DYNAMICS OF A STOCHASTIC EPIDEMIC MODEL WITH MARKOV SWITCHING AND GENERAL INCIDENCE RATE

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Abstract: In this paper, the stochastic SIR epidemic model with Markov switching and general incidence rate is investigated. We classify the model by introducing a threshold value λ. To be more specific, we show that if $\lambda < 0$ then the disease-free is globally asymptotic stable i.e., the disease will eventually disappear while the epidemic is strongly stochastically permanent provided that $\lambda > 0$. We also give some of numerical examples to illustrate our results.

1 Introduction

The idea of using mathematical models to investigate disease transmissions and behavior of epidemics was first introduced by Kermack and McKendrick in [11] [12]. Since then, much attention has been devoted to analyzing, predicting the spread, and designing controls of infectious diseases in host populations (see [2] [3] [4] [13] [14] [16] and the references therein). One of classic epidemic models is the SIR model, which subdivides a homogeneous host population into three epidemiologically distinct types of individuals, the susceptible, the infective, and the removed, with their population sizes denoted by S, I and R, respectively. It is suitable for some infectious diseases of permanent or long immunity, such as chickenpox, smallpox, measles, etc.

As we all know, the incidence rate of a disease is the number of new cases per unit time and it plays an important role in the investigation of mathematical epidemiology. Therefore, during the last few decades, a number of realistic transmission functions have become the focus of considerable attention. Concreterly, in[10], authors studied a deterministic SIR model with the standard bilinear incidence rate and has been extended to stochastic SIR model in [3] [5] [7] [14] [16]. However, there is a variety of reasons why this standard bilinear incidence rate may require modifications. For instance, the underlying assumption of homogeneous mixing and homogeneous environment may be invalid. In this case the necessary population structure and heterogeneous mixing may be incorporated into a model with a specific form of nonlinear transmission. For example, in [2], Capasso and Serio studied the cholera epidemic spread in Bari in 1978. They imposed the saturated incidence rate $\frac{\beta SI}{1+aI}$ in their model of the cholera, where a is positive constant. Anderson et. al. [1] used saturated incidence rate $\frac{\beta SI}{1+aS}$. In [8], authors considered the Beddington-DeAngelis

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functional response $\frac{\beta SI}{1+aS+bI}$. Ruan et. al. [18] considered nonlinear incidence of saturated mass action $\frac{\beta I^m S}{1+\alpha I^n}$, where m, α, n are positive constants. Taking into account the presence of white noise, color noise and both of them, the stochastic SIR models with various incidence rates mentioned above have been studied in [4] [15] [19] [20].

In this paper, we work with the general incidence rate $SIF_1(S, I)$, where F_1 is locally Lipschitz continuous. Thus, our model includes almost incidence rates appeared in the literature. Furthermore, we suppose that the model is perturbed by both white nose and color noise. To be specific, we consider the following model

$$\begin{cases} dS(t) = (-S(t)I(t)F_1(S(t), I(t), r_t) + \mu(r_t)(K - S(t)))\, dt - S(t)I(t)F_2(S(t), I(t), r_t)dB(t) \\ dI(t) = \big(S(t)I(t)F_1(S(t), I(t), r_t) - (\mu(r_t) + \rho(r_t) + \gamma(r_t))I(t)\big)dt \\ \qquad\qquad +S(t)I(t)F_2(S(t), I(t), r_t)dB(t) \\ dR(t) = (\gamma(r_t)I(t) - (\mu(r_t))R(t))dt, \end{cases} \tag{1.1}$$

where $\{r_t, t \geq 0\}$ is a right continuous Markov chain taking values in $\mathcal{M} = \{1, 2, ..., m_0\}$, $F_1(\cdot), F_2(\cdot)$ are positive and locally Lipschitz functions on $[0, \infty)^2 \times \mathcal{M}$, $B(t)$ is a one dimensional standard Brownian motion, $\mu(i), \rho(i), \gamma(i)$ are assumed to be positive for all $i \in \mathcal{M}$.

Our main goal in this paper is to provide a sufficient and almost necessary condition for strongly stochastically permanent and extinction of the disease in the stochastic SIR model (1.1). Concretely, we establish a threshold λ such that the sign of λ determines the asymptotic behavior of the system. If $\lambda < 0$, the disease is eradicated at a disease-free equilibrium $(K, 0)$. In this case, we derive that the density of disease converges to 0 with exponential rate. Meanwhile, in the case $\lambda > 0$, we show that the disease is strongly stochastically permanent.

The rest of the paper is arranged as follows. In section 2, we give and prove our main results. Section 3 is reserved for providing some numerical examples and figures.

2 Main results

Denote $\mathbb{R}^2_+ := \{(x, y) : x \geq 0, y \geq 0\}$, $\mathbb{R}^{2,o}_+ := \{(x, y) : x > 0, y > 0\}$, $\Delta := \{(x, y) \in \mathbb{R}^2_+ : x + y \leq K\}$ and $\mathcal{M} = \{1, 2, ..., m_0\}$ for a positive integer m_0. Let $B(t)$ be an one-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote by $Q = (q_{kl})_{m_0 \times m_0}$ the generator of the Markov chain $\{r_t, t \geq 0\}$ taking values in $\mathcal{M}$. This means that

$$\mathbb{P}\{r_{t+\delta} = l | r_t = k\} = \begin{cases} q_{kl}\delta + o(\delta) & \text{if } k \neq l, \\ 1 + q_{kk}\delta + o(\delta) & \text{if } k = l, \end{cases}$$

as $\delta \to 0$. Here, q_{kl} is the transition rate from k to l and $q_{kl} \geq 0$ if $k \neq l$, while $q_{kk} = -\sum_{k \neq l} q_{kl}$. We assume that the Markov chain r_t is irreducible, under this condition, the Markov chain r_t has a unique stationary distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_{m_0}) \in \mathbb{R}^{m_0}$.

We assume that the Markov chain r_t is independent of the Brownian motion $B(t)$. Because the dynamics of class of recover has no effect on the disease transmission dynamics,

we only consider the reduced system,

$$\begin{cases} dS(t) = \big(- S(t)I(t)F_1(S(t), I(t), r_t) + \mu(r_t)(K - S(t))\big)dt - S(t)I(t)F_2(S(t), I(t), r_t)dB(t) \\ dI(t) = \big(S(t)I(t)F_1(S(t), I(t), r_t) - (\mu(r_t) + \rho(r_t) + \gamma(r_t))I(t)\big)dt \\ \qquad\qquad +S(t)I(t)F_2(S(t), I(t), r_t)dB(t). \end{cases} \tag{2.1}$$

Theorem 2.1. For any given initial value $(S(0), I(0)) \in \mathbb{R}^2_+$, there exists a unique global solution $\{(S(t), I(t)), t \geq 0\}$ of Equation (2.1) and the solution will remains in $\mathbb{R}^2_+$ with probability one. Moreover, if $I(0) > 0$ then $I(t) > 0$ for any $t \geq 0$ with probability 1.

Proof. The proof is almost the same as those in [9]. Hence we obmit.

To simplify notations, we denote by $\Phi(t) = (S(t), I(t))$ the solution of system (2.1), and $\phi = (x, y) \in \mathbb{R}^{2,\circ}_+$.

Lemma 2.1. For any initial value $\phi = (x, y) \in \mathbb{R}^{2,\circ}_+$ the solution $\Phi(t) = (S(t), I(t))$ of Equation (2.1) eventually enters Δ. Further Δ is an invariant set.

Proof. By adding side by side in system (2.1), we have

$$\begin{aligned} \frac{d}{dt}(S(t) + I(t)) &= K\mu(r_t) - \mu(r_t)(S(t) + I(t)) - (\rho(r_t) + \gamma(r_t))I(t) \\ &\leq \mu(r_t)K - \mu(r_t)(S(t) + I(t)). \end{aligned}$$

Using the comparison theorem yields

$$\limsup_{t\to\infty}(S(t) + I(t)) \leq K. \tag{2.2}$$

Therefore $(S(t), I(t))$ eventually enters Δ. Further, if $S(0) + I(0) \leq K$, so is $(S(t) + I(t)$ for $t \geq 0$.

Remark 2.1. Thus, $\Delta = \{(x, y) \in \mathbb{R}^2_+ : x + y \leq K\}$ is an invariant set. By Lemma 2.1,we only need to work with the process $(S(t), I(t))$ on the invariant set Δ.

We are now in position to provide a condition for the extinction and permanence of disease. Let

$$g(x, y, i) = F_1(x, y, i)x - \Big(\mu(i) + \rho(i) + \gamma(i) + \frac{F_2^2(x, y, i)x^2}{2}\Big).$$

We define the threshold

$$\lambda = \sum_{i=1}^{m_0} g(K, 0, i)\pi_i = \sum_{i=1}^{m_0} \Big[F_1(K, 0, i)K - \Big(\mu(i) + \rho(i) + \gamma(i) + \frac{F_2^2(K, 0, i)K^2}{2}\Big)\Big]\pi_i. \tag{2.3}$$

Let $C^2(\mathbb{R}^2 \times \mathcal{M}, \mathbb{R}_+)$ denote the family of all non-negative functions $V(\phi, i)$ on $\mathbb{R}^2 \times \mathcal{M}$ which are twice continuously differentiable in ϕ. The operator $\mathcal{L}$ associated with (2.1) is defined as follows. For $V \in C^2(\mathbb{R}^2 \times \mathcal{M}, \mathbb{R}_+)$, define

$$\mathcal{L}V(\phi, i) = \mathcal{L}_iV(\phi, i) + \sum_{j\in\mathcal{M}} q_{ij}V(\phi, j) \tag{2.4}$$

where $\mathcal{L}_i V(\phi,i) = V_\phi(\phi,i)\widetilde{f}(\phi,i) + \frac{1}{2}\widetilde{g}^\top(\phi,i)V_{\phi\phi}(\phi,i)\widetilde{g}(\phi,i)$, $V_\phi(\phi,i)$ and $V_{\phi\phi}(\phi,i)$ are the gradient and Hessian of $V(\cdot,i)$, $\widetilde{f}$ and $\widetilde{g}$ are the drift and diffusion coefficients of (2.1), respectively; i.e.,

$$\widetilde{f}(\phi,i) = (-xyF_1(x,y,i) + \mu(i)(K-x), xyF_1(x,y,i) - (\mu(i)+\rho(i)+\gamma(i))y)^\top$$

and

$$\widetilde{g}(\phi,i) = (-xyF_2(x,y,i), xyF_2(x,y,i))^\top.$$

Following lemma gives condition for the locally asymptotic stability of free-desease point $(K,0)$.

Lemma 2.2. If $\lambda < 0$, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all initial value $(\phi,i) \in \mathcal{U}_\delta \times \mathcal{M} := (K-\delta, K] \times [0,\delta) \times \mathcal{M}$, we have

$$\mathbb{P}_{\phi,i}\left\{\lim_{t\to\infty} \Phi(t) = \big(K,0\big)\right\} \geq 1-\varepsilon. \tag{2.5}$$

Proof. Since $\lambda < 0$, we can choose sufficiently small $\kappa > 0$ such that

$$\sum_{j\in\mathcal{M}} (g(K,0,j)+\kappa)\pi_j < 0.$$

Consider the Lyapunov function $V(x,y,i) = (K-x)^2 + y^p$, where $p \in (0,1)$ is a constant to be specified. By direct calculation we have for $(x,y,i) \in \Delta \times \mathcal{M}$ that

$$\begin{aligned}
&\mathcal{L}_i V(x,y,i)\\
&= -2(K-x)[-F_1(x,y,i)xy + \mu(i)(K-x)] + py^p g(x,y,i) + x^2y^2F_2^2(x,y,i) + \frac{p^2F_2^2(x,y,i)x^2y^p}{2}\\
&\leq -2\mu(i)(K-x)^2 + py^p g(x,y,i) + y\Big(2(K-x)F_1(x,y,i)x + x^2yF_2^2(x,y,i)\Big) + \frac{p^2F_2^2(x,y,i)x^2y^p}{2}.
\end{aligned}$$

Because of the continuity of $g(\cdot), F_1(\cdot), F_2(\cdot)$, the compactness of $\Delta \times \mathcal{M}$ and the fact that $y^{1-p} \to 0$ as $y \to 0$, we can choose $p \in (0,1)$ and $\delta_1 \in (0,K)$ such that for any $(x,y,i) \in \mathcal{U}_{\delta_1} \times \mathcal{M}$,

$$\begin{aligned}
py^p g(x,y,i) + y\Big(2(K-x)F_1(x,y,i)x + x^2yF_2^2(x,y,i)\Big) + \frac{p^2F_2^2(x,y,i)x^2y^p}{2}\\
\leq p(g(K,0,i)+\kappa)y^p.
\end{aligned}$$

When p is sufficiently small, we also have

$$-2\mu(i)(K-x)^2 \leq p(g(K,0,i)+\kappa)(K-x)^2.$$

Therefore,

$$\mathcal{L}_i V(x,y,i) \leq p[g(K,0,i)+\kappa]V(x,y,i)\ \forall (x,y,i) \in \mathcal{U}_{\delta_1} \times \mathcal{M}.$$

By [17; Theorem 5.36], for any $\varepsilon > 0$, there is $0 < \delta < \delta_1$ such that

$$\mathbb{P}_{\phi,i}\Big\{\lim_{t\to\infty}(S(t), I(t)) = \big(K, 0\big)\Big\} \geq 1 - \varepsilon \text{ for } (\phi, i) \in \mathcal{U}_\delta \times \mathcal{M}. \tag{2.6}$$

The proof is complete. For any $\delta > 0$, $(\phi, i) \in \Delta \times \mathcal{M}$, set the first entrance time of $\Phi(t)$ into the set $\mathcal{U}_\delta$ by

$$\tau_\delta = \inf\{t > 0 : \Phi(t) \in \mathcal{U}_\delta\}.$$

Lemma 2.3. For all $\delta > 0$, for each initial data $(\phi, i) \in \Delta \times \mathcal{M}$, we have $\tau_\delta^{\phi,i} < \infty$ almost surely.

Proof. Consider the Lyapunov function $U(\phi, i) = c_1 - (x+1)^{c_2}$, where c_1 and c_2 are two positive constants to be specified. We have

$$\mathcal{L}U(\phi, i) = -c_2(x+1)^{c_2-2}\big[(x+1)(\mu(i)(K-x) - xyF_1(x, y, i)) + \frac{c_2 - 1}{2}x^2y^2F_2^2(x, y, i)\big].$$

Let $\mu_m = \min\{\mu(i) : i \in \mathcal{M}\}$. Since $(x+1)\mu(i)(K-x) \geq \mu_m\delta$ for any $x \in [0, K-\delta]$ and $\inf\{F_2(x, y, i) : (x, y, i) \in \Delta \times \mathcal{M}\} > 0$, we can find sufficiently large c_2 such that

$$-xyF_1(x, y, i) + \frac{c_2 - 1}{2}x^2y^2F_2(x, y, i) \geq -0.5\mu_m\delta \text{ for } (\phi, i) \in \Delta \times \mathcal{M}, x \leq K - \delta.$$

Hence

$$(x+1)\mu(i)(K-x) - xyF_1(x, y, i)) + \frac{c_2 - 1}{2}x^2y^2F_2(x, y, i) \geq 0.5\mu_m\delta \text{ for } (\phi, i) \in \Delta \times \mathcal{M}, x \leq K - \delta,$$

$$\mathcal{L}U(\phi, i) \leq -0.5c_2\mu_m\delta \text{ given that } (x, y, i) \in \Delta \times \mathcal{M}, x \leq K - \delta.$$

Let $c_1 > 0$ be chosen such that U is positive on Δ. By Dynkin's formula, we obtain

$$\mathbb{E}_{\phi,i}U(\Phi(\tau_\delta \wedge t), r_{\tau_\delta \wedge t}) = U(\phi, i) + \mathbb{E}_{\phi,i}\int_0^{\tau_\delta \wedge t}\mathcal{L}U(\Phi(s), r_s)ds \leq U(\phi, i) - 0.5c_2\mu_m\delta\mathbb{E}_{\phi,i}\tau_\delta \wedge t.$$

Letting $t \to \infty$ and using Fatou's lemma yields that

$$\mathbb{E}_{\phi,i}U(\Phi(\tau_\delta), r_{\tau_\delta}) \leq U(\phi, i) - 0.5c_2\mu_m\delta\mathbb{E}_{\phi,i}\tau_\delta.$$

Since U is positive on $\Delta \times \mathcal{M}$, we deduce that $\mathbb{E}_{\phi,i}\tau_\delta < \infty$. This implies that $\tau_\delta < \infty$ almost surely. The proof is complete. We now provide condition for the disease-free globally asymptotic stability.

Theorem 2.2 (Condition for extinction of disease)**.** If $\lambda < 0$, then $\Phi(t) \to (K, 0)$ a.s. as $t \to \infty$ for all given initial value $(\phi, i) \in \Delta \times \mathcal{M}$, i.e., the disease will be extinct. Moreover,

$$\mathbb{P}_{\phi,i}\Big\{\lim_{t\to\infty}\frac{\ln I(t)}{t} = \lambda < 0\Big\} = 1 \text{ for } (\phi, i) \in \Delta \times \mathcal{M}, y > 0. \tag{2.7}$$

Proof. 2.2, we have, if $\lambda < 0$ then the disease-free is locally stable. Meanwhile Lemma 2.3 implies that for all $\delta > 0$ the first entrance time to $\mathcal{U}_\delta$ of $\Phi(t)$ is finite. Combining these properties and the strong Markov property, we have

$$\mathbb{P}_{\phi,i}\{\lim_{t\to\infty} \Phi(t) = (K, 0)\} \geq 1 - \varepsilon \text{ for } (\phi, i) \in \Delta \times \mathcal{M},$$

for any $\varepsilon > 0$. As a result,

$$\mathbb{P}_{\phi,i}\{\lim_{t\to\infty} \Phi(t) = (K, 0)\} = 1 \text{ for } (\phi, i) \in \Delta \times \mathcal{M}. \tag{2.8}$$

Applying Itô's formula we have

$$\ln I(t) = \ln I(0) - G(t)$$

where

$$G(t) = -\int_0^t g(\Phi(u), r_u)du - \int_0^t S(u)F_2(S(u), I(u), r_u)dB(u).$$

This imlies that

$$\frac{\ln I(t)}{t} = \frac{\ln I(0)}{t} + \frac{1}{t}\int_0^t g(\Phi(u), r_u)du + \frac{1}{t}\int_0^t S(u)F_2(S(u), I(u), r_u)dB(u). \tag{2.9}$$

We derive from the ergodicity r_t, (2.8) and (2.3) that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t g(\Phi(u), r_u)du = \lambda. \tag{2.10}$$

By using Remark 2.1 and the strong law of large numbers for martingales, we get

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t S(u)I(u)F_2(S(u), I(u), r_u)dB(u) = 0 \text{ a.s.} \tag{2.11}$$

Combining (2.9), (2.10) and (2.11) we obtain (2.7). The proof is complete.

We now consider condition for the permanent of disease. As a preparation, we present the following lemma.

Lemma 2.4. Let $\partial\Delta_2 := \{\phi = (x, y) \in \Delta : y = 0\}$. Then there exists $T > 0$ such that for any $(\phi, i) \in \partial\Delta_2 \times \mathcal{M}$,

$$\mathbb{E}_{\phi,i}\int_0^T g(\Phi(u), r_u)du \geq \frac{3\lambda}{4}T. \tag{2.12}$$

Proof. When $I(0) = 0$, we have $I(t) = 0$ for any $t > 0$ and $\lim_{t\to\infty} S(t) = K$ uniformly in the initial values. This and the uniform ergodicity of r_t imply that

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}_{\phi,i}\int_0^t g(\Phi(u), r_u)du = \lambda \text{ uniformly in } (\phi, i) \in \partial_2\Delta \times \mathcal{M}.$$

Thus, we can easily find a T satisfying (2.12).

Theorem 2.3 (Condition for permanent of disease)**.** If $\lambda > 0$, the disease is strongly stochastically permanent in the sense that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\liminf_{t\to\infty} \mathbb{P}_{\phi,i}\{I(t) \geq \delta\} > 1 - \varepsilon \text{ for any } (\phi, i) \in \Delta \times \mathcal{M}, y > 0. \tag{2.13}$$

Proof. Consider the Lyapunov function $V_\theta(\phi, i) = y^\theta$, where θ is a real constant to be determined. We have

$$\mathcal{L}V_\theta(\phi, i) = \theta y^\theta [F_1(x, y, i)x - (\mu(i) + \rho(i) + \gamma(i)) + \frac{\theta - 1}{2} x^2 F_2^2(x, y, i)].$$

It implies that $\mathcal{L}V_\theta(\phi, i) \leq H_\theta V_\theta(\phi, i)$, where $H_\theta = \sup\{\theta[F_1(x, y, i)x - (\mu(i) + \rho(i) + \gamma(i)) + \frac{\theta-1}{2}x^2F_2^2(x, y, i)] : (x, y, i) \in \Delta \times \mathcal{M}\}$. Let $\tau_n = \inf\{t \geq 0 : V_\theta(\Phi(t), r_t) \geq n\}$. By using Itô's formula and taking expectation in both sides, we obtain

$$\begin{aligned}\mathbb{E}_{\phi,i} V_\theta(\Phi(t \wedge \tau_n, r_{t\wedge\tau_n})) &= V_\theta(\phi, i) + \mathbb{E}_{\phi,i} \int_0^{t\wedge\tau_n} \mathcal{L}V_\theta(\Phi(s), r_s)ds \\ &\leq V_\theta(\phi, i) + H_\theta \int_0^t \mathbb{E}_{\phi,i} V_\theta(\Phi(s \wedge \tau_n), r_{s\wedge\tau_n})ds.\end{aligned}$$

By using Gronwall inequality, we have

$$\mathbb{E}_{\phi,i} I^\theta(t \wedge \tau_n) \leq y^\theta \exp\{H_\theta t\}.$$

Letting $n \to \infty$, we get

$$\mathbb{E}_{\phi,i} I^\theta(t) \leq y^\theta \exp\{H_\theta t\} \text{ for any } t \geq 0, (\phi, i) \in (\Delta \setminus \partial_2\Delta) \times \mathcal{M}. \tag{2.14}$$

By the Feller property and (2.12), there exists $\delta_2 > 0$ such that if $\phi = (x, y) \in \Delta$ with $y < \delta_2$ we have

$$\mathbb{E}_{\phi,i} G(T) = -\mathbb{E}_{\phi,i} \int_0^T g(\Phi(t), r_t)dt \leq -\frac{\lambda}{2}T. \tag{2.15}$$

From (2.14) and $G(t) = \ln I(0) - \ln I(t)$, we have

$$\mathbb{E}_{\phi,i} \exp\{G(T)\} + \mathbb{E}_{\phi,i} \exp\{-G(T)\} = \mathbb{E}_{\phi,i} \frac{y}{I(T)} + \mathbb{E}_{\phi,i} \frac{I(T)}{y} \leq \exp\{H_{-1}T\} + \exp\{H_1T\}.$$

Applying [6; Lemma 3.5; pp. 1912], we deduce that

$$\ln \mathbb{E}_{\phi,i} e^{\theta G(T)} \leq -\frac{\lambda\theta}{2}T + \hat{H}\theta^2 \text{ for } \theta \in [0, 0.5],$$

where $\hat{H}$ is a constant depending on T, H_{-1} and H_1. For sufficiently small θ, we have

$$\mathbb{E}_{\phi,i} e^{\theta G(T)} \leq \exp\Big\{-\frac{\lambda\theta}{4}T\Big\} \text{ for } \phi \in \Delta, y < \delta_2, i \in \mathcal{M}.$$

Or equivalently

$$\mathbb{E}_{\phi,i}I^{-\theta}(T) \le qy^{-\theta} \text{ for } q = \exp\Big\{-\frac{\lambda\theta}{4}T\Big\} \text{ for } \phi \in \Delta, y < \delta_2, i \in \mathcal{M}.$$

This and (2.14) imply that

$$\mathbb{E}_{\phi,i}I^{-\theta}(T) \le qy^{-\theta} + C \text{ for } C = \delta_2^{-\theta}\exp\{H_{-\theta}T\} \text{ for } \phi \in \Delta, i \in \mathcal{M}.$$

By the Markov property, we deduce that

$$\mathbb{E}_{\phi,i}I^{-\theta}((k+1)T) \le q\mathbb{E}_{\phi,i}I^{-\theta}(kT) + C \text{ for } \phi \in \Delta, i \in \mathcal{M}, k \in \mathbb{Z}_+.$$

Using this recursively we obtain

$$\mathbb{E}_{\phi,i}I^{-\theta}(nT) \le q^n y^{-\theta} + \frac{C(1-q^n)}{1-q} \text{ for } \phi \in \Delta, i \in \mathcal{M}, n \in \mathbb{Z}_+. \tag{2.16}$$

This and (2.14) imply

$$\mathbb{E}_{\phi,i}I^{-\theta}(t) \le \Big(q^n y^{-\theta} + \frac{C(1-q^n)}{1-q}\Big)\exp\{H_{-\theta}T\} \text{ for } t \in [nT, nT+T]. \tag{2.17}$$

Letting $n \to \infty$ we obtain $\limsup_{t\to\infty}\mathbb{E}_{\phi,i}I^{-\theta}(t) = \dfrac{C}{1-q}\exp\{H_{-\theta}T\}$, which leads to (2.13).

3 Numerical Examples

In this section we providing some numerical examples to illustrate our results. We consider the Holling II functional responses; that is

$$F_1(S,I,r_t) = \frac{\beta_1(r_t)SI}{1+a_1(r_t)S}, \quad F_2(S,I,r_t) = \frac{\beta_2(r_t)SI}{1+a_2(r_t)S}.$$

The process $\{r_t, t \ge 0\}$ is a right continuous Markov chain taking values in $\mathcal{M} = \{1,2\}$. The transition rate from state 1 to 2 is $q_{12} = 0.5$ and state 2 to 1 is $q_{21} = 0.8$, then the stationary distribution $\pi = (\pi_1, \pi_2) = (\frac{8}{13}, \frac{5}{13})$.

Example 3.1. We assume that the capacity of the environment $K = 4$ and the coefficients of the Equation (1.1) are given in Table 1 below.

Coefficients / States	a_1	a_2	β_1	β_2	γ	μ	ρ
1	1.5	0.1	8	1	0.5	2	1
2	1	0.3	5	1.5	0.2	1.5	0.2

Table 1: *Values of the coefficients in Example 3.1*

By using formula (2.3), we have $\lambda = -0.6916 < 0$. As a result of Theorem 2.2, $\lim_{t\to\infty} I(t) = 0$ and $\lim_{t\to\infty} S(t) = 4$. That is, disease will eventually be extinction. This claim is illustrated by Figure 1.

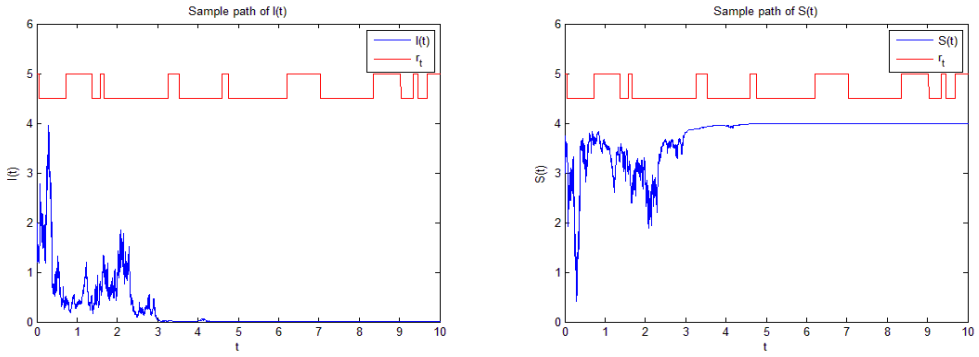

Fig.1: *Sample paths of $I(t)$ (on the left), $S(t)$ (on the right), and r_t in Example 3.1.*

Example 3.2. We assume that the capacity of the environment is $K = 20$. The table of parameter values is given below.

Coefficients / States	a_1	a_2	β_1	β_2	γ	μ	ρ
1	2	1.7	5	1	0.5	1	0.7
0.8	1	3	3	1.5	1	0.5	1

Table 2: *Values of the coefficients in Example 3.2*

Detailed computations give us that $\lambda = 2.2564 > 0$. Thus, by Theorem 2.3, $I(t)$ is permanent. That result can be described in Figure 3.2.

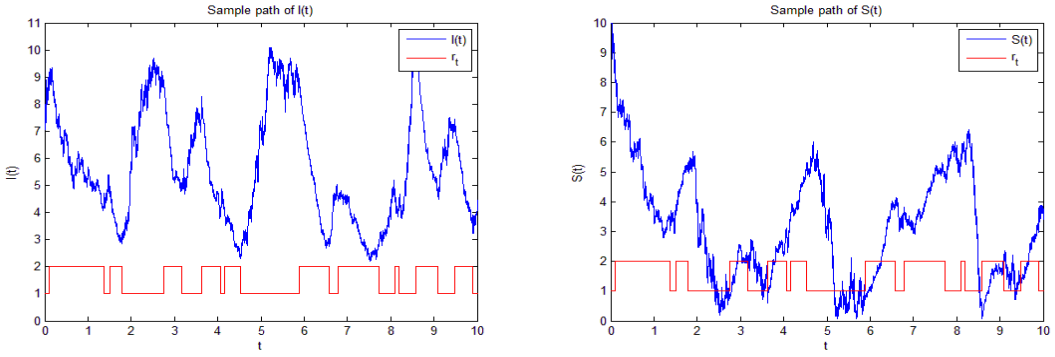

Fig.1: *Sample paths of $I(t)$ (on the left) and $S(t)$ (on the right) and r_t in Example 3.2.*

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TÓM TẮT

ĐỘNG HỌC CỦA MÔ HÌNH DỊCH TỄ NGẪU NHIÊN VỚI BƯỚC CHUYỂN MARKOV VÀ HÀM ĐÁP ỨNG TỔNG QUÁT

Trong bài báo này, mô hình dịch tễ SIR ngẫu nhiên với bước chuyển Markov và hàm đáp ứng dạng tổng quát được quan tâm nghiên cứu. Chúng tôi phân loại mô hình bằng cách đưa ra một giá trị ngưỡng λ. Cụ thể, chúng tôi chỉ ra rằng nếu $\lambda < 0$, điểm cân bằng sạch bệnh ổn định tiệm cận toàn cục nghĩa là về lâu dài mô hình sẽ sạch bệnh; mô hình sẽ tồn tại dịch bệnh theo nghĩa ngẫu nhiên mạnh khi $\lambda > 0$. Chúng tôi cũng đưa ra một vài ví dụ số để minh họa cho kết quả lý thuyết đạt được trong bài báo.