

$$\omega_{-j}(a) - \omega_m(a) \leq d_N(\omega_m, \omega_{-j}) = \sum_{m'=-j+1}^{m} \frac{1}{\sqrt{(j+m')(j-m'+1)}}$$

for all a with $\|[D_N, a]\| \leq 1$, with the supremum saturated on the diagonal element $\hat{a}$ given by (4.6). On substituting $n = j+m$ and $k = j+m'$, we arrive at (4.9). $\blacksquare$

Lemma 4.7. *The derivative $\rho'_N(\theta)$ of (4.9) satisfies $0 \leq \rho'_N(\theta) \leq 1$.*

Proof. From (4.3b) we deduce that $\psi^N_{(0,\theta)}([E,a]) = \frac{\partial}{\partial\theta}\psi^N_{(0,\theta)}(a)$ for all $a \in \mathcal{B}_N$. Using this relation and the equality $\rho_N(\theta) = \psi^N_{(0,\theta)}(\hat{a}) - \psi^N_{(0,0)}(\hat{a})$, with $\hat{a}$ the element in (4.6), we get:

$$\rho'_N(\theta) = \frac{\partial}{\partial\theta}\psi^N_{(0,\theta)}(\hat{a}) = \psi^N_{(0,\theta)}([E,\hat{a}]).$$

Since states are functionals with norm 1, it follows that

$$|\rho'_N(\theta)| = |\psi^N_{(0,\theta)}([E,\hat{a}])| \leq \psi^N_{(0,\theta)}(1)\,\|[E,\hat{a}]\| = 1.$$

On the other hand, since $L := [E,\hat{a}]$ is the ladder operator $|j,m\rangle \mapsto |j,m+1\rangle$, we may compute

$$\begin{aligned}\rho'_N(\theta) &= \psi^N_{(0,\theta)}(L) = (\varphi,\theta \mid L \mid \varphi,\theta)_N \\ &= \sum_{m=-j}^{j-1} \binom{2j}{j+m}^{\frac{1}{2}} \binom{2j}{j+m+1}^{\frac{1}{2}} (\sin\tfrac{\theta}{2})^{2j+2m+1}(\cos\tfrac{\theta}{2})^{2j-2m-1} \geq 0.\end{aligned}$$

Indeed, $\rho'_N(\theta) > 0$ for $0 < \theta < \pi$. ■

The previous lemma has two consequences. First of all, $\rho_N(\theta)$ is strictly increasing on $0 \leq \theta \leq \pi$, for fixed N. Secondly, for $0 < \theta \leq \pi$ the mean value theorem gives ϕ with $0 < \phi < \theta$ such that

$$\rho_N(\theta) = \rho_N(\theta) - \rho_N(0) = \theta\,\rho'_N(\phi) \leq \theta.$$

That is: $\rho_N(\theta)$ is no greater than the geodesic distance on the circle.

5 Spectral distance between coherent states

5.1 $SU(2)$-invariance of the distance

Lemma 5.1. *The distance function $d_N(\psi^N_{(\varphi,\theta)}, \psi^N_{(\varphi',\theta')})$ is $SU(2)$-invariant.*

Proof. Up to now, we have identified the element $a \in \mathcal{A}_N \simeq \mathrm{End}(V_j)$ with the operator $a \otimes 1_2$ acting on $\mathcal{H}_N = V_j \otimes V_{\frac{1}{2}}$, by a slight abuse of notation. In this proof, we shall write explicitly $a \otimes 1_2$ to avoid ambiguities.

For any $g \in SU(2)$ and $a \in \mathcal{A}_N$, we write $a^g := \pi_j(g) a \pi_j(g)^*$. Since $\pi_{\frac{1}{2}}(g)\pi_{\frac{1}{2}}(g)^* = 1_2$ by unitarity of $\pi_{\frac{1}{2}}$, we get

$$a^g \otimes 1_2 = u(a \otimes 1_2)u^* \quad \text{where} \quad u := \pi_j(g) \otimes \pi_{\frac{1}{2}}(g).$$

Since D_N commutes with u, the operator $[D_N, a^g \otimes 1_2] = u[D_N, a \otimes 1_2]u^*$ has the same norm as $[D_N, a \otimes 1_2]$.

Given a state ω on $\mathcal{A}_N$ and $g \in SU(2)$, let $g_*\omega$ be the state defined by $g_*\omega(a) = \omega(a^g)$. For any pair of states ω, ω', we then obtain

$$\begin{aligned} d_N(g_*\omega, g_*\omega') &= \sup_{a \in \mathcal{A}_N} \{\, |\omega(a^g) - \omega'(a^g)| : \|[D_N, a \otimes 1_2]\| \leq 1 \,\} \\ &= \sup_{b = a^g \in \mathcal{A}_N} \{\, |\omega(b) - \omega'(b)| : \|[D_N, b \otimes 1_2]\| \leq 1 \,\} = d_N(\omega,\omega'), \end{aligned}$$

where we used $\|[D_N, b \otimes 1_2]\| = \|[D_N, a \otimes 1_2]\|$.

By construction, the action $\psi^N_{(\varphi,\theta)} \mapsto g_*\psi^N_{(\varphi,\theta)}$ corresponds to the usual rotation action of $SU(2)$ on $\mathbb{S}^2$. ■

5.2 Dependence on the dimension

In this section we prove that, for each pair of points $(\varphi,\theta),(\varphi',\theta')\in\mathbb{S}^2$, the distance $d_N(\psi^N_{(\varphi,\theta)},\psi^N_{(\varphi',\theta')})$ is non-decreasing with $N=2j$.

Using the fuzzy spinor basis (3.4), one defines injections $U_j^\pm\colon V_{j\pm\frac12}\to V_j\otimes V_{\frac12}$ by

$$U_j^+|j+\tfrac12,m+\tfrac12\rangle := |j,m\rangle\!\rangle_+\,,\qquad U_j^-|j-\tfrac12,m+\tfrac12\rangle := |j,m\rangle\!\rangle_-\,,$$

using the same index sets as in (3.4), namely $m=-j-1,\dots,j$ for the range of U_j^+ and $m=-j,\dots,j-1$ for the range of U_j^-. One easily checks that these $U_j^\pm$ are isometries, i.e., $(U_j^\pm)^*U_j^\pm=1$, that intertwine the representations of $\mathfrak{su}(2)$. Also, $V_j\otimes V_{\frac12}$ is the orthogonal direct sum of the ranges of U_j^+ and U_j^-.

Lemma 5.2. *For any $(\varphi,\theta)\in\mathbb{S}^2$,*

$$U_j^+|\varphi,\theta)_{N+1}=|\varphi,\theta)_N\otimes|\varphi,\theta)_1\,.$$

Proof. Note that $|\varphi,\theta)_1=e^{-\frac12 i\varphi}\sin\frac{\theta}{2}\,|\frac12,-\frac12\rangle+e^{\frac12 i\varphi}\cos\frac{\theta}{2}\,|\frac12,\frac12\rangle$. The rest is an easy computation, using (4.2). ∎

We define two injective linear maps

$$\eta_N^\pm:\mathcal{A}_N\to\mathcal{A}_{N\pm1}\,,\qquad \eta_N^\pm(a):=(U_j^\pm)^*(a\otimes 1_2)U_j^\pm\,.$$

They are unital and commute with the involution, but they are neither surjective nor algebra morphisms, since $U_j^+(U_j^+)^*+U_j^-(U_j^-)^*=1$. They are, however, norm decreasing: the norm of $a\otimes 1_2$ on the range of $U_j^\pm$ is no greater than its norm on $V_j\otimes V_{\frac12}$, which equals the norm of a on V_j.

Lemma 5.3. *For any $a\in\mathcal{A}_N$,*

$$\psi^{N+1}_{(\varphi,\theta)}\circ\eta_N^+(a)=\psi^N_{(\varphi,\theta)}(a),\tag{5.1}$$

and

$$\big\|[D_{N\pm1},\eta_N^\pm(a)]\big\|\le\big\|[D_N,a]\big\|.\tag{5.2}$$

Proof. Lemma 5.2 implies the equality

$$(\varphi,\theta\mid\eta_N^+(a)\mid\varphi,\theta)_{N+1}=(\varphi,\theta\mid a\mid\varphi,\theta)_N\,(\varphi,\theta\mid\varphi,\theta)_1=(\varphi,\theta\mid a\mid\varphi,\theta)_N\,,$$

which is exactly the relation (5.1).

Since $U_j^\pm$ intertwines representations of $\mathfrak{su}(2)$, i.e.,

$$U_j^\pm X=(X\otimes 1_2+1_2\otimes X)U_j^\pm\quad\text{for all}\quad X\in\mathfrak{su}(2)$$

(the representation symbols are omitted), we conclude that

$$[X,\eta_N^\pm(a)]=(U_j^\pm)^*\big([X,a]\otimes 1_2\big)U_j^\pm=\eta_N^\pm\big([X,a]\big).$$

Therefore, in view of (4.1),

$$[D_{N\pm1}, \eta_N^\pm(a)] = \eta_N^\pm\big([D_N, a]\big),$$

where $[D_N, a] \in M_2(\mathcal{A}_N)$ and we extend $\eta_N^\pm$ from $\mathcal{A}_N$ to $M_2(\mathcal{A}_N)$ by applying it to each matrix entry. Since both $\eta_N^\pm$ are norm-decreasing maps, this proves (5.2). ■

Proposition 5.4. *For any $N \geq 1$, the following majorization holds:*

$$d_{N+1}(\psi^{N+1}_{(\varphi,\theta)}, \psi^{N+1}_{(\varphi',\theta')}) \geq d_N(\psi^N_{(\varphi,\theta)}, \psi^N_{(\varphi',\theta')}).$$

Proof. We get directly:

$$\begin{aligned}
d_{N+1}(\psi^{N+1}_{(\varphi,\theta)}, \psi^{N+1}_{(\varphi',\theta')}) &= \sup_{a\in\mathcal{A}_{N+1}} \big\{ \big|\psi^{N+1}_{(\varphi,\theta)}(a) - \psi^{N+1}_{(\varphi',\theta')}(a)\big| : \|[D_{N+1}, a]\| \leq 1 \big\} \\
&\geq \sup_{a\in\mathcal{A}_N} \big\{ \big|\psi^{N+1}_{(\varphi,\theta)} \circ \eta_N^+(a) - \psi^{N+1}_{(\varphi',\theta')} \circ \eta_N^+(a)\big| : \big\|[D_{N+1}, \eta_N^+(a)]\big\| \leq 1 \big\} \\
&= \sup_{a\in\mathcal{A}_N} \big\{ \big|\psi^N_{(\varphi,\theta)}(a) - \psi^N_{(\varphi',\theta')}(a)\big| : \big\|[D_{N+1}, \eta_N^+(a)]\big\| \leq 1 \big\} \\
&\geq \sup_{a\in\mathcal{A}_N} \big\{ \big|\psi^N_{(\varphi,\theta)}(a) - \psi^N_{(\varphi',\theta')}(a)\big| : \|[D_N, a]\| \leq 1 \big\} \\
&= d_N(\psi^N_{(\varphi,\theta)}, \psi^N_{(\varphi',\theta')}) \,.
\end{aligned}$$

The first inequality follows since the supremum over the range of η_N^+ in $\mathcal{A}_{N+1}$ is smaller than the supremum over the whole $\mathcal{A}_{N+1}$. In the next line we used (5.1); and then we get the final lower bound from (5.2). ■

Note that a similar proof with U_j^- would not work, since analogues of Lemma 5.2 or of the relation (5.1) are not available. Such a result would in any case make the distance independent of N, contrary to the diameter calculation (4.7).

Remark 5.5. The calculation in the proof of Prop. 5.4 can be adapted to establish that

$$\rho_{N+1}(\theta - \theta') \geq \rho_N(\theta - \theta'), \quad \text{for} \quad \theta, \theta' \in [0, \pi]. \tag{5.3}$$

For that, just restrict $a \in \mathcal{A}_N$ to (be self-adjoint and) lie in the diagonal subalgebra $\mathcal{B}_N$. The only thing to note that is that η_N^+ maps $\mathcal{B}_N$ into a non-diagonal subalgebra of $\mathcal{A}_{N+1}$; but the notion of diagonal subalgebra is in any case basis-dependent. It is enough to replace $\mathcal{B}_{N+1}$ by a conjugate subalgebra that includes $\eta_N^+(\mathcal{B}_N)$, after conjugating $\mathcal{A}_{N+1}$ by a unitary operator commuting with the $SU(2)$ action via $\operatorname{ad}\pi_{j+\frac12}$. This rotates the basis vectors in $V_{j+\frac12}$, in such a way that the coherent states $\psi^{N+1}_{(\varphi,\theta)}$ are unchanged. Thus also, $\rho_{N+1}(\theta - \theta')$ is unchanged, and (5.3) holds.

5.3 Upper and lower bounds and the large N limit

Proposition 5.6. *The following inequalities hold, for all $(\varphi, \theta), (\varphi', \theta') \in \mathbb{S}^2$:*

$$\rho_N(\theta - \theta') \leq d_N(\psi^N_{(\varphi,\theta)}, \psi^N_{(\varphi',\theta')}) \leq d_{\mathrm{geo}}\big((\varphi, \theta), (\varphi', \theta')\big), \tag{5.4}$$

where $\rho_N(\theta)$ is the auxiliary distance (4.8) *and* $d_{\rm geo}$ *is the geodesic distance for the round metric of* $\mathbb{S}^2$*. In particular,*

$$\rho_N(\theta) \leq d_N(\psi^N_{(0,\theta)}, \psi^N_{(0,0)}) \leq \theta. \tag{5.5}$$

Proof. Due to Lemma 5.1, the second inequality in (5.4) involves two $SU(2)$-invariant expressions. It is then enough to prove it when $(\varphi', \theta') = (0, \frac{\pi}{2})$ and $(\varphi, \theta) = (\varphi, \frac{\pi}{2})$. We thus need to prove that

$$d_N(\psi^N_{(\varphi,\frac{\pi}{2})}, \psi^N_{(0,\frac{\pi}{2})}) \leq |\varphi| \quad \text{for all} \quad -\pi < \varphi \leq \pi.$$

Integrating (4.3a), we find

$$\psi^N_{(\varphi,\frac{\pi}{2})}(a) - \psi^N_{(0,\frac{\pi}{2})}(a) = \mathrm{i} \int_0^\varphi \psi^N_{(\alpha,\frac{\pi}{2})}([H,a])\, d\alpha\,,$$

and since $|\omega(A)| \leq \|A\|$ for any state ω and operator A, we obtain, using Lemma 4.1:

$$|\psi^N_{(\varphi,\frac{\pi}{2})}(a) - \psi^N_{(0,\frac{\pi}{2})}(a)| \leq \|[H,a]\| \left| \int_0^\varphi d\alpha \right| = |\varphi|\, \|[H,a]\| \leq |\varphi|\, \|[D_N,a]\|.$$

This proves the upper bound in (5.4). That of (5.5) follows from $d_{\rm geo}\big((0,\theta),(0,0)\big) = \theta$.

A lower bound for the distance is given by the supremum over diagonal matrices:

$$d_N(\psi^N_{(\varphi,\theta)}, \psi^N_{(\varphi',\theta')}) \geq \sup_{a=a^*\in\mathcal{B}_N} \big\{ \big|\psi^N_{(\varphi,\theta)}(a) - \psi^N_{(\varphi',\theta')}(a)\big| : \|[D_N,a]\| \leq 1 \big\}.$$

Since $\psi^N_{(\varphi,\theta)}(a)$ is independent of φ for any diagonal a, we arrive at

$$d_N(\psi^N_{(\varphi,\theta)}, \psi^N_{(\varphi',\theta')}) \geq \sup_{a=a^*\in\mathcal{B}_N} \big\{ \big|\psi^N_{(0,\theta)}(a) - \psi^N_{(0,\theta')}(a)\big| : \|[D_N,a]\| \leq 1 \big\} = \rho_N(\theta - \theta'). \qquad \blacksquare$$

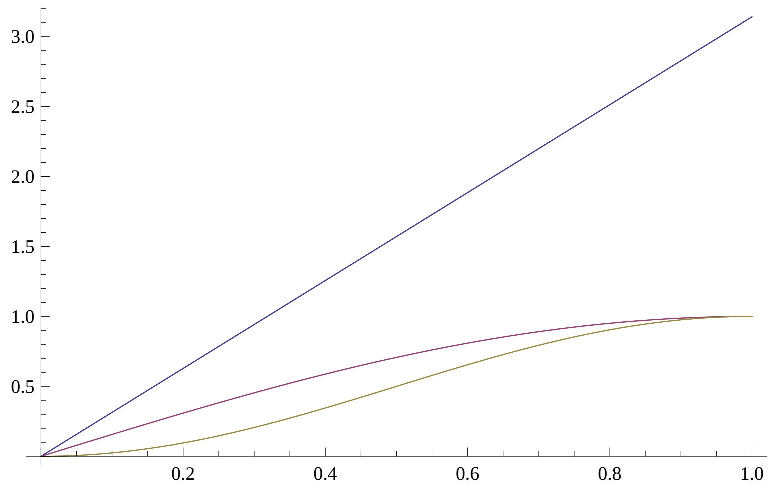
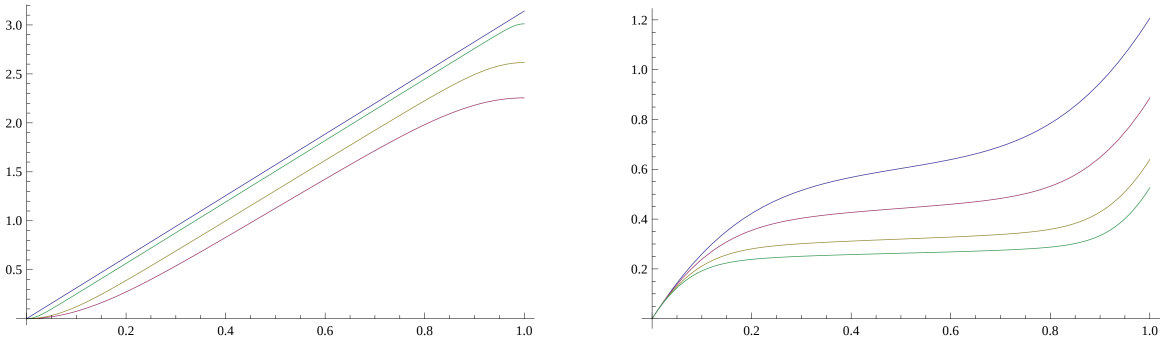


Figure 1: Plot of $\rho_1(\theta)$, $d_1(\psi^1_{(0,\theta)}, \psi^1_{(0,0)})$ and $d_{\rm geo}\big((0,\theta),(0,0)\big)$. The abscissa is $x = \pi\theta$.

Figure 1 contains a plot of the geodesic distance (blue), compared to the spectral distance d_1 (pink) and to the auxiliary distance ρ_1 (yellow). Note that for $0 < \theta < \pi$ neither the upper nor the lower bound is optimal.



(a) Plot of θ and $\rho_N(\theta)$ for $N = 10, 30, 500$. (b) Plot of $\theta - \rho_N(\theta)$ for $N = 5, 10, 20, 30$.

Figure 2: Comparison of $\rho(\theta)$ with θ. The abscissa is $x = \pi\theta$.

On the other hand, $d_N(\psi^N_{(0,\pi)}, \psi^N_{(0,0)}) = \rho_N(\pi)$, since the formula (4.9) coincides with (4.7) when $\theta = \pi$. Thus the lower bound is sharp for $\theta = \pi$ (as can be seen in Figure 1 for $N = 1$).

In Figure 2a we show a plot of the upper bound (blue) and the lower bound ρ_N for $N = 10, 30, 500$ (it is non-decreasing with N). We want to prove that the lower bound is uniformly convergent to the upper bound, for $N \to \infty$. The figure indicates that $\theta - \rho_N(\theta)$ has its maximum at $\theta = \pi$. Figure 2b plots $\theta - \rho_N(\theta)$ for $N = 5, 10, 20, 30$ (again, non-decreasing with N). This suggests how to prove the next proposition.

Proposition 5.7. *As $N \to \infty$, the sequence $\rho_N(\theta)$ is uniformly convergent to θ in $[0, \pi]$. In consequence,*

$$\lim_{N\to\infty} d_N(\psi^N_{(\varphi,\theta)}, \psi^N_{(\varphi',\theta')}) = d_{\text{geo}}\big((\varphi, \theta), (\varphi', \theta')\big).$$

Proof. Let $f_N(\theta) := \theta - \rho_N(\theta)$. Clearly $f_N(0) = 0$, and $f'_N(\theta) \geq 0$ by Lemma 4.7. Hence $f_N(\theta)$ is a non-decreasing positive function for each N, and

$$\|\theta - \rho_N(\theta)\|_\infty = \sup_{\theta\in[0,\pi]} f_N(\theta) \leq f_N(\pi) = \pi - \rho_N(\pi).$$

Therefore, the uniform convergence $\lim_{N\to\infty} \|\theta - \rho_N(\theta)\|_\infty = 0$ holds if and only if the diameter converges to π, i.e., $\lim_{N\to\infty} \rho_N(\pi) = \pi$.

The formula for $\rho_N(\pi)$ is given by (4.7). The sequence $\rho_N(\pi)$ is bounded, $\rho_N(\pi) \leq \pi$, and is non-decreasing by Remark 5.5. Hence it is convergent, and the limit can be computed using any subsequence. Therefore, it is then enough to prove that $\rho_N(\pi) \geq c_N$, where $c_N \to \pi$ as $N \to \infty$.

We consider the subsequence with odd N only. The function $(x(N - x + 1))^{-1/2}$ is positive for $1 < x < N+1$, symmetric about $x = \frac{1}{2}(N+1)$, and monotonically decreasing for $1 \leq x \leq \frac{1}{2}(N+1)$. Hence

$$\rho_N(\pi) = 2 \sum_{k=1}^{\frac{1}{2}(N-1)} \frac{1}{\sqrt{k(N-k+1)}} + \frac{2}{N+1} \geq 2 \int_1^{\frac{1}{2}(N+1)} \frac{dx}{\sqrt{x(N-x+1)}}\,.$$

Substituting $x =: \frac{1}{2}(N+1)(1+\sin\xi)$, so that $d\xi = dx/\sqrt{x(N-x+1)}$, we obtain

$$\rho_N(\pi) \geq 2 \arcsin \frac{N-1}{N+1}\,.$$

The right hand side converges monotonically to π as $N \to \infty$, thus $\lim_{N\to\infty} \rho_N(\pi) = \pi$ through odd N, and so, as noted above, through all N. (A slightly modified estimate gives $\lim_{N\to\infty} \rho_N(\pi) = \pi$ through even N, directly, without using Remark 5.5.) This proves the uniform convergence $\rho_N(\theta) \to \theta$.

The estimate (5.5) now shows that $d_N(\psi^N_{(0,\theta)}, \psi^N_{(0,0)})$ is uniformly convergent to θ, and by $SU(2)$-invariance $d_N(\psi^N_{(\varphi,\theta)}, \psi^N_{(\varphi',\theta')})$ converges to $d_{\mathrm{geo}}\big((\varphi,\theta),(\varphi',\theta')\big)$ uniformly on $\mathbb{S}^2$. ∎

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A Basics on $\mathcal{U}(\mathfrak{su}(2))$

In order to fix notations, we recall some basic facts about the algebra $\mathcal{U}(\mathfrak{su}(2))$.

We denote by J_1, J_2, J_3 its real generators, satisfying

$$[J_k, J_l] = \mathrm{i} \textstyle\sum_m \varepsilon_{klm} J_m$$

where ε_{klm} is the Levi-Civita symbol. The Cartan generators are $H = J_3$, $E = J_1 + \mathrm{i}J_2$, $F = E^* = J_1 - \mathrm{i}J_2$; they satisfy

$$[H,E] = E, \qquad [H,F] = -F, \qquad [E,F] = 2H.$$

For each $j \in \frac{1}{2}\mathbb{N}$, let (V_j, π_j) be the spin-j irreducible representation of $SU(2)$, of dimension $2j+1$. With a slight abuse of notation, we will denote by the same name the corresponding representation of $\mathcal{U}(\mathfrak{su}(2))$. Denoting by $\{\, |j,m\rangle : m = -j, -j+1, \dots, j \,\}$ the standard orthonormal basis of V_j, the representation π_j is given by

$$\begin{aligned}
\pi_j(H)|j,m\rangle &= m|j,m\rangle, \\
\pi_j(E)|j,m\rangle &= \sqrt{(j-m)(j+m+1)}\,|j,m+1\rangle, \\
\pi_j(F)|j,m\rangle &= \sqrt{(j+m)(j-m+1)}\,|j,m-1\rangle.
\end{aligned}$$

The Casimir element is

$$J^2 = \textstyle\sum_k J_k^2 = H(H+1) + FE = H(H-1) + EF,$$

and $\pi_j(J^2)$ is $j(j+1)$ times the identity operator on V_j. The spin-$\frac{1}{2}$ representation gives an algebra homomorphism

$$\pi_{\frac{1}{2}} : \mathcal{U}(\mathfrak{su}(2)) \to \mathrm{C}\ell_{20} \simeq M_2(\mathbb{C})$$

and in matrix notations, under the identifications

$$|\tfrac{1}{2}, +\tfrac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

we get immediately $\pi_{\frac{1}{2}}(J_k) = \frac{1}{2}\,\sigma_k$, where the σ_k are the three Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The universal enveloping algebra $\mathcal{U}(\mathfrak{su}(2))$ is a Hopf $*$-algebra, with coproduct Δ defined by declaring the elements $X \in \mathfrak{su}(2)$ to be primitive: $\Delta(X) := X \otimes 1 + 1 \otimes X$. For the antipode S, it follows that $S(X) = -X$. We shall use Sweedler notation for the coproduct, $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for any $h \in \mathcal{U}(\mathfrak{su}(2))$, with summation understood.

Note in particular, since Δ is an algebra homomorphism, that

$$\Delta(J^2) = \Delta\bigl(\textstyle\sum_k J_k^2\bigr) = \sum_k \Delta(J_k)^2 = \sum_k (J_k \otimes 1 + 1 \otimes J_k)^2. \tag{A.1}$$

References

[1] G. S. Agarwal: Relation between atomic coherent-state representation, state multipoles, and generalized phase-space distributions. Phys. Rev. A **24**, 2889–2896 (1981).

[2] J.-P. Amiet and M. B. Cibils: Description of quantum spin using functions on the sphere $\mathbb{S}^2$. J. Phys. A **24**, 1515–1535 (1991).

[3] A. A. Andrianov and F. Lizzi: Bosonic spectral action induced from anomaly cancellation. JHEP **1005**, 057 (2010).

[4] A. A. Andrianov, M.A. Kurkov and F. Lizzi: Spectral action, Weyl anomaly and the Higgs-dilaton potential. JHEP **1110**, 001 (2011).

[5] F. T. Arecchi, E. Courtens, R. Gilmore and H. Thomas: Atomic coherent states in quantum optics. Phys. Rev. A **6**, 2211–2237 (1972).

[6] A. P. Balachandran, S. Kürkçüoğlu and S. Vaidya: *Lectures on Fuzzy and Fuzzy SUSY Physics*. World Scientific, Singapore, 2007.

[7] A. P. Balachandran and P. Padmanabhan: Spin j Dirac operators on the fuzzy 2-sphere. JHEP **0909**, 120 (2009).

[8] L. C. Biedenharn and J. D. Louck: *Angular Momentum in Quantum Physics: Theory and Applications*, Addison-Wesley, Reading, MA, 1981.

[9] C. Brif and A. Mann: A general theory of phase-space quasiprobability distributions. J. Phys. A: Math. Gen. **31**, L9–L17 (1998).

[10] U. Carow-Watamura and S. Watamura: Chirality and Dirac operator on noncommutative sphere. Commun. Math. Phys. **183**, 365–382 (1997).

[11] U. Carow-Watamura and S. Watamura: Differential calculus on fuzzy sphere and scalar field. Int. J. Mod. Phys. A**13**, 3235–3244 (1998).

[12] C. S. Chu, J. Madore and H. Steinacker: Scaling limits of the fuzzy sphere at one loop. JHEP **0108**, 038 (2001).

[13] A. Connes: Compact metric spaces, Fredholm modules, and hyperfiniteness. Ergodic Thy. Dynam. Sys. **9**, 207–220 (1989).

[14] A. Connes: *Noncommutative Geometry.* Academic Press, London and San Diego, 1994.

[15] A. Connes: Noncommutative geometry and reality. J. Math. Phys. **36**, 6194–6231 (1995).

[16] E. Cagnache, F. D'Andrea, P. Martinetti and J.-C. Wallet: The spectral distance on the Moyal plane. J. Geom. Phys. **61**, 1881–1897 (2011).

[17] T. Friedrich: *Dirac Operators in Riemannian Geometry.* Amer. Math. Soc., Providence, RI, 2000.

[18] J. M. Gracia-Bondía, J. C. Várilly and H. Figueroa: *Elements of Noncommutative Geometry.* Birkhäuser, Boston, 2001.

[19] M. Gromov: *Metric Structures for Riemannian and non-Riemannian Spaces.* Modern Birkhäuser Classics (Birkhäuser, Boston, 2007). Based on the 1981 French original.

[20] H. Grosse, C. Klimčík and P. Prešnajder: Field theory on a supersymmetric lattice. Commun. Math. Phys. **185**, 155–175 (1997).

[21] E. Harikumar, A. R. Queiroz and P. Teotonio-Sobrinho: Dirac operator on the q-deformed fuzzy sphere and its spectrum. JHEP **0609**, 037 (2006).

[22] J-S. Huang and P. Pandžić: *Dirac Operators in Representation Theory.* Birkhäuser, Boston, 2006.

[23] B. Iochum, T. Krajewski and P. Martinetti: Distances in finite spaces from noncommutative geometry. J. Geom. Phys. **37**, 100–125 (2001).

[24] A. B. Klimov and P. Espinoza: Moyal-like form of the star product for generalized $SU(2)$ Stratonovich–Weyl symbols. J. Phys. A **35**, 8435–8447 (2002).

[25] B. Kostant: A cubic Dirac operator and the emergence of Euler number multiplets of representations for equal-rank subgroups. Duke Math. J. **100**, 447–501 (1999).

[26] J. Madore: The fuzzy sphere. Class. Quant. Grav. **9**, 69–87 (1992).

[27] P. Martinetti and L. Tomassini: Noncommutative geometry of the Moyal plane: translation isometries and spectral distance between coherent states. arXiv:1110.6164v3 [math-ph].

[28] A. M. Perelomov: *Generalized Coherent States and their Applications.* Springer, Berlin, 1986.

[29] M. A. Rieffel: Metrics on state spaces. Doc. Math. **4**, 559–600 (1999).

[30] M. A. Rieffel: Compact quantum metric spaces. In *Operator Algebras, Quantization, and Noncommutative Geometry*, R. S. Doran and R. V. Kadison, eds. Amer. Math. Soc., Providence, RI, 2004; pp. 315–330.

[31] M. A. Rieffel: Gromov-Hausdorff distance for quantum metric spaces. Memoirs Amer. Math. Soc. **168**, 1–65 (2004).

[32] M. A. Rieffel: Matrix algebras converge to the sphere for quantum Gromov–Hausdorff distance. Memoirs Amer. Math. Soc. **168**, 67–91 (2004).

[33] M. A. Rieffel: Leibniz seminorms for 'Matrix algebras converge to the sphere'. In *Quanta of Maths*, E. Blanchard et al, eds. Amer. Math. Soc., Providence, RI, 2010; pp. 543–578.

[34] A. Sitarz: Equivariant spectral triples. In *Noncommutative Geometry and Quantum Groups*, Banach Centre Publ. **61**, Warsaw, (2003); pp. 231–263.

[35] J. C. Várilly and J. M. Gracia-Bondía: The Moyal representation for spin. Ann. Phys. **190**, 107–148 (1989).

[36] J.-C. Wallet: Connes distance by examples: Homothetic spectral metric spaces. arXiv: 1112.3285v1 [math-ph].