BÀI GIẢNG MÔN HỌC
TRƯỜNG ĐIỆN TỪ

Credits: 2
Textbook:
Electromagnetic Fields and Waves, Paul Lorrain and Dale R. Corson, W. H. Freeman and Company, New York, 1988

NỘI DUNG

CHƯƠNG 1. ĐIỆN TRƯỜNG TĨNH

COULOMB'S LAW

Experiments show that the force exerted *by* a stationary point charge Q_a *on* a stationary point charge Q_b situated a distance r away is given by

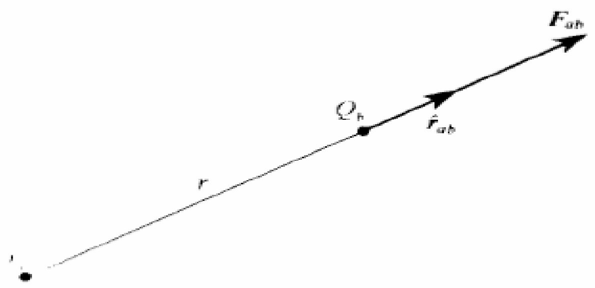

Fig. 3-1. Charges Q_a and Q_b separated by a distance r. Coulomb's law gives the force $\boldsymbol{F}_{ab}$ exerted *by* Q_a *on* Q_b if Q_a is stationary.

$$\boldsymbol{F}_{ab} = \frac{Q_a Q_b}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}}_{ab}, \tag{3-1}$$

where the unit vector $\hat{\boldsymbol{r}}_{ab}$ points from Q_a to Q_b, as in Fig. 3-1. This is *Coulomb's law.*† The force is repulsive if the two charges have the same sign, and attractive if they have different signs. The charges are measured in coulombs, the force in newtons, and the distance in meters. The constant ϵ_0 is the *permittivity of free space* and has the following value:

$$\epsilon_0 = 8.854187817 \times 10^{-12} \text{ farad/meter}. \tag{3-2}$$

Substituting the value of ϵ_0, we find that

$$F_{ab} \approx 9 \times 10^9 \frac{Q_a Q_b}{r^2} \qquad \text{newtons}, \tag{3-3}$$

where the factor of 9 is too large by about one part in a thousand.

We shall not be able to define the coulomb until Chap. 22. For the moment, we may take the value of ϵ_0 to be given, and use this law as a provisional definition of the unit of charge.

To what extent does Coulomb's law remain valid when Q_a and Q_b are not stationary?

(1) If Q_a is stationary and Q_b is not, then Coulomb's law applies to the force on Q_b, whatever the velocity of Q_b. This is an experimental fact. Indeed, the trajectories of charged particles in oscilloscopes, mass spectrographs, and ion accelerators are invariably calculated on that basis.

(2) If Q_a is not stationary, Coulomb's law is no longer strictly valid.

THE ELECTRIC FIELD STRENGTH E

The force between two electric charges Q_a and Q_b results from the interaction of Q_b with the *field* of Q_a at the position of Q_b, or vice versa.

We thus define the *electric field strength* $\boldsymbol{E}$ at a point as the force exerted on a unit test charge situated at that point. Thus, at a distance r from charge Q_a,

$$\boldsymbol{E}_a = \frac{\boldsymbol{F}_{ab}}{Q_b} = \frac{Q_a}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}} \qquad \text{newtons/coulomb, or volts/meter,} \qquad (3\text{-}5)$$

where 1 volt equals 1 joule/coulomb. The field of Q_a is the same, whether the test charge Q_b lies in the field or not, even if Q_b is larger than Q_a.

THE PRINCIPLE OF SUPERPOSITION

If there are several charges, each one imposes its own field, and the resultant $\boldsymbol{E}$ is simply the vector sum of all the individual $\boldsymbol{E}$'s. This is the *principle of superposition*.

For a continuous distribution of charge, as in Fig. 3-2, the electric field strength at (x, y, z) is

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\int_{v'} \frac{\rho\hat{\boldsymbol{r}}}{r^2}\, dv', \qquad (3\text{-}6)$$

where ρ is the volume charge density at the source point (x', y', z'), as in the figure, $\hat{\boldsymbol{r}}$ is the unit vector pointing from the *source* point $P'(x', y', z')$ to the *field* point $P(x, y, z)$, r is the distance between these two points, and dv' is the element of volume $dx'\, dy'\, dz'$. If there exist surface distributions of charge, then we must add a similar integral, with

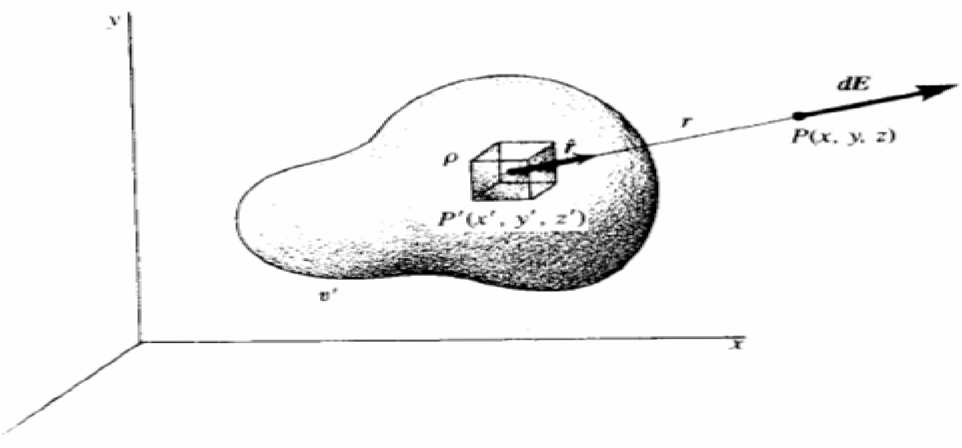

Fig. 3-2. Charge distribution of volume density ρ occupying a volume v'. The element of volume at $P'(x', y', z')$ has a field $d\boldsymbol{E}$ at $P(x, y, z)$.

ρ replaced by the surface charge density σ and v' by the area $\mathscr{A}'$ of the charged surfaces.

THE ELECTRIC POTENTIAL V AND THE CURL OF $\boldsymbol{E}$

Consider a test charge Q' that can move about in an electric field. The energy $\mathscr{E}$ required to move it at a constant velocity from a point A to a point B along a given path is

$$\mathscr{E} = -\int_A^B \boldsymbol{E}Q' \cdot d\boldsymbol{l}. \tag{3-7}$$

Because of the negative sign, $\mathscr{E}$ is the work done *against* the field. We assume that Q' is so small that it does not disturb the charge distributions appreciably.

If the path is closed, the total work done on Q' is

$$\mathscr{E} = -\oint \boldsymbol{E}Q' \cdot d\boldsymbol{l}. \tag{3-8}$$

Let us evaluate this integral. We first consider the electric field of a single stationary point charge Q. Then

$$\oint \boldsymbol{E}Q' \cdot d\boldsymbol{l} = \frac{QQ'}{4\pi\epsilon_0}\oint \frac{\hat{\boldsymbol{r}} \cdot d\boldsymbol{l}}{r^2}. \tag{3-9}$$

Now the term under the integral on the right is simply dr/r^2, or $-d(1/r)$. But the sum of the increments of $1/r$ over a closed path is zero, since r has the same value at the beginning and at the end. So the line integral is zero, and the net work done in moving Q' around any closed path in the field of Q, which is fixed, is zero.

If the electric field is that of some fixed charge distribution, then the line integrals corresponding to each individual charge of the distribution are all zero. Thus, for any distribution of fixed charges,

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = 0. \tag{3-10}$$

An electrostatic field is therefore conservative (Example, Sec. 1.9). This important property follows from the fact that the Coulomb force is a central force: the force in the field of a point charge is radial.

We can now show that the work done in moving a test charge at a constant velocity from a point A to a point B is independent of the path. Let m and n be any two paths leading from A to B. Then these two paths together form a closed curve, and the work done in going from A to B along m and then from B back to A along n is zero. Then the work done in going from A to B is the same along m as it is along n.

Now let us choose a datum point $R(x_0, y_0, z_0)$, and let us define a scalar function V of $P(x, y, z)$ such that

$$V_P = \int_P^R \boldsymbol{E} \cdot \boldsymbol{dl}. \tag{3-11}$$

This definition is unambiguous because the integral is the same for all paths leading from P to R. Then, for any pair of points A and B,

$$-\int_A^B \boldsymbol{\nabla} V \cdot \boldsymbol{dl} = V_A - V_B = \int_A^B \boldsymbol{E} \cdot \boldsymbol{dl}, \tag{3-12}$$

as in Fig. 3-3, and therefore

$$\boldsymbol{E} = -\boldsymbol{\nabla} V. \tag{3-13}$$

The *electric potential* $V(x, y, z)$ describes the field completely. The negative sign makes $\boldsymbol{E}$ point toward a *decrease* in V.

Note that V is not uniquely defined, because point R is arbitrary. In fact, one can add to V any quantity that is independent of the coordinates without affecting $\boldsymbol{E}$.

From Eq. 3-10 and from Stokes's theorem (Sec. 1.9),

$$\boldsymbol{\nabla} \times \boldsymbol{E} = 0. \tag{3-14}$$

This is also obvious from the fact that

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\boldsymbol{\nabla} \times \boldsymbol{\nabla} V = 0. \tag{3-15}$$

Remember that we are dealing here with *static* fields. If there were time-dependent currents, $\boldsymbol{\nabla} \times \boldsymbol{E}$ would not necessarily be zero, and $-\boldsymbol{\nabla} V$ would then describe only part of $\boldsymbol{E}$. We shall investigate these more complicated phenomena later.

The Electric Potential V at a Point

Equation 3-12 shows that $\boldsymbol{E}$ concerns only *differences* between the potentials at two points. When one wishes to speak of the potential at a given point, one must arbitrarily define V in a given region of space to be zero. In the previous section, for instance, we made V equal to zero at point R. When the charges extend over only a finite region, it is usually convenient to choose the potential V at infinity to be zero. Then, at point P,

$$V = \int_P^\infty \boldsymbol{E} \cdot d\boldsymbol{l}. \tag{3-16}$$

The energy $\mathscr{E}$ required to bring a charge Q from a point where V is zero, by definition, to P is VQ. Thus V is $\mathscr{E}/Q$, and the unit of V is 1 joule/coulomb, or 1 *volt.*

If the field is that of a single point charge, then

$$V = \int_r^\infty \frac{Q}{4\pi\epsilon_0} \frac{dr}{r^2} = \frac{Q}{4\pi\epsilon_0 r}. \tag{3-17}$$

The sign of this V is the same as that of Q.

The principle of superposition applies to V as well as to $\boldsymbol{E}$, and for any charge distribution of density ρ,

$$V = \frac{1}{4\pi\epsilon_0} \int_{v'} \frac{\rho \, dv'}{r}, \tag{3-18}$$

with $\boldsymbol{r}$ as in Fig. 3-2. The volume v' encloses all the charges. If there are surface charges, one adds a surface integral.

GAUSS'S LAW

Gauss's law relates the flux of $\boldsymbol{E}$ through a closed surface to the total charge enclosed within that surface.

Consider Fig. 3-4, in which a finite volume v bounded by a surface $\mathscr{A}$ encloses a charge Q. We can calculate the outward flux of $\boldsymbol{E}$ through $\mathscr{A}$ as follows. The flux of $\boldsymbol{E}$ through the element of area $\boldsymbol{d\mathscr{A}}$ is

$$\boldsymbol{E} \cdot \boldsymbol{d\mathscr{A}} = \frac{Q}{4\pi\epsilon_0} \frac{\hat{\boldsymbol{r}} \cdot \boldsymbol{d\mathscr{A}}}{r^2}. \tag{3-19}$$

Now $\hat{\boldsymbol{r}} \cdot \boldsymbol{d\mathscr{A}}$ is the projection of $\boldsymbol{d\mathscr{A}}$ on a plane normal to $\hat{\boldsymbol{r}}$. Then

$$\boldsymbol{E} \cdot \boldsymbol{d\mathscr{A}} = \frac{Q}{4\pi\epsilon_0} d\Omega, \tag{3-20}$$

where $d\Omega$ is the solid angle subtended by $\boldsymbol{d\mathscr{A}}$ at the point P'.

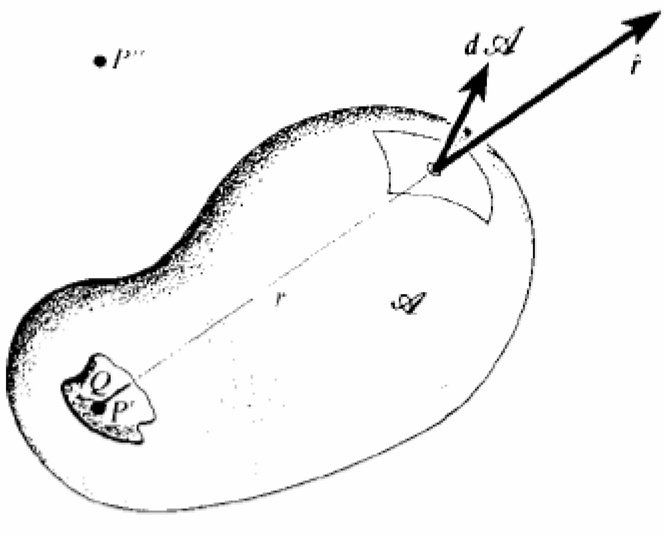

Fig. 3-4. A point charge Q located inside a volume v bounded by the surface of area $\mathscr{A}$. Gauss's law states that the surface integral of $\boldsymbol{E} \cdot \boldsymbol{d\mathscr{A}}$ over $\mathscr{A}$ is equal to Q/ϵ_0. The vector $\boldsymbol{d\mathscr{A}}$ points *outward*.

To find the outward flux of $\boldsymbol{E}$, we integrate over the area $\mathscr{A}$, or over a solid angle of 4π. Thus

$$\int_{\mathscr{A}} \boldsymbol{E} \cdot \boldsymbol{d\mathscr{A}} = \frac{Q}{\epsilon_0}. \tag{3-21}$$

If Q is outside the surface at P'', the integral is equal to zero. The solid angle subtended by any closed surface (or set of closed surfaces) is 4π at a point P' inside and zero at a point P'' outside.

If more than one charge resides within v, the fluxes add algebraically and the total flux of $\boldsymbol{E}$ leaving v is equal to the total enclosed charge Q divided by ϵ_0:

$$\int_{\mathscr{A}} \boldsymbol{E} \cdot \boldsymbol{d\mathscr{A}} = \frac{Q}{\epsilon_0}. \tag{3-22}$$

This is *Gauss's law* in integral form.†

If the charge occupies a finite volume, then

$$\int_{\mathscr{A}} \boldsymbol{E} \cdot \boldsymbol{d\mathscr{A}} = \frac{1}{\epsilon_0} \int_v \rho \, dv, \tag{3-23}$$

where $\mathscr{A}$ is the area of the surface bounding the volume v, and ρ is the electric charge density. We assumed that there are no surface charges on the bounding surface.

If we apply the divergence theorem to the left-hand side, we have that

$$\int_v \boldsymbol{\nabla} \cdot \boldsymbol{E} \, dv = \frac{1}{\epsilon_0} \int_v \rho \, dv. \tag{3-24}$$

Since this equation applies to any finite volume v, the integrands are equal and

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \tag{3-25}$$

at every point in space.

THE EQUATIONS OF POISSON AND OF LAPLACE

Let us replace $\boldsymbol{E}$ by $-\boldsymbol{\nabla}V$ in Eq. 3-25. Then

$$\boldsymbol{\nabla}^2 V = -\frac{\rho}{\epsilon_0}. \tag{4-1}$$

This is *Poisson's equation*. It relates the space charge density ρ *at a given point* to the second space derivatives of V *in the region of that point.*

In a region where the charge density ρ is zero,

$$\boldsymbol{\nabla}^2 V = 0, \tag{4-2}$$

which is *Laplace's equation.*

The general problem of finding V in the field of a given charge distribution amounts to finding a solution to either Laplace's or Poisson's equation that will satisfy the given boundary conditions.

THE POTENTIAL ENERGY $\mathscr{E}$ OF A CHARGE DISTRIBUTION EXPRESSED IN TERMS OF CHARGES AND POTENTIALS

The Potential Energy of a Set of Point Charges

Assume that the charges remain in equilibrium under the action of both the electric forces and restraining mechanical forces.

The potential energy of the system is equal to the work performed by the electric forces in the process of dispersing the charges out to infinity. After dispersal, the charges are infinitely remote from each other, and there is zero potential energy.

First, let Q_1 recede to infinity slowly, keeping the electric and the mechanical forces in equilibrium. There is zero acceleration and zero kinetic energy. The other charges remain fixed. The decrease in potential energy $\mathscr{E}_1$ is equal to Q_1 multiplied by the potential V_1 due to the other charges at the original position of Q_1:

$$\mathscr{E}_1 = \frac{Q_1}{4\pi\epsilon_0}\left(\frac{Q_2}{r_{12}} + \frac{Q_3}{r_{13}} + \cdots + \frac{Q_N}{r_{1N}}\right). \tag{6-1}$$

All the charges except Q_1 appear in the series between parentheses.

With Q_1 removed, let Q_2 recede to infinity, to some point infinitely distant from Q_1. The decrease in potential energy is now

$$\mathscr{E}_2 = \frac{Q_2}{4\pi\epsilon_0}\left(\frac{Q_3}{r_{23}} + \frac{Q_4}{r_{24}} + \cdots + \frac{Q_N}{r_{2N}}\right). \tag{6-2}$$

The series for $\mathscr{E}_2$ has $N-2$ terms. We continue the process for all the remaining charges, until finally the Nth charge can stay in position, since it lies in a zero field.

and the potential energy of the initial charge configuration is

$$\mathscr{E} = \frac{1}{2}\sum_{i=1}^{N} Q_i V_i. \tag{6-7}$$

The Potential Energy of a Continuous Charge Distribution

For a continuous electric charge distribution, we replace Q_i by $\rho\, dv$ and the summation by an integration over any volume v that contains all the charge:

$$\mathscr{E} = \tfrac{1}{2}\int_v V\rho\, dv. \tag{6-8}$$

This integral is equal to the work performed by the electric forces in going from the given charge distribution to the situation where $\rho = 0$ everywhere, by dispersing all the charge to infinity, or by letting positive and negative charges coalesce, or by both processes combined.

Observe that the potential V under the integral sign does not include the part that originates in the element of charge $\rho\, dv$ itself. We saw in Sec. 3.5 that the infinitesimal element of charge at a given point contributes nothing to V.

If there are surface charge densities σ, then their stored energy is

$$\mathscr{E} = \tfrac{1}{2}\int_{\mathscr{A}} \sigma V\, d\mathscr{A}, \tag{6-9}$$

where $\mathscr{A}$ includes all the surfaces carrying charge.

THE POTENTIAL ENERGY $\mathscr{E}$ OF AN ELECTRIC CHARGE DISTRIBUTION EXPRESSED IN TERMS OF $\boldsymbol{E}$

We have expressed the potential energy $\mathscr{E}$ of a charge distribution in terms of the charge density ρ and the potential V. Now both ρ and V are related to $\boldsymbol{E}$. So it should be possible to express $\mathscr{E}$ solely in terms of $\boldsymbol{E}$. This is what we shall do here. We shall find that

$$\mathscr{E} = \int_v \frac{\epsilon_0 E^2}{2}\, dv, \tag{6-11}$$

where the volume v includes all the regions where $\boldsymbol{E}$ exists. Thus we can calculate $\mathscr{E}$ by assigning to each point in space an *electric energy density* of $\epsilon_0 E^2/2$.

THE CONTINUITY CONDITIONS AT AN INTERFACE

The Potential V

The potential V is continuous across the boundary between two media. Otherwise, a discontinuity would imply an infinitely large E, which is physically impossible.

The Normal Component of $\boldsymbol{D}$

Consider a short imaginary cylinder spanning the interface, and of cross section $\mathcal{A}$ as in Fig. 10-4. The top and bottom faces of the cylinder are parallel to the boundary and close to it. The interface carries a free surface charge density σ_f.

According to Gauss's law (Sec. 9.5), the net flux of $\boldsymbol{D}$ coming out of the cylinder is equal to the enclosed free charge. Now the only flux of $\boldsymbol{D}$ is that through the top and bottom faces because the height of the cylinder is small. If now the area $\mathcal{A}$ is not too large, $\boldsymbol{D}$ is approximately uniform over it, and then

$$(D_{2n} - D_{1n})\mathcal{A} = \sigma_f \mathcal{A}, \qquad (\boldsymbol{D}_2 - \boldsymbol{D}_1) \cdot \hat{\boldsymbol{n}} = \sigma_f, \tag{10-9}$$

where $\hat{\boldsymbol{n}}$ is the unit vector normal to the interface and pointing from medium 2 to medium 1.

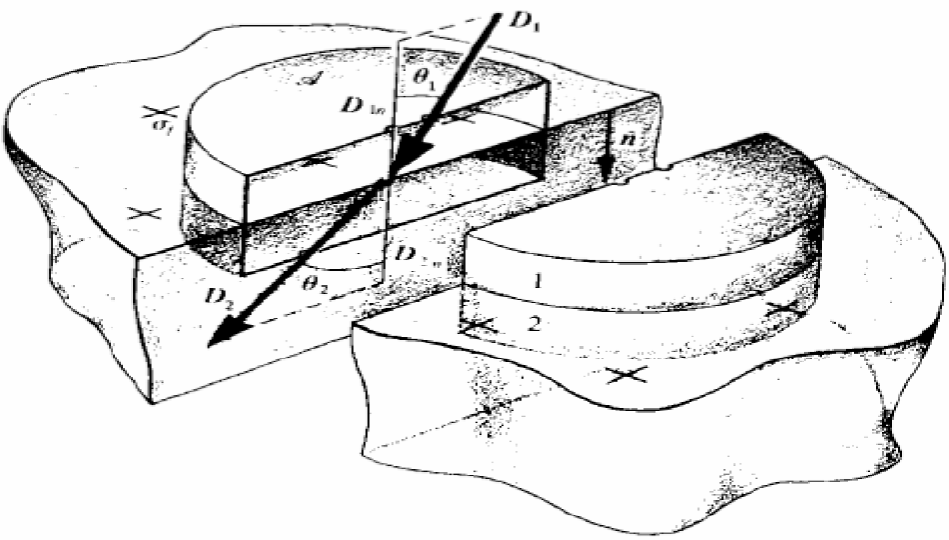

Fig. 10-4. Imaginary cylinder straddling the interface between media 1 and 2 and delimiting an area $\mathcal{A}$. The difference $D_{2n} - D_{1n}$ between the *normal* components of $\boldsymbol{D}$ is equal to the free surface charge density σ_f.

As a rule, the boundary between two dielectrics does not carry *free* charges, and then the normal component of $\boldsymbol{D}$ is continuous across the interface. Thus the normal component of $\boldsymbol{E}$ is discontinuous.

On the other hand, if one medium is a conductor and the other a dielectric, and if $\boldsymbol{D}$ is not a function of the time, then $\boldsymbol{D} = 0$ in the conductor and $D_n = \sigma_f$ in the dielectric. If $\boldsymbol{D}$ is a function of the time, Eq. 10.9 still applies, but $\boldsymbol{D}$ is not zero in the conductor.

The Tangential Component of $\boldsymbol{E}$

Consider now the path shown in Fig. 10-5, with two sides of length L parallel to the boundary and close to it. The other two sides are infinitesimal. If L is short, $\boldsymbol{E}$ does not vary significantly over that distance, and integrating over the path yields

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = E_{1t}L - E_{2t}L. \qquad (10\text{-}10)$$

Now, from Sec. 3.4 this line integral is zero, and thus

$$E_{1t} = E_{2t}\dagger, \qquad \text{or} \qquad (\boldsymbol{E}_1 - \boldsymbol{E}_2) \times \hat{\boldsymbol{n}} = 0, \qquad (10\text{-}11)$$

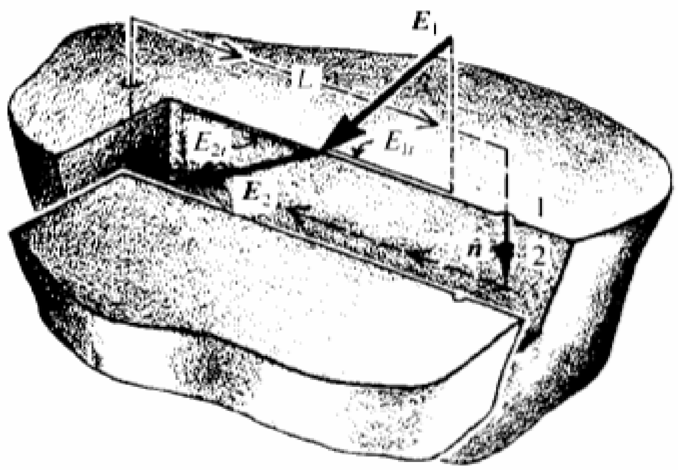

Fig. 10-5. Closed path of integration spanning the interface between media 1 and 2. The tangential components of $\boldsymbol{E}$ are equal: $E_{1t} = E_{2t}$.

with $\hat{\boldsymbol{n}}$ defined as above. The tangential component of $\boldsymbol{E}$ is continuous across any interface.

IMAGES

If an electric charge distribution lies in a uniform dielectric that is in contact with a conducting body, then the method of images often provides the simplest route for calculating the electric field. The method is best explained by examples such as the two given below, but the principle is the following.

Call the charge distribution Q, the dielectric D, and the conductor C. One replaces C, on paper, by more dielectric D' and by a second charge distribution Q' such that the original boundary conditions are not disturbed. Then the field in D is left undisturbed, according to the uniqueness theorem. The charge distribution Q' is said to be the *image* of Q. Of course, the dielectric D can be simply air or a vacuum.

CHƯƠNG 2. DÒNG ĐIỆN

THE LAW OF CONSERVATION OF ELECTRIC CHARGE

Consider a closed surface of area $\mathcal{A}$ enclosing a volume v. The volume charge density inside is ρ. Charges flow in and out, and the current density at a given point on the surface is $\boldsymbol{J}$ amperes/meter^2.

It is a well-established experimental fact that there is never any net creation of electric charge. Then any net outflow depletes the enclosed charge Q: at any given instant,

$$\int_{\mathcal{A}} \boldsymbol{J} \cdot d\boldsymbol{\mathcal{A}} = -\frac{d}{dt}\int_v \rho \, dv = -\frac{dQ}{dt}, \tag{4-24}$$

where the vector $d\boldsymbol{\mathcal{A}}$ points *outward*, according to the usual sign convention.

Applying now the divergence theorem on the left, we find that

$$\int_v \boldsymbol{\nabla} \cdot \boldsymbol{J} \, dv = -\int_v \frac{\partial \rho}{\partial t} dv. \tag{4-25}$$

We have transferred the time derivative under the integral sign, but then we must use a partial derivative because ρ can be a function of x, y, z, as well as of t.

Now the volume v is of any shape or size. Therefore

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t}. \tag{4-26}$$

Equations 4-24 and 4-26 are, respectively, the integral and differential forms of the *law of conservation of electric charge*.

CONDUCTION

Semiconductors may contain two types of mobile charges: conduction electrons and positive holes. A *hole* is a vacancy left by an electron liberated from the valence bond structure in the material. A hole behaves as a free particle of charge $+e$, and it moves through the semiconductor much as an air bubble rises through water.

In most good conductors and semiconductors, the current density $\boldsymbol{J}$ is proportional to $\boldsymbol{E}$:

$$\boldsymbol{J} = \sigma \boldsymbol{E}, \qquad (4\text{-}27)$$

where σ is the *electric conductivity* of the material expressed in siemens per meter, where 1 *siemens*† is 1 ampere/volt. This is *Ohm's law* in a more general form. As we shall see later, an electric conductivity can be complex. We shall find a still more general form of Ohm's law in Chap. 23.

Table 4-1 shows the conductivities of some common materials.

Ohm's law does not always apply. For example, in a certain type of ceramic semiconductor, J is proportional to the fifth power of E. Also some conductors are not isotropic.

Conduction in a Steady Electric Field

For simplicity, we assume that the charge carriers are conduction electrons.

The detailed motion of an individual conduction electron is exceedingly complex because, every now and then, it collides with an atom and rebounds. The atoms, of course, vibrate about their equilibrium positions, because of thermal agitation, and exchange energy with the conduction electrons.

However, on the average, each electron has a kinetic energy of $\frac{3}{2}kT$, where k is Boltzmann's constant and T is the temperature in kelvins. Thus, at room temperature, the velocity $v_{\rm th}$ associated with thermal agitation is given by

$$\frac{mv_{\rm th}^2}{2} = \tfrac{3}{2}kT = \tfrac{3}{2}(1.38 \times 10^{-23} \times 300) \approx 6 \times 10^{-21} \text{ joule}, \qquad (4\text{-}32)$$

and

$$v_{\rm th} \approx \left(\frac{12 \times 10^{-21}}{9.1 \times 10^{-31}}\right)^{1/2} \approx 10^5 \text{ meters/second}. \qquad (4\text{-}33)$$

Under the action of a steady electric field, the cloud of conduction electrons drifts at a constant velocity $\boldsymbol{v}_d$ such that

$$\boldsymbol{J} = \sigma \boldsymbol{E} = -Ne\boldsymbol{v}_d, \tag{4-34}$$

where $\boldsymbol{v}_d$ points in the direction opposite to $\boldsymbol{J}$ and to $\boldsymbol{E}$, and N is the number of conduction electrons per cubic meter.

The drift velocity is low. In copper, $N = 8.5 \times 10^{28}$. If a current of 1 ampere flows through a wire having a cross section of 1 millimeter^2, $J = 10^6$ and v_d works out to about 10^{-4} meter/second, or about 300 millimeters/*hour*! Then the drift velocity is smaller than the thermal agitation velocity by *nine* orders of magnitude!

In Eq. 4-34 v_d is small, but Ne is very large. In copper,

$$Ne = 8.5 \times 10^{28} \times 1.6 \times 10^{-19} \approx 10^{10} \text{ coulombs/meter}^3. \tag{4-35}$$

The low drift velocity of conduction electrons is the source of many paradoxes. For example, a radio transmitting antenna is about 75 meters long and operates at about 1 megahertz. How can conduction electrons go from one end to the other and back in 1 microsecond? The answer is that they do not. They drift back and forth by a distance of the order of 1 atomic diameter, and that is enough to generate the required current.

The Mobility $\mathcal{M}$ of Conduction Electrons

The mobility of conduction electrons

$$\mathcal{M} = \frac{|v_d|}{E} = \frac{\sigma}{Ne} \tag{4-36}$$

is, by definition, a positive quantity.† It is independent of $\boldsymbol{E}$ in linear conductors. Thus

$$\sigma = Ne\mathcal{M} \tag{4-37}$$

where, as usual, we have taken e to be the *magnitude* of the electronic charge.

The Volume Charge Density ρ in a Conductor

(1) Assume steady-state conditions and a homogeneous conductor. Then $\partial\rho/\partial t = 0$ and, from Sec. 4.2, $\nabla \cdot \boldsymbol{J} = 0$. If $\boldsymbol{J}$ is the conduction current density in a homogeneous conductor that satisfies Ohm's law $\boldsymbol{J} = \sigma\boldsymbol{E}$, then

$$\nabla \cdot \boldsymbol{J} = \nabla \cdot \sigma\boldsymbol{E} = \sigma\nabla \cdot \boldsymbol{E} = 0, \qquad \nabla \cdot \boldsymbol{E} = 0. \tag{4-47}$$

But the divergence of $\boldsymbol{E}$ is proportional to the volume charge density ρ, from Sec. 3.7. Thus, under steady-state conditions and in homogeneous conductors (σ independent of the coordinates), ρ is zero.

As a rule, the *surface* charge density on a conducting body carrying a current is not zero.

(2) Now suppose that one injects charge into a piece of copper by bombarding it with electrons. What happens to the charge density? In that case, from Sec. 4.2,

$$\nabla \cdot \boldsymbol{J} = -\frac{\partial\rho}{\partial t}. \tag{4-48}$$

But, from Sec. 3.7,

$$\nabla \cdot \boldsymbol{J} = \sigma\nabla \cdot \boldsymbol{E} = \frac{\sigma\rho}{\epsilon_r\epsilon_0}, \tag{4-49}$$

where ϵ_r is the relative permittivity of the material (Sec. 9.9). Thus

$$\frac{\partial\rho}{\partial t} = -\frac{\sigma\rho}{\epsilon_r\epsilon_0}, \qquad \rho = \rho_0 \exp\left(-\frac{\sigma t}{\epsilon_r\epsilon_0}\right), \tag{4-50}$$

and ρ decreases exponentially with time.

The relative permittivity ϵ_r of a good conductor is not measurable because conduction completely overshadows polarization. One may presume that ϵ_r is of the order of 3, as in common dielectrics.

The inverse of the coefficient of t in the above exponent is the *relaxation time*.

We have neglected the fact that σ is frequency-dependent and is thus itself a function of the relaxation time. Relaxation times in good conductors are, in fact, short; and ρ may be set equal to zero, in practice. For example, the relaxation time for copper at room temperature is about 4×10^{-14} second, instead of $\approx 10^{-19}$ second according to the above calculation.

(3) In a homogeneous conductor carrying an alternating current, ρ is zero because Eq. 4-47 applies.

(4) In a nonhomogeneous conductor carrying a current, ρ is not zero. For example, under steady-state conditions,

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} = \boldsymbol{\nabla} \cdot (\sigma \boldsymbol{E}) = (\boldsymbol{\nabla}\sigma) \cdot \boldsymbol{E} + \sigma \boldsymbol{\nabla} \cdot \boldsymbol{E} = 0 \tag{4-51}$$

and

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_r \epsilon_0} = -\frac{(\boldsymbol{\nabla}\sigma) \cdot \boldsymbol{E}}{\sigma}. \tag{4-52}$$

(5) If there are magnetic forces on the charge carriers, then $\boldsymbol{J} = \sigma \boldsymbol{E}$ does not apply and there can exist a volume charge density. See Sec. 22.4.1.

The Joule Effect

What is the kinetic energy gained by the conduction electrons? Consider a cube of the conductor, with side a. Apply a voltage V between opposite faces. The current is I. Then the kinetic energy gained is VI, and the power dissipated as heat per cubic meter is

$$P' = \frac{VI}{a^3} = \left(\frac{V}{a}\right)\left(\frac{I}{a^2}\right) = EJ \tag{4-53}$$

$$= \sigma E^2 = \frac{J^2}{\sigma} \qquad \text{watts/meter}^3. \tag{4-54}$$

If E and J are sinusoidal functions of the time,

$$P'_{av} = E_{rms} J_{rms} = \sigma E_{rms}^2 = \frac{J_{rms}^2}{\sigma}. \tag{4-55}$$

CHƯƠNG 3. TỪ TRƯỜNG TĨNH

MAGNETIC FIELDS

Imagine a set of charges moving around in space.[†] At any point $\boldsymbol{r}$ in space and at any time t there exists an electric field strength $\boldsymbol{E}(\boldsymbol{r}, t)$ and a magnetic flux density $\boldsymbol{B}(\boldsymbol{r}, t)$ that are defined as follows. If a charge Q moves at velocity $\boldsymbol{v}$ at $(\boldsymbol{r}, t)$ in this field, then it suffers a Lorentz force

$$\boldsymbol{F} = Q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}). \qquad (18\text{-}1)$$

The *electric force* $Q\boldsymbol{E}$ is proportional to Q but independent of $\boldsymbol{v}$, while the *magnetic force* $Q\boldsymbol{v} \times \boldsymbol{B}$ is orthogonal to both $\boldsymbol{v}$ and $\boldsymbol{B}$.

MAGNETIC MONOPOLES

We assume here that magnetic fields arise solely from the motion of electric charges.

However, Dirac postulated in 1931 that magnetic fields can also arise from magnetic "charges," called *magnetic monopoles*. Such particles have not been observed to date (1987). The theoretical value of the elementary magnetic charge is

$$\frac{h}{e} = 4.1356692 \times 10^{-15} \text{ weber,}^{\dagger} \qquad (18\text{-}2)$$

where h is Planck's constant and e is the charge of the electron. See the table inside the back cover.

At a distance r from a stationary magnetic monopole of "charge" Q^*, we would have that

$$\boldsymbol{B} = \frac{Q^*}{4\pi r^2}\hat{\boldsymbol{r}}. \qquad (18\text{-}3)$$

Also, the force of attraction or repulsion between two monopoles Q_a^* and Q_b^* would be

$$\boldsymbol{F} = \frac{Q_a^* Q_b^*}{4\pi\mu_0 r^2}\hat{\boldsymbol{r}}. \qquad (18\text{-}4)$$

A magnetic field would exert a force $Q^*\boldsymbol{B}/\mu_0$ on a monopole in free space.

THE MAGNETIC FLUX DENSITY *B*. THE BIOT-SAVART LAW

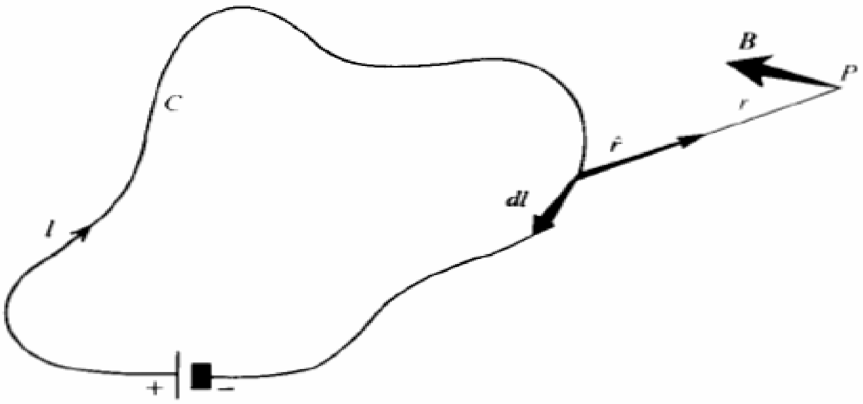

Fig. 18-1. Circuit C carrying a current I and a point P in its field. At P the magnetic flux density is $\boldsymbol{B}$.

$$\boldsymbol{B} = \frac{\mu_0 I}{4\pi} \oint_C \frac{\boldsymbol{dl}' \times \hat{\boldsymbol{r}}}{r^2}. \tag{18-5}$$

As usual, the unit vector $\hat{\boldsymbol{r}}$ points *from* the source *to* the point of observation P. This is the *Biot-Savart law*. The integration can be carried out analytically only for the simplest geometries. See below for the definition of μ_0.

This integral applies to the fields of alternating currents, as long as the time r/c, where c is the speed of light, is a small fraction of one period (Sec. 37.4).

The unit of magnetic flux density is the *tesla*. We can find the dimensions of the tesla as follows. As we saw in the introduction to this chapter, vB has the dimensions of E. Then

$$\text{Tesla} = \frac{\text{volt}}{\text{meter}}\frac{\text{second}}{\text{meter}} = \frac{\text{weber}}{\text{meter}^2}. \tag{18-6}$$

One volt-second is defined as 1 weber.

By definition,

$$\mu_0 = 4\pi \times 10^{-7}\ \text{weber/ampere-meter}. \tag{18-7}$$

This is the *permeability of free space*.

We have assumed a current I flowing through a thin wire. If the current flows over a finite volume, we substitute $J\,d\mathcal{A}'$ for I, $\boldsymbol{J}$ being the current density in amperes per square meter at a point and $d\mathcal{A}'$ an element of area, as in Fig. 18-2. Then $J\,d\mathcal{A}'\,dl'$ is $J\,dv'$ and, at a point P,

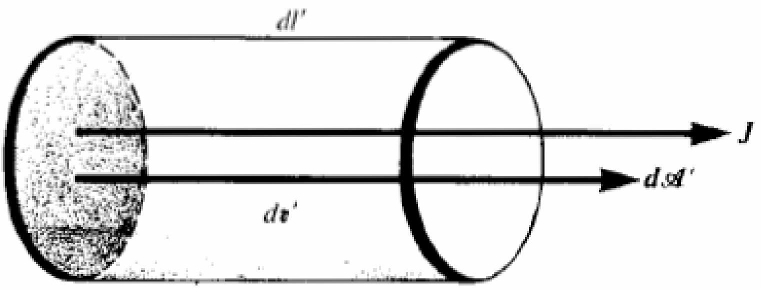

Fig. 18-2. At a given point in a volume distribution of current, the current density is $\boldsymbol{J}$. The vector $\boldsymbol{d\mathcal{A}}'$ specifies the magnitude and orientation of the shaded area. Shifting this element of area to the right by the distance $\boldsymbol{dl}'$ along $\boldsymbol{J}$ sweeps out a volume $d\mathcal{A}'\,dl' = dv'$.

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\int_{v'} \frac{\boldsymbol{J} \times \hat{\boldsymbol{r}}}{r^2}\,dv', \tag{18-8}$$

in which v' is any volume enclosing all the currents and r is the distance between the element of volume dv' and the point P.

The current density $\boldsymbol{J}$ encompasses moving free charges, polarization currents in dielectrics (Sec. 9.3.3), and equivalent currents in magnetic materials (Sec. 20.3).

Can this integral serve to calculate $\boldsymbol{B}$ at a point *inside* a current distribution? The integral appears to diverge because r goes to zero when dv' is at P. The integral does not, in fact, diverge: it does apply even if the point P lies inside the conducting body. We encountered the same problem when we calculated the value of $\boldsymbol{E}$ inside a charge distribution in Sec. 3.5.

Lines of $\boldsymbol{B}$ point everywhere in the direction of $\boldsymbol{B}$. They prove to be just as useful as lines of $\boldsymbol{E}$. The density of lines of $\boldsymbol{B}$ is proportional to the magnitude of $\boldsymbol{B}$.

As with electric fields again, a great deal of convenience attends the use of the concept of flux. The *magnetic flux* through a surface of area $\mathcal{A}$ is

$$\Phi = \int_{\mathcal{A}} \boldsymbol{B} \cdot \boldsymbol{d\mathcal{A}} \text{ webers.} \tag{18-9}$$

The surface is usually open; if it is closed, then $\Phi = 0$, as we shall see below.

The Principle of Superposition

The above integrals for $\boldsymbol{B}$ imply that the net magnetic flux density at a point is the sum of the $\boldsymbol{B}$'s of the elements of current $I\,d\boldsymbol{l}'$, or $\boldsymbol{J}\,dv'$. The principle of superposition applies to magnetic fields as well as to electric fields (Sec. 3.3): if there exist several current distributions, then the net $\boldsymbol{B}$ is the vector sum of the individual $\boldsymbol{B}$'s

THE DIVERGENCE OF $\boldsymbol{B}$

Assuming that magnetic monopoles do not exist (Sec. 18.1), or at least that the net magnetic charge density is everywhere zero, all magnetic fields result from electric currents, and the lines of $\boldsymbol{B}$ for each element of current are circles, as in Fig. 18-3. Thus the net outward flux of $\boldsymbol{B}$ through any closed surface is zero:

$$\int_{\mathcal{A}} \boldsymbol{B} \cdot d\boldsymbol{\mathcal{A}} = 0. \tag{18-18}$$

Applying the divergence theorem, it follows that

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0. \tag{18-19}$$

These are alternate forms of one of Maxwell's equations. Observe that Eq. 18-19 establishes a relation between the space derivatives of $\boldsymbol{B}$ at a given *point.* Equation 18-18, on the contrary, concerns the magnetic flux over a *closed surface.*

THE VECTOR POTENTIAL $\boldsymbol{A}$

We have just seen that $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$. It is convenient to set

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}, \tag{18-20}$$

where $\boldsymbol{A}$ is the *vector potential,* as opposed to V, which is the *scalar potential.* The divergence of $\boldsymbol{B}$ is then automatically equal to zero because the divergence of a curl is zero.

Note the analogy with the relation

$$\boldsymbol{E} = -\boldsymbol{\nabla} V \tag{18-21}$$

of electrostatics.

The vector potential is an important quantity; we shall use it as often as V.

Notice also that $\boldsymbol{B}$ is a function of the space derivatives of $\boldsymbol{A}$, just as $\boldsymbol{E}$ is a function of the space derivatives of V. Thus, to deduce the value of $\boldsymbol{B}$ from $\boldsymbol{A}$ at a given point P, one must know the value of $\boldsymbol{A}$ in the *region* around P.

We now deduce the integral for $\boldsymbol{A}$, starting from the Biot-Savart law of Sec. 18.2:

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\int_{v'} \frac{\boldsymbol{J} \times \hat{\boldsymbol{r}}}{r^2}\, dv' = \frac{\mu_0}{4\pi}\int_{v'} \left(\boldsymbol{\nabla}\frac{1}{r}\right) \times \boldsymbol{J}\, dv', \tag{18-22}$$

from Identity 16 inside the back cover. Applying now Identity 11, we find that

$$\left(\boldsymbol{\nabla}\frac{1}{r}\right) \times \boldsymbol{J} = \boldsymbol{\nabla} \times \frac{\boldsymbol{J}}{r} - \frac{\boldsymbol{\nabla} \times \boldsymbol{J}}{r}, \tag{18-23}$$

where the second term on the right is zero because $\boldsymbol{J}$ is a function of x', y', z', while $\boldsymbol{\nabla}$ involves derivatives with respect to x, y, z. Thus

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\int_{v'} \left(\boldsymbol{\nabla} \times \frac{\boldsymbol{J}}{r}\right) dv' = \boldsymbol{\nabla} \times \left(\frac{\mu_0}{4\pi}\int_{v'} \frac{\boldsymbol{J}}{r}\, dv'\right), \tag{18-24}$$

and

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\int_{v'} \frac{\boldsymbol{J}}{r}\, dv'. \tag{18-25}$$

This expression for $\boldsymbol{A}$ has a definite value for a given current distribution.

This integral, like that for $\boldsymbol{B}$, appears to diverge inside a current-carrying conductor, because of the r in the denominator. Actually, it is well behaved, like the integral for V inside a charge distribution.

If a current I flows in a circuit C that is not necessarily closed, then, at a point $P(x, y, z)$ in space,

$$\mathbf{A} = \frac{\mu_0 I}{4\pi} \int_C \frac{\boldsymbol{dl'}}{r}, \tag{18-26}$$

where the element $\boldsymbol{dl'}$ of circuit C is at $P'(x', y', z')$, and r is the distance between P and P'.

These two integrals apply to the fields of alternating currents if the time delay r/c is a small fraction of one period.

THE LINE INTEGRAL OF $\mathbf{A} \cdot \boldsymbol{dl}$ AROUND A CLOSED CURVE

Consider first a simple closed curve, as in Fig. 19-1(a). The line integral of $\mathbf{A} \cdot \boldsymbol{dl}$ around C is equal to the magnetic flux linking C:

$$\oint_C \mathbf{A} \cdot \boldsymbol{dl} = \int_{\mathcal{A}} (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \boldsymbol{d\mathcal{A}} = \int_{\mathcal{A}} \boldsymbol{B} \cdot \boldsymbol{d\mathcal{A}} = \Phi, \tag{19-1}$$

where $\mathcal{A}$ is the area of any surface bounded by C. We have used Stokes's theorem.

Now suppose the coil has N turns wound close together, as in Fig. 19-1(b). Over any cross section of the coil, say at P, the various turns are all exposed to approximately the same $\mathbf{A}$. Then

$$\oint_C \mathbf{A} \cdot \boldsymbol{dl} = N \int_{\mathcal{A}} \boldsymbol{B} \cdot \boldsymbol{d\mathcal{A}} = N\Phi = \Lambda, \tag{19-2}$$

where Λ is the *flux linkage* and $\mathcal{A}$ is the area of any surface bounded by the coil.

The unit of flux linkage is the *weber turn*.

What if one has a circuit such as that of Fig. 19-1(c)? Then

$$\oint_C \mathbf{A} \cdot \boldsymbol{dl} = \int_{\mathcal{A}} \boldsymbol{B} \cdot \boldsymbol{d\mathcal{A}} = \Lambda, \tag{19-3}$$

except that now it is difficult to devise a surface bounded by C. Luckily enough, this surface is of no interest because the flux linkage Λ is easily measurable (Sec. 24.2).

THE LAPLACIAN OF A

You will recall from Secs. 3.4.1 and 4.1 that

$$V = \frac{1}{4\pi\epsilon_0}\int_{v'} \frac{\rho}{r}\, dv', \qquad \boldsymbol{\nabla}^2 V = -\frac{\rho}{\epsilon_0}. \tag{19-7}$$

The first equation relates the potential V at the point $P(x, y, z)$ to the complete charge distribution, ρ being the total volume charge density at $P'(x', y', z')$ and r the distance PP'. The second equation expresses the relation between the *space derivatives* of V at any point to the volume charge density ρ *at that point.*

There exists an analogous pair of equations for the vector potential $\boldsymbol{A}$. We have already found the integral for $\boldsymbol{A}$ in Sec. 18.4:

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{v'} \frac{\boldsymbol{J}}{r}\, dv', \tag{19-8}$$

where v' is any volume enclosing all the currents. The x component of this equation is

$$A_x = \frac{\mu_0}{4\pi}\int_{v'} \frac{J_x}{r}\, dv'. \tag{19-9}$$

Then, by analogy with Eq. 19-7,

$$\boldsymbol{\nabla}^2 A_x = -\mu_0 J_x. \tag{19-10}$$

Of course, similar equations apply to the y- and z-components, and

$$\boldsymbol{\nabla}^2 \mathbf{A} = -\mu_0 \boldsymbol{J}. \tag{19-11}$$

This equation applies only to *static* fields.

THE DIVERGENCE OF *A*

We can prove that, for *static* fields and for currents of finite extent, the divergence of **A** is zero. First,

$$\boldsymbol{\nabla}\cdot\boldsymbol{A}=\boldsymbol{\nabla}\cdot\frac{\mu_0}{4\pi}\int_{v'}\frac{\boldsymbol{J}}{r}dv'=\frac{\mu_0}{4\pi}\int_{v'}\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{J}}{r}\right)dv',\tag{19-12}$$

where the del operator acts on the unprimed coordinates (x, y, z) of the field point, while $\boldsymbol{J}$ is a function of the source point (x', y', z'). The integral operates on the primed coordinates. As usual, r is the distance between these two points, and the integration covers any volume enclosing all the currents.

We now use successively Identities 15, 16, and 6 from the back of the front cover:

$$\boldsymbol{\nabla}\cdot\boldsymbol{A}=\frac{\mu_0}{4\pi}\int_{v'}\left(\boldsymbol{\nabla}\frac{1}{r}\right)\cdot\boldsymbol{J}\,dv'=-\frac{\mu_0}{4\pi}\int_{v'}\left(\boldsymbol{\nabla}'\frac{1}{r}\right)\cdot\boldsymbol{J}\,dv'\tag{19-13}$$

$$=\frac{\mu_0}{4\pi}\int_{v'}\left(-\boldsymbol{\nabla}'\cdot\frac{\boldsymbol{J}}{r}+\frac{\boldsymbol{\nabla}'\cdot\boldsymbol{J}}{r}\right)dv'.\tag{19-14}$$

In a time-independent field, $\partial\rho/\partial t=0$ and, from the conservation of charge (Sec. 4.2), $\boldsymbol{\nabla}'\cdot\boldsymbol{J}=0$. Therefore

$$\boldsymbol{\nabla}\cdot\boldsymbol{A}=-\frac{\mu_0}{4\pi}\int_{v'}\boldsymbol{\nabla}'\cdot\frac{\boldsymbol{J}}{r}dv'=-\frac{\mu_0}{4\pi}\int_{\mathscr{A}'}\frac{\boldsymbol{J}}{r}\cdot d\boldsymbol{\mathscr{A}}'=0,\tag{19-15}$$

where $\mathscr{A}'$ is the area of the surface enclosing the volume v'. We have used the divergence theorem to transform the first integral into the second. The second integral is zero because, over $\mathscr{A}'$, $\boldsymbol{J}$ is either zero or tangential.

THE CURL OF *B*

From Definitions 5, 10, and 15 on the back of the front cover

$$\boldsymbol{\nabla}\times\boldsymbol{B}=\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A})=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A})-\boldsymbol{\nabla}^2\boldsymbol{A}.\tag{19-16}$$

Thus, from Secs. 19.2 and 19.3,

$$\boldsymbol{\nabla}\times\boldsymbol{B}=\mu_0\boldsymbol{J}.\tag{19-17}$$

This equation is valid only for *static fields.*

AMPÈRE'S CIRCUITAL LAW

The line integral of $\boldsymbol{B} \cdot d\boldsymbol{l}$ around a closed curve C is important:

$$\oint_C \boldsymbol{B} \cdot d\boldsymbol{l} = \int_{\mathscr{A}} (\boldsymbol{\nabla} \times \boldsymbol{B}) \cdot d\boldsymbol{\mathscr{A}} = \mu_0 \int_{\mathscr{A}} \boldsymbol{J} \cdot d\boldsymbol{\mathscr{A}} = \mu_0 I. \qquad (19\text{-}18)$$

In this set of equations we first used Stokes's theorem, $\mathscr{A}$ being the area of any surface bounded by C. Then we used the relation $\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$ that we found above. Finally, I is the net current that crosses any surface bounded by the closed curve C. The right-hand screw rule applies to the direction of I and to the direction of integration around C, as in Fig. 19-3(a).

This is *Ampère's circuital law:* the line integral of $\boldsymbol{B} \cdot d\boldsymbol{l}$ around a closed curve C is equal to μ_0 times the current linking C. This result is again valid only for constant fields.

Sometimes the same current crosses the surface bounded by C several times. For example, with a solenoid, the closed curve C could follow the axis and return outside the solenoid, as in Fig. 19-3(b). The total current linking C is then the current in one turn, multiplied by the number of turns, or the number of *ampere-turns.*

The circuital law can be used to calculate $\boldsymbol{B}$, when $\boldsymbol{B}$ is uniform along the path of integration. This law is analogous to Gauss's law, which we used to calculate an $\boldsymbol{E}$ that is uniform over a surface.

THE LAPLACIAN OF *B*

We can deduce the value of the Laplacian of $\boldsymbol{B}$ from that of the Laplacian of $\mathbf{A}$ (Sec. 19.2). Since

$$\boldsymbol{\nabla}^2 \mathbf{A} = -\mu_0 \boldsymbol{J}, \qquad (19\text{-}22)$$

then

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla}^2 \mathbf{A}) = -\mu_0 \boldsymbol{\nabla} \times \boldsymbol{J}. \qquad (19\text{-}23)$$

Now the curl of a Laplacian is equal to the Laplacian of a curl and thus

$$\boldsymbol{\nabla}^2 (\boldsymbol{\nabla} \times \mathbf{A}) = -\mu_0 \boldsymbol{\nabla} \times \boldsymbol{J}. \qquad (19\text{-}24)$$

Finally,

$$\boldsymbol{\nabla}^2 \boldsymbol{B} = -\mu_0 \boldsymbol{\nabla} \times \boldsymbol{J}, \qquad (19\text{-}25)$$

again for *static* fields.

THE MAGNETIC FIELD STRENGTH $\boldsymbol{H}$. THE CURL OF $\boldsymbol{H}$

In Sec. 19.4 we found that, for static fields in the absence of magnetic materials,

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}_f. \tag{20-10}$$

Henceforth we shall use $\boldsymbol{J}_f$, instead of the unadorned $\boldsymbol{J}$, for the current density related to the motion of free charges.

In the presence of magnetized materials,

$$\nabla \times \boldsymbol{B} = \mu_0(\boldsymbol{J}_f + \boldsymbol{J}_e). \tag{20-11}$$

This equation, of course, applies only in regions where the space derivatives exist, that is, inside magnetized materials, but not at their surfaces. Then

$$\nabla \times \boldsymbol{B} = \mu_0(\boldsymbol{J}_f + \nabla \times \boldsymbol{M}), \tag{20-12}$$

$$\nabla \times \left(\frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}\right) = \boldsymbol{J}_f. \tag{20-13}$$

The vector within the parentheses, whose curl equals the free current density, is the *magnetic field strength*:

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}. \tag{20-14}$$

Both $\boldsymbol{H}$ and $\boldsymbol{M}$ are expressed in amperes/meter. Thus

$$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}) \tag{20-15}$$

and, even inside magnetized materials,

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_f \tag{20-16}$$

for *static* fields.

AMPÈRE'S CIRCUITAL LAW IN THE PRESENCE OF MAGNETIC MATERIAL

Let us integrate Eq. 20-16 over an open surface of area $\mathscr{A}$ bounded by a curve C:

$$\int_{\mathscr{A}} (\boldsymbol{\nabla} \times \boldsymbol{H}) \cdot \boldsymbol{d\mathscr{A}} = \int_{\mathscr{A}} \boldsymbol{J}_f \cdot \boldsymbol{d\mathscr{A}}, \tag{20-18}$$

or, using Stokes's theorem on the left-hand side,

$$\oint_C \boldsymbol{H} \cdot \boldsymbol{dl} = I_f, \tag{20-19}$$

where I_f is the current of free charges linking C. The right-hand screw rule applies to the direction of integration and to the direction of z. Note that I_f does not include the equivalent currents. The term on the left is the *magnetomotance*.

This is a more general form of *Ampère's circuital law* of Sec. 19.5, in that it can serve to calculate $\boldsymbol{H}$ even in the presence of magnetic materials. It is rigorously valid, however, only for steady currents.

THE MAGNETIC SUSCEPTIBILITY χ_m AND THE RELATIVE PERMEABILITY μ_r

It is convenient to define a *magnetic susceptibility* χ_m such that†

$$\boldsymbol{M} = \chi_m \boldsymbol{H}. \tag{20-21}$$

Then

$$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}) = \mu_0(1 + \chi_m)\boldsymbol{H} = \mu_0\mu_r\boldsymbol{H} = \mu\boldsymbol{H}, \tag{20-22}$$

where

$$\mu_r = 1 + \chi_m \tag{20-23}$$

is the *relative permeability* and

$$\mu = \mu_r\mu_0 \tag{20-24}$$

is the *permeability* of a material. Both χ_m and μ_r are pure numbers.

Thus

$$\boldsymbol{M} = \chi_m \frac{\boldsymbol{B}}{\mu}. \tag{20-25}$$

BOUNDARY CONDITIONS

Both $\boldsymbol{B}$ and $\boldsymbol{H}$ obey boundary conditions at the interface between two media. We proceed as in Sec. 10.2.

Figure 20-6(a) shows a short Gaussian volume at an interface. From Gauss's law, the flux leaving through the top equals that entering the bottom and

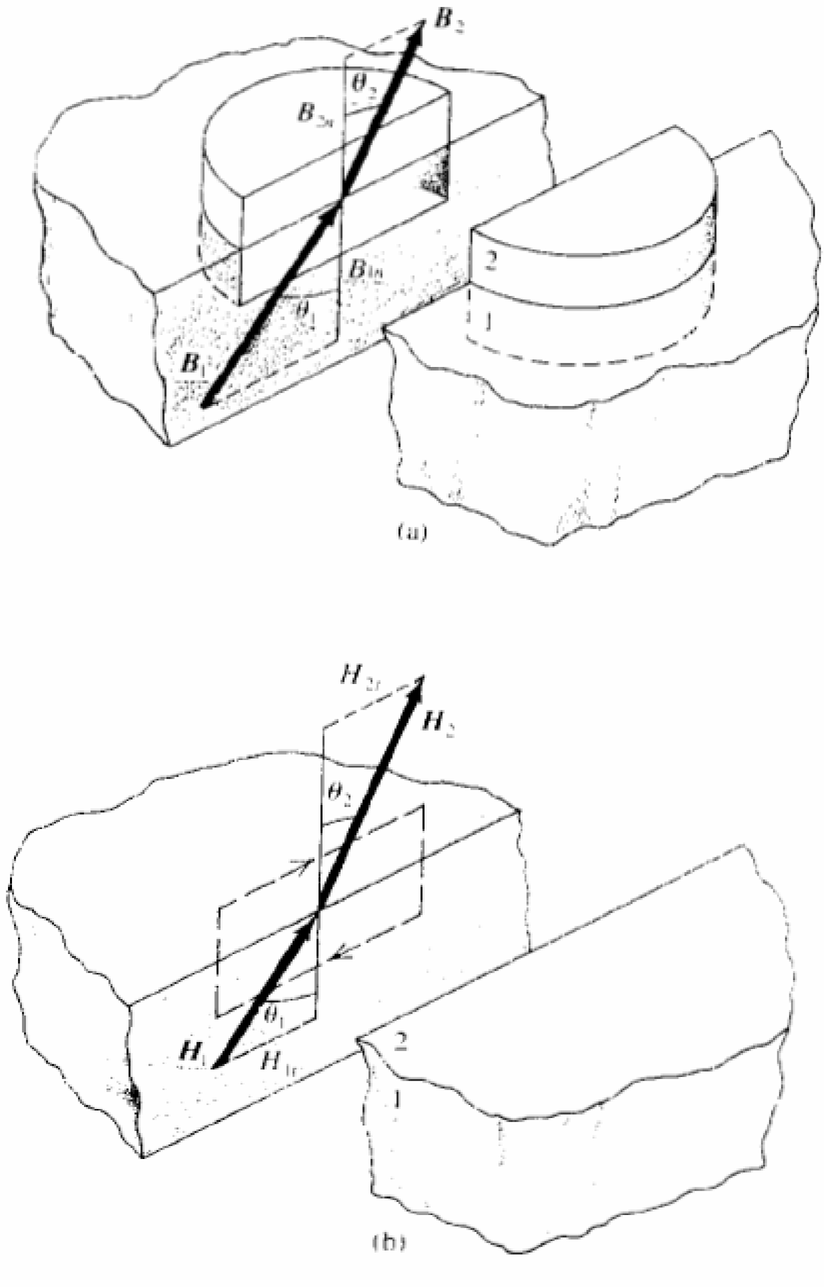

Fig. 20-6. (a) Gaussian surface straddling the interface between media 1 and 2. The normal components of the $\boldsymbol{B}$'s are equal. (b) Closed path piercing the interface. The tangential components of the $\boldsymbol{H}$'s are equal.

$$B_{1n} = B_{2n}. \tag{20-26}$$

The normal component of $\boldsymbol{B}$ is therefore continuous across an interface.

Consider now Fig. 20-6(b). The small rectangular path pierces the interface. From the circuital law of Sec. 20.6, the line integral of $\boldsymbol{H} \cdot \boldsymbol{dl}$ around the path is equal to the current I linking the path. With the two long sides of the path infinitely close to the interface, I is zero and the tangential component of $\boldsymbol{H}$ is continuous across the interface:

$$H_{1t} = H_{2t}. \tag{20-27}$$

These two equations are general.

Setting $\boldsymbol{B} = \mu \boldsymbol{H}$ for both media, the permeabilities being those that correspond to the actual fields, and assuming that the materials are isotropic, then the above two equations imply that

$$\frac{\tan \theta_1}{\tan \theta_2} = \frac{\mu_{r1}}{\mu_{r2}}. \tag{20-28}$$

We therefore have the following rule for linear and isotropic media: lines of $\boldsymbol{B}$ lie farther away from the normal in the medium possessing the larger permeability. In other words, the lines "prefer" to pass through the more permeable medium, as in Fig. 20-7. You will recall from Sec. 10.2.4 that we had a similar situation with dielectrics.

THE MAGNETIC ENERGY DENSITY $\mathscr{E}'_m$ EXPRESSED IN TERMS OF $\boldsymbol{H}$ AND $\boldsymbol{B}$

To express the magnetic energy in terms of $\boldsymbol{H}$ and $\boldsymbol{B}$, we use Eq. 26-9 and apply it to the loop of Fig. 26-2. The loop lies in a homogeneous,

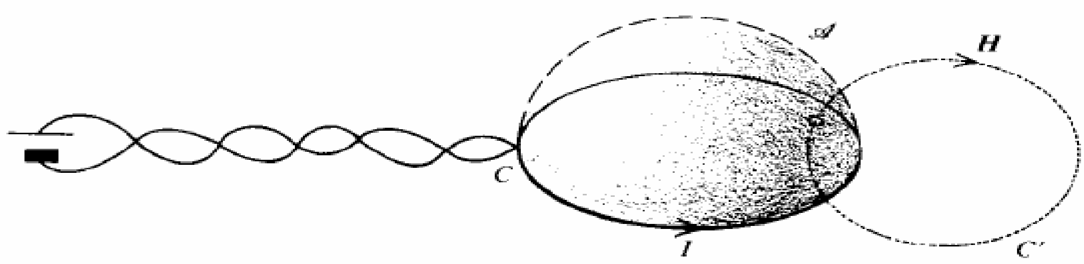

Fig. 26-2. Single-turn loop of wire C bearing a current I. The dotted line is a typical line of $\boldsymbol{H}$. The open surface, of area $\mathscr{A}$, is bounded by C, and it is everywhere orthogonal to $\boldsymbol{H}$.

isotropic, linear, and stationary (HILS) magnetic medium. This excludes ferromagnetic media. From Ampère's circuital law,

$$I = \oint_{C'} H\, dl, \tag{26-20}$$

where C' is any line of $\boldsymbol{H}$.

Also, let $\mathscr{A}$ be the area of any open surface bounded by the loop C and orthogonal to the lines of $\boldsymbol{H}$ and of $\boldsymbol{B}$. Then

$$\Lambda = \Phi = \int_{\mathscr{A}} \boldsymbol{B} \cdot \boldsymbol{d\mathscr{A}} \tag{26-21}$$

and

$$\mathscr{E}_m = \tfrac{1}{2} I \Lambda = \frac{1}{2} \oint_{C'} H\, dl \int_{\mathscr{A}} B\, d\mathscr{A}. \tag{26-22}$$

Now the lines of $\boldsymbol{H}$ and the set of open surfaces define a coordinate system in which $\boldsymbol{dl} \cdot \boldsymbol{d\mathscr{A}}$ is an element of volume with $\boldsymbol{dl}$ and $\boldsymbol{d\mathscr{A}}$ both parallel to $\boldsymbol{H}$. Also, for each element $\boldsymbol{dl}$ along the chosen line of $\boldsymbol{H}$, one integrates over all the corresponding surface. Since the field extends to infinity, this double integral is the volume integral of $\boldsymbol{H} \cdot \boldsymbol{B}$ over all space, and

$$\mathscr{E}_m = \frac{1}{2} \int_{\infty} \boldsymbol{H} \cdot \boldsymbol{B}\, dv. \tag{26-23}$$

The *magnetic energy density* in nonferromagnetic media is thus

$$\mathscr{E}'_m = \frac{\boldsymbol{H} \cdot \boldsymbol{B}}{2} = \frac{B^2}{2\mu} = \frac{\mu H^2}{2}. \tag{26-24}$$

The magnetic energy density varies as B^2. Thus, after superposing several fields, the total field energy is *not* equal to the sum of the individual energies. See Eq. 26-12.

CHƯƠNG 4. TRƯỜNG ĐIỆN TỪ BIẾN THIÊN

MOTIONAL ELECTROMOTANCE. THE FARADAY INDUCTION LAW FOR $\boldsymbol{v} \times \boldsymbol{B}$ FIELDS

Consider a closed circuit C that moves as a whole and distorts in some arbitrary way in a constant magnetic field, as in Fig. 23-1. Then, by definition, the *induced*, or *motional, electromotance* is

$$\mathcal{V} = \oint_C (\boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{l} = -\oint_C \boldsymbol{B} \cdot (\boldsymbol{v} \times d\boldsymbol{l}). \tag{23-2}$$

The negative sign comes from the fact that we have altered the cyclic order of the terms under the integral sign.

Now $\boldsymbol{v} \times d\boldsymbol{l}$ is the area swept by the element $d\boldsymbol{l}$ in 1 second. Thus $\boldsymbol{B} \cdot (\boldsymbol{v} \times d\boldsymbol{l})$ is the rate at which the magnetic flux linking the circuit increases because of the motion of the element $d\boldsymbol{l}$. Integrating over the complete circuit, we find that the induced electromotance is proportional to the time rate of change of the magnetic flux linking the circuit:

$$\mathcal{V} = -\frac{d\Phi}{dt}. \tag{23-3}$$

The positive directions for $\mathcal{V}$ and for Φ satisfy the right-hand screw rule. The current is the same as if the circuit comprised a battery of voltage $\mathcal{V}$.

This is the *Faraday induction law for* $\boldsymbol{v} \times \boldsymbol{B}$ *fields*. This law is important. As far as our demonstration goes, it applies only to constant B's, but it is, in fact, general, as we see in Sec. 23.4. Quite often Φ is difficult to define; then we can integrate $\boldsymbol{v} \times \boldsymbol{B}$ around the circuit to obtain $\mathcal{V}$.

If C is open, as in Fig. 23-2, then current flows until the electric field resulting from the accumulations of charge exactly cancels the $\boldsymbol{v} \times \boldsymbol{B}$ field.

FARADAY'S INDUCTION LAW FOR TIME-DEPENDENT $\boldsymbol{B}$'s. THE CURL OF $\boldsymbol{E}$

Imagine now two closed and rigid circuits as in Fig. 23-6. The active circuit a is stationary, while the passive circuit b moves in some arbitrary way, say in the direction of a as in the figure. The current I_a is constant.

From Sec. 23.2, the electromotance induced in circuit b is

$$\mathcal{V} = \oint_b (\boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{dl} = -\frac{d\Phi}{dt}, \qquad (23\text{-}27)$$

where Φ is the magnetic flux linking b. This seems trivial, but it is not, because $d\Phi/dt$ could be the same if both circuits were stationary and if I_a changed appropriately. This means that the *Faraday induction law*

$$\mathcal{V} = -\frac{d\Phi}{dt} \qquad (23\text{-}28)$$

applies whether there are moving conductors in a constant $\boldsymbol{B}$ *or stationary conductors in a time-varying* $\boldsymbol{B}$. However, our argument is no more than plausible. A proper demonstration follows at the end of this chapter. It requires relativity.

Assuming the correctness of the above result, the electromotance induced in a rigid and stationary circuit C lying in a time-varying magnetic field is

$$\mathcal{V} = \oint_C \boldsymbol{E} \cdot \boldsymbol{dl} = \int_{\mathcal{A}} (\boldsymbol{\nabla} \times \boldsymbol{E}) \cdot \boldsymbol{d\mathcal{A}} = -\frac{d\Phi}{dt} = -\int_{\mathcal{A}} \frac{\partial \boldsymbol{B}}{\partial t} \cdot \boldsymbol{d\mathcal{A}}. \qquad (23\text{-}29)$$

We have used Stokes's theorem in going from the first to the second integral, $\mathcal{A}$ being an arbitary surface bounded by C. Also, we have a partial derivative under the last integral sign, to take into account the fact that the magnetic field can be a function of the coordinates as well as of the time. The right-hand screw rule applies.

The path of integration need not lie in conducting material.

Observe that the above equation involves only the integral of $\boldsymbol{E} \cdot \boldsymbol{dl}$. It does *not* give $\boldsymbol{E}$ as a function of the coordinates, except for simple geometries, and only after integration.

Since the surface of area $\mathcal{A}$ chosen for the surface integrals is arbitrary, the equality of the third and last terms above means that

$$\boxed{\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}.} \qquad (23\text{-}30)$$

This is yet another of Maxwell's equations. This equation, like the other two (Eqs. 9-15 and 18-19), is valid on the condition that all the variables relate to the same reference frame.

THE ELECTRIC FIELD STRENGTH $\boldsymbol{E}$ EXPRESSED IN TERMS OF THE POTENTIALS V AND $\boldsymbol{A}$

An arbitrary, rigid, and stationary closed circuit C lies in a time-dependent $\boldsymbol{B}$. Then, from Sec. 23.4,

$$\oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d}{dt}\int_{\mathscr{A}} \boldsymbol{B} \cdot d\boldsymbol{\mathscr{A}}, \tag{23-41}$$

where $\mathscr{A}$ is the area of any open surface bounded by C.

Now, from Sec. 19.1, we can replace the surface integral on the right by the line integral of the vector potential $\boldsymbol{A}$ around C:

$$\oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d}{dt}\oint_C \boldsymbol{A} \cdot d\boldsymbol{l} = -\oint_C \frac{\partial \boldsymbol{A}}{\partial t} \cdot d\boldsymbol{l}. \tag{23-42}$$

There is no objection to inserting the time derivative under the integral sign, but then it becomes a partial derivative because $\boldsymbol{A}$ is normally a function of the coordinates as well as of the time.

Thus

$$\oint_C \left(\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t}\right) \cdot d\boldsymbol{l} = 0, \tag{23-43}$$

where C is a closed curve, as stated above. Then, from Sec. 1.9.1, the expression enclosed in parentheses is equal to the gradient of some function:

$$\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} = -\boldsymbol{\nabla} V, \tag{23-44}$$

$$\boldsymbol{E} = -\boldsymbol{\nabla} V - \frac{\partial \boldsymbol{A}}{\partial t}, \tag{23-45}$$

where V is, of course, the electric potential.

So $\boldsymbol{E}$ is the sum of two terms, $-\boldsymbol{\nabla} V$ that results from accumulations of charge and $-\partial \boldsymbol{A}/\partial t$ whenever there are time-dependent fields in the given reference frame.

This is an important equation; we shall use it repeatedly. Observe that it expresses $\boldsymbol{E}$ itself, *not* its derivatives or its integral, at a given point in terms of the derivatives of V and of $\boldsymbol{A}$ in the region of that point. Its magnetic equivalent is $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ (Sec. 18.4).

The Faraday induction law, in differential form (Eq. 23-30), relates space derivatives of $\boldsymbol{E}$ to the time derivative of $\boldsymbol{B}$ at a given point.

Observe that $\boldsymbol{\nabla} V$ is a function of V, which depends on the positions of the charges. However, $\partial \boldsymbol{A}/\partial t$ is a function of the time derivative of the current density $\boldsymbol{J}$, hence of the acceleration of the charges.

The relations

$$\boldsymbol{E} = -\boldsymbol{\nabla} V - \frac{\partial \boldsymbol{A}}{\partial t} \qquad \text{and} \qquad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \tag{23-46}$$

are always valid in any given inertial reference frame.[†]

In a time-dependent $\boldsymbol{B}$, the electromotance induced in a circuit C is

$$\mathcal{V} = -\int_C \frac{\partial \boldsymbol{A}}{\partial t} \cdot d\boldsymbol{l}. \tag{23-47}$$

SIX KEY EQUATIONS

It is useful at this stage to group the following six equations:

$$\text{(G)} \qquad \boldsymbol{E} = -\boldsymbol{\nabla} V - \frac{\partial \boldsymbol{A}}{\partial t}, \tag{23-46}$$

$$\text{(G)} \qquad \oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = -\int_{\mathcal{A}} \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{\mathcal{A}}, \tag{23-29}$$

$$\text{(G)} \qquad \boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{23-30}$$

$$\text{(G)} \qquad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}, \qquad \text{(Sec. 18.4)} \qquad \text{and} \qquad \text{(23-46)}$$

$$\oint \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 \int_{\mathcal{A}} \boldsymbol{J} \cdot d\boldsymbol{\mathcal{A}}, \qquad \text{(Sec. 19.5)}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}. \qquad \text{(Sec. 19.4)}$$

The four equations preceded by (G) are general, while the other two apply only to slowing varying fields (Sec. 27.1). In each equation all the terms concern the *same* reference frame.

MAXWELL'S EQUATIONS IN DIFFERENTIAL FORM

Let us group Maxwell's four equations; we discuss them at length below. We found them successively in Secs. 9.5, 23.4, 20.4, and 17.4:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}, \quad (27\text{-}1) \qquad \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0, \quad (27\text{-}2)$$

$$\nabla \cdot \boldsymbol{B} = 0, \quad (27\text{-}3) \qquad \nabla \times \boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = \mu_0 \boldsymbol{J}.^{+} \quad (27\text{-}4)$$

The above equations are general in that the media can be *nonhomogeneous, nonlinear,* and *nonisotropic.* However, (1) they apply only to media that are stationary with respect to the coordinate axes,† and (2) the coordinate axes must not accelerate and must not rotate.

These are the four fundamental equations of electromagnetism. They form a set of simultaneous partial differential equations relating certain time and space derivatives at a point to the charge and current densities at that point. They apply, whatever be the number or diversity of the sources.

We have followed the usual custom of writing the field terms on the left and the source terms on the right. However, this is somewhat illusory because ρ and $\boldsymbol{J}$ are themselves functions of $\boldsymbol{E}$ and $\boldsymbol{B}$. As usual,

$\boldsymbol{E}$ is the electric field strength, in volts/meter;

$\rho = \rho_f + \rho_b$ is the total electric charge density, in coulombs/meter^3;

ρ_f is the free charge density;

$\rho_b = -\nabla \cdot \boldsymbol{P}$ is the bound charge density;

$\boldsymbol{P}$ is the electric polarization, in coulombs/meter^2;

$\boldsymbol{B}$ is the magnetic flux density, in teslas;

$\boldsymbol{J} = \boldsymbol{J}_f + \partial \boldsymbol{P}/\partial t + \nabla \times \boldsymbol{M}$ is the total current density, in amperes/meter^2;‡

$\boldsymbol{J}_f$ is the current density resulting from the motion of free charge;

$\partial \boldsymbol{P}/\partial t$ is the polarization current density in a dielectric;

$\nabla \times \boldsymbol{M}$ is the equivalent current density in magnetized matter;

$\boldsymbol{M}$ is the magnetization, in amperes/meter;

c is the speed of light, about 300 megameters per second;

ϵ_0 is the permittivity of free space, about 8.85×10^{-12} farad/meter.

In isotropic, linear, and stationary media,

$$\boldsymbol{J}_f = \sigma \boldsymbol{E}, \qquad \boldsymbol{P} = \epsilon_0 \chi_e \boldsymbol{E}, \qquad \boldsymbol{M} = \chi_m \boldsymbol{H}, \qquad (27\text{-}5)$$

where σ is the conductivity, χ_e is the electric susceptibility, and χ_m is the magnetic susceptibility. Also,

$$\epsilon_r = 1 + \chi_e, \qquad \mu_r = 1 + \chi_m, \qquad (27\text{-}6)$$

where ϵ_r is the relative permittivity and μ_r is the relative permeability. Inside a source, such as a battery or a Van de Graaff generator, electric charges are "pumped" by the locally generated electric field $\boldsymbol{E}_s$ against the electric field $\boldsymbol{E}$ of other sources, and $\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{E}_s)$.

Writing out ρ and $\boldsymbol{J}$ in full, Maxwell's equations become

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho_f - \boldsymbol{\nabla} \cdot \boldsymbol{P}}{\epsilon_0}, \tag{27-7}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0, \tag{27-8}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{27-9}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = \mu_0\left(\boldsymbol{J}_f + \frac{\partial \boldsymbol{P}}{\partial t} + \boldsymbol{\nabla} \times \boldsymbol{M}\right). \tag{27-10}$$

This *Amperian formulation* expresses the field in terms of the four vectors ***E***, ***B***, ***P***, and ***M***.

With homogeneous, isotropic, linear, and stationary (HILS) media,

$$\rho = \frac{\rho_f}{\epsilon_r} \qquad \text{(Sec. 9.9)} \tag{27-11}$$

$$\boldsymbol{P} = (\epsilon_r - 1)\epsilon_0 \boldsymbol{E} \qquad \text{(Sec. 9.9)} \tag{27-12}$$

$$\boldsymbol{M} = \frac{(\mu_r - 1)}{\mu_r \mu_0}\boldsymbol{B} \qquad \text{(Sec. 20.7)} \tag{27-13}$$

and

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho_f}{\epsilon}, \tag{27-14}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0, \tag{27-15}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{27-16}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} - \epsilon\mu \frac{\partial \boldsymbol{E}}{\partial t} = \mu \boldsymbol{J}_f. \tag{27-17}$$

Recall that $\epsilon = \epsilon_r \epsilon_0$ and $\mu = \mu_r \mu_0$, ϵ_r is frequency-dependent, and μ_r is hardly definable in ferromagnetic materials. The expressions for ***P*** and for ***M*** are not symmetrical, but ***P***, ***E***, and ***D*** point in the same direction, like ***M***, ***H***, and ***B***, in isotropic and linear media.

Observe that the above set of equations follows from Eqs. 27-1 to 27-4 with the following substitutions:

$$\epsilon_0 \to \epsilon, \qquad \mu_0 \to \mu, \tag{27-18}$$

$$\rho \to \rho_f, \qquad \boldsymbol{J} \to \boldsymbol{J}_f. \tag{27-19}$$

This is a general rule for transforming an equation in terms of ϵ_0, μ_0, ρ, $\boldsymbol{J}$ *to another one in terms of* ϵ, μ, ρ_f, $\boldsymbol{J}_f$.

The *Minkowski formulation* of Maxwell's equations is often useful. It expresses the same relations, but in terms of the four vectors $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$, $\boldsymbol{H}$:

$$\nabla \cdot \boldsymbol{D} = \rho_f, \qquad (27\text{-}20) \qquad \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0, \qquad (27\text{-}21)$$

$$\nabla \cdot \boldsymbol{B} = 0, \qquad (27\text{-}22) \qquad \nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t} = \boldsymbol{J}_j. \qquad (27\text{-}23)$$

In the following chapters we shall be mostly concerned with electric and magnetic fields that are sinusoidal functions of the time. Then, for isotropic, linear, and stationary media, not necessarily homogeneous,

$$\nabla \cdot \epsilon \boldsymbol{E} = \rho_f, \qquad (27\text{-}24) \qquad \nabla \times \boldsymbol{E} + j\omega\mu \boldsymbol{H} = 0, \qquad (27\text{-}25)$$

$$\nabla \cdot \mu \boldsymbol{H} = 0, \qquad (27\text{-}26) \qquad \nabla \times \boldsymbol{H} - j\omega\epsilon \boldsymbol{E} = \boldsymbol{J}_f. \qquad (27\text{-}27)$$

It is worthwhile to discuss Maxwell's equations further, but first let us rewrite them in integral form.

MAXWELL'S EQUATIONS IN INTEGRAL FORM

Integrating Eq. 27-1 over a finite volume v and then applying the divergence theorem, we find the integral form of Gauss's law (Sec. 9.5):

$$\int_{\mathscr{A}} \boldsymbol{E} \cdot d\boldsymbol{\mathscr{A}} = \frac{1}{\epsilon_0} \int_v \rho \, dv = \frac{Q}{\epsilon_0}, \qquad (27\text{-}28)$$

where $\mathscr{A}$ is the area of the surface bounding the volume v and Q is the total charge enclosed within v. See Fig. 27-1.

Similarly, Eq. 27-3 says that the net outward flux of $\boldsymbol{B}$ through any closed surface is zero, as in Fig. 27-2:

$$\int_{\mathscr{A}} \boldsymbol{B} \cdot d\boldsymbol{\mathscr{A}} = 0. \qquad (27\text{-}29)$$

Equation 27-2 is the differential form of the Faraday induction law for time-dependent magnetic fields. Integrating over an open surface of area $\mathscr{A}$ bounded by a curve C gives the integral form, as in Sec. 23.4:

$$\oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d}{dt} \int_{\mathscr{A}} \boldsymbol{B} \cdot d\boldsymbol{\mathscr{A}} = -\frac{d\Lambda}{dt}, \qquad (27\text{-}30)$$

where Λ is the linking flux. See Fig. 27-3. The electromotance induced around a closed curve C is equal to minus the time derivative of the flux linkage. The positive directions for Λ and around C satisfy the right-hand screw convention.

$$\oint_C \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 \int_{\mathscr{A}} \left(\boldsymbol{J} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right) \cdot d\boldsymbol{\mathscr{A}}. \tag{27-31}$$

We found two less general forms of this law in Secs. 19.5 and 20.6. The closed curve C bounds a surface of area $\mathscr{A}$ through which flows a current of density $\boldsymbol{J} + \epsilon_0 \partial \boldsymbol{E}/\partial t$. See Fig. 27-4.

THE LAW OF CONSERVATION OF CHARGE

In Sec. 4.2 we saw that free charges are conserved. At that time we were using the symbol $\boldsymbol{J}$ for the current density of free charges instead of $\boldsymbol{J}_f$.

Let us calculate the divergence of $\boldsymbol{J}$ as defined in Sec. 27.1. We will need the value of this divergence in the next section. First,

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} = \boldsymbol{\nabla} \cdot \left(\boldsymbol{J}_f + \frac{\partial \boldsymbol{P}}{\partial t} + \boldsymbol{\nabla} \times \boldsymbol{M} \right) = \boldsymbol{\nabla} \cdot \boldsymbol{J}_f + \frac{\partial}{\partial t} (\boldsymbol{\nabla} \cdot \boldsymbol{P}), \tag{27-41}$$

the divergence of a curl being equal to zero. Thus

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} = -\frac{\partial \rho_f}{\partial t} - \frac{\partial \rho_b}{\partial t} = -\frac{\partial (\rho_f + \rho_b)}{\partial t} = -\frac{\partial \rho}{\partial t}. \tag{27-42}$$

This is a more general form of the law of conservation of charge of Sec. 4.2.

MAXWELL'S EQUATIONS ARE REDUNDANT

Maxwell's four equations are redundant. We saw in Secs. 17.3 and 17.4 that the equation for $\boldsymbol{\nabla} \times \boldsymbol{E}$ follows from the one for $\boldsymbol{\nabla} \cdot \boldsymbol{B}$, and the equation for $\boldsymbol{\nabla} \times \boldsymbol{B}$ from the one for $\boldsymbol{\nabla} \cdot \boldsymbol{E}$. These are, respectively, the first and second pairs.

The two equations of the first pair are also related as follows. If we take the divergence of Eq. 27-2 and remember that the divergence of a curl is zero, we find that

$$\boldsymbol{\nabla} \cdot \frac{\partial \boldsymbol{B}}{\partial t} = 0 \qquad \text{or} \qquad \frac{\partial}{\partial t} \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0. \tag{27-43}$$

So $\nabla \cdot \boldsymbol{B}$ is a constant at every point in space. Then we can set $\nabla \cdot \boldsymbol{B} = 0$ everywhere and for all time if we assume that, for each point in space, $\nabla \cdot \boldsymbol{B}$ is zero at some time, in the past, at present, or in the future. With this assumption, Eq. 27-3 follows from Eq. 27-2.

Similarly, taking the divergence of Eq. 27-4 and applying the law of conservation of charge, we find

$$\epsilon_0 \nabla \cdot \frac{\partial \boldsymbol{E}}{\partial t} = -\nabla \cdot \boldsymbol{J} = \frac{\partial \rho}{\partial t}, \tag{27-44}$$

$$\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{E}) = \frac{\partial}{\partial t}\left(\frac{\rho}{\epsilon_0}\right), \qquad \nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} + C. \tag{27-45}$$

The constant of integration C can be a function of the coordinates.

If we now assume that, at every point in space, at some time, $\nabla \cdot \boldsymbol{E}$ and ρ are simultaneously equal to zero, then C is zero and we have Eq. 27-1.

So there are really only two independent equations.

DUALITY

Imagine a field $\boldsymbol{E}$, $\boldsymbol{B}$ that satisfies Maxwell's equations with $\rho_f = 0$, $\boldsymbol{J}_f = 0$ in a given region. The medium is homogeneous, isotropic, linear, and stationary (HILS). Now imagine a different field

$$\boldsymbol{E}' = -K\boldsymbol{B} = -K\mu \boldsymbol{H}, \tag{27-46}$$

$$\boldsymbol{H}' = +K\boldsymbol{D} = +K\epsilon \boldsymbol{E}, \tag{27-47}$$

where the constant K has the dimensions of a velocity and is independent of x, y, z, t. This other field *also* satisfies Maxwell's equations, as you can check by substitution into Eqs. 27-20 to 27-23.

Figure 27-5 illustrates this duality property of electromagnetic fields. One field is said to be the *dual*, or the *dual field*, of the other. Therefore, if one field can exist, then its dual can also exist.

THE WAVE EQUATIONS FOR $\boldsymbol{E}$ AND FOR $\boldsymbol{B}$

Taking the curl of Eq. 27-2 and remembering that

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{E} = -\boldsymbol{\nabla}^2 \boldsymbol{E} + \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{E}), \tag{27-63}$$

then, from Eq. 27-4,

$$\boldsymbol{\nabla}^2 \boldsymbol{E} - \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{E}) = \frac{\partial}{\partial t} \boldsymbol{\nabla} \times \boldsymbol{B} = \frac{\partial}{\partial t}\left(\mu_0 \boldsymbol{J} + \epsilon_0 \mu_0 \frac{\partial \boldsymbol{E}}{\partial t}\right). \tag{27-64}$$

Substituting now the value of the divergence of $\boldsymbol{E}$ from Eq. 27-1 and rearranging,

$$\boldsymbol{\nabla}^2 \boldsymbol{E} - \epsilon_0 \mu_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = \frac{\boldsymbol{\nabla} \rho}{\epsilon_0} + \mu_0 \frac{\partial \boldsymbol{J}}{\partial t}. \tag{27-65}$$

This is the *nonhomogeneous wave equation for* $\boldsymbol{E}$. The source terms are on the right.

Outside the sources,

$$\boldsymbol{\nabla}^2 \boldsymbol{E} - \epsilon_0 \mu_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0. \tag{27-66}$$

This is the usual *wave equation*. The speed of propagation, which is the *speed of light*, is

$$c = \frac{1}{(\epsilon_0 \mu_0)^{1/2}}. \tag{27-67}$$

Similarly, taking the curl of Eq. 27-4 and substituting Eqs 27-2 and 27-3, we find that

$$\nabla^2 \boldsymbol{B} - \epsilon_0 \mu_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2} = -\mu_0 \nabla \times \boldsymbol{J}, \tag{27-68}$$

which is the *nonhomogeneous wave equation for* ***B***. The source term is again on the right.

Outside the sources,

$$\nabla^2 \boldsymbol{B} - \epsilon_0 \mu_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2} = 0. \tag{27-69}$$

According to the rule given in Sec. 27-1, the wave equations for a HILS medium are as follows:

$$\nabla^2 \boldsymbol{E} - \epsilon \mu \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = \frac{\nabla \rho_f}{\epsilon} + \mu \frac{\partial \boldsymbol{J}_f}{\partial t}, \tag{27-70}$$

$$\nabla^2 \boldsymbol{B} - \epsilon \mu \frac{\partial^2 \boldsymbol{B}}{\partial t^2} = -\mu \nabla \times \boldsymbol{J}_f. \tag{27-71}$$

We therefore have a wave equation for the field ***E***, and a separate wave equation for ***B***. Within the wave, however, ***E*** and ***B*** are inextricably related through Maxwell's equations. In other words, purely electric, or purely magnetic, waves are impossible. The fact remains that, in some waves, the energy density can be either mostly magnetic or mostly electric.

If σ is constant,

$$\nabla^2 \boldsymbol{E} - \epsilon \mu \frac{\partial^2 \boldsymbol{E}}{\partial t^2} - \sigma \mu \frac{\partial \boldsymbol{E}}{\partial t} = \frac{\nabla \rho_f}{\epsilon}, \tag{27-72}$$

$$\nabla^2 \boldsymbol{B} - \epsilon \mu \frac{\partial^2 \boldsymbol{B}}{\partial t^2} - \sigma \mu \frac{\partial \boldsymbol{B}}{\partial t} = 0. \tag{27-73}$$

CHƯƠNG 5. SÓNG ĐIỆN TỪ PHẲNG

UNIFORM PLANE ELECTROMAGNETIC WAVES IN A GENERAL MEDIUM

A *wave front* is a surface of uniform phase. The wave fronts of a *plane wave* are planar. A wave is *uniform* if a wave front is a surface of uniform phase and uniform amplitude. We shall not be concerned with nonuniform waves until Chap. 31.

Uniform plane electromagnetic waves in unbounded media possess several general properties that apply whether the wave travels in free space or in matter. To avoid needless repetition, we start with a general medium ϵ_r, μ_r, σ that is homogeneous, isotropic, linear, and stationary (HILS).

We assume a sinusoidal wave traveling in the positive direction of the z-axis. We also assume that the $\boldsymbol{E}$ vectors are all parallel to a given direction. In other words, we assume that the wave is *linearly polarized.* If the plane wave is not linearly polarized, then it is the sum of linearly polarized waves.[†] The *plane of polarization* is parallel to $\boldsymbol{E}$.

In a linearly polarized plane wave, $\boldsymbol{E}$ and $\boldsymbol{H}$ are of the form

$$\boldsymbol{E} = \boldsymbol{E}_m \exp j(\omega t - kz), \qquad \boldsymbol{H} = \boldsymbol{H}_m \exp j(\omega t - kz)^{\ddagger} \qquad (28\text{-}1)$$

where $\boldsymbol{E}_m$ and $\boldsymbol{H}_m$ are vectors that are independent of the time and of the coordinates. If there is no attenuation, the *wave number* is real:

$$k = \frac{\omega}{v} = \frac{2\pi}{\lambda} = \frac{1}{\bar{\lambda}}, \qquad (28\text{-}2)$$

where v is the *phase velocity,* λ is the *wavelength,* and $\bar{\lambda}$ (pronounced "lambda bar") is the *radian length.* You can easily show by substitution that Eqs. 28-1 are solutions of Eqs. 27-66 and 27-69.

The Relative Orientations of $\boldsymbol{E}$, $\boldsymbol{H}$, and $\boldsymbol{k}$

For this particular field,

$$\frac{\partial}{\partial t} = j\omega, \qquad \boldsymbol{\nabla} = \frac{\partial}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial}{\partial z}\hat{\boldsymbol{z}} = \frac{\partial}{\partial z}\hat{\boldsymbol{z}} = -jk\hat{\boldsymbol{z}}. \tag{28-3}$$

We set $\rho_f = 0$. We also set

$$\boldsymbol{J}_f = \sigma \boldsymbol{E}, \tag{28-4}$$

on the assumption that $\boldsymbol{v} \times \boldsymbol{B}$ is negligible compared to $\boldsymbol{E}$, where $\boldsymbol{v}$ is the velocity of a conduction electron.

Then Maxwell's equations 27-24 to 27-27 reduce to

$$-jk\hat{\boldsymbol{z}} \cdot \boldsymbol{E} = 0, \qquad -jk\hat{\boldsymbol{z}} \times \boldsymbol{E} = -j\omega\mu\boldsymbol{H}, \tag{28-5}$$

$$-jk\hat{\boldsymbol{z}} \cdot \boldsymbol{H} = 0, \qquad -jk\hat{\boldsymbol{z}} \times \boldsymbol{H} = \sigma\boldsymbol{E} + j\omega\epsilon\boldsymbol{E} \tag{28-6}$$

and then to

$$\hat{\boldsymbol{z}} \cdot \boldsymbol{E} = 0, \qquad \boldsymbol{E} = -\frac{k}{\omega\epsilon + j\sigma}\hat{\boldsymbol{z}} \times \boldsymbol{H}, \tag{28-7}$$

$$\hat{\boldsymbol{z}} \cdot \boldsymbol{H} = 0, \qquad \boldsymbol{H} = \frac{k}{\omega\mu}\hat{\boldsymbol{z}} \times \boldsymbol{E}. \tag{28-8}$$

It follows that $\boldsymbol{E}$ and $\boldsymbol{H}$ are transverse and orthogonal. Figure 28-2 shows the relative orientations of $\boldsymbol{E}$, $\boldsymbol{H}$, and $\boldsymbol{k} = k\hat{\boldsymbol{z}}$. Observe that $\boldsymbol{E} \times \boldsymbol{H}$ points in the direction of propagation.

The Characteristic Impedance Z of a Medium

The ratio E/H is the *characteristic impedance* Z of the medium of propagation:

$$Z = \frac{E}{H} = \frac{k}{\omega\epsilon - j\sigma} = \frac{\omega\mu}{k}. \tag{28-9}$$

The Wave Number k

The value of k^2 follows from the above equation:

$$k^2 = \omega^2 \epsilon\mu - j\omega\sigma\mu = \omega^2\epsilon\mu\left(1 - j\frac{\sigma}{\omega\epsilon}\right), \tag{28-10}$$

$$= \omega^2\epsilon_0\mu_0\epsilon_r\mu_r\left(1 - j\frac{\sigma}{\omega\epsilon}\right). \tag{28-11}$$

The σ terms account for Joule losses and attenuation.

The Wave Equations

We found the nonhomogenous wave equations for $\boldsymbol{E}$ and $\boldsymbol{B}$ in Sec. 27.9:

$$\nabla^2\boldsymbol{E} - \epsilon_0\mu_0\frac{\partial^2\boldsymbol{E}}{\partial t^2} = \frac{\nabla\rho}{\epsilon_0} + \mu_0\frac{\partial\boldsymbol{J}}{\partial t}, \tag{28-12}$$

$$\nabla^2\boldsymbol{B} - \epsilon_0\mu_0\frac{\partial^2\boldsymbol{B}}{\partial t^2} = \mu_0\nabla\times\boldsymbol{J}. \tag{28-13}$$

We now apply the rule of Sec. 27.1 and Eq. 28-4 to obtain the equivalent equations for a medium ϵ, μ, σ. We again set $\rho_f = 0$. Then

$$\nabla^2\boldsymbol{E} - \epsilon\mu\frac{\partial^2\boldsymbol{E}}{\partial t^2} = \mu\frac{\partial\boldsymbol{J}_f}{\partial t} = \mu\sigma\frac{\partial\boldsymbol{E}}{\partial t}, \tag{28-14}$$

$$\nabla^2\boldsymbol{B} - \epsilon\mu\frac{\partial^2\boldsymbol{B}}{\partial t^2} = -\mu\sigma\nabla\times\boldsymbol{E} = \mu\sigma\frac{\partial\boldsymbol{B}}{\partial t}. \tag{28-15}$$

It is the custom to write these wave equations in the form

$$\nabla^2 \boldsymbol{E} - \epsilon\mu \frac{\partial^2 \boldsymbol{E}}{\partial t^2} - \mu\sigma \frac{\partial \boldsymbol{E}}{\partial t} = 0, \tag{28-16}$$

$$\nabla^2 \boldsymbol{B} - \epsilon\mu \frac{\partial^2 \boldsymbol{B}}{\partial t^2} - \mu\sigma \frac{\partial \boldsymbol{B}}{\partial t} = 0, \tag{28-17}$$

or,

$$\nabla^2 \boldsymbol{H} - \epsilon\mu \frac{\partial^2 \boldsymbol{H}}{\partial t^2} - \mu\sigma \frac{\partial \boldsymbol{H}}{\partial t} = 0. \tag{28-18}$$

Then, from Sec. 28.2.1,

$$(-k^2 + \omega^2\epsilon\mu - j\omega\sigma\mu)\boldsymbol{E} = 0, \tag{28-19}$$

and similarly for $\boldsymbol{H}$. The expression enclosed in parentheses is equal to zero, from Eq. 28-10.

UNIFORM PLANE WAVES IN FREE SPACE

In free space, $\epsilon_r = 1$, $\mu_r = 1$, $\sigma = 0$, there is no attenuation, and from Eq. 28-19,

$$k = \frac{1}{\lambda\!\!\!^{-}_0} \tag{28-20}$$

$$= \omega(\epsilon_0\mu_0)^{1/2}. \tag{28-21}$$

From Eq. 28-2 the speed of light is

$$c = \frac{\omega}{k} = \frac{1}{(\epsilon_0\mu_0)^{1/2}} = 2.99792458 \times 10^8 \text{ meters/second.} \tag{28-22}$$

This equation is remarkable. It links three basic constants of electromagnetism: the speed of light c, the permittivity of free space ϵ_0 that appears in the expression for the Coulomb force, and the permeability of free space μ_0 from the magnetic force law.

Since μ_0 is, by definition, *exactly* equal to $4\pi \times 10^{-7}$, the value of ϵ_0 follows from the value of c:

$$\epsilon_0 = \frac{1}{\mu_0 c^2} = 8.854187817 \times 10^{-12} \text{ farad/meter.} \tag{28-23}$$

The *characteristic impedance of the vacuum* is

$$Z_0 = \frac{E}{H} = \frac{k}{\omega\epsilon_0} = \frac{\omega\mu_0}{k} = \frac{1}{\epsilon_0 c} = \mu_0 c = \left(\frac{\mu_0}{\epsilon_0}\right)^{1/2} \tag{28-24}$$

$$= 3.767303 \times 10^2 \approx 377 \text{ ohms}. \tag{28-25}$$

Thus, since $B = \mu_0 H$ in free space,

$$\frac{E}{B} = \frac{1}{(\epsilon_0 \mu_0)^{1/2}} = c, \qquad \text{or} \qquad E = Bc. \tag{28-26}$$

The $\boldsymbol{E}$ and $\boldsymbol{H}$ vectors in free space are in phase because the characteristic impedance of free space is real.

The electric and magnetic energy densities† are equal:

$$\frac{\epsilon_0 E^2/2}{\mu_0 H^2/2} = \frac{\epsilon_0}{\mu_0}\left(\frac{\mu_0}{\epsilon_0}\right) = 1. \tag{28-27}$$

At any instant the total energy density fluctuates with z as in Fig. 28-3, and its time-averaged value at any point is

$$\mathscr{E}' = \frac{\epsilon_0 E_{\text{rms}}^2}{2} + \frac{\mu_0 H_{\text{rms}}^2}{2} = \epsilon_0 E_{\text{rms}}^2 = \mu_0 H_{\text{rms}}^2. \tag{28-28}$$

Abandoning the phasor notation for a moment,

$$\boldsymbol{E} = \boldsymbol{E}_m \cos(\omega t - kz), \qquad \boldsymbol{H} = \boldsymbol{H}_m \cos(\omega t - kz). \tag{29-29}$$

The magnitude of the *Poynting vector* is

$$|\mathscr{S}| = |\boldsymbol{E} \times \boldsymbol{H}| = E_m H_m \cos^2(\omega t - kz). \tag{28-30}$$

We shall see in Sec. 28.6 that the Poynting vector, when integrated over a surface, yields the power flow through that surface. Power flows in the direction of $\mathcal{S}$.

Returning to phasors, the time-averaged Poynting vector is (Sec. 2.4)

$$\mathcal{S}_{av} = \tfrac{1}{2}\,\mathrm{Re}\,(\boldsymbol{E} \times \boldsymbol{H}^*) \tag{28-31}$$

and, for a uniform plane wave in free space,

$$\mathcal{S}_{av} = \tfrac{1}{2}\,\mathrm{Re}\,(EH^*)\hat{\boldsymbol{z}} \tag{28-32}$$

$$= \tfrac{1}{2}c\epsilon_0\,|E_m|^2\,\hat{\boldsymbol{z}} = c\epsilon_0 E_{rms}^2\hat{\boldsymbol{z}} = \frac{E_{rms}^2}{Z_0}\,\hat{\boldsymbol{z}} \tag{28-33}$$

$$\approx \frac{E_{rms}^2}{377}\,\hat{\boldsymbol{z}} \qquad \text{watts/meter}^2. \tag{28-34}$$

This is the time-averaged total energy density $\epsilon_0 E_{rms}^2$, multiplied by the speed of light c.

UNIFORM PLANE WAVES IN NONCONDUCTORS

The situation here is the same as in free space, with ϵ and μ replacing ϵ_0 and μ_0. The phase velocity is now

$$v = \frac{1}{(\epsilon\mu)^{1/2}} = \frac{c}{(\epsilon_r\mu_r)^{1/2}} = \frac{c}{n}, \tag{28-38}$$

where n is the *index of refraction*:

$$n = (\epsilon_r\mu_r)^{1/2}. \tag{28-39}$$

The phase velocity v is less than in free space, since both ϵ_r and μ_r are larger than unity. In nonmagnetic media,

$$n = \epsilon_r^{1/2}. \tag{28-40}$$

As we saw in Sec. 10.1.2, ϵ_r is a function of the frequency, so n is also frequency-dependent. As a rule, tables of n apply to optical frequencies ($\approx 10^{15}$ hertz), whereas tables of ϵ_r apply at much lower frequencies, at best up to about 10^{10} hertz. Pairs of values drawn from such tables do not therefore satisfy the above equation.

The characteristic impedance of the medium is

$$Z = \frac{E}{H} = \left(\frac{\mu}{\epsilon}\right)^{1/2} = 377\left(\frac{\mu_r}{\epsilon_r}\right)^{1/2} \qquad \text{ohms.} \tag{28-41}$$

The electric and magnetic energy densities are again equal:

$$\frac{\epsilon E^2/2}{\mu H^2/2} = 1, \tag{28-42}$$

and the time-averaged energy density is

$$\mathscr{E}'_{av} = \frac{\epsilon E^2_{rms}}{2} + \frac{\mu H^2_{rms}}{2} = \epsilon E^2_{rms} = \mu H^2_{rms}. \tag{28-43}$$

The Poynting vector $\boldsymbol{E} \times \boldsymbol{H}$ points again in the direction of propagation, and

$$\mathscr{S}_{av} = \tfrac{1}{2}\,\text{Re}\,(EH^*)\hat{\boldsymbol{z}} = \left(\frac{\epsilon}{\mu}\right)^{1/2} E^2_{rms}\hat{\boldsymbol{z}} \tag{28-44}$$

$$\mathscr{S}_{av} \approx \frac{(\epsilon_r/\mu_r)^{1/2} E^2_{rms}}{377}\,\hat{\boldsymbol{z}} \qquad \text{watts/meter}^2 \tag{28-45}$$

$$= \frac{1}{\epsilon\mu}\,\epsilon E^2_{rms}\hat{\boldsymbol{z}} = v\epsilon E^2_{rms}\hat{\boldsymbol{z}}. \tag{28-46}$$

The time-averaged Poynting vector is again equal to the phase velocity multiplied by the time-averaged energy density.

UNIFORM PLANE WAVES IN CONDUCTORS

The Complex Wave Number $k = \beta - j\alpha$

In Sec. 28.2.3 we found that in a conducting medium

$$k^2 = \frac{\epsilon_r \mu_r}{\bar{\lambda}_0^2}\left(1 - j\frac{\sigma}{\omega\epsilon}\right), \tag{28-52}$$

so k is complex. It is the custom to set

$$k = \beta - j\alpha \qquad \text{and then} \qquad \boldsymbol{E} = \boldsymbol{E}_m \exp(-\alpha z) \exp j(\omega t - \beta z), \tag{28-53}$$

where both α and β are positive.

The quantity $1/\alpha$ is the *attenuation distance* or the *skin depth* δ over which the amplitude decreases by a factor of e. The real part β of k is the inverse of $\bar{\lambda}$:

$$\alpha = \frac{1}{\delta}, \tag{28-54}$$

$$\beta = \frac{1}{\bar{\lambda}} = \frac{2\pi}{\lambda}, \tag{28-55}$$

and the phase velocity is

$$v = \frac{\omega}{\beta}. \tag{28-56}$$

Let us find α and β in terms of ϵ_r, μ_r, σ, and $\bar{\lambda}_0$. First we set

$$\mathscr{D} = \frac{\sigma}{\omega\epsilon} = \left|\frac{\sigma E}{\epsilon\, \partial E/\partial t}\right| = \left|\frac{\sigma E}{\partial D/\partial t}\right| \approx 377\,\frac{\sigma \bar{\lambda}_0}{\epsilon_r}. \tag{28-57}$$

This is the magnitude of the conduction current density, divided by the magnitude of the displacement current density. As a rule, $\mathscr{D}$ (for dissipation"), is written $\tan l$, as in Sec. 10.1.1:

$$\mathscr{D} = \tan l, \tag{28-58}$$

where l is here the *loss angle* of the medium, but we use $\mathscr{D}$ for conciseness.

The permittivity ϵ that appears above is the real part $\epsilon_r'\epsilon_0$ (Sec. 10.1.1). One can account for conductivity *either* by means of a complex permittivity $(\epsilon_r' - j\epsilon_r'')\epsilon_0$ *or* by means of a real permittivity and a conductivity σ, where $\sigma = \omega\epsilon_r''\epsilon_0$, again as in Sec. 10.1.1. Thus

$$\mathscr{D} = \frac{\epsilon_r''}{\epsilon_r'}. \tag{28-59}$$

This quantity, like ϵ_r' and ϵ_r'', is always positive.

If $\mathscr{D} \ll 1$, the medium is a good dielectric; if $\mathscr{D} \gg 1$, the medium is a good conductor. For common types of good conductor, $\sigma \approx 10^7$ ($\sigma = 5.8 \times 10^7$ for copper) and $\epsilon_r \approx 1$ (Sec. 4.3.6). You will remember from Sec. 4.3.6 that ϵ/σ is the relaxation time of a medium.

Thus

$$k^2 = (\beta - j\alpha)^2 = \left(\frac{\epsilon_r\mu_r}{\lambda\!\!\!^{-}{}_0^2}\right)(1 - j\mathscr{D}), \tag{28-60}$$

and

$$\alpha = \frac{1}{\lambda\!\!\!^{-}{}_0}\left(\frac{\epsilon_r\mu_r}{2}\right)^{1/2}[(1 + \mathscr{D}^2)^{1/2} - 1]^{1/2}, \tag{28-61}$$

$$\beta = \frac{1}{\lambda\!\!\!^{-}{}_0}\left(\frac{\epsilon_r\mu_r}{2}\right)^{1/2}[(1 + \mathscr{D}^2)^{1/2} + 1]^{1/2}, \tag{28-62}$$

$$k = \frac{(\epsilon_r\mu_r)^{1/2}}{\lambda\!\!\!^{-}{}_0}(1 + \mathscr{D}^2)^{1/4} \exp\left(-j \arctan\frac{\alpha}{\beta}\right). \tag{28-63}$$

The argument of the exponential function is correct because β is positive (Sec. 2.1).

In a *low-loss dielectric* $\mathscr{D}$ is small, and

$$\alpha \approx \frac{(\epsilon_r\mu_r)^{1/2}\mathscr{D}}{2\lambda\!\!\!^{-}{}_0} = \left(\frac{\mu_r}{\epsilon_r}\right)^{1/2}\frac{\sigma c\mu_0}{2}, \tag{28-64}$$

$$\beta \approx \frac{(\epsilon_r\mu_r)^{1/2}}{\lambda\!\!\!^{-}{}_0}, \qquad v = \frac{\omega}{\beta} \approx \frac{c}{(\epsilon_r\mu_r)^{1/2}}. \tag{28-65}$$

In such media the conductivity hardly affects the phase velocity, but it gives rise to an attenuation that is independent of the frequency.

In a *good conductor* $\mathcal{D} \gg 1$ and

$$k^2 = -j\mathcal{D}\frac{\epsilon_r\mu_r}{\lambda_0^2} = -j\frac{\sigma}{\omega\epsilon}\epsilon_r\mu_r\omega^2\epsilon_0\mu_0 = -j\sigma\mu\omega, \tag{28-66}$$

$$k = \left(\frac{\sigma\mu\omega}{2}\right)^{1/2}(1-j), \tag{28-67}$$

$$\alpha = \beta = \left(\frac{\sigma\mu\omega}{2}\right)^{1/2}. \tag{28-68}$$

The index of refraction of a *good conductor*

$$n = \frac{c}{\omega/\beta} = \frac{c\beta}{\omega} = c\left(\frac{\sigma\mu}{2\omega}\right)^{1/2} \tag{28-69}$$

is a large quantity. It is 1.1×10^8 for copper at 1 megahertz.

The Characteristic Impedance Z of a Conductor

The characteristic impedance of a conducting medium is complex:

$$Z = \frac{E}{H} = \frac{k}{\omega\epsilon - j\sigma} = \frac{\omega\mu}{k} \tag{28-70}$$

$$= \left(\frac{\mu}{\epsilon}\right)^{1/2}\frac{\exp j\arctan(\alpha/\beta)}{(1+\mathcal{D}^2)^{1/4}} \approx 377\left(\frac{\mu_r}{\epsilon_r}\right)^{1/2}\frac{\exp j\arctan(\alpha/\beta)}{(1+\mathcal{D}^2)^{1/4}} \text{ ohms} \tag{28-71}$$

as we saw in Sec. 28.2.2. This means that E and H are not in phase:

$$\frac{E}{H} = \frac{\omega\mu}{\beta - j\alpha}, \tag{28-72}$$

where α and β are both positive. So $\boldsymbol{E}$ leads $\boldsymbol{H}$ by the angle

$$\theta = \arctan \frac{\alpha}{\beta}. \qquad (28\text{-}73)$$

Figure 28-4 shows θ as a function of $\mathscr{D}$.

Therefore,

$$\boldsymbol{E} = \boldsymbol{E}_m \exp(-\alpha z) \exp j(\omega t - \beta z), \qquad (28\text{-}74)$$

$$\boldsymbol{H} = \boldsymbol{H}_m \exp(-\alpha z) \exp j(\omega t - \beta z - \theta), \qquad (28\text{-}75)$$

with

$$\frac{E_m}{H_m} = \frac{\omega\mu}{k} = \left(\frac{\mu}{\epsilon}\right)^{1/2} \frac{1}{(1+\mathscr{D}^2)^{1/4}} = \left(\frac{\mu_0}{\epsilon_0}\right)^{1/2} \left(\frac{\mu_r}{\epsilon_r}\right)^{1/2} \frac{1}{(1+\mathscr{D}^2)^{1/4}} \qquad (28\text{-}76)$$

$$\approx 377\left(\frac{\mu_r}{\epsilon_r}\right)^{1/2} \frac{1}{(1+\mathscr{D}^2)^{1/4}} \text{ ohms}. \qquad (28\text{-}77)$$

From Eq. 28-8, $\boldsymbol{E}$ and $\boldsymbol{H}$ are orthogonal in a linearly polarized wave. If the wave is *not* linearly polarized, then the vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ are not necessarily orthogonal.

The Energy Densities

The time-averaged electric and magnetic energy densities are in the ratio

$$\frac{\mathscr{E}_e'}{\mathscr{E}_m'} = \frac{\epsilon E_{\text{rms}}^2/2}{\mu H_{\text{rms}}^2/2} = \frac{1}{(1+\mathscr{D}^2)^{1/2}}. \qquad (28\text{-}78)$$

There is less electric energy than magnetic energy because the conductivity both decreases $\boldsymbol{E}$ and adds a conduction current to the displacement current, which increases $\boldsymbol{H}$.

The time-averaged total energy density is

$$\tfrac{1}{2}(\epsilon E_{\text{rms}}^2 + \mu H_{\text{rms}}^2) \exp(-2\alpha z) = \tfrac{1}{2}(\epsilon E_{\text{rms}}^2)[1 + (1+\mathscr{D}^2)^{1/2}] \exp(-2\alpha z). \qquad (28\text{-}79)$$

THE POYNTING THEOREM

We referred to the Poynting vector

$$\mathcal{S} = \boldsymbol{E} \times \boldsymbol{H} \tag{28-80}$$

in previous sections, but we said very little about it. We only stated that it is equal to the power density in an electromagnetic wave, and that it points in the direction of propagation.

The Poynting vector is of great theoretical and practical interest. Its significance follows from the Poynting theorem that we now prove.

First, we have the vector identity

$$\boldsymbol{\nabla} \cdot (\boldsymbol{E} \times \boldsymbol{H}) = \boldsymbol{H} \cdot (\boldsymbol{\nabla} \times \boldsymbol{E}) - \boldsymbol{E} \cdot (\boldsymbol{\nabla} \times \boldsymbol{H}). \tag{28-81}$$

In a HILS medium, Eqs. 27-20 to 27-23 apply, and then

$$\boldsymbol{\nabla} \cdot (\boldsymbol{E} \times \boldsymbol{H}) = -\boldsymbol{H} \cdot \mu \frac{\partial \boldsymbol{H}}{\partial t} - \boldsymbol{E} \cdot \left(\epsilon \frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{J}_f \right) \tag{28-82}$$

$$= -\frac{\partial}{\partial t} \left(\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2} \right) - \boldsymbol{E} \cdot \boldsymbol{J}_f. \tag{28-83}$$

We now change the signs, integrate over a volume v of finite extent and of surface area $\mathcal{A}$, and finally apply the divergence theorem on the left. This yields the *Poynting theorem*:

$$-\int_{\mathcal{A}} (\boldsymbol{E} \times \boldsymbol{H}) \cdot \boldsymbol{d\mathcal{A}} = \frac{d}{dt} \int_v \left(\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2} \right) dv + \int_v \boldsymbol{E} \cdot \boldsymbol{J}_f \, dv. \tag{28-84}$$

The first integral on the right gives the increase in the electric and magnetic energy densities inside the volume v, per unit time. The second gives that part of the field energy that dissipates as heat, again per unit time. Then the term on the left, with its negative sign, must represent the rate at which electromagnetic energy flows *into* the volume v.

Then the integral

$$\int_{\mathcal{A}} \mathcal{S} \cdot \boldsymbol{d\mathcal{A}} = \int_{\mathcal{A}} (\boldsymbol{E} \times \boldsymbol{H}) \cdot \boldsymbol{d\mathcal{A}} \tag{28-85}$$

is the total power flowing *out* of a closed surface of area $\mathcal{A}$.

The Poynting theorem therefore simply states that there is conservation of energy in electromagnetic fields. It is a proof of the validity of Eq. 27-23, and hence of Eq. 27-4.

For a uniform, plane, and linearly polarized wave in conducting material, the time-averaged magnitude of the Poynting vector is

$$\mathcal{S}_{av} = \tfrac{1}{2}\,\mathrm{Re}\,\{[E_m \exp(-\alpha z)\exp j(\omega t - \beta z) \times H_m \exp(-\alpha z)\exp j(-\omega t + \beta z + \theta)]\} \tag{28-86}$$

$$= \tfrac{1}{2}E_m H_m \cos\theta \exp(-2\alpha z), \tag{28-87}$$

where θ is defined as in Sec. 28.5.2 and

$$\cos\theta = \frac{\beta}{(\alpha^2 + \beta^2)^{1/2}}. \tag{28-88}$$

We found the ratio E_m/H_m in the previous section. If we eliminate H_m, then

$$\mathcal{S}_{av} = \left(\frac{\epsilon}{\mu}\right)^{1/2}(1 + \mathcal{D}^2)^{1/4}E_{rms}^2 \cos\theta \exp(-2\alpha z) \tag{28-89}$$

$$\approx \frac{1}{377}\left(\frac{\epsilon_r}{\mu_r}\right)^{1/2}(1 + \mathcal{D}^2)^{1/4}E_{rms}^2 \cos\theta \exp(-2\alpha z). \tag{28-90}$$

You can easily show that

$$\mathcal{S}_{av} = (\text{time-averaged energy density}) \times (\text{phase velocity}) \tag{28-91}$$

UNIFORM PLANE ELECTROMAGNETIC WAVES IN GOOD CONDUCTORS. THE SKIN EFFECT

Recall from Sec. 28.5.1 that, in a linearly polarized, uniform plane wave propagating in a conductor in the positive direction of the z-axis,

$$\boldsymbol{E} = \boldsymbol{E}_m \exp(-\alpha z) \exp j(\omega t - \beta z). \tag{29-1}$$

We define a good conductor as a material such that, in the expressions for α and β given in Sec. 28.5.1,

$$[(1 + \mathscr{D}^2)^{1/2} + 1]^{1/2} \approx \mathscr{D}^{1/2}. \tag{29-2}$$

This condition is satisfied within 1% if

$$\mathscr{D} \equiv \frac{\sigma}{\omega\epsilon} \equiv \left|\frac{\sigma E}{\partial D/\partial t}\right| \geq 50, \tag{29-3}$$

or if the conduction current density is at least 50 times larger than the displacement current density. But note here that σ and ϵ are functions of ω, especially at optical and x-ray frequencies. So $\mathscr{D}$ does not decrease indefinitely as $1/f$, as the above equation appears to indicate.

In good conductors the wave equation 28-16 reduces to†

$$\boldsymbol{\nabla}^2\boldsymbol{E} - \mu\sigma\frac{\partial \boldsymbol{E}}{\partial t} = 0, \tag{29-4}$$

and Eq. 28-10 for the wave number to

$$k^2 = -j\omega\sigma\mu. \tag{29-5}$$

Thus

$$k = \beta - j\alpha = \left(\frac{\omega\sigma\mu}{2}\right)^{1/2}(1 - j) = \frac{1 - j}{\delta}, \qquad n = \frac{c}{v} = \frac{c}{\omega\lambda\!\!\!^{-}} = \lambda\!\!\!^{-}_0 k = \frac{\lambda\!\!\!^{-}_0}{\delta}(1 - j) \tag{29-6}$$

$$\beta = \frac{1}{\lambda\!\!\!^{-}} = \alpha = \frac{1}{\delta} = \left(\frac{\omega\sigma\mu}{2}\right)^{1/2}, \tag{29-7}$$

where n is the index of refraction, $\lambda\!\!\!^{-} = \lambda/2\pi$, as usual, and where δ is the attenuation distance, defined in Sec. 28.5.1 as the distance over which the amplitude decreases by a factor of e.

From Eq. 28-70, the characteristic impedance of a good conductor is

$$Z = \frac{E}{H} = \frac{\omega\mu}{k} = \left(\frac{\omega\mu}{\sigma}\right)^{1/2} \exp\frac{j\pi}{4}, \tag{29-8}$$

and $\boldsymbol{E}$ leads $\boldsymbol{H}$ by $\pi/4$ radian. Compare with nonconductors in which $\boldsymbol{E}$ and $\boldsymbol{H}$ are in phase (Sec. 28.4). The difference comes from the fact that the current that is associated with $\boldsymbol{H}$ in good conductors is the conduction current, which is in phase with $\boldsymbol{E}$, and not the displacement current of nonconductors, which leads $\boldsymbol{E}$ by 90°.

Therefore

$$E = E_m \exp\left[j\left(\omega t - \frac{z}{\delta}\right) - \frac{z}{\delta}\right], \tag{29-9}$$

$$H = \left(\frac{\sigma}{\omega\mu}\right)^{1/2} E_m \exp\left[j\left(\omega t - \frac{z}{\delta} - \frac{\pi}{4}\right) - \frac{z}{\delta}\right]. \tag{29-10}$$

The vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ are transverse and orthogonal, say $\boldsymbol{E}$ is parallel to the x-axis and $\boldsymbol{H}$ to the y-axis. In terms of cosine functions,

$$E = E_m \exp\left(-\frac{z}{\delta}\right) \cos\left(\omega t - \frac{z}{\delta}\right), \tag{29-11}$$

$$H = \left(\frac{\sigma}{\omega\mu}\right)^{1/2} E_m \exp\left(-\frac{z}{\delta}\right) \cos\left(\omega t - \frac{z}{\delta} - \frac{\pi}{4}\right) \tag{29-12}$$

$$= H_m \exp\left(-\frac{z}{\delta}\right) \cos\left(\omega t - \frac{z}{\delta} - \frac{\pi}{4}\right). \tag{29-13}$$

Figure 29-1 shows E/E_m and H/H_m as functions of z/λ at $t = 0$.

The amplitude of the wave decreases by a factor of $(1/e)^{2\pi} \approx 2 \times 10^{-3}$ in one wavelength, and the Poynting vector by $(1/e)^{4\pi} \approx 3 \times 10^{-6}$. This is the *skin effect.*

The attenuation distance δ in conductors is termed the *skin depth,* or the *depth of penetration.* The skin depth *decreases* if the conductivity, the relative permeability, or the frequency *increases.* Good conductors are therefore opaque to light, except in the form of extremely thin films. It does not follow, however, that substances that are nonconducting at low frequencies are transparent at optical frequencies.

Table 29-1 shows the skin depth δ for various conductors at four typical frequencies. The attenuation in iron is much larger than in silver, despite the fact that iron is a relatively poor conductor.

The phase velocity

$$v_p = \frac{\omega}{\beta} = \omega \lambda\!\!\!^{-} = \left(\frac{2\omega}{\sigma\mu}\right)^{1/2} \tag{29-14}$$

is proportional to the square root of the frequency.

In good conductors the group velocity (App. C) is twice as large as the phase velocity:

$$v_g \equiv \frac{1}{d\beta/d\omega} = 2v_p, \tag{29-15}$$

if σ and μ are not frequency-dependent.

The ratio of the time-averaged electric to the time-averaged magnetic energy densities is

$$\frac{\epsilon E_{\text{rms}}^2/2}{\mu H_{\text{rms}}^2/2} = \frac{\omega\epsilon}{\sigma} = \frac{1}{\mathscr{D}} \leq \frac{1}{50}. \tag{29-16}$$

The energy is thus essentially all magnetic. This results from the large conductivity σ, which causes E/J_f to be small. The electric field strength is weak, but the current density and hence H are relatively large.

From Eqs. 29-11 and 29-13, the time-averaged value of the Poynting vector is

$$\mathscr{S}_{\text{av}} = \tfrac{1}{2}\,\text{Re}\,(\boldsymbol{E} \times \boldsymbol{H}^*) = \frac{1}{2}\left(\frac{\sigma}{2\omega\mu}\right)^{1/2} \exp\left(-\frac{2z}{\delta}\right) E_m^2 \hat{\boldsymbol{z}}. \tag{29-17}$$

REFLECTION AND REFRACTION

Medium 1 carries the incident and reflected waves. Medium 2 carries the refracted wave. For simplicity, we assume in Secs. 30-1 and 30-2 that the incident wave is linearly polarized. Then, in the incident wave,

$$\boldsymbol{E}_I = \boldsymbol{E}_{Im} \exp j(\omega_I t - \boldsymbol{k}_I \cdot \boldsymbol{r}), \tag{30-1}$$

where the vector wave number $\boldsymbol{k}_I$ is real and points in the direction of propagation of the incident wave. The magnitude of $\boldsymbol{k}_I$ is $n_1 k_0$, or $n_1/\lambda\!\!\!^-_0$, n_1 being the index of refraction of medium 1 and $\lambda\!\!\!^-_0$ the radian length of a wave of the same frequency in a vacuum. For convenience, we set the origin of $\boldsymbol{r}$ in the interface, as in Fig. 30-1, and we take $\boldsymbol{E}_{Im}$ to be real.

This equation defines a plane wave for all values of t and $\boldsymbol{r}$, and thus a wave that extends throughout all time and space. However, it applies only in medium 1.

Since the incident wave is plane, all the incident rays are parallel. By hypothesis, the interface is plane. Now the laws of reflection and of refraction must be the same at all points on the interface. It follows that the reflected rays are parallel to each other. Similarly, the refracted rays are parallel to each other. Further, since a wave front is by definition perpendicular to a ray, we can expect the reflected and transmitted waves to be of the form

$$\boldsymbol{E}_R = \boldsymbol{E}_{Rm} \exp j(\omega_R t - \boldsymbol{k}_R \cdot \boldsymbol{r}), \tag{30-2}$$

$$\boldsymbol{E}_T = \boldsymbol{E}_{Tm} \exp j(\omega_T t - \boldsymbol{k}_T \cdot \boldsymbol{r}). \tag{30-3}$$

What do we know about $\boldsymbol{k}_R$ and $\boldsymbol{k}_T$? From the wave equation 27-72 applied to medium 1, with $\sigma = 0$, $\rho_f = 0$,

$$\nabla^2 \boldsymbol{E}_R + \epsilon_1 \mu_1 \omega^2 \boldsymbol{E}_R = \nabla^2 \boldsymbol{E}_R + k_1^2 \boldsymbol{E}_R = 0, \tag{30-4}$$

where

$$k_1 = \frac{1}{\lambda\!\!\!^-_1} = \frac{n_1}{\lambda\!\!\!^-_0} = n_1 k_0 = \omega(\epsilon_1 \mu_1)^{1/2}. \tag{30-5}$$

A similar string of equations applies to k_2. Also,

$$k_{Ix}^2 + k_{Iy}^2 + k_{Iz}^2 = k_{Rx}^2 + k_{Ry}^2 + k_{Rz}^2 = k_1^2, \qquad k_{Tx}^2 + k_{Ty}^2 + k_{Tz}^2 = k_2^2. \tag{30-6}$$

The wave numbers k_1 and k_2 are real, but $\boldsymbol{k}_R$ and $\boldsymbol{k}_T$ are vectors that can be complex.

The tangential component of $\boldsymbol{E}$ is continuous at the interface. This means that the tangential component of $\boldsymbol{E}_I + \boldsymbol{E}_R$ in medium 1, at the interface, is equal to the tangential component of $\boldsymbol{E}_T$ in medium 2, at the interface. The same applies to $\boldsymbol{H}$. These continuity conditions will permit us to find all the unknowns in Eqs. 30-2 and 30-3.

Some relation must exist between $\boldsymbol{E}_I$, $\boldsymbol{E}_R$, $\boldsymbol{E}_T$ at the interface for all t and for all points r_{int} on the interface. Such a relation is possible only if the three vectors are identical functions of t and r_{int}. Then

$$\omega_I = \omega_R = \omega_T. \tag{30-7}$$

All three waves are of the same frequency. This is obvious because the waves are all superpositions of the wave emitted by the source and of those waves emitted by the electrons executing forced vibrations in media 1 and 2. Recall from mechanics that forced vibrations are of the same frequency as the applied force.

Also, from the above equations for the $\boldsymbol{E}$'s,

$$\boldsymbol{k}_I \cdot \boldsymbol{r}_{\text{int}} = \boldsymbol{k}_R \cdot \boldsymbol{r}_{\text{int}} = \boldsymbol{k}_T \cdot \boldsymbol{r}_{\text{int}}. \tag{30-8}$$

Then the $\boldsymbol{k}$'s are oriented in such a way that their components parallel to the interface are equal. In particular, if $k_{Iy} = 0$ as in Fig. 30-1, then

$$k_{Ry} = 0, \qquad k_{Ty} = 0, \tag{30-9}$$

and $\boldsymbol{k}_I$, $\boldsymbol{k}_R$, $\boldsymbol{k}_T$ are *coplanar*. The plane containing these three vectors is called the *plane of incidence*. The x components of the $\boldsymbol{k}$'s are thus all equal:

$$k_{Rx} = k_{Tx} = k_{Ix} = k_1 \sin \theta_I, \tag{30-10}$$

where θ_I is the angle of incidence shown in Fig. 30-1.

It is now easy to find $\boldsymbol{k}_R$:

$$k_{Rx}^2 + k_{Rz}^2 = k_{Ix}^2 + k_{Iz}^2 = k_1^2 \tag{30-11}$$

and

$$k_{Rz}^2 = k_{Iz}^2, \qquad k_{Rz} = -k_{Iz}. \tag{30-12}$$

We choose the negative sign because the reflected wave travels away from the interface. It follows that, if $\boldsymbol{k}_I$ is real, as we assumed at the beginning (there is zero attenuation in medium 1), then $\boldsymbol{k}_R$ is also real, the reflected wave is uniform, and

$$\theta_I = \theta_R. \tag{30-13}$$

The angle of reflection is equal to the angle of incidence.

Therefore the incident, reflected, and transmitted rays are coplanar, and the angle of reflection is equal to the angle of incidence. These are the *laws of reflection*.

SNELL'S LAW

Now return to Eq. 30-10. It says that

$$k_{Tx} = k_1 \sin \theta_I. \tag{30-14}$$

Then

$$k_{Tz}^2 = k_2^2 - k_{Tx}^2 = k_2^2 - k_1^2 \sin^2 \theta_I = k_0^2(n_2^2 - n_1^2 \sin^2 \theta_I). \tag{30-15}$$

If the term in parentheses is negative, then there is total reflection. We disregard this possibility until Chap. 31. Otherwise, k_{Tz} is real, k_T is real, and the transmitted wave is plane and uniform. If θ_T is the angle of refraction as in the figure,

$$k_{Tz} = -k_2 \cos \theta_T, \qquad k_{Tz} = k_2 \sin \theta_T. \tag{30-16}$$

From Eqs. 30-14 and 30-16,

$$k_2 \sin \theta_T = k_1 \sin \theta_I, \qquad \text{or} \qquad n_2 \sin \theta_T = n_1 \sin \theta_I. \tag{30-17}$$

When an electromagnetic wave crosses an interface, there is conservation of the quantity $n \sin \theta$. This is *Snell's law*.

Therefore, choosing axes as in Fig. 30-1, we find that

$$\boldsymbol{E}_I = \boldsymbol{E}_{Im} \exp j[\omega t - k_1(x \sin \theta_I - z \cos \theta_I)], \tag{30-18}$$

$$\boldsymbol{E}_R = \boldsymbol{E}_{Rm} \exp j[\omega t - k_1(x \sin \theta_I + z \cos \theta_I)], \tag{30-19}$$

$$\boldsymbol{E}_T = \boldsymbol{E}_{Tm} \exp j[\omega t - k_2(x \sin \theta_T - z \cos \theta_T)]. \tag{30-20}$$

The laws of reflection and Snell's law are general. They apply to any two homogeneous, isotropic, linear, and stationary (HILS) media, whether conducting or not, with either real or complex k's, provided that one allows complex angles as in the next chapter.

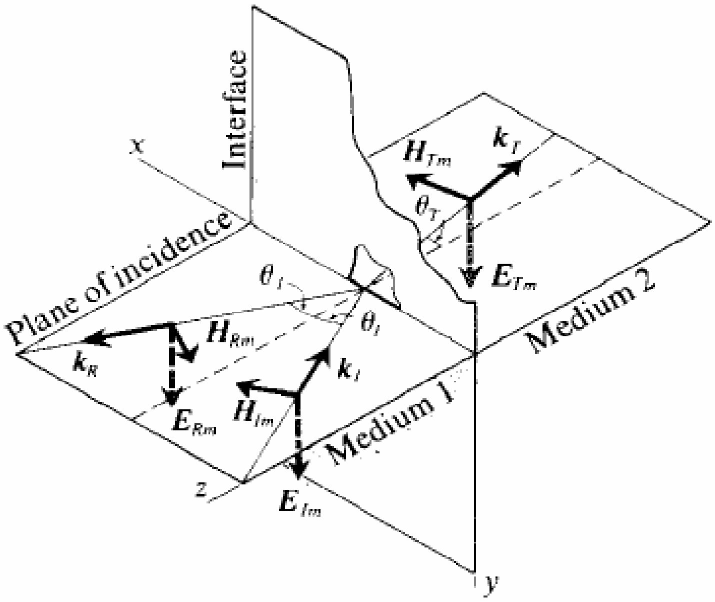

Fig. 30-2. The incident, reflected, and transmitted waves for an incident wave polarized with its $\boldsymbol{E}$ field *normal* to the plane of incidence. The arrows show the directions in which the vectors are taken to be positive *at the interface*. The vectors $\boldsymbol{E} \times \boldsymbol{H}$ point everywhere in the direction of propagation.

$$H_{Ix} + H_{Rx} = H_{Tx}, \qquad H_{Iy} + H_{Ry} = H_{Ty}. \tag{30-22}$$

Since the relation

$$\boldsymbol{H} = \frac{\boldsymbol{k} \times \boldsymbol{E}}{\omega\mu} \tag{30-23}$$

of Sec. 28.2.1 applies to all three waves, we first find $\boldsymbol{E}_R$ and $\boldsymbol{E}_T$ and then deduce $\boldsymbol{H}_R$ and $\boldsymbol{H}_T$.

It will be convenient to divide the discussion into two parts. We consider successively incident waves polarized with their $\boldsymbol{E}$ vectors normal and then parallel to the plane of incidence. Any uniform plane incident wave is the sum of two such components.

We now define our sign conventions. See Figs. 30-2 and 30-3. Observe that the two figures agree at normal incidence. We utilize the continuity of E_y and H_x in Fig. 30-2, and the continuity of E_x and H_y in Fig. 30-3. This will yield relations that apply again to any pair of HILS media and to any angle of incidence.

E Normal to the Plane of Incidence

The $\boldsymbol{E}$ and $\boldsymbol{H}$ vectors of the incident wave point as in Fig. 30-2. The media being isotropic, the $\boldsymbol{E}$ vectors of the other two waves are also normal to the plane of incidence. This is because the electrons in both media oscillate in the direction normal to the plane of incidence and reradiate waves polarized with $\boldsymbol{E}$ normal to the plane of incidence.

If the $\boldsymbol{E}$ vectors point in the directions shown, at the interface, then the $\boldsymbol{H}$ vectors point as shown, to orient the Poynting vectors $\boldsymbol{E} \times \boldsymbol{H}$ (Sec. 28.6) in the proper directions.

The continuity of the tangential component of $\boldsymbol{E}$ at the interface requires that

$$E_{Im} + E_{Rm} = E_{Tm} \tag{30-24}$$

at any given point on the interface. Similarly, for continuity of the tangential component of $\boldsymbol{H}$,

$$H_{Im} \cos \theta_I - H_{Rm} \cos \theta_I = H_{Tm} \cos \theta_T \tag{30-25}$$

or, from Sec. 28.4,

$$\frac{(E_{Im} - E_{Rm}) \cos \theta_I}{Z_1} = \frac{E_{Tm} \cos \theta_T}{Z_2}, \tag{30-26}$$

where Z is the characteristic impedance of a medium

$$Z = \frac{E}{H} = \frac{\omega\mu}{k} = \frac{\omega\mu}{nk_0} = \frac{\omega\mu}{n(\omega/c)} = \frac{c\mu}{n}, \tag{30-27}$$

n being the index of refraction.

Solving,

$$\left(\frac{E_{Rm}}{E_{Im}}\right)_\perp = \frac{Z_2 \cos \theta_I - Z_1 \cos \theta_T}{Z_2 \cos \theta_I + Z_1 \cos \theta_T}, \tag{30-28}$$

$$\left(\frac{E_{Tm}}{E_{Im}}\right)_\perp = \frac{2Z_2 \cos \theta_I}{Z_2 \cos \theta_I + Z_1 \cos \theta_T}. \tag{30-29}$$

E Parallel to the Plane of Incidence

The ***E***'s are now all in the plane of incidence, as in Fig. 30-3, and

$$H_{Im} - H_{Rm} = H_{Tm}, \tag{30-30}$$

or

$$\frac{E_{Im} - E_{Rm}}{Z_1} = \frac{E_{Tm}}{Z_2}. \tag{30-31}$$

Also,

$$(E_{Im} + E_{Rm}) \cos \theta_I = E_{Tm} \cos \theta_T. \tag{30-32}$$

Then

$$\left(\frac{E_{Rm}}{E_{Im}}\right)_{\parallel} = \frac{Z_2 \cos \theta_T - Z_1 \cos \theta_I}{Z_2 \cos \theta_T + Z_1 \cos \theta_I}, \tag{30-33}$$

$$\left(\frac{E_{Tm}}{E_{Im}}\right)_{\parallel} = \frac{2Z_2 \cos \theta_I}{Z_2 \cos \theta_T + Z_1 \cos \theta_I}. \tag{30-34}$$

This is the second pair of *Fresnel's equations*.

At normal incidence $\theta_I = \theta_R = \theta_T = 0$, the plane of incidence is undefined, and the two pairs of Fresnel's equations are identical:

$$\frac{E_{Rm}}{E_{Im}} = \frac{Z_2 - Z_1}{Z_2 + Z_1}, \tag{30-35}$$

$$\frac{E_{Tm}}{E_{Im}} = \frac{2Z_2}{Z_2 + Z_1}. \tag{30-36}$$

THE COEFFICIENTS OF REFLECTION *R* AND OF TRANSMISSION *T*

The coefficients of reflection and of transmission concern the flow of energy across the interface. The average energy flux per unit area in a wave is equal to the average value of the Poynting vector, as in Eq. 28-31. We exclude total reflection as well as reflection from conducting media. Setting $\mu_r = 1$, we find that

$$\mathscr{S}_{I,\text{av}} = \frac{1}{2}\left(\frac{\epsilon_1}{\mu_0}\right)^{1/2} E_{Im}^2 \hat{\boldsymbol{n}}_I, \tag{30-51}$$

$$\mathscr{S}_{R,\text{av}} = \frac{1}{2}\left(\frac{\epsilon_1}{\mu_0}\right)^{1/2} E_{Rm}^2 \hat{\boldsymbol{n}}_R, \tag{30-52}$$

$$\mathscr{S}_{T,\text{av}} = \frac{1}{2}\left(\frac{\epsilon_2}{\mu_0}\right)^{1/2} E_{Tm}^2 \hat{\boldsymbol{n}}_T, \tag{30-53}$$

where $\hat{\boldsymbol{n}}_I$ is normal to a wave front of the incident wave:

$$\hat{\boldsymbol{n}}_I = \frac{\boldsymbol{k}_I}{k_1}, \tag{30-54}$$

and similarly for $\hat{\boldsymbol{n}}_R$ and $\hat{\boldsymbol{n}}_T$.

The *coefficients of reflection* R and *of transmission* T are the ratios of the average energy fluxes per unit time and per unit area at the interface:

$$R = \left|\frac{\mathscr{S}_{R,\text{av}} \cdot \hat{\boldsymbol{n}}}{\mathscr{S}_{I,\text{av}} \cdot \hat{\mathbf{n}}}\right| = \frac{E_{Rm}^2}{E_{Im}^2}, \tag{30-55}$$

where $\hat{\boldsymbol{n}}$ is the unit vector normal to the interface;

$$T = \left|\frac{\mathscr{S}_{T,\text{av}} \cdot \hat{\boldsymbol{n}}}{\mathscr{S}_{I,\text{av}} \cdot \hat{\boldsymbol{n}}}\right| = \left(\frac{\epsilon_{r2}}{\epsilon_{r1}}\right)^{1/2} \frac{E_{Tm}^2}{E_{Im}^2} \frac{\cos\theta_T}{\cos\theta_I} = \frac{n_2 E_{Tm}^2 \cos\theta_T}{n_1 E_{Im}^2 \cos\theta_I}. \tag{30-56}$$

Then, from Fresnel's equations for nonconductors,

$$R_\perp = \left[\frac{(n_1/n_2)\cos\theta_I - \cos\theta_T}{(n_1/n_2)\cos\theta_I + \cos\theta_T}\right]^2, \tag{30-57}$$

$$T_\perp = \frac{4(n_1/n_2)\cos\theta_I \cos\theta_T}{[(n_1/n_2)\cos\theta_I + \cos\theta_T]^2}, \tag{30-58}$$

$$R_\parallel = \left[\frac{-\cos\theta_I + (n_1/n_2)\cos\theta_T}{\cos\theta_I + (n_1/n_2)\cos\theta_T}\right]^2, \tag{30-59}$$

$$T_\parallel = \frac{4(n_1/n_2)\cos\theta_I \cos\theta_T}{[\cos\theta_I + (n_1/n_2)\cos\theta_T]^2}. \tag{30-60}$$

CHƯƠNG 6. CƠ SỞ BỨC XẠ ĐIỆN TỪ

6.1 PHƯƠNG TRÌNH HELMHOLTZ

By taking the curl of Eq. (2.3*a*), using Eq. (2.3*b*) and the constitutive relations (2.4), it will be found that

$$\nabla \times \nabla \times \mathbf{E} = k_0^2\mathbf{E} = -j\omega\mu_0\mathbf{J} \tag{2.12}$$

where $k_0 = \omega(\mu_0\epsilon_0)^{1/2}$ is the free-space wave number. This is the equation that must be solved to find the electric field directly in terms of the specified current source **J**. In practice a simpler equation to solve is obtained by introducing the vector potential **A** and scalar potential Φ.

Since the divergence of **B** is identically zero, **B** can be expressed as

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{2.13}$$

because $\nabla \cdot \nabla \times \mathbf{A} \equiv 0$. **A** is called the *vector potential*. By using Eq. (2.13) in Eq. (2.3*a*), we obtain

$$\nabla \times (\mathbf{E} + j\omega\mathbf{A}) = 0$$

Any function with zero curl can be expressed as the gradient of a scalar function; thus we can assume that

$$\mathbf{E} + j\omega\mathbf{A} = -\nabla\Phi \tag{2.14}$$

In order that Eq. (2.3*b*) will hold, we require

$$\begin{aligned}\nabla \times \mu_0\mathbf{H} &= \nabla \times \nabla \times \mathbf{A}\\ &= j\omega\mu_0\epsilon_0\mathbf{E} + \mu_0\mathbf{J}\\ &= j\omega\mu_0\epsilon_0(-j\omega\mathbf{A} - \nabla\Phi) + \mu_0\mathbf{J}\end{aligned}$$

We can now use the expansion $\nabla \times \nabla \times \mathbf{A} = \nabla\nabla \cdot \mathbf{A} - \nabla^2\mathbf{A}$ to obtain, after a rearrangement of terms,

$$\nabla^2\mathbf{A} + k_0^2\mathbf{A} = -\mu_0\mathbf{J} + \nabla(\nabla \cdot \mathbf{A} + j\omega\mu_0\epsilon_0\Phi)$$

So far only the curl of **A** is fixed by the relation (2.13). Thus, we are still free to specify the divergence of **A**. In order to simplify the equation for **A** we choose

$$\nabla \cdot \mathbf{A} = -j\omega\mu_0\epsilon_0\Phi \tag{2.15}$$

which is known as the *Lorentz condition*. Our equation for **A** now becomes the inhomogeneous Helmholtz equation:

$$\nabla^2\mathbf{A} + k_0^2\mathbf{A} = -\mu_0\mathbf{J} \tag{2.16}$$

If Eqs. (2.14) and (2.15) are used in Eq. (2.3*c*), it will be found that Φ satisfies a similar equation, namely,

$$\nabla^2\Phi + k_0^2\Phi = -\frac{\rho}{\epsilon_0} \tag{2.17}$$

However, the charge is not an independent source term for time-varying fields, since it is related to the current by the continuity equation (2.3*e*), and it is not necessary to solve for the scalar potential Φ. By using the Lorentz condition in Eq. (2.14), we can find the electric field in terms of the vector potential **A** alone by means of the relation:

$$\mathbf{E} = -j\omega\mathbf{A} + \frac{\nabla\nabla \cdot \mathbf{A}}{j\omega\mu_0\epsilon_0} \tag{2.18}$$

The simplification obtained by introducing the vector potential **A** may be appreciated by considering the case of a z-directed current source $\mathbf{J} = J_z\mathbf{a}_z$ in which case $\mathbf{A} = A_z\mathbf{a}_z$ and A_z is a solution of the scalar equation

$$(\nabla^2 + k_0^2)A_z = -\mu_0 J_z \tag{2.19}$$

The equation satisfied by the electric field is a vector equation even when the current has only a single component.

RADIATION FROM A SHORT CURRENT FILAMENT

Figure 2.5 shows a short, thin filament of current located at the origin and oriented along the z axis. For this source the vector potential has only a z component and is a solution of Eq. (2.19), that is,

$$(\nabla^2 + k_0^2)A_z = -\mu_0 J_z$$

where $J_z = I/dS$ and dS is the cross-sectional area of the current filament of length dl. The volume $dV = dS\,dl$ occupied by the current is of infinitesimal size so the source term can be considered as located at a point. There is spherical symmetry in the source distribution, so A_z will be a function only of the radial distance r away from the source. A_z will not be a function of the polar angle θ or the azimuth angle ϕ shown in Fig. 2.5. For values of r not

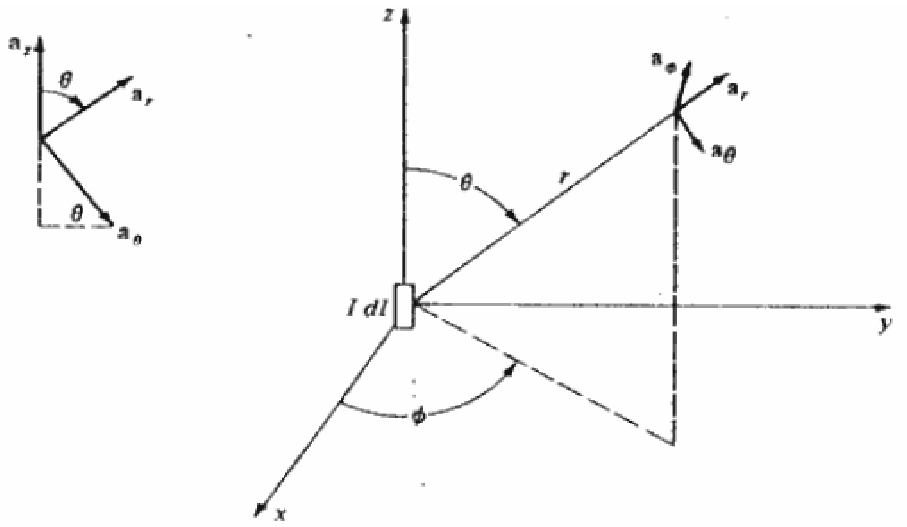

Figure 2.5 The short current filament and the spherical coordinate system.

equal to zero, A_z satisfies the equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial A_z}{\partial r}+k_0^2A_z=0 \tag{2.20}$$

as obtained by expressing the Laplace operator ∇^2 in spherical coordinates and dropping the derivatives with respect to θ and ϕ. If we make the substitution $A_z=\psi/r$, then $dA_z/dr=r^{-1}\,d\psi/dr-r^{-2}\psi$, and the equation obtained from Eq. (2.20) for ψ becomes

$$\frac{d^2\psi}{dr^2}+k_0^2\psi=0 \tag{2.21}$$

This is a simple harmonic-motion equation with solutions $C_1\,e^{-jk_0r}$ and $C_2\,e^{jk_0r}$, where C_1 and C_2 are constants. If we choose the first solution and restore the time factor we obtain

$$\psi(r,t)=C_1\,e^{-jk_0r+j\omega t}$$

Now $k_0=\omega/c$, where $c=(\mu_0\epsilon_0)^{-1/2}$ is the speed of light in free space, so

$$\psi(r,t)=C_1\,e^{j\omega(t-r/c)} \tag{2.22}$$

This is a wave solution corresponding to an outward propagating wave, since the phase is retarded by the factor k_0r and the corresponding time delay is r/c. The other solution with the constant C_2 corresponds to an inward propagating spherical wave and is not present as part of the solution for radiation from a current element located at $r=0$. Our solution for A_z is now seen to be of the form

$$A_z=C_1\frac{e^{-jk_0r}}{r} \tag{2.23}$$

In order to relate the constant C_1 to the source strength, we integrate both sides of Eq. (2.19) over a small spherical volume of radius r_0. We note that $\nabla^2 A_z = \nabla \cdot \nabla A_z$, so upon using the divergence theorem we obtain

$$\int_V \nabla^2 A_z \, dV = \int_V \nabla \cdot \nabla A_z \, dV$$

$$= \oint_S \nabla A_z \cdot \mathbf{a}_r r_0^2 \sin\theta \, d\theta \, d\phi$$

$$= -k_0^2 \int_V A_z \, dV - \mu_0 \int_V J_z \, dV$$

Now $dV = r^2 \sin\theta \, d\theta \, d\phi \, dr$ and A_z varies as $1/r$; consequently, if we choose r_0 vanishingly small the volume integral of A_z, which is proportional to r_0^2, vanishes. The volume integral of J_z gives $J_z \, dS \, dl = I \, dl$, which is the total source strength. Also

$$\nabla A_z \cdot \mathbf{a}_r = \frac{\partial A_z}{\partial r} = -(1 + jk_0 r) C_1 \frac{e^{-jk_0 r}}{r^2}$$

so

$$\lim_{r_0 \to 0} \int_0^{2\pi} \int_0^{\pi} -(1 + jk_0 r_0) C_1 \, e^{-jk_0 r_0} \sin\theta \, d\theta \, d\phi = -4\pi C_1 = -\mu_0 I \, dl$$

Our final solution for the vector potential is

$$\mathbf{A} = \mu_0 I \, dl \frac{e^{-jk_0 r}}{4\pi r} \mathbf{a}_z \tag{2.24}$$

The vector potential is an outward propagating spherical wave with an amplitude that decreases inversely with distance. The surfaces of constant phase or constant time delay are spheres of fixed radius r centered on the source. The phase velocity of the wave is the speed of light c, or 3×10^8 m/s. The distance that corresponds to a phase change of 2π is the wavelength λ_0 and may be found from the relationship $k_0 \lambda_0 = 2\pi$; thus

$$\lambda_0 = \frac{2\pi}{k_0} = \frac{c}{\omega/2\pi} = \frac{c}{f} \tag{2.25}$$

From our solution for the vector potential we can readily find the electromagnetic field by using Eqs. (2.13) and (2.18). This evaluation is best done in spherical coordinates, so we first express **A** in terms of components in spherical coordinates by noting that (see Fig. 2.5)

$$\mathbf{a}_z = \mathbf{a}_r \cos\theta - \mathbf{a}_\theta \sin\theta$$

and consequently

$$\mathbf{A} = \frac{\mu_0 I\,dl}{4\pi r} e^{-jk_0 r}(\mathbf{a}_r \cos\theta - \mathbf{a}_\theta \sin\theta) \tag{2.26}$$

We now use Eq. (2.13) to obtain

$$\mathbf{H} = \frac{1}{\mu_0}\nabla\times\mathbf{A} = \frac{I\,dl \sin\theta}{4\pi}\left(\frac{jk_0}{r} + \frac{1}{r^2}\right) e^{-jk_0 r}\mathbf{a}_\phi \tag{2.27}$$

and use Eq. (2.18) to obtain

$$\begin{aligned}
\mathbf{E} &= -j\omega\mathbf{A} + \frac{\nabla\nabla\cdot\mathbf{A}}{j\omega\mu_0\epsilon_0} \\
&= \frac{jZ_0 I\,dl}{2\pi k_0}\cos\theta\left(\frac{jk_0}{r^2} + \frac{1}{r^3}\right) e^{-jk_0 r}\mathbf{a}_r \\
&\quad - \frac{jZ_0 I\,dl}{4\pi k_0}\sin\theta\left(-\frac{k_0^2}{r} + \frac{jk_0}{r^2} + \frac{1}{r^3}\right) e^{-jk_0 r}\mathbf{a}_\theta \\
&= E_r\mathbf{a}_r + E_\theta\mathbf{a}_\theta
\end{aligned} \tag{2.28}$$

When r is large relative to the wavelength λ_0, the only important terms are those that vary as $1/r$. These terms make up the far zone, or radiation field, and are

$$\mathbf{E} = jZ_0 I\,dl\,k_0 \sin\theta \frac{e^{-jk_0 r}}{4\pi r}\mathbf{a}_\theta \tag{2.29a}$$

$$\mathbf{H} = jI\,dl\,k_0 \sin\theta \frac{e^{-jk_0 r}}{4\pi r}\mathbf{a}_\phi \tag{2.29b}$$

We note that in the far zone the radiation field has transverse components only; that is, both **E** and **H** are perpendicular to the radius vector as well as perpendicular to each other. The ratio of E_θ to H_ϕ equals the intrinsic impedance $Z_0 = (\mu_0/\epsilon_0)^{1/2}$ of free space. This is a general feature of the radiation field from any antenna. In vector form, one always finds that the radiation field in the far-zone region satisfies the relations

$$\mathbf{E} = -Z_0\mathbf{a}_r \times \mathbf{H} \tag{2.30a}$$

$$\mathbf{H} = Y_0\mathbf{a}_r \times \mathbf{E} \tag{2.30b}$$

where $Y_0 = Z_0^{-1}$. This spatial relationship is illustrated in Fig. 2.6.

We also note that both E_θ and H_ϕ vary as $\sin\theta$. Thus the radiated field is not a spherically symmetric outward-propagating wave as was found for the vector potential. This is also a general feature of all radiation fields—the electromagnetic radiation field can never have complete spherical symmetry.

The complex Poynting vector for the radiation field is

$$\tfrac{1}{2}\mathbf{E}\times\mathbf{H}^* = II^* Z_0 (dl)^2 k_0^2 \sin^2\theta \frac{\mathbf{a}_r}{32\pi^2 r^2} \tag{2.31}$$

and is pure real, and directed radially outward. The radiated power per unit area decreases as $1/r^2$, as expected because of the spreading out of the field as it propagates radially outward. This is the inverse-square-law attenuation behavior discussed in Chap. 1.

RADIATION FROM ARBITRARY CURRENT DISTRIBUTIONS

In this section we will present some useful formulas for calculating the far-zone radiation field from an arbitrary distribution of current. Consider a volume V with a current distribution $\mathbf{J}(\mathbf{r}')$, as shown in Fig. 2.9. The current element $\mathbf{J}(\mathbf{r}')\,dV'$ will contribute an amount

$$\frac{\mu_0 \mathbf{J}(\mathbf{r}')\,dV'}{4\pi R} e^{-jk_0 R}$$

to the total vector potential where $R = |\mathbf{r}-\mathbf{r}'|$. In the far-zone region $|\mathbf{r}| \gg |\mathbf{r}'|$ for all $\mathbf{r}'$ in V. Thus all rays from the various current elements to the far-zone field point can be considered to be parallel to each other, as shown in Fig. 2.9. Thus a useful approximation for R is

$$R \approx r - \mathbf{a}_r \cdot \mathbf{r}' \tag{2.47}$$

We can replace R by r in the amplitude term for the vector potential, since this has a negligible effect on the amplitude of each elementary contribution when $r \gg r'$. Hence in the far zone we obtain

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 e^{-jk_0 r}}{4\pi r}\int_V \mathbf{J}(\mathbf{r}')\, e^{jk_0 \mathbf{a}_r\cdot\mathbf{r}'}\,dV' \tag{2.48}$$

This equation superimposes the effects of each current element and takes into

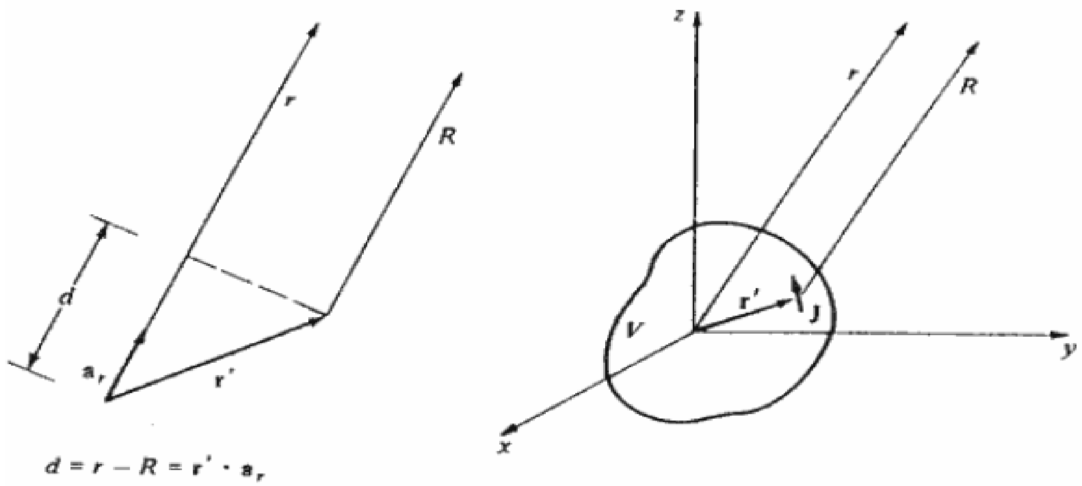

Figure 2.9 An arbitrary distribution of current.

account the relative phase angle or path-length phase delay of each contribution. Since the current elements do not, in general, contribute in phase, interference effects are produced that may be exploited to control the shape of the radiation pattern. In the next chapter we will examine the use of such interference effects to produce high-gain directive radiation beams.

We can find the fields **E** and **H** from Eq. (2.48) by using the relations (2.13) and (2.18). When only the terms varying as $1/r$ are retained, it is found that

$$\mathbf{E}(\mathbf{r}) = \frac{jk_0 Z_0 e^{-jk_0 r}}{4\pi r} \int_V [\mathbf{a}_r \cdot \mathbf{J}(\mathbf{r}')\mathbf{a}_r - \mathbf{J}(\mathbf{r}')] \, e^{jk_0 \mathbf{a}_r \cdot \mathbf{r}'} \, dV' \tag{2.49a}$$

$$\mathbf{H} = Y_0 \mathbf{a}_r \times \mathbf{E} \tag{2.49b}$$

The form of the integrand in this expression shows that in a given direction, as specified by the unit vector $\mathbf{a}_r$, it is only the current perpendicular to $\mathbf{a}_r$ that contributes to the radiation field. The reason for this is that the radiation field along the axis of a current element is zero.

When the current is a line current I along a contour C, then Eq. (2.49a) can be expressed in the form

$$\mathbf{E}(\mathbf{r}) = \frac{jk_0 Z_0 e^{-jk_0 r}}{4\pi r} \int_C [(\mathbf{a}_r \cdot \mathbf{a})\mathbf{a}_r - \mathbf{a}] I(l') \, e^{jk_0 \mathbf{a}_r \cdot \mathbf{r}'} \, dl' \tag{2.50}$$

where **a** is a unit vector along C in the direction of the current.

From Eqs. (2.49a) and (2.50) we see that the electric field has the form

$$\mathbf{E}(\mathbf{r}) = \frac{jk_0 Z_0 e^{-jk_0 r}}{4\pi r} \mathbf{f}(\theta, \phi) \tag{2.51}$$

where $\mathbf{f}(\theta, \phi)$, which is given by the integral, describes the radiation amplitude pattern or the angular dependence of the radiation distribution in space. The other factor $e^{-jk_0 r}/4\pi r$ is the outward-propagating spherical wave function.

CHƯƠNG 7. CƠ SỞ SÓNG ĐIỆN TỪ TRONG CÁC HỆ ĐỊNH HƯỚNG

GENERAL PROPERTIES OF AN ELECTROMAGNETIC WAVE PROPAGATING IN A STRAIGHT LINE

To simplify, we assume the six following conditions.

(1) The medium of propagation is homogeneous, isotropic, linear, and stationary (HILS).

(2) It is nonconducting. This does *not* exclude metallic guides, because the wave propagates *along* a metallic guide.

(3) The free charge density is zero. This makes $\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0$.

(4) Propagation occurs in a straight line, in the positive direction of the z-axis. There is no reflected wave traveling in the $-z$ direction.

(5) The wave is sinusoidal.

(6) There is zero attenuation. If the guide is metallic, then its conductivity must be infinite to avoid Joule losses. We shall see in Sec. 34.8 how to calculate attenuation with real conductors.

We may therefore write that

$$\boldsymbol{E} = \boldsymbol{E}_m \exp j(\omega t - k_z z) = (E_{mx}\hat{\boldsymbol{x}} + E_{my}\hat{\boldsymbol{y}} + E_{mz}\hat{\boldsymbol{z}}) \exp j(\omega t - k_z z), \quad (33\text{-}1)$$

$$\boldsymbol{H} = \boldsymbol{H}_m \exp j(\omega t - k_z z) = (H_{mx}\hat{\boldsymbol{x}} + H_{my}\hat{\boldsymbol{y}} + H_{mz}\hat{\boldsymbol{z}}) \exp j(\omega t - k_z z), \quad (33\text{-}2)$$

where the coefficients E_{mx}, E_{my}, E_{mz}, H_{mx}, . . . are unspecified functions of x and y. The dependence on z and t appears only in the exponential function. The wave number k_z for the guided wave is real, since there is zero attenuation. It is equal to $2\pi/\lambda_z$, where λ_z is the wavelength of the guided wave.

Let us substitute the above expressions for $\boldsymbol{E}$ and $\boldsymbol{H}$ into Maxwell's equations. Since $\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0$,

$$\frac{\partial E_{mx}}{\partial x} + \frac{\partial E_{my}}{\partial y} - jk_z E_{mz} = 0. \tag{33-3}$$

Similarly, $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$, and

$$\frac{\partial H_{mx}}{\partial x} + \frac{\partial H_{my}}{\partial y} - jk_z H_{mz} = 0. \tag{33-4}$$

From the fact that $\boldsymbol{\nabla} \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$,

$$\frac{\partial E_{mz}}{\partial y} + jk_z E_{my} = -j\omega\mu H_{mx}, \tag{33-5}$$

$$-jk_z E_{mx} - \frac{\partial E_{mz}}{\partial x} = -j\omega\mu H_{my}, \tag{33-6}$$

$$\frac{\partial E_{my}}{\partial x} - \frac{\partial E_{mx}}{\partial y} = -j\omega\mu H_{mz}. \tag{33-7}$$

From $\boldsymbol{\nabla} \times \boldsymbol{H} = \partial \boldsymbol{D}/\partial t$,

$$\frac{\partial H_{mz}}{\partial y} + jk_z H_{my} = j\omega\epsilon E_{mx}, \tag{33-8}$$

$$-jk_z H_{mx} - \frac{\partial H_{mz}}{\partial x} = j\omega\epsilon E_{my}, \tag{33-9}$$

$$\frac{\partial H_{my}}{\partial x} - \frac{\partial H_{mx}}{\partial y} = j\omega\epsilon E_{mz}. \tag{33-10}$$

The Transverse Components are Functions of the Longitudinal Components

We can now show that the four transverse components E_{mx}, E_{my}, H_{mx}, H_{my} are functions of the longitudinal components E_{mz}, H_{mz}. From Eqs. 33-6 and 33-8,

$$E_{mx} = \frac{j}{k_z^2 - k^2}\left(k_z \frac{\partial E_{mz}}{\partial x} + \omega\mu \frac{\partial H_{mz}}{\partial y}\right). \tag{33-11}$$

Here

$$k = \omega(\epsilon\mu)^{1/2} = \frac{1}{\lambda} \tag{33-12}$$

is the wave number of a uniform plane wave of wavelength λ traveling in the medium.

We have assumed that $k_z \neq k$ for the moment. Both k and k_z are real and positive. Similarly,

$$E_{my} = \frac{j}{k_z^2 - k^2}\left(k_z \frac{\partial E_{mz}}{\partial y} - \omega\mu \frac{\partial H_{mz}}{\partial x}\right), \tag{33-13}$$

$$H_{mx} = \frac{j}{k_z^2 - k^2}\left(-\omega\epsilon \frac{\partial E_{mz}}{\partial y} + k_z \frac{\partial H_{mz}}{\partial x}\right), \tag{33-14}$$

$$H_{my} = \frac{j}{k_z^2 - k^2}\left(\omega\epsilon \frac{\partial E_{mz}}{\partial x} + k_z \frac{\partial H_{ms}}{\partial y}\right). \tag{33-15}$$

We use the subscript $\perp$ to identify components that are perpendicular to the direction of propagation. Thus

$$\boldsymbol{E}_{m\perp} = E_{mx}\hat{\boldsymbol{x}} + E_{my}\hat{\boldsymbol{y}}, \qquad \boldsymbol{H}_{m\perp} = H_{mx}\hat{\boldsymbol{x}} + H_{my}\hat{\boldsymbol{y}}. \tag{33-16}$$

More succinctly,

$$\boldsymbol{E}_{m\perp} = \frac{j}{k_z^2 - k^2}(k_z \boldsymbol{\nabla}_\perp E_{mz} + \omega\mu \boldsymbol{\nabla} \times H_{mz}\hat{\boldsymbol{z}}), \tag{33-17}$$

$$\boldsymbol{H}_{m\perp} = \frac{j}{k_z^2 - k^2}(k_z \boldsymbol{\nabla}_\perp H_{mz} - \omega\epsilon \boldsymbol{\nabla} \times E_{mz}\hat{\boldsymbol{z}}). \tag{33-18}$$

So we need to solve the wave equation and apply the boundary conditions only for the two longitudinal components. Once that is done, the other four components will follow immediately.

The longitudinal component of $\boldsymbol{E}$ satisfies the wave equation 27-70 with $\rho_f = 0$, $\boldsymbol{J}_f = 0$. So

$$\frac{\partial^2 E_{mz}}{\partial x^2} + \frac{\partial^2 E_{mz}}{\partial y^2} - k_z^2 E_{mz} = -\epsilon\mu\omega^2 E_{mz} = -k^2 E_{mz}, \tag{33-19}$$

or

$$\frac{\partial^2 E_{mz}}{\partial x^2} + \frac{\partial^2 E_{mz}}{\partial y^2} + (k^2 - k_z^2)E_{mz} = 0, \tag{33-20}$$

$$(\nabla_\perp^2 + k^2 - k_z^2)E_{mz} = 0. \tag{33-21}$$

Similarly,

$$\frac{\partial^2 H_{mz}}{\partial x^2} + \frac{\partial^2 H_{mz}}{\partial y^2} + (k^2 - k_z^2)H_{mz} = 0, \tag{33-22}$$

$$(\nabla_\perp^2 + k^2 - k_z^2)H_{mz} = 0. \tag{33-23}$$

TE and TM Waves

It is convenient to consider separately three types of wave: (1) transverse electric (TE) waves, in which $E_{mz} = 0$; (2) transverse magnetic (TM) waves, with $H_{mz} = 0$; (3) transverse electric and magnetic (TEM) waves, with $E_{mz} = 0$, $H_{mz} = 0$.

With either TE or TM waves, it follows from Eqs. 33-11 to 33-15 that

$$\frac{E_{mx}}{H_{my}} = -\frac{E_{my}}{H_{mx}}. \tag{33-24}$$

If k_z is real and positive, as we have assumed, these ratios are also real and positive. Then the components

$$E_x = E_{mx} \exp j(\omega t - k_z z) \qquad \text{and} \qquad H_y = H_{my} \exp j(\omega t - k_z z) \tag{33-25}$$

are in phase, and so are E_y and $-H_x$. This fact, together with Eq. 33-24, implies that

$$\frac{\text{Re}\, E_x}{\text{Re}\, H_y} = \frac{\text{Re}\, E_y}{\text{Re}\,(-H_x)}, \tag{33-26}$$

and that the real parts of $E_\perp$ and $H_\perp$ are mutually orthogonal in both TE and TM waves.

The ratio $E_{m\perp}/H_{m\perp}$ is the *wave impedance*. This is a real positive quantity if there is no dissipation:

$$Z_{\text{TE}} = \frac{E_{m\perp}}{H_{m\perp}} = \frac{\omega\mu}{k_z} = \left(\frac{\mu}{\epsilon}\right)^{1/2} \frac{\lambda_z}{\lambda_0} \tag{33-27}$$

$$= 3.76731 \times 10^2 \frac{\lambda_z}{\lambda_0} \approx 377 \frac{\lambda_z}{\lambda_0} \qquad \text{ohms} \qquad (\epsilon_r = 1,\ \mu_r = 1), \tag{33-28}$$

$$Z_{\text{TM}} = \frac{E_{m\perp}}{H_{m\perp}} = \frac{k_z}{\omega\epsilon} = \left(\frac{\mu}{\epsilon}\right)^{1/2} \frac{\lambda_0}{\lambda_z} \tag{33-29}$$

$$\approx 377 \frac{\lambda_0}{\lambda_z} \qquad \text{ohms} \qquad (\epsilon_r = 1,\ \mu_r = 1). \tag{33-30}$$

Here λ and λ_0 are the wavelengths v/f of a plane wave of the same frequency f, and λ_z is the wavelength of the guided wave.

TEM Waves

If $k_z = k$ in Eqs. 33-11 to 33-15, the items in parentheses must be zero. The simplest way of satisfying this condition is to set both E_{mz} and H_{mz} equal to zero. We then have a TEM wave.

With TEM waves the wavelength λ_z of the guided wave is the same as that of a uniform plane wave in the same medium of propagation because k_z is equal to k, so

$$\lambda_z = \lambda. \tag{33-31}$$

If the medium is air, then the phase velocity is c, whatever the geometry of the guide and whatever the frequency. Such a guide is *distortionless* because the various frequency components of a complex waveform all travel at the same velocity.[†]

Setting $E_{mz} = 0$, $H_{mz} = 0$ in Eqs. 33-6 and 33-9 gives

$$E_{mx} = \left(\frac{\mu}{\epsilon}\right)^{1/2} H_{my}, \qquad E_{my} = -\left(\frac{\mu}{\epsilon}\right)^{1/2} H_{mx}. \tag{33-32}$$

The wave impedance is now

$$\frac{E_m}{H_m} = \frac{(E_{mx}^2 + E_{my}^2)^{1/2}}{(H_{mx}^2 + H_{my}^2)^{1/2}} = \left(\frac{\mu}{\epsilon}\right)^{1/2} \tag{33-33}$$

$$\approx 377 \text{ ohms} \qquad (\epsilon_r = 1, \mu_r = 1). \tag{33-34}$$

The ratio $(\mu/\epsilon)^{1/2}$ is the *characteristic impedance of the medium* (Sec. 28.5.2).

The electric and magnetic energy densities are equal:

$$\frac{\epsilon E^2}{2} = \frac{\mu H^2}{2}. \tag{33-35}$$

Also, the average Poynting vector is

$$\mathscr{S}_{av} = \tfrac{1}{2} \operatorname{Re} (\boldsymbol{E} \times \boldsymbol{H}^*) = \frac{1}{2}\left(\frac{\epsilon}{\mu}\right)^{1/2} E_m^2 \hat{\boldsymbol{z}} \tag{33-36}$$

$$= \left(\frac{\epsilon}{\mu}\right)^{1/2} E_{rms}^2 \hat{\boldsymbol{z}} = 2.65441 \times 10^{-3} \left(\frac{\epsilon_r}{\mu_r}\right)^{1/2} E_{rms}^2 \hat{\boldsymbol{z}} \quad \text{watts/meter}^2 \tag{33-37}$$

$$= v\epsilon E_{rms}^2 \hat{\boldsymbol{z}} = v\mu H_{rms}^2 \hat{\boldsymbol{z}}, \tag{33-38}$$

where

$$v = \frac{1}{(\epsilon\mu)^{1/2}} = \frac{c}{(\epsilon_r\mu_r)^{1/2}} \qquad (33\text{-}39)$$

is the *phase velocity*.

The magnitude of the time-averaged Poynting vector is equal to the energy density multiplied by the phase velocity.

THE FIELD COMPONENTS OF A TE WAVE IN A RECTANGULAR METALLIC WAVEGUIDE

We use the coordinate system of Fig. 34-2. The wave propagates in the positive direction of the z-axis by multiple reflection on the upper and lower walls. The figure also shows a wave front of a plane wave incident on the top face at the angle θ.

With *this* mode of propagation,

$$E_{mz} = 0, \qquad E_{mx} = 0, \qquad H_{my} = 0, \qquad \frac{\partial}{\partial y} = 0. \qquad (34\text{-}1)$$

We require the three other components E_{my}, H_{mx}, H_{mz}.

We proceed as indicated at the end of Sec. 33.1.1. First we solve the wave equation for H_{mz} for the given boundary conditions. This will give us both H_{mz} and k_z. Then the values of E_{my} and H_{mx} will follow, from Eqs. 33-13 and 33-14.

From Eq. 33-22,

$$\frac{\partial^2 H_{mz}}{\partial x^2} = (k_z^2 - k_0^2)H_{mz}, \qquad (34\text{-}2)$$

where

$$k_0 = \frac{1}{\lambda\!\!\!^-_0} = \frac{2\pi}{\lambda_0} = \frac{2\pi f}{c} = \frac{\omega}{c} \qquad (34\text{-}3)$$

is known, for a given frequency. However,

$$k_z = \frac{1}{\lambda\!\!\!^-_z} \qquad (34\text{-}4)$$

is unknown, λ_z being the wavelength of the guided wave.

We expect an interference pattern of some sort in the x direction. So H_{mz} is a sinusoidal function of x, and this requires that the expression in parentheses in Eq. 34-2 be negative. So we know that $k_z < k_0$ and hence that $\lambda_z > \lambda_0$, or that the wavelength measured along the guide is longer than the wavelength of a plane wave in air. This makes the phase velocity larger than c, which is correct; the group velocity will turn out to be smaller than c.

Thus

$$H_{mz} = M \cos (k_x x + \alpha), \qquad k_x = +(k_0^2 - k_z^2)^{1/2}, \tag{34-5}$$

where M is an arbitrary constant that defines the amplitude of the wave.

We now apply the boundary conditions of Sec. 33.1.4:

$$\frac{\partial H_{mz}}{\partial x} = 0 \qquad \text{at } x = 0, a, \tag{34-6}$$

$$\frac{\partial H_{mz}}{\partial y} = 0 \qquad \text{at } y = 0, b. \tag{34-7}$$

The second condition is already satisfied because $\partial/\partial y = 0$. From the first condition,

$$k_z \sin \alpha = 0 \qquad \text{and} \qquad k_x \sin (k_x a + \alpha) = 0. \tag{34-8}$$

Now k_x is a positive number. Therefore

$$\alpha = 0 \qquad \text{and} \qquad k_x a = n\pi, \tag{34-9}$$

where n is an integer.

Observe that $k_x a$ can take on only discrete or *eigen* values and that $n = 0$ is forbidden:

$$n = 1, 2, 3, \ldots \tag{34-10}$$

So

$$H_{mz} = M \cos \frac{n\pi x}{a} \tag{34-11}$$

and, from Eq. 34-5,

$$k_z = \left[k_0^2 - \left(\frac{n\pi}{a} \right)^2 \right]^{1/2} = \frac{\{1 - [n\lambda_0/(2a)]^2\}^{1/2}}{\lambda_0}. \tag{34-12}$$

Then, from Eqs. 33-13 and 33-14, remembering that both E_{mz} and $\partial/\partial y$ are zero,

$$E_{my} = \frac{-j\omega\mu_0}{-k_x^2} \frac{\partial H_{mz}}{\partial x} = \frac{j\omega\mu_0}{k_x^2} (-k_x M \sin k_x x) \tag{34-13}$$

$$= -\frac{j\omega\mu_0}{k_x} M \sin k_x x = -\frac{j\omega\mu_0 a}{n\pi} M \sin \frac{n\pi x}{a}, \tag{34-14}$$

$$H_{mx} = \frac{jk_z a}{n\pi} M \sin \frac{n\pi x}{a}. \tag{34-15}$$

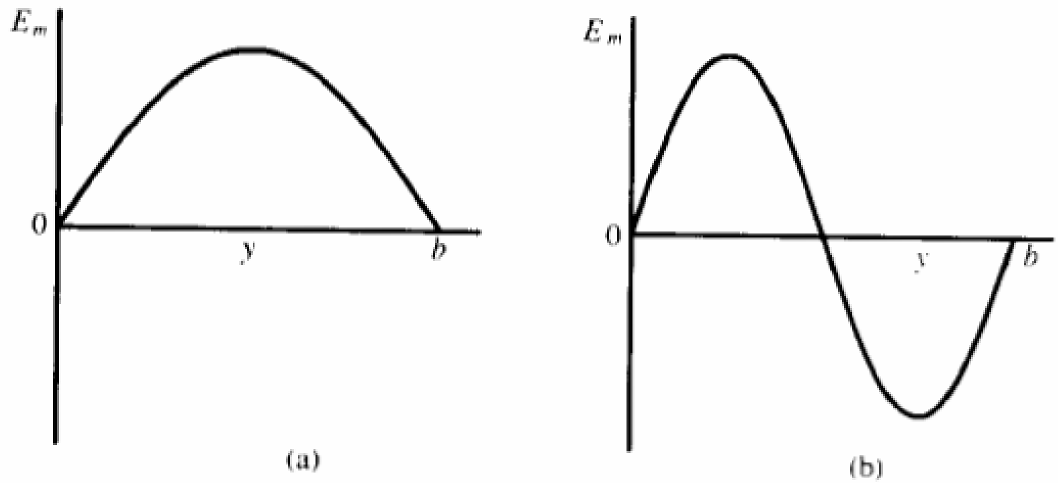

Fig. 34-3. The amplitude $E_m = E_{my}$ of the electric field strength for a TE wave. (a) TE_1 mode. (b) TE_2 mode.

Figure 34-3(a) shows E_{my} as a function of x for $n = 1$: $\boldsymbol{E}$ is zero along the walls and maximum in the plane $x = a/2$. With $n = 2$, E_{my} is zero at $x = a/2$.

The various values of n thus correspond to different modes of propagation, denoted as TE_1, TE_2, etc. As we shall see below, TE_1 is the only useful mode.

Summarizing,

$$E_{mx} = 0, \qquad E_{my} = -\frac{j\omega\mu_0 a}{n\pi} M \sin\frac{n\pi x}{a}, \qquad E_{mz} = 0, \tag{34-16}$$

$$H_{mx} = \frac{jk_z a}{n\pi} M \sin\frac{n\pi x}{a} = \frac{k_z}{\omega\mu_0} E_{my}, \qquad H_{my} = 0, \qquad H_{mz} = M \cos\frac{n\pi x}{a}, \tag{34-17}$$

$$k_z = \frac{\{1 - [n\lambda_0/(2a)]^2\}^{1/2}}{\lambda_0}. \tag{34-18}$$

THE CUTOFF WAVELENGTH. NONPROPAGATING FIELDS

From Eq. 34-18,

$$k_z = \left[k_0^2 - \left(\frac{n\pi}{a}\right)^2\right]^{1/2} = \left[\left(\frac{\omega}{c}\right)^2 - \left(\frac{n\pi}{a}\right)^2\right]^{1/2}. \tag{34-19}$$

For

$$\omega > \frac{n\pi c}{a}, \qquad \text{or} \qquad \lambda_0 < \frac{2a}{n}, \tag{34-20}$$

k_z is real and a wave can propagate unattenuated down the guide.

The wavelength $2a/n$ is the *cutoff wavelength* for the TE_n mode. This corresponds to the condition $\omega = \omega_p$ for propagation in an ionized gas. At that wavelength $k_x = 0$ and $\lambda_z \to \infty$.

At wavelengths larger than $2a/n$, k_z is imaginary, there is no wave, and the field decreases exponentially with z. There is zero power flow once the field is established. At these longer wavelengths the field amplitude decreases rapidly with z. For example, at twice the cutoff wavelength, where the frequency is too low by a factor of 2,

$$\frac{\omega}{c} = \frac{n\pi}{2a}, \qquad k_z = \left[\left(\frac{\omega}{c}\right)^2 - \left(\frac{2\omega}{c}\right)^2\right]^{1/2} = -j\frac{\omega}{c}3^{1/2}. \tag{34-21}$$

We choose the negative sign before the square root so that the amplitude will decrease exponentially with z, and then

$$\exp(-jk_z z) = \exp\left(-\frac{2\pi 3^{1/2} z}{\lambda_0}\right) = \exp\left(-10.88\frac{z}{\lambda_0}\right). \tag{34-22}$$

The amplitude decreases by a factor of 5×10^4 in one free-space wavelength λ_0!

The waveguide thus acts as a *high-pass* filter, with the lower frequency limit fixed by the width a, and not by b.

The free-space wavelength λ_0 must be shorter than twice the distance between the reflecting walls. For example, if $a = 100$ millimeters, then λ_0 must be less than 200 millimeters and the frequency must be higher than 1.5 gigahertz (1.5×10^9 hertz).

THE TE_1 MODE

In practice, one selects first the operating frequency, and then a guide whose dimensions are such that it can carry only the $n = 1$ mode. This condition requires that $2a$ be larger than λ_0, as above. But a must be less than λ_0 to make TE_2, TE_3, . . . forbidden modes. Thus the dimension a must be such that

$$a < \lambda_0 < 2a. \tag{34-23}$$

With single-mode propagation the field configuration is well defined. Rectangular metallic waveguides are narrow band devices: for a given a, λ_0 can vary by at most a factor of 2.

The antennas of Fig. 34-1 launch an assortment of modes, but only the TE_1 survives.

We now write out the field components for the $n = 1$ mode. We simplify the notation by setting

$$E'_{my} = -\frac{j\omega\mu_0 aM}{\pi}. \tag{34-24}$$

Then

$$E_x = 0, \qquad E_y = E'_{my}\sin\frac{\pi x}{a}\exp j(\omega t - k_z z), \qquad E_z = 0, \tag{34-25}$$

$$H_x = -\frac{E'_{my}k_z}{\omega\mu_0}\sin\frac{\pi x}{a}\exp j(\omega t - k_z z), \qquad H_y = 0, \tag{34-26}$$

$$H_z = \frac{\pi E'_{my}}{\omega\mu_0 a}\cos\frac{\pi x}{a}\exp j\left[\omega t - k_z\left(z - \frac{\lambda_z}{4}\right)\right], \tag{34-27}$$

where

$$k_z = \frac{2\pi}{\lambda_z} = \frac{\{1 - [\lambda_0/(2a)]^2\}^{1/2}}{\lambda_0} = \left[1 - \left(\frac{\lambda_0}{2a}\right)^2\right]^{1/2} k_0, \tag{34-28}$$

$$\lambda_z = \frac{\lambda_0}{\{1 - [\lambda_0/(2a)]^2\}^{1/2}}. \tag{34-29}$$

THE PHASE, SIGNAL, AND GROUP VELOCITIES

The phase velocity is

$$v_p = f\lambda_z = \omega \lambda\!\!\!^{-}_z = \frac{\omega}{k_z} \tag{34-38}$$

$$= \frac{c}{\{1 - [\lambda_0/(2a)]^2\}^{1/2}} = \frac{c}{\sin\theta} > c\,. \tag{34-39}$$

This is the velocity at which the phase propagates down the guide. It is larger than the speed of light c because the TE_1 wave is the superposition of two plane waves whose $\boldsymbol{k}$'s are inclined as in Fig. 34-2. For example, when the wave front AB of Fig. 34-7 moves downward at speed c to $A'B'$, point A moves to the right at a speed $v_p > c$.

At what velocity does a signal progress down the line? From Fig. 34-8 this is $c \sin\theta$. So the *signal velocity* is

$$v_s = c \sin\theta < c, \qquad v_p v_s = c^2. \tag{34-40}$$

THE TRANSMITTED POWER

To calculate the average power transmitted through the guide we integrate the time-averaged Poynting vector over the cross section of the guide:

$$P_{T,\mathrm{av}} = \int_0^b \int_0^a \mathscr{S}_{\mathrm{av}}\, dx\, dy = \int_0^b \int_0^a \tfrac{1}{2}\,\mathrm{Re}\,|(\boldsymbol{E} \times \boldsymbol{H}^*)|\, dx\, dy. \tag{34-41}$$

Here

$$\mathscr{S}_{\mathrm{av}} = \tfrac{1}{2}\,\mathrm{Re} \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ 0 & E_y & 0 \\ H_x^* & 0 & H_z^* \end{vmatrix} = \tfrac{1}{2}\,\mathrm{Re}\,(E_y H_z^* \hat{\boldsymbol{x}} - E_y H_x^* \hat{\boldsymbol{z}}), \tag{34-42}$$

where the components of $\boldsymbol{E}$ and of $\boldsymbol{H}$ are as in Eqs. 34-25 to 34-27. After substituting 2π for $k_z\lambda_z$, the expression for H_z becomes

$$H_z = \frac{j\pi E'_{my}}{\omega\mu_0 a} \cos\frac{\pi x}{a} \exp j(\omega t - k_z z). \tag{34-43}$$

Thus the x component of the time-averaged Poynting vector is zero. The net power flows in the direction of the z-axis and

$$\mathcal{S}_{av} = \frac{E'^2_{my} k_z}{2\omega\mu_0} \sin^2 \frac{\pi x}{a} \hat{z}. \tag{34-44}$$

We assume that E'_{my} is real.

The average power density $\mathcal{S}_{av}$ is independent of y, as expected, since both $\boldsymbol{E}$ and $\boldsymbol{H}$ are independent of y. It is zero at $x = 0$, $x = a$ where $\boldsymbol{E}$ is zero and maximum at $x = a/2$.

The time-averaged transmitted power is thus

$$P_{T,av} = \frac{E'^2_{my} k_z}{2\omega\mu_0} \int_0^a \sin^2 \frac{\pi x}{a} b \, dx \tag{34-45}$$

$$= \frac{E'^2_{my} k_z ab}{4\omega\mu_0} = \frac{E'^2_{my} k_0 ab}{4\omega\mu_0} \left[1 - \left(\frac{\lambda_0}{2a}\right)^2\right]^{1/2} \tag{34-46}$$

$$P_{T,av} = E'^2_{my} \frac{ab}{4} \left(\frac{\epsilon_0}{\mu_0}\right)^{1/2} \left[1 - \left(\frac{\lambda_0}{2a}\right)^2\right]^{1/2}. \tag{34-47}$$