An Introduction to Financial Mathematics in Continuous Time

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2007

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Introduction

In stochastic analysis in continuous time one usually considers $\mathbb{R}_+$ instead of $\mathbb{N}$ as the index set describing time, i.e. at every time point $t \in \mathbb{R}$ one observes a random variable X_t. Whereas in discrete time one is dealing with random sequences $X(t) : \mathbb{N} \to \mathbb{R}$, in continuous time one is working with stochastic functions $X(t) : \mathbb{R}_+ \to \mathbb{R}$. Though many results obtained in discrete time have a continuous time analogue we have to modify several notions and results. Stochastic calculus in continuous time is a very sophisticated area and usually it requires an extensive treatment. Here we want to summarise the most important results (mostly without any proofs) in order to be able to apply them to continuous time financial mathematics later.

1 Stochastic Calculus

1.1 Stochastic Processes

Modern probability theory has become the natural language for formulating quantitative models of financial markets. This chapter presents, in the form of a crash course, some of its tools and concepts that will be important for us in the sequel.

Recall that in discrete time we modeled the flow of information by a filtration. In continuous time we need filtrations which are right-continuous. One calls this the usual hypothesis. Right-continuity has the effect that there exists no event for which we cannot determine whether it has occured at time t or not.

Definition 1.1 *Let* $(\Omega, \mathcal{F}, P)$ *denote a probability space. A* filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ *is an increasing, right-continuous family of* σ*-algebras* $\mathcal{F}_t \subset \mathcal{F}$*, i.e.* $\mathcal{F}_s \subset \mathcal{F}_t$ *for* $s \leq t$ *and* $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$ *for* $t \geq 0$*. We call* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$ *a filtered probability space.*

From now on we assume that we are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$. This is also a key concept in financial theory: it is the flow of information available on a financial market and underlying the traders' decisions.

Many of the stochastic processes we will consider later will satisfy some specified continuity properties. Usually we will assume that the sample paths $X(\omega) : \mathbb{R}_+ \to \mathbb{R}$ are càdlàg in the sense of the following definition.

Definition 1.2

1. *A* stochastic process $X = \{X_t\}_{t\in\mathbb{R}_+}$ *is a family of* $\mathbb{R}^d$*-valued random variables.*

2. *A function* $f : \mathbb{R}_+ \to \mathbb{R}^d$ *is said to be càdlàg (continu à droite avec des limites à gauche), if it is right-continuous with left limits, i.e. the limits* $f(t-) = \lim_{s \uparrow t} f(s)$ *and* $f(t+) = \lim_{s \downarrow t} f(s)$ *exist for every* $t > 0$ *and* $f(t) = f(t+)$.

3. *We call a process* X *continuous, càdlàg, left-continuous etc., if almost every sample path* $X(\omega) : \mathbb{R}_+ \to \mathbb{R}^d$ *has this property.*

4. *A process* X *is* adapted *if* X_t *is* $\mathcal{F}_t$*-measurable for all* $t \in \mathbb{R}_+$.

Remark 1.3 Obviously, any continuous function is càdlàg but càdlàg functions can have discontinuities. If t is a discontinuity point we denote by

$$\Delta f(t) = f(t) - f(t-)$$

the *jump* of f at t.

Notation 1.4 *For a càdlàg process* X *we define*

$$X_{t-} := \lim_{s \uparrow t} X_s \quad \text{for} \quad t > 0,$$

$$X_{0-} := 0,$$

$$\Delta X_t = X_t - X_{t-} \quad \text{for} \quad t \geq 0.$$

For σ*-algebras we write*

$$\mathcal{F}_{t-} := \sigma(\cup_{s<t} \mathcal{F}_s) \quad \text{for} \quad t > 0 \quad \text{and} \quad \mathcal{F}_{0-} := \mathcal{F}_0.$$

In probability theory many results are only true outside null sets and one identifies versions of random variables, which take different values only on null sets. In the theory of stochastic processes one works with the notions of evanescent sets and indistinguishable processes.

Definition 1.5 *A set* $A \subset \Omega \times \mathbb{R}_+$ *is said to be* evanescent *if*

$$\{\omega \in \Omega : \text{ there exists } t \in \mathbb{R}_+ \text{ such that } (\omega, t) \in A\}$$

is a P*-null set. Two processes* X *and* Y *are said to be* indistinguishable *if the set* $\{X \neq Y\} \subset \Omega \times \mathbb{R}_+$ *is evanescent.*

Remark 1.6

1. If two processes X and Y are right-continuous (or left-continuous) and $X_t = Y_t$ almost everywhere for all $t \in \mathbb{R}_+$, then X and Y are indistinguishable.

2. From now on terms like "unique" or "equal" are meant in the sense of indistinguishability.

In discrete time a process X is predictable if $X_t \in \mathcal{F}_{t-1}$. However, the continuous time analogue $X_t \in \mathcal{F}_{t-}$ is not sufficient for our purposes.

Definition 1.7 *The* predictable σ-algebra *is the σ-algebra $\mathcal{P}$ generated on $\Omega \times \mathbb{R}_+$ by all real-valued, adapted and left-continuous processes.*

A mapping $X : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ which is measurable with respect to $\mathcal{P}$ is called a predictable process.

Definition 1.8 *A filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ is said to be generated by a process X if*

$$\mathcal{F}_t = \bigcap_{s>t} \sigma(X_r : \; r \leq s)$$

for all $t \in \mathbb{R}_+$.

We will often deal with events happening at random times. A random time is nothing else than a positive random variable $T \geq 0$ which represents the time at which some event is going to take place. Given an information flow $(\mathcal{F}_t)$, a natural question is whether, given the information in $\mathcal{F}_t$, one can determine if the event has happened $(T \leq t)$ or not $(T > t)$. If the answer is yes, the random time T is called an adapted random time or stopping time. In other words:

Definition 1.9 *A* stopping time *is a mapping $T : \Omega \to \mathbb{R}_+ \cup \{\infty\}$ such that*

$$\{T \leq t\} \in \mathcal{F}_t$$

for all $t \in \mathbb{R}_+$.

Examples of stopping times are hitting times.

Theorem 1.10 *Given an adapted càdlàg process X, the* hitting time *of a Borel set $A \in \mathcal{B}(\mathbb{R}^d)$, that is the first time when X reaches A, i.e.*

$$T_A = \inf\{t \in \mathbb{R}_+ : \; X_t \in A\}$$

is a stopping time.

Definition 1.11 *Given a stopping time T and an adapted process X one can define the process X stopped at T by*

$$X_t^T := X_{T \wedge t}.$$

Definition 1.12 *For stopping times S and T we define stochastic intervals $[[S,T]]$, $]]S,T]]$ etc. as follows*

$$[[S,T]] := \{(\omega, t) \in \Omega \times \mathbb{R}_+ : \; S(\omega) \le t \le T(\omega)\}$$

$$]]S,T]] := \{(\omega, t) \in \Omega \times \mathbb{R}_+ : \; S(\omega) < t \le T(\omega)\}$$

etc.

Given an information flow $(\mathcal{F}_t)$ and a stopping time T, the information set $\mathcal{F}_T$ can be defined as the information obtained by observing all adapted (càdlàg) processes at T, i.e. the σ-algebra generated by these observations, that is $\mathcal{F}_T = \sigma(X_T, \; X \text{ adapted càdlàg process})$. It can be shown that this definition is equivalent to the following

Definition 1.13

$$\mathcal{F}_T := \{A \in \mathcal{F} : \; A \cap \{T \le t\} \in \mathcal{F}_t \; \textit{for all} \; t \in \mathbb{R}_+\}$$

1.2 Martingales

Martingales in continuous time are defined analogously to martingales in discrete time. As in the previous chapter we consider a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration (i.e. an information flow) $(\mathcal{F}_t)$.

Definition 1.14 (Martingale) *A càdlàg process* $\{X_t\}_{t\in\mathbb{R}_+}$ *is said to be a* martingale *if* X *is adapted (with respect to* $\mathcal{F}_t$*),* $E[|X_t|] < \infty$ *for any* $t \in \mathbb{R}_+$ *(i.e.* X *is integrable) and*

$$E[X_t \,|\, \mathcal{F}_s] = X_s, \quad \text{for all } s,t \in \mathbb{R}_+ \text{ such that } s \le t. \tag{1.1}$$

If

$$E[X_t \,|\, \mathcal{F}_s] \ge (or \ \le) X_s, \quad \text{for all } s,t \in \mathbb{R}_+ \text{ such that } s \le t,$$

X *is called a submartingale (or supermartingale).*

Hence, the best prediction of a martingale's future value is its present value. A familiar example of a martingale is Brownian motion $\{B_t\}_{t\ge 0}$. Notice that the definition of a martingale makes sense only when the underlying filtration $(\mathcal{F}_t)_{t\ge 0}$ and the probability measure P have been specified. To avoid confusion we shall sometimes use the term P-martingale to emphasize that the notion of martingale depends on the probability measure P. A typical method to construct a martingale is the following.

Theorem 1.15 *Given a real-valued, integrable random variable* Y *there exists a unique martingale* X *such that*

$$X_t = E[Y \,|\, \mathcal{F}_t]$$

for $t \in \mathbb{R}_+$*. We will refer to* X *as the martingale generated by the process* Y*.*

Example 1.16 *Let* $Q \sim P$ *be an equivalent probability measure. Then the martingale* Z *generated by* $\frac{dQ}{dP}$ *is called the density process of* Q *with respect to* P*.*

An obvious consequence of (1.1) is that a martingale has constant expectation, i.e. $E[X_t] = E[X_0]$ for all $t \in \mathbb{R}_+$. One can wonder whether any driftless process is a martingale. The answer is no. For instance, if $\{B_t\}_{t\ge 0}$ is a Brownian motion, B_t^3 has constant expectation $E[B_t^3] = 0$ but is not a martingale. In fact, if $s < t$

$$\begin{aligned}
E[B_t^3 \,|\, \mathcal{F}_s] &= E[(B_t - B_s + B_s)^3 \,|\, \mathcal{F}_s] \\
&= E[(B_t - B_s)^3 + B_s^3 + 3(B_s - B_t)B_s^2 + 3(B_s - B_t)^2 B_t \,|\, \mathcal{F}_s] \\
&= E[(B_t - B_s)^3] + B_s^3 + 3B_s^2 E[B_s - B_t] + 3E[(B_s - B_t)^2] B_s \\
&= 0 + B_s^3 + 0 + 3(s - t)B_s \ne B_s^3.
\end{aligned}$$

However, if one asks the process to be driftless when computed at random times, then this property actually characterises martingales. Indeed, it can be shown that if $E[X_T] = E[X_0]$ for any stopping time T, then X is a martingale.

A fundamental property of martingales is the sampling property: the martingale property (1.1) is also true when the deterministic times s and t are replaced by stopping times.

Theorem 1.17 (Sampling theorem) *If X is a martingale (or submartingale, supermartingale) and S and T are stopping times with $0 \leq S \leq T$ a.s. then*

$$E[X_T \mid \mathcal{F}_S] = X_S \ (\text{or } \geq, \leq).$$

In particular, the process X^T is a martingale (or submartingale, supermartingale) for arbitrary stopping times T.

In continuous time mathematics of finance a very important technique for proving results is localisation.

Definition 1.18 *We call a class C of stochastic processes stable with respect to stopping if $X^T \in C$ for all $X \in C$ and all stopping times T. The corresponding localised class C_{loc} is then defined to be the set of all processes X for which there exists an increasing sequence $(T_n)_{n\in\mathbb{N}}$ of stopping times such that $X^{T_n} \in C$ for all $n \in \mathbb{N}$ and $T_n \uparrow \infty$ a.s.*

Example 1.19 *In the sense of the above definition the class of* local martingales *belongs to the class of martingales. X is said to be a local martingale, if X^{T_n} is a martingale for a sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ converging a.s. to ∞.*

Unfortunately the set of adapted processes is too large for stochastic analysis, e.g. to define a stochastic integral. Indeed, in stochastic analysis one therefore works with two classes of processes - the martingales and processes of finite total variation. The latter class is very much related to ordinary deterministic analysis. In fact every differentiable function is of finite total variation.

Definition 1.20 *A function $f : \mathbb{R}_+ \to \mathbb{R}$ is said to be of* finite total variation *if it can be represented as the difference of two monotone increasing functions.*

A real-valued càdlàg process X is montone increasing, monotone decreasing or of finite total variation, respectively, if its trajectories $X(\omega) : \mathbb{R}_+ \to \mathbb{R}$ a.s. have this property.

The following decomposition is one of the main results in continuous time stochastic analysis.

Theorem 1.21 (Doob-Meyer decomposition) *Every submartingale (or supermartingale) X has a unique decomposition*

$$X = X_0 + M + A, \tag{1.2}$$

where M is a martingale and A is a predictable, monotone increasing (or monotone decreasing) process satisfying $M_0 = A_0 = 0$. The process A is referred to as the compensator *of the process X.*

Note that the Doob-Meyer decomposition is unique only because of the montonicity assumption, since there exist non-trivial predictable martingales like e.g. Brownian motion.

We will now introduce the notion of semimartingales. From a first point of view the class of semimartingales seems to be somehow artificial. However, this class is fairly wide and defines the most general setup in stochastic analysis. The class of semimartingales is stable with respect to many transformations: changes of measure, time changes, integration and so on. Second, there exists a well-developed machinery of stochastic calculus of semimartingales. In a certain sense the crucial factor of the success of stochastic calculus of semimartingales is the fact that it is possible to define stochastic integrals with respect to semimartingales.

Definition 1.22 (Semimartingale) *A* semimartingale *is defined to be a real-valued stochastic process having a representation*

$$X = X_0 + M + A, \tag{1.3}$$

where X_0 is an $\mathcal{F}_0$-measureable finite random variable, M is a local martingale with $M_0 = 0$ and A is an adapted process of finite total variation with $A_0 = 0$.

X is called special semimartingale *if A can be chosen to be predictable.*

Theorem 1.23 *Every continuous semimartingale is a special semimartingale.*

Remark 1.24 In general the decomposition (1.3) of a semimartingale is not unique, whereas in the case of a special semimartingale (with predictable process A) it is. In the latter case we refer to A again as the compensator of X.

In order to define Itô's formula we will need the following result.

Theorem 1.25 *Every local martingale M has a unique decomposition*

$$M = M_0 + M^c + M^d,$$

where M^c is a continuous local martingale and M^d is a purely discontinuous local martingale satisfying $M_0^c = M_0^d = 0$.

Here a local martingale M^d (with $M_0^d = 0$) is said to be purely discontinuous *if $M^d N$ defines a local martingale for every continuous martingale N.*

Definition 1.26 *If X is a semimartingale with decomposition $X = X_0 + M + A$ then $X^c := M^c$ (in the sense of the preceding theorem) is called the* continuous martingale part *of X.*

Remark 1.27 Note that X^c does not depend on the choice of M.

1.3 The Stochastic Integral

Our aim in this chapter is to present some useful results in stochastic integration theory and stochastic calculus, using an elementary approach accessible to the nonspecialist. Stochastic calculus for continuous processes is usually presented in the framework of martingales. However, since we want to consider jump processes we need to introduce the more general and complicated framework of semimartingales.

We consider the class $\mathcal{S}$ of simple predictable processes of the form

$$H(t) = H_0 1_{\{0\}}(t) + \sum_{i=1}^{n} H_i 1_{[T_i, T_{i+1})}(t),$$

where $0 = T_1 \leq \ldots \leq T_{n+1} < \infty$ is a finite sequence of stopping times, $H_i \in \mathcal{F}_{T_i}$ with $|H_i| < \infty$ a.s., $0 \leq i \leq n+1$.

Definition 1.28 (ucp convergence) *A sequence of processes* $(Y^n)_{n\geq 1}$ *converges to a process* Y uniformly on compacts in probability (ucp)*, if for each* $t > 0$, $\sup_{0\leq s\leq t} |Y^n_s - Y_s|$ *converges to* 0 *in probability.*

By $\mathbb{D}$ *we denote the space of all càdlàg processes. Then if* $Y^n \in \mathbb{D}$ *we write* $\mathbb{D}_{ucp}$ *to denote the respective space endowed with the ucp topology. Analogously, we define the spaces* $\mathbb{L}_{ucp}$ *and* $\mathcal{S}_{ucp}$*, where* $\mathbb{L}$ *denotes the space of adapted processes with càglàd paths.*

Theorem 1.29 *The space* $\mathcal{S}_{ucp}$ *is dense in* $\mathbb{L}_{ucp}$*.*

Definition 1.30 (Stochastic integral) *For* $H \in \mathcal{S}$ *and a semimartingale* X *we define the linear mapping* $H \bullet X : \mathcal{S} \to \mathbb{D}$ *by*

$$H \bullet X = H_0 X_0 + \sum_{i=1}^{n} H_i (X^{T_i+1} - X^{T_i}).$$

Then the continuous linear mapping $H \bullet X : \mathbb{L}_{ucp} \to \mathbb{D}_{ucp}$ *obtained as the extension of* $H \bullet X : \mathcal{S} \to \mathbb{D}$ *is called the* stochastic integral of H with respect to X.

This stochastic integral can be extended to predictable integrands (Protter (2004, Theorem 15, IV.2)).

Theorem 1.31 *Let* X *be a semimartingale and let* $H \in \mathcal{P}$ *be locally bounded, then the stochasic integral* $H \bullet X$ *exists and the mapping*

$$H \mapsto H \bullet X = \int_0^{\cdot} H_t \, dX_t. \tag{1.4}$$

is unique.

Theorem 1.32 *The following properties hold*

1. $H \bullet X_t = Y(X_{s\wedge t} - X_{r\wedge t})$ *if* $H_t(\omega) = Y(\omega)1_{]r,s]}(t)$ *for a bounded* $\mathcal{F}_r$*-measurable random variable* Y*.*

2. $H \bullet X_t = 0$ *if* $H_t(\omega) = Y(\omega)1_{\{0\}}(t)$ *for a bounded* $\mathcal{F}_0$*-measurable random variable* Y

3. $H \bullet X$ *is a semimartingale.*

In the multivariate setting the stochastic integral is defined as a sum of one-dimensional stochastic integrals.

Definition 1.33 *Let X be an $\mathbb{R}^d$-valued semimartingale and H a predictable, locally bounded, $\mathbb{R}^d$-valued stochastic process. Then we define the stochastic integral of H with respect to X by*

$$H \bullet X := \sum_{i=1}^{d} H^i \bullet X^i.$$

Unfortunately, locally bounded integrands are not sufficient for all purposes in mathematical finance. Therefore we have to extend the notion of the stochastic integral to a broader class of integrands.

Definition 1.34 *Let X be a semimartingale on $\mathbb{R}^d$. An $\mathbb{R}^d$-valued, predictable process H is said to be integrable with respect to X if there exists a sequence $(D_n)_{n\in\mathbb{N}}$ of predictable sets $D_n \subset \Omega \times \mathbb{R}_+$ and a semimartingale Y such that*

1. *$D_n \uparrow \Omega \times \mathbb{R}_+$ for $n \to \infty$,*
2. *$H1_{D_n}$ is bounded for all n,*
3. *$(H1_{D_n}) \bullet X = 1_{D_n} \bullet Y$ for all n.*

In this case the stochastic integral of H with respect to X is defined by

$$H \bullet X := Y.$$

We will denote the class of all processes which are intregrable with respect to X by $L(X)$.
Note that for integrands of finite total variation the stochastic integral can be defined path-by-path as a Lebesgue-Stieltjes integral. Since the most important process in financial mathematics - Brownian motion - has sample paths of a.s. infinite total variation, we cannot simply work with the Lebesgue-Stieltjes integral. However, if X is of finite total variation, we can define the integral $H \bullet X$ pathwise without restricting the process H to be predictable.

In mathematical finance integrals with respect to martingales are of particular importance. In general these integrals are not local martingales but only so-called σ-martingales. This is the reason why we will introduce the notion of σ-martingales in this course. The σ-localisation can be understood as a generalisation of the localisation using stopping times.

Definition 1.35 *A real-valued semimartingale X is said to be a σ-martingale if there exists a sequence $(D_n)_{n\in\mathbb{N}}$ of predictable sets $D_n \subset \Omega \times \mathbb{R}_+$ such that*

1. $D_n \uparrow \Omega \times \mathbb{R}_+$ *for* $n \to \infty$,
2. $1_{D_n} \bullet X$ *is a martingale for all* n.

Remark 1.36 Every martingale is a local martingale and every local martingale is a σ-martingale.

Theorem 1.37 *Every non-negative σ-martingale is a supermartingale. Every bounded σ-martingale is a martingale. Every continuous σ-martingale is a local martingale.*

Theorem 1.38 *Suppose X and Y to be real-valued semimartingales and H and K real-valued predictable processes with $H \in L(X)$. Then*

1. *The mappings* $H \mapsto H \bullet X$ *and* $X \mapsto H \bullet X$ *are linear.*
2. $K \in L(H \bullet X)$ *if and only if* $HK \in L(X)$. *In this case* $K \bullet (H \bullet X) = (KH) \bullet X$.
3. $H \bullet X_0 = 0$
4. $\Delta(H \bullet X) = H\Delta X$
5. $H \bullet X = H \bullet (X - X_0)$
6. *For the identitiy process* $I_t = t$, $t \in \mathbb{R}_+$ *we have* $H \bullet I_t = \int_0^t H_s\,ds$ *which is the ordinary Lebesgue integral.*
7. *If X is a local martingale then $H \bullet X$ is a σ-martingale.*
8. *If X is a continuous local martingale then $H \bullet X$ is a local martingale.*
9. *If X is a local martingale and H is locally bounded then $H \bullet X$ is a local martingale.*

10. *For stopping times T we have* $X^T = X_0 + 1_{[[0,T]]} \bullet X$ *and* $(H \bullet X)^T = (H1_{[[0,T]]}) \bullet X$.

Remark 1.39 The above properties are also true for vector processes, e.g. $K \bullet (H \bullet X) = (KH) \bullet X$, if $H, X \in \mathbb{R}^d$.

The covariaton and predictable covariation can be defined by *partial integration*. The notion of covariation plays a central role in stochastic analysis.

Definition 1.40 *Let X and Y be two real-valued semimartingales. The* covariation *of X and Y is the semimartingale defined by*

$$[X,Y] := XY - X_0Y_0 - X_- \bullet Y - Y_- \bullet X. \tag{1.5}$$

If $X = Y$ we call $[X,X]$ the quadratic variation *of X and simply write $[X]$.*

Note that the integrals in (1.5) are well-defined because the left-continuous processes X_- and Y_- are locally bounded. Furthermore, observe that from (1.5) it follows that the product XY of two semimartingales X and Y is itself a semimartingale.

Definition 1.41 *Let X and Y be real-valued semimartingales such that $[X,Y]$ is a special semimartingale. Then the compensator of $[X,Y]$ is called* predictable covariation *of X and Y and and we denote it by $< X,Y >$. If $X = Y$ we call $< X,X > = < X >$ the* predictable quadratic variation *of X.*

Theorem 1.42 *Let X and Y be real-valued semimartingales. Then*

1. $[X,Y]_t = < X^c, Y^c >_t + \sum_{s\le t} \Delta X_s \Delta Y_s$, *where we set by definition* $< X^c, Y^c >_t = < X, Y >_t$.

2. *If X is continuous then* $[X,Y] = [X^c, Y^c] = < X,Y >$.

3. *If X is continuous and either X or Y is of finite total variation then* $[X,Y] = 0$.

4. $[X,Y]$ *and* $< X,Y >$ *are semimartingales of finite total variation. They are continuous if X or Y is continuous.*

5. *The mappings* $X \mapsto [X,Y]$ *and* $X \mapsto < X,Y >$ *are linear.*

6. *For* $H \in L(X)$, $K \in L(Y)$ *we have* $[H \bullet X, K \bullet Y] = (HK) \bullet [X,Y]$ *and (eventually)* $< H \bullet X, K \bullet Y >= (HK) \bullet < X,Y >$.

In continuous time stochastic analysis Itô's formula is of foremost importance.

Theorem 1.43 (Itô formula) *Let* X *be an* $\mathbb{R}^d$*-valued semimartingale and* $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ *be twice continuously differentiable, i.e.* $f \in C^2(\mathbb{R})$. *Then* $f(\cdot, X)$ *is a semimartingale and*

$$\begin{aligned} f(t, X_t) = f(0, X_0) &+ D_t f(\cdot, X) \bullet I_t + D_x f(\cdot, X_-) \bullet X_t \\ &+ \frac{1}{2} \sum_{i,j=1}^{d} D_{xx}^{ij} f(\cdot, X_-) \bullet [X^{i,c}, X^{j,c}]_t \\ &+ \sum_{s \le t} \left(f(s, X_s) - f(s, X_{s-}) - \sum_{i=1}^{d} D_x^i f(s, X_{s-}) \Delta X_s^i \right). \end{aligned}$$

Remark 1.44

1. For X continuous we have ($d = 1$):

$$f(t, X_t) = f(0, X_0) + D_t f(\cdot, X) \bullet I_t + D_x f(\cdot, X) \bullet X_t + \frac{1}{2} D_{xx} f(\cdot, X) \bullet [X]_t.$$

2. Itô's formula is also true for functions $f : U \to \mathbb{R}$, $f \in C^2(\mathbb{R})$, which are defined on an open subset $U \subset \mathbb{R}^d$ if X and X_- take values in U.

3. Itô's formula is also true for complex-valued functions if stochastic processes, integrals etc. are extended to the complex numbers in a natural manner.

We will now introduce the stochastic exponential.

Theorem 1.45 *For every real-valued semimartingale* X *there exists a unique real-valued semimartingale* Z *which is the solution to the equation*

$$Z = 1 + Z_- \bullet X. \tag{1.6}$$

It is given by

$$Z_t = \exp\left\{ X_t - X_0 - \frac{1}{2}[X^c]_t \right\} \prod_{s \le t} (1 + \Delta X_s) e^{\Delta X_s}.$$

Proof. We only consider the case $\Delta X > -1$, in which Z is of the form

$$Z_t = \exp\{Y_t\}$$

with

$$Y_t = X_t - X_0 - \frac{1}{2}[X^c]_t + \sum_{s \leq t} (\log(1 + \Delta X_s) - \Delta X_s).$$

Making use of the above defintions and theorems we obtain

$$Y^c = X^c, \quad \Delta Y_s = \log(1 + \Delta X_s)$$

and

$$\sum_{s \leq t} Z_{s-}(e^{Y_s} - 1 - \Delta Y_s) = Z_- \bullet (\sum_{s \leq \cdot} (e^{Y_s} - 1 - \Delta Y_s))_t.$$

An application of Itô's formula to $Z = \exp(Y)$ gives

$$\begin{aligned}
Z &= Z_0 + Z_- \bullet Y + \frac{1}{2} Z_- \bullet [Y^c] + \sum_{s \leq \cdot} (Z_{s-} e^{\Delta Y_s} - Z_{s-} - Z_{s-} \Delta Y_s) \\
&= 1 + Z_- \bullet (Y + \frac{1}{2}[Y^c] + \sum_{s \leq \cdot} (e^{\Delta Y_s} - 1 - \Delta Y_s)) \\
&= 1 + Z_- \bullet (X - X_0 - \frac{1}{2}[X^c] + \sum_{s \leq \cdot} (\log(1 + \Delta X_s) - \Delta X_s) \\
&\quad + \frac{1}{2}[X^c] + \sum_{s \leq \cdot} (1 + \Delta X_s - 1 - \log(1 + \Delta X_s)) \\
&= 1 + Z_- \bullet X,
\end{aligned}$$

i.e. Z solves (1.6). □

Definition 1.46 *The process Z in the preceding theorem is called* stochastic exponential *of X and we will denote it by $\mathcal{E}(X)$*

Remark 1.47 For X continuous we have

$$\mathcal{E}(X)_t = \exp\left\{ X_t - X_0 - \frac{1}{2}[X]_t \right\}.$$

Theorem 1.48 (Yor's formula) *For real-valued semimartingales X and Y we have*

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X + Y + [X, Y]).$$

Proof. Since,

$$
\begin{aligned}
\mathcal{E}(X)\mathcal{E}(Y) &= \mathcal{E}(X)_0\mathcal{E}(Y)_0 + \mathcal{E}(X)_- \bullet \mathcal{E}(Y) + \mathcal{E}(Y)_- \bullet \mathcal{E}(X) + [\mathcal{E}(X), \mathcal{E}(Y)] \\
&= 1 + \mathcal{E}(X)_- \bullet (\mathcal{E}(Y)_- \bullet Y) + \mathcal{E}(Y)_- \bullet (\mathcal{E}(X)_- \bullet X) \\
&\quad + [\mathcal{E}(X)_- \bullet X, \mathcal{E}(Y)_- \bullet Y] \\
&= 1 + (\mathcal{E}(X)\mathcal{E}(Y))_- \bullet (Y + X + [X, Y])
\end{aligned}
$$

the assertion follows from the definition of the stochastic exponential. □

The following results concerning change of measures are analogous to the discrete time case.

Theorem 1.49 *Let $Q \sim P$ be a probability measure with density process Z and let X be an adapted càdlàg process. Then*

1. *X is a Q-martingale if and only if XZ is a P-martingale.*
2. *X is a Q-local martingale if and only if XZ is a P-local martingale.*
3. *X is a Q-σ-martingale if and only if XZ is a P-σ-martingale.*

Theorem 1.50 (Girsanov) *Let $Q \sim P$ be a probability measure with density process Z. Moreover, supppose X is a local martingale satisfying $X_0 = 0$ such that $[Z, X]$ is a special semimartingale. Then*

$$X - \frac{1}{Z_-} \bullet < Z, X >$$

is a Q-local martingale, where predictable covariation is defined with respect to the measure P.

Theorem 1.51 *The set of semimartingales, the set $L(X)$ of processes which are integrable with respect to a semimartingale X, stochastic integrals $H \bullet X$ and the covariation $[X, Y]$ do not change if we substitute the probability measure P by an equivalent probability measure Q.*

Lévy processes play a fundamental role in stochastic analysis. They can be considered as an analogue of linear functions in ordinary deterministic analysis. On the one hand they represent constant growth, though constant is to understand only in a stochastic sense. On the other hand, like a linear function, they can be uniquely characterised by a only a few parameters.

Definition 1.52 *A* Lévy process *(process with independent and stationary increments) is a semimartingale X such that*

1. $X_0 = 0$
2. $X_t - X_s$ *is independent of* $\mathcal{F}_s$ *for* $s \leq t$.
3. *The distribution of* $X_t - X_s$ *depends only on* $t - s$.

The most important Lévy process (besides deterministic linear functions) is standard Brownian motion. The next theorem states the surprising fact that every continuous process with constant growth in the above sense can be represented as a linear combination of a linear function and a standard Brownian motion. This shows the importance of Brownian motion in stochastic analysis. Moreover, we see that objects with constant growth can be uniquely characterised using only a few parameters, namely μ and σ (like the drift μ in the deterministic case).

Definition 1.53 *A real-valued Lévy process is called a* standard Bownian motion *if X_t is $N(0,t)$-distributed for all $t \geq 0$.*

Analogously an $\mathbb{R}^d$-valued Lévy process is a standard Brownian motion on $\mathbb{R}^d$ if X_t is $N(0, tI_d)$-distributed for all $t \geq 0$, where I_d denotes the identity matrix on $\mathbb{R}^d$.

Remark 1.54 The distribution of standard Brownian motion is uniquely determined. It is called *Wiener measure*.

Theorem 1.55 *A real-valued Lévy process has continuous sample paths if and only if there exists a standard Brownian motion W and constants $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}_+$ such that*

$$X_t = \mu t + \sigma W_t, \quad t \in \mathbb{R}_+.$$

An analogous result holds for $\mathbb{R}^d$-valued Lévy processes with $\mu \in \mathbb{R}^d$, a positive semidefinite matrix $\Sigma \in \mathbb{R}^{d\times d}$ and a standard Brownian motion W on $\mathbb{R}^d$.

Definition 1.56 *Due to the preceding theorem we call a continuous Lévy process* Brownian motion with drift.

Theorem 1.57 *A real-valued semimartingale X is a standard Brownian motion if it is a continuous local martingale satisfying $X_0 = 0$ and $[X]_t = t$ for all $t \in \mathbb{R}_+$.*

The same is true for a standard Brownian motion W on $\mathbb{R}^d$ with $[W^i, W^j]_t = \delta_{ij}t$.

Theorem 1.58 *Let X be a Lévy process. If X (or $\mathcal{E}(X)$ or $\exp\{X\}$, respectively) is a local martingale, then X (or $\mathcal{E}(X)$ or $\exp\{X\}$, respectively) is a martingale.*

In financial mathematics price processes have a multiplicative rather than an additive structure. This motivates the following definition.

Definition 1.59 *A* geometric Brownian motion *is a process of the form $\mathcal{E}(X)$ or equivalently $\exp\{X\}$ for a Brownian motion X with drift.*

Remark 1.60

1. Analogously we can introduce geometric Lévy processes. However, in this case considering the stochastic exponential leads to a larger class of processes, since for an arbitrary Lévy process, $\mathcal{E}(X)$ can become negative.

2. In mathematics of finance the multivariate analogue $(\mathcal{E}(X^1), ..., \mathcal{E}(X^d))$ or $(\exp\{X^1\}, ..., \exp\{X^d\})$, respectively, for an $\mathbb{R}^d$-valued Brownian motion (or Lévy process) with drift is of interest, too.

For a Brownian motion Girsanov's theorem can be formulated in a more tractable way.

Theorem 1.61 (Girsanov) *Let $Q \sim P$ be a probability measure with density process $Z = \mathcal{E}(N)$ for a semimartingale N. If W is a standard Brownian motion with respect to P then $W - [W, N]$ is a standard Brownian motion with respect to Q.*

Proof. Since $[W,N]$ is continuous and hence predictable, we have $< W, N >= [W,N]$. Hence, it follows from Theorem 1.50 that

$$\begin{aligned} W - \frac{1}{Z_-} \bullet < W, Z > &= W - \frac{1}{\mathcal{E}(N)_-} \bullet (\mathcal{E}(N)_- \bullet < W, N >) \\ &= W - 1 \bullet < W, N > \\ &= W - 1 \bullet [W, N] \\ &= W - [W, N] \end{aligned}$$

is a Q-local martingale. Hence, from $(W - [W,N])^c = W^c = W$ follows $[W - [W,N], W - [W,N]] = [W,W] = I$. Therefore, applying Theorem 1.57 we obtain that $W - [W,N]$ is a standard Brownian motion with respect to Q. □

Remark 1.62 The preceding theorem is also true for standard Brownian motions on $\mathbb{R}^d$ if the Q-Brownian motion is defined componentwise by

$$(W - [W,N])^i := (W^i - [W^i, N]).$$

Notation 1.63 *By I we denote the identity process $I_t := t$, $t \in \mathbb{R}_+$.*

We are now in the position to introduce Itô processes.

Definition 1.64 (Itô process) *Let W be a standard Brownian motion on $\mathbb{R}^d$. For every $\mu \in L(I)$ and $\sigma \in L(W)$ we call a semimartingale X having representation*

$$X = X_0 + \mu \bullet I + \sigma \bullet W \tag{1.7}$$

an Itô *process.*

Like in one dimension, also in the multivariate case we can formulate integrability of a predictable process with respect to a standard Brownian motion in terms of an explicit condition.

Theorem 1.65 *1. For a real-valued predictable process H we have $H \in L(I)$ if and only if*

$$\int_0^t |H_s|\, ds < \infty, \quad t \in \mathbb{R}_+.$$

2. *Let W be a standard Brownian motion on $\mathbb{R}^d$ and H be an $\mathbb{R}^d$-valued predictable process. Then $H \in L(W)$ if and only if*

$$\int_0^t H_s^T H_s \, ds < \infty, \quad t \in \mathbb{R}_+.$$

Theorem 1.66 (Martingale representation theorem) *Let W be a standard Brownian motion on $\mathbb{R}^d$ and $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ be the filtration generated by W. Then for every local martingale M there exists a process $H \in L(W)$ such that*

$$M = M_0 + H \bullet W.$$

An analogous martinale representation theorem also exists for other processes, provided they satisfy the assumptions of the following theorem. Examples of such processes are, besides Brownian motion, (for bounded time intervals $[0,T]$) compensated Poisson processes. However, the set of processes having such a martingale representation is very small.

Theorem 1.67 *Let X be an $\mathbb{R}^d$-valued martingale with $E(\sup_{t\in\mathbb{R}_+} |Xt|) < \infty$. If besides P there exists no other probability measure $Q \sim P$ such that X is a Q-locale martingale, then for every bounded martingale M there exists a process $H \in L(X)$ such that*

$$M = M_0 + H \bullet X.$$

In the case of deterministic integrands the distribution of integrals with respect to Brownian motion is known explicitly.

Theorem 1.68 *Let W be a standard Brownian motion on $\mathbb{R}^d$.*

1. *If the processes $m \in L(I)$ and $\sigma \in L(W)$ are both deterministic then*

$$\int_0^t m_s \, ds + \int_0^t \sigma_s \, dW_s = m \bullet I_t + \sigma \bullet W_t$$

is normally distributed with mean $\int_0^t m_s \, ds$ and variance $\int_0^t \sigma_s^T \sigma_s \, ds$ for every $t \in \mathbb{R}_+$.

2. *If $\sigma \in L(W)$ is deterministic then*

$$\sigma \bullet W_T - \sigma \bullet W_t$$

is independent of $\mathcal{F}_t$ for every $t \leq T$.

Fubini's theorem can be formulated also for stochastic integrals. The main consequence is that we can interchange the order of stochastic and Lebesgue integration.

Theorem 1.69 (Fubini's theorem for stochastic integrals) *Let X be a semimartingale, $(\Gamma, \mathcal{G}, \mu)$ be a measure space with σ-finite measure μ and $H : \Omega \times \mathbb{R}_+ \times \Gamma \to \mathbb{R}$, $(\omega, t, \gamma) \mapsto H_t^\gamma(\omega)$ be a $\mathcal{P} \otimes \mathcal{G}$-measurable mapping such that*

$$\left(\sqrt{\int_\Gamma H_t^\gamma(\omega)\, \mu(d\gamma)} \right)_{t \in \mathbb{R}_+} \in L(X).$$

Then

1. *$H \in L(X)$ for μ-a.e. $\gamma \in \Gamma$.*
2. *There exists a $\mathcal{F} \otimes \mathcal{B} \otimes \mathcal{G}$-measurable version of the mapping*

$$(\omega, t, \gamma) \mapsto Z_t^\gamma(\omega) := H^\gamma \bullet X_t(\omega).$$

3. *$(\int_\Gamma Z_t^\gamma\, \mu(d\gamma))_{t \in \mathbb{R}_+}$ exists and is a semimartingale.*
4. *$(\int_\Gamma H_t^\gamma\, \mu(d\gamma))_{t \in \mathbb{R}_+} \in L(X)$.*
5. $$\left(\int_\Gamma H_t^\gamma\, \mu(d\gamma) \right) \bullet X = \int_\Gamma Z_t^\gamma\, \mu(d\gamma) =: \int_\Gamma (H^\gamma \bullet X)\, \mu(d\gamma).$$

We conclude this chapter with an example that shows the distinction between martingales and local martingales.

Example 1.70 Let W be a standard Brownian motion. For $n \in \mathbb{N}$ fixed define the process K^n by

$$K_t^n = 1_{[0, \arctan n]}(t) \sqrt{1 + \tan^2 t}.$$

Since K^n is predictable and bounded, we have $K^n \in L(W)$. Define stopping times

$$T_n := \inf\{t \in \mathbb{R}_+ : \; K^n \bullet W_t \geq 1\}$$

and the process H by

$$H_t := \begin{cases} K_t^n 1_{[[0,T_n]]}(t) & \text{if } t < \arctan n \text{ for } \; n \in \mathbb{N} \\ 0 & \text{if } t \geq \frac{\pi}{2}. \end{cases}$$

First, we show that H is well-defined. Therefore, assume that $m, n \in \mathbb{N}$ with $t < \arctan m \wedge \arctan n$. We have $K^m = K^n$ on $[0,t]$ and hence $K^m \bullet W = K^n \bullet W$ on $[0,t]$. Thus

$$\begin{aligned} \{t \leq T_m\} &= \{K^m \bullet W < 1 \text{ for all } s \in [0,t)\} \\ &= \{K^n \bullet W < 1 \text{ for all } s \in [0,t)\} \\ &= \{t \leq T_n\} \end{aligned}$$

yields

$$K_t^m 1_{[[0,T_m]]}(t) = K_t^n 1_{[[0,T_n]]}(t).$$

Furthermore, H is left-continuous and adapted and thus predictable.

Now, we show hat $\int_0^\infty H_t^2 \, dt$ is a.s. finite, since then it follows from Theorem 1.65 that H is integrable with respect to W. Obviously,

$$\begin{aligned} N :=& \left\{ \int_0^\infty H_t^2 \, dt = \infty \right\} \\ &\subset \{\text{there is no } n \text{ such that } T_n < \frac{\pi}{2}\} \\ &\subset \cap_{n \in \mathbb{N}} \{K^n \bullet W_t < 1 \text{ for all } t \in [0, \arctan n]\}. \end{aligned}$$

For $\vartheta \in \mathbb{R}_+$ define $\mathcal{G}_\vartheta := \mathcal{F}_{\arctan \vartheta}$, $B_\vartheta := K^n \bullet W_{\arctan \vartheta}$, where $n > \arctan \vartheta$. As before one can show that B_ϑ is well-defined. Moreover, $(\mathcal{G}_\vartheta)_{\vartheta \in \mathbb{R}_+}$ is a filtration and $B = \{B_\vartheta\}_{\vartheta \in \mathbb{R}_+}$ is a continuous and adapted (with respect to this filtration) process. For $\vartheta \geq \upsilon$ we have from Theorem 1.68 that

$$B_\vartheta - B_\upsilon = K^n \bullet W_{\arctan \vartheta} - K^n \bullet W_{\arctan \upsilon}$$

is independent of $\mathcal{F}_{\arctan \upsilon} = \mathcal{G}_\upsilon$, i.e. B has independent increments. Furthermore, it follows from Theorem 1.68 that

$$B_\vartheta - B_\upsilon = (K^n 1_{]]\arctan \upsilon, \infty[[}) \bullet W_{\arctan \vartheta}$$

is normally distributed with zero mean and variance

$$\begin{aligned}\int_0^{\arctan\vartheta} (K_t^n)^2 1_{]]\arctan\upsilon,\infty[[}\,dt &= \int_{\arctan\upsilon}^{\arctan\vartheta} (1+\tan^2 t)\,dt \\ &= \int_\upsilon^\vartheta (1+\tan^2(\arctan\xi))\arctan'\xi\,d\xi \\ &= \vartheta-\upsilon\end{aligned}$$

for $\vartheta \geq \upsilon$. In particular, the above independence yields

$$E[B_\vartheta - B_\upsilon \,|\, \mathcal{G}_\upsilon] = E[B_\vartheta - B_\upsilon] = 0,$$

i.e. B is a martingale and thus a semimartingale. Therefore, B is a standard Brownian motion. A Brownian motion reaches a.s. the value 1. Hence, continuity from above yields

$$P(K^n \bullet W_t < 1 \text{ for all } t \in [0,\arctan n]) = P(B_\vartheta < 1 \text{ for all } \vartheta \leq n) \to 0$$

as $n \to \infty$. Consequently N is a null set. Since K^n reaches the value 1 for sufficiently large n we have $H \bullet W_{\frac{\pi}{2}} = 1$. Moreover, it is obvious that $H \bullet W = (K^n \bullet W)^{T_n} \leq 1$ on $[0,\arctan n]$ and therefore $H \bullet W \leq 1$. Consequently, for $M := H \bullet W$:

- M is an integral with respect to a continuous martingale.
- M is therefore a local martingale.
- M is not a martingale, since $E[M_{\frac{\pi}{2}}] = 1 \neq 0 = E[M_0]$.
- M is a submartingale, since $1 - M$ is a non-negative local martingale and thus a supermartingale.

2 Mathematical Modeling of Financial Markets

2.1 Trading and Arbitrage

We consider a financial market of $d+1$ assets $S = (S^0, \dots, S^d)$ that operates in uncertain conditions of the probabilistic character described by a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, where $(\mathcal{F}_t)_{t\geq 0}$ is the flow of incoming information. For simplicity we assume that $\mathcal{F}_0 = (\emptyset, \Omega)$. Our main assumption about price process $S = (S^0, \dots, S^d)$ is that it is an $\mathbb{R}^{d+1}$-valued semimartingale.

Definition 2.1 *A* trading strategy *(often refered to as portfolio) is an $\mathbb{R}^{d+1}$-valued predictable process $\phi = (\phi^0, \dots, \phi^d)$. The* value process *of the portfolio is given by*

$$V(\phi) := \phi^T S.$$

A trading strategy ϕ is called self-financing *if $\phi \in L(S)$ and*

$$V(\phi) = V_0(\phi) + \phi \bullet S, \tag{2.1}$$

where $V_0(\phi)$ is the initial capital.

The random variable ϕ_t^i describes the number of assets of type i in the portfolio at time t. The stochastic integral describes the profits of the trading strategy ϕ. The self-financing condition means that all changes of the value V must be the results of changes in the market value (price) of the assets S^i. In particular, it means that after time 0 no in- or outflows of capital are possible.

Since the stochastic integral $\phi \bullet S$ describes the trading profits of ϕ, equation (2.1) shows that the portfolio value only changes according to these profits.

Comparing the prices of different assets one usually distinguishes a basis asset and valuates other assets in its terms. The reason is that considering discounted quantities often simplifies calculations. In particular, we chose S^0 as the basis asset, usually called the numeraire, where we assume that S^0 as well as S^0_- is positive

Definition 2.2 *The process*

$$\hat{S} := \frac{1}{S^0} S = \left(1, \frac{S^1}{S^0}, \dots, \frac{S^d}{S^0}\right)$$

is called discounted price process. *Moreover, we refer to*

$$\hat{V}(\phi) := \frac{1}{S^0} V(\phi) = \phi^T \hat{S}$$

as the discounted value process *of the trading strategy ϕ.*

Observe that the self-financing condition does not change when considering discounted quantities.

Lemma 2.3 *A trading strategy ϕ is self-financing if and only if $\phi \in L(\hat{S})$ and*

$$\hat{V}(\phi) = \hat{V}_0(\phi) + \phi \bullet \hat{S}. \tag{2.2}$$

Proof. $\Leftarrow$: Applying partial integration we obtain

$$\begin{aligned}
\phi^T S &= (\phi^T \hat{S}) S^0 \\
&= \phi_0^T S_0 + (\phi^T \hat{S})_- \bullet S^0 + S^0_- \bullet (\phi^T \hat{S}) + [\phi^T \hat{S}, S^0].
\end{aligned}$$

From (2.2) follows $\phi^T \hat{S} = \phi_0^T \hat{S}_0 + \phi \bullet \hat{S}$ and hence $\Delta(\phi^T \hat{S}) = \Delta(\phi \bullet \hat{S}) = \phi \Delta \hat{S}$, which implies $(\phi^T \hat{S})_- = \phi^T \hat{S}_-$. Therefore,

$$\begin{aligned}
\phi^T S &= \phi_0^T S_0 + (\phi^T \hat{S}_-) \bullet S^0 + S^0_- \bullet (\phi_0^T \hat{S}_0 + \phi \bullet \hat{S}) + [\phi_0^T \hat{S}_0 + \phi \bullet \hat{S}, S^0] \\
&= \phi_0^T S_0 + (\phi^T \hat{S}_-) \bullet S^0 + (\phi S^0_-) \bullet \hat{S} + \phi \bullet [\hat{S}, S^0] \\
&= \phi_0^T S_0 + \phi \bullet (\hat{S}_- \bullet S^0 + S^0_- \bullet \hat{S} + [\hat{S}, S^0]) \\
&= \phi_0^T S_0 + \phi \bullet (\hat{S} S^0 - \hat{S}_0 S^0_0) \\
&= \phi_0^T S_0 + \phi \bullet S.
\end{aligned}$$

$\Rightarrow$: The assertion is proven analogously by interchanging the roles of S and $\hat{S}$ and with $1/S^0$ instead of S^0. □

The following lemma shows that the self-financing condition uniquely determines the amount of the numeraire ϕ^0 in the portfolio. However, in most calculations ϕ^0 plays no role, since the discounted trading gains $\phi \bullet \hat{S}$ do not depend on ϕ^0.

Lemma 2.4 *For every predictable process* $(\phi^1, \dots, \phi^d) \in L((\hat{S}^1, \dots, \hat{S}^d))$ *and every* $V_0 \in \mathbb{R}$ *there exists a unqiue predictable process* ϕ^0 *such that* $\phi = (\phi^0, \dots, \phi^d) \in L(S)$ *is self-financing with* $V_0(\phi) = V_0$.

Proof. Due to the preceding lemma ϕ is self-financing if and only if

$$\begin{aligned} \phi_t^0 \hat{S}_t^0 + (\phi^1, \dots, \phi^d)_t^T (\hat{S}^1, \dots, \hat{S}^d)_t &= \hat{V}_t(\phi) \\ = \hat{V}_0 + \phi \bullet \hat{S}_t &= \hat{V}_0 + (\phi^1, \dots, \phi^d) \bullet (\hat{S}^1, \dots, \hat{S}^d)_t, \end{aligned}$$

i.e. if and only if

$$\begin{aligned} \phi_t^0 &= \hat{V}_0 + (\phi^1, \dots, \phi^d) \bullet (\hat{S}^1, \dots, \hat{S}^d)_t - (\phi^1, \dots, \phi^d)_t^T (\hat{S}^1, \dots, \hat{S}^d)_t \\ &= \hat{V}_0 + (\phi^1, \dots, \phi^d) \bullet (\hat{S}^1, \dots, \hat{S}^d)_{t-} - (\phi^1, \dots, \phi^d)_t^T (\hat{S}^1, \dots, \hat{S}^d)_{t-}. \end{aligned}$$

Obviously this is a predictable process. We omit the technical proof of the integrability of ϕ. □

In markets in continuous time trading can take place infinitely often even in bounded time intervals. It is therefore not surprising that in many markets trading strategies exist which a.s. lead to a profit, like the following example shows.

Example 2.5 *We consider a market consisting of the constant numeraire* $S^0 = 1$ *and a second asset* S^1 *satisfying*

$$S_t^1 = \mathcal{E}(W)_t = \exp\left\{W_t - \frac{1}{2}t\right\},$$

where W *is a standard Browian motion. We assume moreover that the terminal time point is* $T = \pi/2$*. Now, we construct a self-financing trading strategy* $\phi = (\phi^0, \phi^1)$ *such that* $V_0(\phi) = 0$ *and* $V_T(\phi) = 1$*, i.e. without any initial capital we obtain a final profit equal to* 1.

Due to Lemma 2.4 it is enough to determine the amount ϕ^1 *of the asset in the portfolio. We set* $\phi^1 := H/S^1$*, where* H *is the integrand of Example 1.70. Since* $\phi^1 S^1 = H \in L(W)$ *and* $S^1 \in L(W)$ *it follows from Theorem 1.38(2) that* $\phi^1 \in L(S^1 \bullet W) = L(S^1 - 1) = L(S^1)$.

Moreover, $\phi^1 \bullet S^1 = \frac{H}{S^1} \bullet (S^1 \bullet W) = H \bullet W$*. Hence, for the self-financing strategy* $\phi = (\phi^0, \phi^1)$ *which corresponds to* $V_0 = 0$ *and* ϕ^1 *(Lemma 2.4), we have* $V_T = 1$.

This is the reason why we will restrict the set of trading possibility by an admissibility condition. Usually in literature one only allows for strategies whose value is bounded from below by a multiple of the numeraire. This means that one cannot make arbitrarily high debts. However, then the admissibility condition depends on the numeraire. We will therefore work with an admissibility condition which is more general and allows for debts up to a linear combination of the traded securities.

From now on we assume that all securities $S^0, \dots, S^d$ are non-negative. Furthermore, we suppose that the intitial prices S_0 are deterministic.

Definition 2.6 *We call a self-financing strategy* admissible *if there exist* $c_1, \dots, c_d \in \mathbb{R}_+$ *such that*

$$V(\phi) \geq -\sum_{i=0}^{d} c_i S^i.$$

Moreover, from now on we assume to be given a fixed terminal time point $T \in \mathbb{R}_+$, i.e. the index set equals $[0, T]$ instead of $\mathbb{R}_+$. For simplicity we also assume $\mathcal{F} = \mathcal{F}_T$.

Up to the adimissibility condition we define arbtitrage analogously to the discrete time case.

Definition 2.7 *An admissible strategy* ϕ *is called an* arbitrage possibility *if*

$$V_0(\phi) = 0, \quad V_T(\phi) \geq 0$$

P*-almost surely and* $P(V_T(\phi) > 0) > 0$*. The market is called* arbitrage free*, if no arbitrage strategy exists.*

Remark 2.8 Obviously we can replace $c_1, \dots, c_d$ by $c := \max(c_1, \dots, c_d)$.

Similar to the discrete time case existence of an equivalent martingale measure guarantees arbitrage freeness of the market. However, this is stongly based on the admissibility restriction of the set of self-financing strategies.

Theorem 2.9 *If there exisits an equivalent martingale measure (i.e. a probability measure $Q \sim P$ such that $\hat{S}$ is a Q-martingale), then the market is arbitrage free.*

Proof. Let ϕ be an admissible strategy with initial value $V_0(\phi) = 0$. Since $\hat{S}$ is a Q-martingale, $\hat{V}(\phi) = \phi \bullet \hat{S}$ is a Q-σ-martingale. Define

$$M := \hat{V}(\phi) + c \sum_{i=0}^{d} \hat{S}^i$$

for sufficiently large c. Applying Theorem 1.37 it is a Q-supermartingale. Furthermore, since $\sum_{i=0}^{d} \hat{S}^i$ is a Q-martingale, $\hat{V}(\phi)$ is a Q-supermartingale, too. In particular,

$$E_Q(\hat{V}_T(\phi)) \leq E_Q(\hat{V}_0(\phi)) = 0.$$

In the case $\hat{V}(\phi) \geq 0$ we hence obtain $\hat{V}(\phi) = 0$, i.e. the market is arbitrage free. □

For the existence of an equivalent martingale measure the absence of arbitrage is not sufficient. However, we obtain a fundamental pricing theorem if we do not only assume the absence of arbitrage but also the absence of trading strategies which are arbitrary close to this idealistic notion, i.e. the so-called "No Free Luch With Vanishing Risk". The free lunch $X \geq 0$ with $P(X > 0) > 0$, which equals the terminal value of the arbitrage strategy, is not available by a trading strategy with initial value zero, but nevertheless with arbitrary small initial value.

Definition 2.10 *We call a random variable $X \geq 0$ with $P(X > 0) > 0$* free lunch with vanishing risk *if there exists a sequence $(\phi^{(n)})_{n \in \mathbb{N}}$ of admissible strategies and a sequence $(v_n)_{n \in \mathbb{N}}$ in $\mathbb{R}_+$ which converges towards zero such that $V_0(\phi^{(n)}) \leq v_n$ and $V_T(\phi^{(n)}) \geq X$ for all $n \in \mathbb{N}$. We say the market satisfies the NFLVR (no free lunch with vanishing risk) condition if there exisits no such random variable.*

In fact, now the absence of these almost riskless returns is equivalent to the existence of an equivalent martingale measure. The proof is based on the theorem of Hahn-Banach and very technical. For this reason we only show how from the result (proven by Delbaen & Schachermayer (1994)) for bounded price processes the general (non-bounded) case can be derived.

Theorem 2.11 (Fundamental theorem of asset pricing) *A market satisfies the NFLVR condition if and only if there exists and equivalent martingale measure Q.*

Proof. $\Leftarrow$: Let X be a free lunch with vanishing risk and define $\hat{X} := X/S_T^0$. Similar to the proof of Theorem 2.9 it follows that

$$E_Q(\hat{X}) \leq E_Q(\hat{V}_T(\phi^{(n)})) \leq E_Q(\hat{V}_0(\phi^{(n)})) \leq \frac{v_n}{S_0^0} \to 0$$

for $n \to \infty$. Since $\hat{X} \geq 0$ we have $\hat{X} = 0$ a.s., contradicting $P(X > 0) > 0$.

$\Rightarrow$: This is shown in Delbaen & Schachermayer (1994) for bounded $\hat{S}$. We now put down the general to the bounded case. Therefore we define $S^\Sigma := \sum_{i=0}^d S^i$ to be the numeraire instead of S^0. Analogously we define $\tilde{S} := S/S^\Sigma$. It is easy to verify that the results above are true also for $\tilde{S}$ instead of $\hat{S}$. Since the admissibility condition is independent of the numeraire, we have the absence of arbitrage for $\tilde{S}$, too. Applying Delbaen & Schachermayer (1994) there exisits an equivalent martingale measure $\tilde{Q}$ for $\tilde{S}$ with density process denoted by Y. Due to Theorem 1.49 $\frac{S^i}{S^\Sigma} Y = \tilde{S}^i Y$ is a P-martingale for $i = 0, \ldots, d$. In particular, $Z := \frac{S_0^\Sigma S^0}{S_0^0 S^\Sigma} Y$ is a positive P-martingale satisfying $E(Z_T) = E(Z_0) = E(Y_0) = 1$, i.e. Z is the density process of a probability measure $Q \sim P$. Moreover,

$$\hat{S}^i Z = \frac{S^i S_0^\Sigma S^0}{S^0 S_0^0 S^\Sigma} Y = \frac{S^i S_0^\Sigma}{S_0^0 S^\Sigma} Y$$

is a P-martingale for $i = 0, \ldots, d$. Again from Theorem 1.49 follows that Q is hence an equivalent martingale measure for $\hat{S}$. □

Sometimes the following result is very helpful.

Lemma 2.12 *If ϕ is an admissible strategy then under every equivalent martingale measure Q, the corresponding discounted value process $\hat{V}(\phi)$ is a Q-supermartingale.*

Proof. Let $V(\phi) \geq -c\sum_{i=0}^{d} S^i$ for some $c \in \mathbb{R}_+$. Then $\hat{V}(\phi) + c\sum_{i=0}^{d} \hat{S}^i$ is a non-negative Q-σ-martingale and hence a Q-supermartingale according to Theorem 1.38 and Theorem 1.37. Since $\hat{S}^0, \dots, \hat{S}^d$ are Q-martingales, the assertion follows. □

Similar to the discrete time case the absence of arbitrage guarantees that the value process $V(\phi)$ of a self-financing portfolio is already uniquely determined by its terminal value $V_T(\phi)$.

The following example shows that such a so-called "law of one price" in continuous time is not valid, if one considers arbitrary admissible strategies.

Example 2.13 *In the market discussed in Example 2.5, ϕ is not admissible, whereas $-\phi$ is admissible, since the corresponding value process $V(-\phi)$ is bounded by -1. $-\phi$ is a "fortune destroying" strategy, which is not excluded in theory. The trading strategy $-\phi$ has the same terminal value -1 like a trivial constant strategy $\psi = (-1, 0)$, but its initial value is one unit higher, i.e. $V_T(\phi) = -1 = V_T(\psi)$ and $V_0(\phi) = 0 > -1 = V_0(\psi)$. This shows that in arbitrage free markets (in contrast to the discrete time case) the terminal value of an admissible strategy does not determine uniquely the initial capital.*

One possibility to circumvent this problem is to reduce again the set of admissible strategies, e.g. in the sense of the following definition.

Definition 2.14 *We call a trading strategy "double-admissible" if both, ϕ and $-\phi$ are admissible.*

In fact, for such double-admissible strategies we have the above law of one price. On the other hand, the reduction of the portfolio value to be bounded from above makes from an economic point of view no sense. Therefore we will only work with the notion of double-admissible strategies if absolutely necessary.

Lemma 2.15 (Law of one price) *Suppose ϕ and ψ are double-admissible trading strategies satisfying $V_T(\phi) = V_T(\psi)$. If the market is arbitrage free then $V(\phi) = V(\psi)$ on $[0, T]$.*

In particular, in arbitrage free markets $S^i = V(\phi)$, if ϕ is a double-admissible strategy and $S^i_T = V_T(\phi)$ for $i \in \{0, \dots, d\}$.

Proof. Otherwise there exists a time point $t \in [0,T]$ such that $P(V_t(\phi) \neq V_t(\psi)) > 0$, i.e. w.l.o.g. $P(A) > 0$ for $A := \{V_t(\psi) > V_t(\phi)\}$. Assume

$$(\vartheta^1, \ldots, \vartheta^d)_s := \begin{cases} 0 & \text{for } s \leq t \\ ((\phi^1, \ldots, \phi^d)_s - (\psi^1, \ldots, \psi^d)_s)1_A & \text{for } s > t \end{cases}$$

where ϑ is supposed to be the self-financing strategy for $V_0 = 0$ of Lemma 2.4. Then $\hat{V}_0(\vartheta) = 0$ and

$$\begin{aligned} \hat{V}_s(\vartheta) &= \big((\phi - \psi)1_{A\times]t,T]}\big) \bullet \hat{S}_s \\ &= 1_{A\times]t,T]} \bullet ((\phi - \psi) \bullet \hat{S})_s \\ &= \Big((\phi - \psi) \bullet \hat{S}_s - (\phi - \psi) \bullet \hat{S}_t\Big)\, 1_A \\ &= \Big(\hat{V}_s(\phi) - \hat{V}_s(\psi) - \hat{V}_t(\phi) + \hat{V}_t(\psi)\Big)\, 1_A \\ &\geq \Big(\hat{V}_s(\phi) - \hat{V}_s(\psi)\Big)\, 1_A \end{aligned}$$

for $s \geq t$ which implies the admissibility of ϑ. For $s = T$ we have, in particular,

$$\begin{aligned} \hat{V}_T(\vartheta) &= \Big(\hat{V}_T(\phi) - \hat{V}_T(\psi) - \hat{V}_t(\phi) + \hat{V}_t(\psi)\Big)\, 1_A \\ &= \Big(\hat{V}_t(\psi) - \hat{V}_t(\phi)\Big)\, 1_A. \end{aligned}$$

This is everywhere non-negative and positive on A. Hence, ϑ is an arbitrage.

The second assertion follows when considering

$$\psi^j := \begin{cases} 1 & \text{for } j = i \\ 0 & \text{else.} \end{cases}$$

□

Remark 2.16 In general we have

$$V_T(\phi) \geq V_T(\psi) \Longrightarrow V(\phi) \geq V(\psi)$$

in an arbitrage free market, if ϕ is admissible and ψ is double-admissible.

After having introduced these general notions, we want to consider now a concrete market model with 2 or $d+1$ securities, respectively.

This standard model, which is often referred to as the Black and Scholes model, is analogous to the discrete time case. The idea is that relative changes in prices evolve homogenous in time (i.e. stationary) and independent of the past. Due to these assumptions the model is - up to a few parameters - uniquely determined, in contrast to the discrete time case, where the distribution of daily price changes could be chosen arbitrarily. This is a consequence of the surprising fact (see Theorem 1.55) that every continuous process with independent and stationary increments is a Brownian motion with drift. Hence, the term "*standard model*" is indeed appropriate.

We model the riskless asset (bank account or *bond*) by

$$S_t^0 = S_0^0 \exp\{rt\} = S_0^0 \mathcal{E}(rI)_t$$

with a constant interest rate $r \in \mathbb{R}$.

Moreover, we consider a *risky asset* whose price process is assumed to equal

$$S_t^1 = S_0^1 \exp\{\mu t + \sigma W_t\} = S_0^1 \mathcal{E}(\tilde{\mu} I + \sigma W)_t,$$

where $\mu, \sigma \in \mathbb{R},\ \tilde{\mu} = \mu + \frac{\sigma^2}{2}$ and W is a standard Brownian motion.

Thus, S^1 is a geometric Brownian motion. The generalisation to d assets is then given by

$$S_t^i = S_0^i \exp\left\{\mu^i t + \sum_{j=1}^{d} \sigma^{ij} W_t^j\right\} = S_0^i \mathcal{E}\left(\tilde{\mu}^i I + \sum_{j=1}^{d} \sigma^{ij} W_t^j\right),$$

where $\mu \in \mathbb{R}^d$, $\sigma \in \mathbb{R}^{d\times d}$, $\tilde{\mu}^i = \mu^i + \frac{1}{2}(\sigma\sigma^T)^{ii}$ and W is a standard Brownian motion on $\mathbb{R}^d$.

2.2 Pricing and Hedging of Derivatives

In this section we consider the problem of pricing and hedging random payments using duplication strategies and arbitrage arguments. The notions and results resemble those of the discrete time case.

Again we assume that the market model we are working with is based on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ with terminal time point $T \in \mathbb{R}_+$, $\mathcal{F}_0 = (\emptyset, \Omega)$ and $\mathcal{F}_T = \mathcal{F}$.

As above let $S = (S^0, \ldots, S^d)$ be an $\mathbb{R}^{d+1}$-valued, componentwise non-negative asset process such that the numeraire S^0 and S^0_- are positive. The market is assumed to be arbitrage free in the sense of Definition 2.7.

We call $\mathcal{F}_T$-measurable random variables X pay-off functions (sometimes also options or derivatives). The random variable $\hat{X} := \frac{X}{S^0_T}$ is referred to as the discounted pay-off. Our aim is to allocate to a random pay-off function a unique and fair initial price as well as deriving hedging strategies. For replicable pay-off functions this can be done in a satisfying way.

Definition 2.17 *We call a random pay-off function (option, derivative)* X replicable *if there exists a double-admissible strategy* ϕ *having terminal value* $V_T(\phi) = X$*. In this case we call* ϕ *the duplication strategy of* X*.*

Introducing options (pay-off functions) into financial markets means allowing for additional trading possibilities. Pricing of options is based on the assumption that the market is arbitrage free. The price of the option is then equivalent to the expected value under an equivalent martingale measure.

Theorem 2.18 *Let* X *denote a non-negative, replicable random pay-off function with duplication strategy* ϕ*. Then*

1. *there exists a unique process* S^{d+1} *with* $S^{d+1}_T = X$ *such that the extended market* $(S^0, \ldots, S^{d+1})$ *is arbitrage free, namely* $S^{d+1} = V(\phi)$*. In this case even the NFLVR condition is satisfied for* $(S^0, \ldots, S^{d+1})$*.*

2. *if* Q *is an equivalent martingale measure then* $\hat{X}$ *is integrable under the measure* Q *and*

$$\hat{S}^{d+1}_t = E_Q(\hat{X} \mid \mathcal{F}_t)$$

for the price process S^{d+1}*.*

Proof. 1. To show existence we set $S^{d+1} := V(\phi)$. Then from Remark 2.16 follows that S^{d+1} is non-negative. If Q is an equivalent martingale measure for the market $(S^0, \ldots, S^d)$, then $\hat{S}^{d+1} = \hat{V}(\phi)$ as well as $-\hat{S}^{d+1} = \hat{V}(-\phi)$ are Q-supermartingales (see Lemma 2.12). Hence, $\hat{S}^{d+1}$ is a Q-martingale and thus (see Theorem 2.9) the market $(S^0, \ldots, S^{d+1})$ is arbitrage free and even satisfies the NFLVR condition (see Theorem 2.11). Uniqueness of S^{d+1} is a direct consequence of Lemma 2.15.

2. Since $\hat{S}^{d+1}$ is a Q-martingale, we can conclude that $\hat{X} = \hat{S}^{d+1}_T \in L^1(Q)$ and

$$\hat{S}^{d+1}_t = E_Q(\hat{S}^{d+1}_T \mid \mathcal{F}_t) = E_Q(\hat{X} \mid \mathcal{F}_t).$$

□

Notation 2.19 *We call S^{d+1} in the preceding theorem the* fair (hedging) price *of X and refer to ϕ as the corresponding (perfect)* hedging strategy*.*

Theorem 2.18 is inadequate for forward contracts.

Example 2.20 (Forward contract) *Forward contracts are the simplest derivatives. Here one contracting party obtains at the terminal time point T the underlying asset S^1 whereas the other party obtains a fixed forward price K when signing the contract (e.g. at time point 0). The forward price K is chosen such that at time zero the contract has no value. Hence, the pay-off at time T is given by*

$$X := S^1_T - K$$

from the point of view of the first party. Such a payment can be replicated by the underlying asset S^1 and a bond S^0 falling due at time T, where we assume that the bond S^0 has value 1 at time point T. Obviously, a constant portfolio of the form

$$\phi = (\phi^0, \phi^1) = (-K, 1) \tag{2.3}$$

has value X at time T and thus replicates the payment. Its initial value $V_0(\phi) = -KS^0_0 + S^1_0$ is zero if and only if

$$K = \frac{S^1_0}{S^0_0}, \tag{2.4}$$

which is then the unique hedging price of the forward contract.

Theorem 2.18 cannot be applied in this case, since the pay-off X can be negative. Moreover, Theorem 2.18 assumes that the forward (like all other derivatives) is traded continuously. However, forward contracts are usually traded OTC, i.e. contracts between the two parties take place only at time zero. Therefore in this case we have no stock exchange trading which could serve as a basis for arbitrage arguments.

However, in some market models every pay-off function is replicable.

Definition 2.21 *A financial market is called* complete *if every bounded discounted pay-off function X (i.e. with $\hat{X}$ bounded) is replicable.*

Similar to the absence of arbitrage, market completeness can be characterised in terms of martingale measures.

Theorem 2.22 (Second fundamental theorem of asset pricing) *The following are equivalent*

1. *The market satisfies the NFLVR condition and is complete.*
2. *There exists a unique equivalent martingale measure Q.*

In this case every pay-off function can be replicated, which satisfies

$$|X| \leq c \sum_{i=0}^{d} S_T^i$$

for a constant $c \in \mathbb{R}_+$.

Proof. 1. $\Rightarrow$ 2.: Existence of an equivalent martingale measure Q is a consequence of Theorem 2.11. Suppose $A \in \mathcal{F}$ and $\hat{X} := 1_A$. It follows from Theorem 2.18 that

$$E_Q(\hat{X} \,|\, \mathcal{F}_0) = E_Q(\hat{X}) = Q(A)$$

does not depend on which measure Q we choose.

2. $\Rightarrow$ 1.: Similar to the proof of Theorem 2.11 we define $S^\Sigma := \sum_{i=0}^d S^i$ and $\tilde{S} := S/S^\Sigma$. We denote the discounted value (with respect to S^Σ) of a strategy ϕ by $\tilde{V}(\phi) := V(\phi)/S^\Sigma$. Assume that $\tilde{Q} \sim P$ is a martingale measure such that $\tilde{S}$ is a Q-local martingale. Since $\tilde{S}$ is bounded, $\tilde{S}$ is even a $\tilde{Q}$-martingale (see Theorem 1.37). Furthermore, in the proof of Theorem 2.11 we showed that

$$\frac{dQ}{dP} := \frac{S_0^\Sigma S_T^0}{S_0^0 S_T^\Sigma} \frac{d\tilde{Q}}{dP}$$

is the density of an equivalent martingale measure Q with respect to the numeraire S^0. Since there exists only one such equivalent martingale measure Q it follows that $\tilde{Q}$ is unique, too. On the other hand, from existence of an equivalent martingale measure Q follows existence of a measure $\tilde{Q}$ as defined above.

Now, assume that X is a random pay-off function such that $\hat{X} := X/S_T^0$ or more general $\tilde{X} := X/S_T^\Sigma$ is bounded, i.e. $|\tilde{X}| \leq c \in \mathbb{R}_+$. Moreover, suppose that M is the $\tilde{Q}$-martingale generated by $\tilde{X}$, i.e. $M_t := E_{\tilde{Q}}[\tilde{X} \,|\, \mathcal{F}_t]$. Due to Theorem 1.67 there exists $H \in L(\tilde{S})$ such that

$$\tilde{X} = M_T = M_0 + H \bullet \tilde{S}_T. \tag{2.5}$$

Since $\sum_{i=0}^d \tilde{S}^i = 1$ we have $K \bullet (\sum_{i=0}^d \tilde{S}^i) = 0$ for any arbitrary predictable and real K.

Consequently, $H + (K, \ldots, K)$ instead of H satisfies equality (2.5), too. K can be chosen such that $\phi := H + (K, \ldots, K)$ is a self-financing strategy with $\tilde{V}_0(\phi) = M_0$ (see the proof of Lemma 2.4). In fact,

$$\phi_t^T \tilde{S}_t - (M_0 - \phi \bullet \tilde{S}_t) = H_t^T \tilde{S}_t + K_t - M_0 - H \bullet \tilde{S}_t,$$

becomes zero if we set

$$K_t = M_0 + H \bullet \tilde{S}_t - H^T \tilde{S}_t = M_0 + H \bullet \tilde{S}_{t-} - H_t^T \tilde{S}_{t-}.$$

Then $\tilde{X} = \tilde{V}_0(\phi) + \phi \bullet \tilde{S}_T = \tilde{V}_T(\phi)$ and thus

$$X = \tilde{X} S_T^\Sigma = \tilde{V}_T(\phi) S_T^\Sigma = V_T(\phi).$$

Since $|\tilde{V}(\phi)| = |M| \leq c$ we obtain $V(\phi) = \tilde{V}(\phi) S^\Sigma \geq -cS^\Sigma$ and $V(-\phi) = -\tilde{V}(\phi) S^\Sigma \geq -cS^\Sigma$, i.e. ϕ is a double-admissible strategy. Note that the self-financing property does not depend on the numeraire (see Lemma 2.3). Consequently, X is replicable in the sense of Definition 2.17. $\square$

3 The Black and Scholes Model

The Black and Scholes formula plays a central role in modern mathematical finance. Besides Black and Scholes also Merton was involved in developing this fundamental formula. However, Merton published his results in a separate paper, which is the reason why his name was not included. At least, in 1997 Merton, as well as Scholes won the Nobel prize for his achievements (Black had already died). Basically, the Black and Scholes model is the continuous time analogue of the discrete binomial model.

The starting point of the Black and Scholes model is the standard market model considered in the preceding chapter which consits of one riskless asset (bond) and one risky asset (stock), i.e.

$$S_t^0 = e^{rt},$$

$$S_t^1 = S_0^1 \exp(\mu t + \sigma W_t) = S_0^1 \mathcal{E}(\tilde{\mu} I + \sigma W)_t$$

with constants $S_0^1 > 0$, $r \in \mathbb{R}$, $\mu \in \mathbb{R}$, $\sigma \neq 0$, $\tilde{\mu} = \mu + \frac{\sigma^2}{2}$ and a standard Brownian motion W. The filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is assumed to be generated by S^1, i.e. all randomness in the model comes from the stock price process.

3.1 Market Completeness and the Black and Scholes Formula

We first show the completeness of the standard market model with two assets, which then implies (as in the binomial model) that derivatives are replicable and thus can be priced in a fair manner.

Theorem 3.1 *1. By*

$$\frac{dQ}{dP} := \mathcal{E}\left(-\frac{\tilde{\mu}-r}{\sigma}W\right)_T = \exp\left(-\frac{\tilde{\mu}-r}{\sigma}W_T - \frac{1}{2}\left(\frac{\tilde{\mu}-r}{\sigma}\right)^2 T\right)$$

an equivalent martingale measure $Q \sim P$ is defined.

2. We have

$$\hat{S}_t^1 = \hat{S}_0^1 \exp\left(\widetilde{W}_t - \frac{\sigma^2}{2}t\right) = \hat{S}_0^1 \mathcal{E}(\sigma\widetilde{W})_t$$

for the Q-standard Brownian motion $\widetilde{W}_t = W_t + \frac{\tilde{\mu}-r}{\sigma}t$.

3. The market satisfies the NFLVR condition and is complete.

Proof. 1., 2. The geometric Brownian motion

$$Z := \mathcal{E}\left(-\frac{\tilde{\mu}-r}{\sigma}W\right)$$

is a martingale due to Theorem 1.58. Since $E(Z_T) = E(Z_0) = 1$, it follows that Z is the density process of a probability measure $Q \sim P$. Applying Theorem 1.61 the following process is a Q-standard Brownian motion:

$$W - [W, -\frac{\tilde{\mu}-r}{\sigma}W] = W + \frac{\tilde{\mu}-r}{\sigma}[W, W] = \widetilde{W}.$$

Since,

$$\begin{aligned} S_t^1 &= \hat{S}_0^1 \exp((\mu - r)t + \sigma W_t) \\ &= \hat{S}_0^1 \exp((\mu - \tilde{\mu})t + \sigma \widetilde{W}_t) \\ &= \hat{S}_0^1 \exp(\sigma \widetilde{W}_t - \frac{\sigma^2}{2}t) \\ &= \hat{S}_0^1 \, \mathcal{E}(\sigma\widetilde{W})_t \end{aligned}$$

we can conclude furthermore (Theorem 1.58) that $\hat{S}^1$ is a Q-martingale, since it is a Q-locale martingale and a geometric Lévy process.

3. The NFLVR condition is satisfied due to the first fundamental theorem of asset pricing (Theorem 2.11). Since,

$$S_t^1 = S_0^1 \exp\left((r - \frac{\sigma^2}{2})t + \sigma\widetilde{W}_t\right)$$

is a measurable function of $\widetilde{W}_t$ and vice versa, the filtration is also generated by $\widetilde{W}$. Now assume that X is a random pay-off function satisfying $|\hat{X}| \leq m \in \mathbb{R}_+$, and denote by $\hat{S}^2$ the Q-martingale generated by $\hat{X}$. Due to the martingale representation theorem (Theorem 1.66) there exists $H \in L(\widetilde{W})$ such that

$$\hat{S}^2 = \hat{S}^2_0 + H \bullet \widetilde{W}.$$

For $\phi^1 := \frac{H}{\sigma \hat{S}^1}$ we then have

$$\hat{S}^2 = \hat{S}^2_0 + \phi^1 \bullet (\sigma \hat{S}^1 \bullet \widetilde{W}) = \hat{S}^2_0 + \phi^1 \bullet \hat{S}^1.$$

Suppose ϕ is the self-financing strategy of Lemma 2.4 corresponding to ϕ^1 and initial capital $\hat{S}^2_0$. Since, $\hat{V}(\phi) = \hat{S}^2$ it follows $|\hat{V}(\phi)| \leq m$, i.e. ϕ is double-admissible. Therefore, ϕ is a duplication strategy for X. □

For pay-off functions which only depend on the price process of the stock at some specific time point, the price of the option can be calculated in terms of an integral with respect to the normal distribution. The process is Markovian in the sense that it only depends on the present but not on the past price process of the stock. The option pricing function f can also be expressed in terms of a solution to a partial differential equation (PDE). The duplication strategy is then derived by differentiating the option price with respect to the price process of the underlying stock.

Theorem 3.2 *Suppose we are given a random pay-off function $X = g(S^1_T)$, where $g : (0,1) \mapsto \mathbb{R}_+$ is a function satisfying $g(x) \leq c(1+x)$ for some $c \in \mathbb{R}_+$. Then*

1. *X is replicable.*
2. *The fair price process S^2 of X has the representation*

   $$S^2_t = f(t, S^1_t)$$

   for

   $$f(t,x) = e^{-r(T-t)} \int g(e^y) N\left(\log(x) + (r - \frac{\sigma^2}{2})(T-t), \sigma^2(T-t)\right)(dy).$$

3. *f is twice continuously differentiable on $[0,T) \times (0,1)$.*

4. *The strategy* $\phi = (\phi^0, \phi^1)$ *defined by*

$$\phi_t^1 = D_2 f(t, S_t^1),$$

$$\phi_t^0 = e^{-rt}\left(f(t, S_t^1) - S_t^1 D_2 f(t, S_t^1)\right)$$

is a duplication strategy for X.

5. f *satisfies the PDE*

$$D_1 f(t,x) + rxD_2 f(t,x) + \frac{1}{2}x^2\sigma^2 D_{22} f(t,x) - rf(t,x) = 0.$$

Proof. 1. Since, $0 \le X \le c(1 + S_T^1) \le c(S_T^0 + S_T^1)$ it follows from Theorem 2.22 that X is replicable.

2. From Theorem 3.1 follows that $\hat{S}^2$ is a Q-martingale, thus

$$\begin{aligned}\hat{S}_t^2 &= E_Q(\hat{X} \mid \mathcal{F}_t)\\ &= e^{-rT}\, E_Q(g(S_T^1) \mid \mathcal{F}_t)\\ &= e^{-rT}\, E_Q\left(g\left(S_t^1 \exp\left\{(r - \frac{\sigma^2}{2})(T-t) + \sigma(\widetilde{W}_T - \widetilde{W}_t)\right\}\right) \Big| \mathcal{F}_t\right).\end{aligned}$$

Since S^1 is $\mathcal{F}_t$-measurable and since $\widetilde{W}_T - \widetilde{W}_t$ is independent of $\mathcal{F}_t$ and under Q is $N(0, T-t)$-distributed, it follows that

$$\begin{aligned}\hat{S}_t^2 &= e^{-rT}\int g\left(S_t^1 \exp\left\{(r - \frac{\sigma^2}{2})(T-t) + \sigma y\right\}\right) N(0, T-t)\,(dy)\\ &= e^{-rT}\int g(e^y) N\left(\log S_t^1 + (r - \frac{\sigma^2}{2})(T-t), \sigma^2(T-t)\right)(dy).\end{aligned}$$

3. This follows from the theorem of interchanging integration and differentiation, as well as dominated convergence.

4. From 2. follows that $\hat{S}_t^2 = \hat{f}(t, \hat{S}_t^1)$ with $\hat{f}(t,x) = e^{-rt} f(t, xe^{rt})$. Since $f \in C^2$, also $\hat{f} \in C^2$. An application of Itô's formula yields

$$\begin{aligned}\hat{S}_t^2 &= \hat{f}(t, \hat{S}_t^1) = e^{-rt} f(t, \hat{S}_t^1 e^{rt})\\ &= \hat{S}_0^2 + D_1\hat{f}(I, \hat{S}^1) \bullet I_t + D_2\hat{f}(I, \hat{S}^1) \bullet \hat{S}_t^1 + \frac{1}{2} D_{22}\hat{f}(I, \hat{S}^1) \bullet [\hat{S}^1, \hat{S}^1]_t\\ &= \hat{S}_0^2 + D_2\hat{f}(I, \hat{S}^1) \bullet \hat{S}_t^1 + \left(D_1\hat{f}(I, \hat{S}^1) + \frac{1}{2} D_{22}\hat{f}(I, \hat{S}^1)\sigma^2(\hat{S}^1)^2\right) \bullet I_t.\end{aligned}$$

$\hat{S}^2$ is a Q-martingale and thus a Q-special semimartingale whose compensator is 0. Since this compensator equals exactly the last term in the above Itô process representation of $\hat{S}^2$, we can conclude that

$$\int_0^t \left(D_1\hat{f}(s,\hat{S}_s^1) + \frac{1}{2} D_{22}\hat{f}(s,\hat{S}_s^1)\sigma^2(\hat{S}_s^1)^2 \right) ds = 0 \tag{3.1}$$

and

$$\hat{S}_t^2 = \hat{S}_0^2 + D_2\hat{f}(I,\hat{S}^1) \bullet \hat{S}_t^1$$

for all $t \in [0,T]$. Suppose ϕ is the self-financing strategy corresponding to $\phi_t^1 := D_2\hat{f}(I,\hat{S}^1)$ and initial capital S_0^2. Since $0 \leq V(\phi) = S^2 \leq S^1$, the strategy ϕ is double admissible. Furthermore,

$$D_2\hat{f}(t,x) = e^{-rt}e^{rt}D_2 f(t,xe^{rt}) = D_2 f(t,xe^{rt}),$$

i.e. $\phi_t^1 = D_2 f(t,S_t^1)$. The amount of the numeraire in the duplication portfolio follows from

$$\begin{aligned}\phi_t^0 &= \hat{V}_t(\phi) - \phi_t^1\hat{S}_t^1 = \hat{f}(t,\hat{S}_t^1) - \hat{S}_t^1 D_2 f(t,S_t^1)\\ &= e^{-rt}\left(f(t,S_t^1) - S_t^1 D_2 f(t,S_t^1)\right).\end{aligned}$$

5. From (3.1) we conclude that

$$D_1\hat{f}(t,\hat{S}_t^1) + \frac{1}{2}D_{22}\hat{f}(t,\hat{S}_t^1)\sigma^2(\hat{S}_t^1)^2 = 0$$

for Lebesgue-almost every $t \in [0,T]$ and even for all t since the expression is continuous. Thus, for fixed t we have

$$D_1\hat{f}(t,x) + \frac{1}{2}D_{22}\hat{f}(t,x)\sigma^2 x^2 = 0$$

for $P^{\hat{S}_t^1}$-almost every $x > 0$ and even for all $x > 0$, since the distribution of $\hat{S}_t^1$ is equivalent to the Lebesgue measure and the expression is continuous in x. Consequently,

$$\begin{aligned}0 &= D_1\hat{f}(t,x) + \frac{1}{2}D_{22}\hat{f}(t,x)\sigma^2x^2\\ &= -re^{-rt}f(t,xe^{rt}) + e^{-rt}D_1 f(t,xe^{rt}) + rxD_2 f(t,xe^{rt}) + \frac{1}{2}\sigma^2x^2e^{rt}D_{22}f(t,xe^{rt})\\ &= e^{-rt}\left(-rf(t,\tilde{x}) + D_1 f(t,\tilde{x}) + r\tilde{x}D_2 f(t,\tilde{x}) + \frac{1}{2}\sigma^2\tilde{x}^2 D_{22}f(t,\tilde{x})\right),\end{aligned}$$

where $\tilde{x} = xe^{rt}$. □

For concrete pay-off functions like the Europian Call the integral can be calculated explicitly. Note that the Black and Scholes formula resembles the option pricing formula in the discrete time Cox-Ross-Rubinstein-model. Instead of a binomial distribution we have now a standard normal distribution. However, in the Black and Scholes model even for the duplication strategy a simple explicit expression is available.

Theorem 3.3 (Black and Scholes formula) *Let $X = (S^1 - K)^+$ for a $K \in \mathbb{R}_+$. Then for the fair price process S^2 of X we have*

$$S_t^2 = S_t^1 \Phi\left(\frac{\log(S_t^1/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) - e^{-r(T-t)} K \Phi\left(\frac{\log(S_t^1/K) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right),$$

where Φ is the distribution function of the standard normal distribution. For the duplication strategy $\phi = (\phi^0, \phi^1)$ we have

$$\phi_t^1 = \Phi\left(\frac{\log(S_t^1/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right),$$

$$\phi_t^0 = -e^{rT} K \Phi\left(\frac{\log(S_t^1/K) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right).$$

Proof. We have $X = g(S_T^1)$ with $g(x) = (x-K)^+$. From Theorem 3.2 follows $S_t^2 = f(t, S_t^1)$ with

$$\begin{aligned} f(t,x) &= e^{-r(T-t)} \int g(e^y) N\left(\log x + (r - \frac{\sigma^2}{2})(T-t), \sigma^2(T-t)\right)(dy) \\ &= e^{-r\tau} \int g(x \exp(\tilde{r}\tau + \sigma\sqrt{\tau}y)) N(0,1)\,(dy) \\ &= e^{-r\tau} \int_z^\infty (x \exp(\tilde{r}\tau + \sigma\sqrt{\tau}y) - K) N(0,1)\,(dy) \\ &= xe^{-\frac{\sigma^2}{2}\tau} \int_z^\infty e^{y\sigma\sqrt{\tau}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy - e^{-r\tau} K\, N(0,1)([z,\infty)) \end{aligned}$$

$$
\begin{aligned}
&= x \int_z^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\sigma\sqrt{\tau})^2}{2}}\, dy - e^{-r\tau} K\, N(0,1)((-\infty, -z]) \\
&= x N(0,1)([z - \sigma\sqrt{\tau}, \infty)) - e^{-r\tau} K\, N(0,1)((-\infty, -z]) \\
&= x\Phi(-z + \sigma\sqrt{\tau}) - e^{-r\tau} K\Phi(-z),
\end{aligned}
$$

$\tau := T - t$, $\tilde{r} := r - \frac{\sigma^2}{2}$, $z := \frac{\log(K/x) - \tilde{r}\tau}{\sigma\sqrt{\tau}}$. Moreover, also due to Theorem 3.2 we have

$$
\begin{aligned}
\phi_t^1 &= D_2 f(t,x) \\
&= \Phi\left(\frac{\log(S_t^1/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \\
&\quad + x\Phi'\left(\frac{\log(S_t^1/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \frac{1}{x\sigma\sqrt{T-t}} \\
&\quad - e^{-r(T-t)} K\Phi'\left(\frac{\log(S_t^1/K) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \frac{1}{x\sigma\sqrt{T-t}}.
\end{aligned}
$$

Since, $\Phi'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ it follows

$$
\begin{aligned}
&\Phi'\left(\frac{\log(S_t^1/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \Big/ \Phi'\left(\frac{\log(S_t^1/K) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= \exp(\log(K/x) - r(T-t)),
\end{aligned}
$$

which yields the assertion for ϕ^1. Moreover,

$$
\phi_t^0 = e^{-rt}(S_t^2 - \phi_t^1 S_t^1) = -e^{rT} K\Phi\left(\frac{\log(S_t^1/K) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right).
$$

□

Note that the drift has no influence on the price of the option. Thus, the price of the Call does not depend on whether the stock on average heavily increases or decreases, though the probability that the Call is in the money is very much influenced by this parameter. This result, which contradicts our intuition, is based on the fact that the option is priced only relatively with respect to the underlying stock. An extreme drift parameter indicates that the underlying stock has already a price far away from the expected future stock price. This property is then reflected in the price of the option which is correlated with the stock. Hence, the Black and Scholes formula has finally only one (not directly observable) parameter, namely *volatility* σ. If the model coincides well with reality, volatility can be estimated from finance data. The availability of explicit formulas, as well as the dependence on only one parameter, lead to the wide spreading of the Black and Scholes formula. However, like the underlying standard market model, when compared with real market data, the Black and Scholes formula shows many shortcomings.

Remark 3.4 *For a European Put $X = (K - S^1)^+$ we obtain analogously:*

$$S_t^2 = e^{-r(T-t)} K \Phi \left(\frac{\log(K/S_t^1) - (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \right) - S_t^1 \left(\frac{\log(K/S_t^1) - (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \right).$$

For the duplication strategy $\phi = (\phi^0, \phi^1)$ we then have

$$\phi_t^1 = -\Phi \left(\frac{\log(K/S_t^1) - (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \right),$$

$$\phi_t^0 = e^{rT} K \, \Phi \left(\frac{\log(K/S_t^1) - (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \right).$$

Proof. We can proof the assertions analogously to Theorem 3.3 or using the Call-Put-parity. A Put can be rewritten in terms of the bond S^0, the stock S^1 and the Call:

$$(K - S_T^1)^+ = (S_T^1 - K)^+ - S_T^1 + K e^{-rT} S_T^0.$$

The Call itself can then be replicated and priced as described in Theorem 3.3, which yields the assertion. □

3.2 Practical Applications

In order to apply the Black and Scholes formula one has to determine the parameters. If we assume that the interest rate r is continuously constant, we can observe all paramters except the volatility σ. Now suppose we are given price data $(S^1_{t_i})_{i=0,\ldots,n}$ at equidistant time points $t_i = t_0 + i\delta$. Then the sample variance, given by

$$\hat{\sigma} := \frac{1}{(n-1)\delta} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

is a natural estimator for σ, where $X_i := \log(S^1_{t_i}) - \log(S^1_{t_{i-1}})$ and

$$\bar{X} := \frac{1}{n} \sum_{i=1}^{n} X_i.$$

The reason for this is that the random variables $X_1, \ldots, X_n$ are indepent and $N(\mu\delta, \sigma\delta)$-distributed.

However, in practical applications one often faces the problem that the values obtained from this estimator are not as constant as the model supposes them to be.

An alternative approach is to use one concrete Call option which is traded on some stock exchange, in order to use its current market value to determine the implicit volatility by solving the Black and Scholes formula with respect to σ. If all assumptions of the theory are met the σ derived by this method should neither depend on the strike price K nor on the expiry date T of the option. Furthermore this value should remain constant over time and coincide with the volatility parameter of the observed stock prices.

In reality however, implicit volatility depends strongly on the srike price and the expiry date, which implies that real prices and the prices obtained from the Black and Scholes model are inconsistent.

In the context of applying the Black and Scholes Model people consider various partial derivatives, which are usually denoted by Greek letter and therefore are refered to as the "Greeks".

We consider a hedging portfolio having a value process

$$V_t(\phi) = f(t, S^1_t; r, \sigma)$$

like it appeared e.g. in Theorems 3.2 and 3.3 for concrete options. Often discussed are the following derivatives.

$$\begin{aligned}\Delta &= D_2 f(t, S_t^1; r, \sigma),\\ \Gamma &= D_{22} f(t, S_t^1; r, \sigma),\\ \varrho &= D_3 f(t, S_t^1; r, \sigma),\\ \Theta &= D_1 f(t, S_t^1; r, \sigma),\\ \nu &= D_4 f(t, S_t^1; r, \sigma).\end{aligned}$$

The *Delta* of the portfolio is the partial derivative with respect to the stock, which due to Theorem 3.2 gives the amount of stocks in the hedging portfolio. If this amount is strongly dependent on the current stock price, then usually high transaction costs arise, since the portfolio has to be adjusted very often. Therefore, the total portfolio should have a small *Gamma*. Large values of ϱ or ν (the so-called *Vega*) indicate high sensibility to the parameters r or σ, respectively.

3.3 American Options

American options are derivatives where the option writer can choose the date of exercise. In the mathematical world they are treated using the theory of optimal stopping. However, optimal stopping is a complex area and therefore here we will only discuss some aspects.

Again we are working in the setting we established in Chapter 2. Unlike Europian options (as discussed in Section 2.2), American options are defined in terms of a whole stochastic process $\{X_t\}_{t\in\mathbb{R}_+}$, where the random variable X_t describes the amount the option writer recieves when exercising the option at time point t. We denote the discounted pay-off process again by $\hat{X} := X/S^0$.

From now on we consider the Black and Scholes model and an American option with pay-off process $0 \le X \le S^1$.

For instance for the *American call* we have $X_t = (S_t^1 - K)^+$. In the case of an *American put* we have $X_t = (K - S_t^1)^+$. For simplicity we assume that for every $t \in [0, T]$ there exists an *optimal stopping time* τ with values in $[t, T]$ such that

$$\hat{S}_t^2 := E_Q(\hat{X}_\tau \,|\, \mathcal{F}_t) \ge E_Q(\hat{X}_\sigma \,|\, \mathcal{F}_t) \tag{3.2}$$

for every stopping time σ with values in $[t, T]$. Here Q denotes the equivalent martingale measure of Theorem 3.1. For concrete options this assumption is usually met.

Lemma 3.5 *The process $\hat{S}^2$ is the smallest Q-supermartingale greater (or equal) than $\hat{X}$ (the so-called* Snell envelope *of $\hat{X}$).*

Proof. Adaptedness and integrability are obvious. We omit the proof of the càdlàg property. Let $s \leq t$ and σ, τ denote the optimal stopping times corresponding to s, t. The Q-supermartingale property follows from

$$E_Q(\hat{S}^2_t \,|\, \mathcal{F}_s) = E_Q\left(E_Q(\hat{X}_\tau \,|\, \mathcal{F}_t)\Big|\, \mathcal{F}_s\right) = E_Q(\hat{X}_\tau \,|\, \mathcal{F}_s) \leq E_Q(\hat{X}_\sigma \,|\, \mathcal{F}_s) = \hat{S}^2_s.$$

Moreover, from (3.2) we conclude $\hat{S}^2_t \geq \hat{X}_t$ for $\sigma := t$.

Now assume that $S \geq \hat{X}$ is also a Q-supermartingale. Then

$$\hat{S}^2_t = E_Q(\hat{X}_\tau \,|\, \mathcal{F}_t) \leq E_Q(S_\tau \,|\, \mathcal{F}_t) \leq S_t.$$

□

Similar to Example 2.20 we now want to show, why the unique (arbitrage free) price of the American option at time t is S^2_t of all prices. In the following argument all prices are assumed to be discounted with respect to the numeraire S^0.

If the discounted market price $(\pi_t)_{t\in[0,T]}$ of the American option does not agree with the above Snell envelope, then there exists a $t \in [0, T]$ with $P(\pi_t \neq \hat{S}^2_t) > 0$. Since $\hat{S}^2$ is a Q-supermartingale it has a Doob-Meyer-decomposition $\hat{S}^2 = M + A$, where M is a Q-martingale und A is a predictable, monotone decreasing process (Theorem 1.21).

If $P(C) > 0$ for $C = \{\pi_t < \hat{S}^2_t\} \in \mathcal{F}_t$, a riskless profit can be generated as follows. At time t if C happens we buy an American option at the price of π_t and at the same time we sell at the price of $E_Q(\hat{X}_\tau \,|\, \mathcal{F}_t) = \hat{S}^2_t$ the duplication strategy of the option with terminal pay-off $\hat{X}_\tau$, where τ is the optimal stopping time corresponding to t in the sense of (3.2). The remaining difference $\hat{S}^2_t - \pi_t > 0$ is invested in the numeraire. At time τ we exercise the option. The proceeds $\hat{X}_\tau$ balance the liabilities of the short sold duplication portfolio. Hence, the investment in the numeraire remains as a riskless profit.

If $P(C) > 0$ for $C = \{\hat{S}^2_t < \pi_t\} \in \mathcal{F}_t$ we sell at time t if C happens an American option and buy the replicating strategy of an option with terminal pay-off M_T. The difference $\pi_t - M_t = \pi_t - \hat{S}^2_t > A_t$ is invested in the numeraire. If the option writer exercises the option at time $s > t$, (s)he recieves $\hat{X}_s \le \hat{S}^2_s = M_s - A_s$. In order to finance this amount, one sells the duplication portfolio and thus recieves M_s. The shortfall $A_s \le A_t$ is withdrawn from the numeraire account. It remains a positive profit on the numeraire account contradicting absence of arbitrage.

For the American call the fair price in the case $r \ge 0$ can be calculated explicitly. It coincides with the price of the corresponding Europian call.

Theorem 3.6 *Let $X_t = g(S^1_t)$ for a non-negative, monotone increasing, convex function g (e.g. $g(x) = (x-K)^+$ in the case of the American call). If $r \ge 0$ the above fair price process is exactly the fair price process of the random pay-off function $X_T = g(S^1_T)$ in the sense of Chapter 2 (i.e. for instance of the Europian call $(S^1_T - K)^+$).*

Proof. Let $M_t := E_Q(\hat{X}_T \,|\, \mathcal{F}_t)$ denote the Q-martingale generated by $\hat{X}_T$. Hence, M_t is the discounted fair price process of the Europian call with pay-off X_T. Obviously, $g(cx) \ge cg(x)$ for $x \ge 0, c \ge 1$. Applying Jensen's inequality we obtain for all $t \in [0,T]$

$$\begin{aligned} M_t &= E_Q(g(\hat{S}^1_T e^{rT}) e^{-rT} \,|\, \mathcal{F}_t) \\ &\ge E_Q(g(\hat{S}^1_T e^{rt}) e^{-rt} \,|\, \mathcal{F}_t) \\ &\ge g(E_Q(\hat{S}^1_T e^{rt} \,|\, \mathcal{F}_t)) e^{-rt} \\ &= g(\hat{S}^1_t e^{rt}) e^{-rt} \\ &= \hat{X}_t. \end{aligned}$$

Now fix $t \in [0,T]$. For every stopping time σ taking values in $[t,T]$ it follows

$$E_Q(\hat{X}_\sigma \,|\, \mathcal{F}_t) \le E_Q(M_\sigma \,|\, \mathcal{F}_t) = E_Q(E_Q(M_T \,|\, \mathcal{F}_\sigma) \,|\, \mathcal{F}_t) = E_Q(\hat{X}_T \,|\, \mathcal{F}_t).$$

Therefore $\tau := T$ is the optimal stopping time in (3.2). Finally, this yields

$$\hat{S}^2_t = E_Q(\hat{X}_T \,|\, \mathcal{F}_t) = M_t.$$

□

For the American put no closed formulae are known. Only for the put with infinite maturity there exist explicit results. In this case the option is exercised if the stock price falls beyond a critical threshold value. In the more realistic case of an American put with finite maturity the structure is similar; however the exercise border depends on time to maturity and is numerically determinable only. In order to obtain this numerical solution one has to solve a *free boundary value problem.*

Theorem 3.7 *Suppose that the pay-off process X is given by $X_t = g(t, \hat{S}^1_t)$ for a bounded function $g : [0,T] \times (0,1) \to \mathbb{R}$. Moreover, assume that $f : [0,T] \times (0,1) \to \mathbb{R}$ is a bounded, twice continuously differentiable function such that*

1. $$D_1 f + rxD_2 f + \frac{1}{2}x^2\sigma^2 D_{22} f - rf \leq 0 \quad \textit{and} \quad f \geq g$$
   on $[0,T] \times (0,\infty)$,

2. $$\left(D_1 f + rxD_2 f + \frac{1}{2}x^2\sigma^2 D_{22} f - rf\right)(f - g) = 0$$
   on $[0,T] \times (0,\infty)$,

3. $$f(T,x) = g(T,x), \quad x \in (0,\infty).$$

Then the fair price process of the American option with pay-off X has the form

$$S^2_t = f(t, S^1_t).$$

Proof. We first define $\hat{f}(t,x) := e^{-rt} f(t, xe^{rt})$ and $\hat{g}(t,x) := e^{-rt} g(t, xe^{rt})$. It is easy to show that $\hat{f}$ and $\hat{g}$ satisfy the above conditions for $r = 0$. An application of Itô's formula yields

$$\begin{aligned}\hat{f}(t, \hat{S}^1_t) &= \hat{f}(0, \hat{S}^1_0) + D_1\hat{f}(I, \hat{S}^1) \bullet I_t + D_2\hat{f}(I, \hat{S}^1) \bullet \hat{S}^1_t + \frac{1}{2}D_{22}\hat{f}(I, \hat{S}^1) \bullet [\hat{S}^1]_t \\ &= \hat{f}(0, \hat{S}^1_0) + D_2\hat{f}(I, \hat{S}^1) \bullet \hat{S}^1_t + \left(D_1\hat{f}(I, \hat{S}^1) + \frac{1}{2}D_{22}\hat{f}(I, \hat{S}^1)\sigma^2(\hat{S}^1)^2\right) \bullet I_t.\end{aligned}$$

Since $\hat{S}^1$ is a Q-martingale, it follows that

$$M_t := \hat{f}(t, \hat{S}^1_t) - \left(D_1 \hat{f}(I, \hat{S}^1) + \frac{1}{2} D_{22} \hat{f}(I, \hat{S}^1) \sigma^2 (\hat{S}^1)^2 \right) \bullet I_t$$

is a local Q-martingale which is bounded from below. Hence, Theorem 1.37 yields that M_t is a Q-supermartingale. Now assume that $t \in [0,T]$ and τ is a stopping time with values in $[t,T]$. Then

$$\begin{aligned} \hat{f}(t, \hat{S}^1_t) &= M_t + \int_0^t \left(D_1 \hat{f}(I, \hat{S}^1) + \frac{1}{2} D_{22} \hat{f}(I, \hat{S}^1) \sigma^2 (\hat{S}^1)^2 \right) ds \\ &\geq E_Q(M_\tau \,|\, \mathcal{F}_t) + \int_0^t \left(D_1 \hat{f}(I, \hat{S}^1) + \frac{1}{2} D_{22} \hat{f}(I, \hat{S}^1) \sigma^2 (\hat{S}^1)^2 \right) ds \\ &= E_Q(\hat{f}(\tau, \hat{S}^1_\tau) \,|\, \mathcal{F}_t) - \int_t^\tau \left(D_1 \hat{f}(I, \hat{S}^1) + \frac{1}{2} D_{22} \hat{f}(I, \hat{S}^1) \sigma^2 (\hat{S}^1)^2 \right) ds \qquad (3.3) \\ &\geq E_Q(\hat{f}(\tau, \hat{S}^1_\tau) \,|\, \mathcal{F}_t) \\ &\geq E_Q(\hat{g}(\tau, \hat{S}^1_\tau) \,|\, \mathcal{F}_t). \end{aligned}$$

If in particular the stopping time $\tau := \inf\{s \in [t,T] : \ \hat{f}(s, \hat{S}^1_s) = \hat{g}(s, \hat{S}^1_s)\}$ is used, then the integral in (3.3) becomes zero. Moreover, $\hat{f}(\tau, \hat{S}^1_\tau) = \hat{g}(\tau, \hat{S}^1_\tau)$. Finally

$$M_t = M^\tau_t = E_Q(M^\tau_\tau \,|\, \mathcal{F}_t) = E_Q(M_\tau \,|\, \mathcal{F}_t),$$

since M^τ is even a bounded local Q-martingale and hence a Q-martingale. Thus the above three inequalities turn to equalities, i.e.

$$\hat{f}(t, \hat{S}^1_t) = E_Q(\hat{g}(\tau, \hat{S}^1_\tau) \,|\, \mathcal{F}_t).$$

Due to (3.2) the process $(\hat{f}(t, \hat{S}^1_t))_{t \in [0,T]}$ is the Q-Snell envelope of $\hat{X}$. Consequently, Lemma 3.5 yields the assertion. ∎

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