COMPARISON THEOREM OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS AND APPLICATION

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Abstract: *In this paper, we first establish a comparison theorem for the nonlinear factional differential equations, then derive a local asymptotical stability theorem for the nonlinear differential equation under some suitable conditions.*

Keywords: Caputo derivative; Equilibrium; Asympticalability.

1. INTRODUCTION

In recent decades, fractional calculus and fractional differential equations attract much attention and increasing interests due to their potential applications in sciences and engineering ([1, 6, 9, 11, 12] and many references cited therein). Fractional differenital/integral operator is one kind of the pseudo-differential operators [10]. It plays an important role in fractional equations. In general three kinds of fractional derivative operators are used, including Gr¨unwald-Letnikov fractional derivative operator, Riemann -Liouville fractional derivative operator and Caputo fractional derivative operator. They are not equivalent but have close relations [6, 7, 12]. In applied sciences and engineering, the Caputo derivative is often used. In this paper the involved fractional derivative indicates the Caputo derivative, which is defined as

$$D_*^{\alpha} z = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} z^{(m)}(\tau)d\tau, \alpha > 0$$

in which m = $\lceil \alpha \rceil$, i.e., m is the first integer not less than α, $z^{(m)}(\tau)$ is the usual mth-order derivative with respect to τ . In real applications, the fractional order α is often less than 1, here we restrict α ∈ (0, 1) as usual. For the case α > 1, we can often translate the fractional systems into systems with the same actional order which lies in (0,1) provided some suitable conditions are satisfied [2].

The Cauchy problem of the fractional differential equation reads as

$$D_*^{\alpha} x = f(t, x),\ x(0) = x_0,\ \alpha \in (0,1) \tag{1}$$

If we request the vector function f is continuous and satisfies a Lipschitz condition with respect to the second argument x on a suitable set G, then the initial value problem (1) determines a unique solution on some interval [0, T] [4]. Throughout the paper, we always assume f fulfils the above condition on a suitable set which we call G, so equation (1) exists one and only one solution defined on [0, T].

In [8], Matignon studied stability of zero solution to the following fractional system

$$D_*^{\alpha} X(t) = AX$$

with X(0) = $X_0 = (x_{10}, x_{20}, \cdots, x_{n0})^T$, where X = $(x_1, x_2, \cdots, x_{n0})^T$, α ∈ (0, 1), A ∈ $R^{n\times n}$. Afterwards, Li, et al., further studied stability of zero solution of the above system with multiple time-delays [3]. All this stability belongs to linear stability. In this paper, we mainly study nonlinear stability of fractional differential equation.

The outline of the present paper is arranged as follows. In Section 2, some basic notions are introduced and the Comparison Theorem is established. In the last section, we derive a symptotical stability theorem of the fractional nonlinear differential equation.

2. COMPARISON THEOREM

The autonomous differential equation with Caputo derivative is in the following form,

$$D_*^{\alpha} x = f(x),\ \alpha \in (0,1) \tag{2}$$

with initial condition x(0) = x_0. In this paper, we mainly consider stability of equilibrium to equation (2). We assume that the initial problem (2.1) has one and only one solution. This solution is denoted by x(t) = φ(t; x_0).

In the following, we give some definitions.

2.1. Definition 2.1. If there exists a constant e such that f(e) = 0, then e is called an equilibrium point to equation (2).

2.2. Definition 2.2. The equilibrium point e to (2.1) is said to be: 1)locally stable if $\forall\varepsilon > 0$, there exists a δ > 0 such that |x(t) − e| < ε holds for $\forall x_0 \in$ {z : |z − e| < δ} and ∀t ≥ 0; 2) local asymptotically stability if x(t) is locally stable and $\lim_{t\to\infty} x(t)$ = e.

The theorem below can be found in [4].

2.3. Theorem 2.1. If f is continuous and α ∈ (0, 1), then the Cauchy problem $\begin{cases} D_*^{\alpha}x = f(x) \\ x(t_0) = x_0 \end{cases}$ is equivalent to a nonlinear integral equation of the second kind,

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(x(\tau))d\tau.$$

Next, we establish a comparison theorem for the fractional differential equations. Consider the following fractional equations,

$$D_*^{\alpha}x = f(t, x),\ \alpha \in (0,1) \tag{3}$$

$$D_*^{\alpha}x = F(t, x),\ \alpha \in (0,1) \tag{4}$$

2.4. Theorem 2.2. Consider equations (3) and (4) with initial value condition (t_0, x(t_0)) = (0, x_0). If f, F are continuous then (3) and (4) have one and only one solution x(t), y(t) passing through (0, x_0), respectively. Besides, if the vector functions satisfy f(t, x) ≤ F(t, x), then x(t) ≤ y(t) for t > 0.

Proof. The solutions of systems (5) and (6) can be expressed in the following form:

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau))d\tau \tag{5}$$

And

$$y(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} F(\tau, x(\tau))d\tau \tag{6}$$

Subtracting Eq. (6) from Eq. (5), one has

$$y(t) - x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[F(\tau, x(\tau)) - f(\tau, x(\tau))]d\tau \tag{7}$$

Since $(t-\tau)^{\alpha-1}$ is a nonnegative function, $F(\tau, x(\tau)) - f(\tau, x(\tau)) \geq 0$ it then follows from Eq. (7) that x(t) $\leq y(t), \forall t \geq 0$. The proof is completed.

Similar to the ordinary differential equation, the solution of equation (1) can be extended to +∞. This fact means that the conclusions of Theorem 2.2 hold for (0,+∞). In the last section, we discuss the stability of equilibrium to the autonomous nonlinear differential equation with Caputo derivative. Since the non-zero equilibrium of differential equation with Caputo derivative can be moved to the origin, we only study stability of the zero solution of the considered equation.

3. ASYMPTOTICAL STABILITY ANALYSIS

In this section, we derive the following stability theorem.

Theorem 3.1. If the nonlinear equation $D_*^{\alpha}x$ = f(x) with α ∈ (0, 1) satisfies following conditions: (i) f(0) = 0, (ii) the 2th-order derivative f’’(x) of f(x) is continuous and f’(0) < 0, (iii) if its non-zero solution x(t) = φ(t; x_0) has a zero point but its derivative value at this zero point is non-zero, then the zero solution is locally asymptotically stable.

Proof. Set f’(0) = λ, s(x)x = f(x) – f’(0)x, then near the zero solution one has
f(x) = f’(0)x + o(x) = (λ + s(x))x,

where s(x) = O(x).

Since f'(0) < 0 therefore, there exit some constant numbers M > m > 0 such that −M < λ+s(x) < −m, futhermore −Mx < f(x) < −mx if x > 0, or −Mx > f(x) > −mx if x < 0, for all x ∈ (−δ, δ) with δ > 0 small enough. Now, we assert that for the initial value $|x_0|$ small enough then |x(t)| < $|x_0|$ for all t > 0. Because λ < 0, there exits a positive constant δ_1 small enough such that $\forall x_0 \in \{z : |z| < \delta_1\}$, λ + s($x_0$) < 0. Throughout the proof, set x_0 > 0. For $x_0 \leq 0$, this case can be similarly proved so it is omitted here.

Firstly, following Theorem 2.1 we have that equation (2.1) is equivalent to

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\lambda + s\big(x(\tau)\big)\right)x(\tau)d\tau.$$

It is obvious that x(t) is continuous with respect to t, so we can take $\varepsilon > 0$ small enough such that

$$|x(t) - x_0| < \frac{x_0}{2} \text{ và } \lambda + s\big(x(t)\big)\lambda < 0, \forall t \in [0,\varepsilon],$$

i.e.,

$$x(t) > \frac{x_0}{2} > 0 \text{ và } \lambda + s\big(x(t)\big) < 0, \forall t \in [0,\varepsilon].$$

Therefore,

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\lambda + s\big(x(\tau)\big)\right)x(\tau)d\tau < 0, \forall t \in [0,\varepsilon].$$

Thus, there exits a small $\varepsilon_0 > 0$ such that the solution x(t) of (2.1) satisfies

$$0 < x(t) < x_0, \forall t \in (0,\varepsilon_0].$$

Now, we shall prove that before x(t) passes through time-axis t, that is, before it becomes negative if it does indeed, it is always less than x_0. Infact, if x(t) equals to x_0 in the first time, say, at time T_0 > 0, x(T_0) = x_0 and x_0 > x(t) ≥ 0 for all t ∈ (0, T_0), then

$$x_0 = x(T_0) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^{T_0} (T_0 - \tau)^{\alpha-1}\left(\lambda + s\big(x(\tau)\big)\right)x(\tau)d\tau.$$

From the above formula, we know

$$\frac{1}{\Gamma(\alpha)}\int_0^{T_0} (T_0 - \tau)^{\alpha-1}\left(\lambda + s\big(x(\tau)\big)\right)x(\tau)d\tau = 0.$$

but, clearly

$$\frac{1}{\Gamma(\alpha)}\int_0^{T_0} (T_0 - \tau)^{\alpha-1}\left(\lambda + s\big(x(\tau)\big)\right)x(\tau)d\tau < 0$$

due to assumption. So the conclusion holds.

Next, we show that x(t) never transverses the time-axis t, i.e., it is always non-negative.

Assume x(t) transverses the time-axis t in the first time T_0, then for ε_1 > 0 small enough,

$$x(T_0) = 0 \; and - x_0 < x(t) < 0 \; for \; \forall t \in (T_0, T_0 + \varepsilon_1].$$

Thus, we get

$$x(T_0) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^{T_0} (T_0 - \tau)^{\alpha-1}\left(\lambda + s\big(x(\tau)\big)\right)x(\tau)d\tau,$$

$$x(T_0 + \varepsilon) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^{T_0+\varepsilon} (T_0 + \varepsilon - \tau)^{\alpha-1}\left(\lambda + s\big(x(\tau)\big)\right)x(\tau)d\tau,$$

In which $\varepsilon \in (0,\varepsilon_1]$, then

$$x(T_0+\varepsilon) = x(T_0+\varepsilon) - x(T_0) = \frac{1}{\Gamma(\alpha)}\int_0^{T_0+\varepsilon} (T_0 + \varepsilon - \tau)^{\alpha-1}\left(\lambda + s\big(x(\tau)\big)\right)x(\tau)d\tau - \frac{1}{\Gamma(\alpha)}\int_0^{T_0} (T_0 - \tau)^{\alpha-1}\left(\lambda + s\big(x(\tau)\big)\right)x(\tau)d\tau$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^{T_0} ((T_0 + \varepsilon - \tau)^{\alpha-1} - (T_0 - \tau)^{\alpha-1}) \left(\lambda + s\big(x(\tau)\big)\right) x(\tau) d\tau$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{T_0}^{T_0+\varepsilon} (T_0 + \varepsilon - \tau)^{\alpha-1} \left(\lambda + s\big(x(\tau)\big)\right) x(\tau) d\tau . \ (8)$$

For the right hand of (8),

$$((T_0 + \varepsilon - \tau)^{\alpha-1} - (T_0 - \tau)^{\alpha-1}) \left(\lambda + s\big(x(\tau)\big)\right) x(\tau) > 0, \forall \tau \epsilon (0, T_0),$$

And

$$(T_0 + \varepsilon - \tau)^{\alpha-1} \left(\lambda + s\big(x(\tau)\big)\right) x(\tau) > 0, \forall \tau \in (T_0, T_0 + \varepsilon).$$

So

$$\frac{1}{\Gamma(\alpha)} \int_0^{T_0} ((T_0 + \varepsilon - \tau)^{\alpha-1} - (T_0 - \tau)^{\alpha-1}) \left(\lambda + s\big(x(\tau)\big)\right) x(\tau) d\tau$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{T_0}^{T_0+\varepsilon} (T_0 + \varepsilon - \tau)^{\alpha-1} \left(\lambda + s\big(x(\tau)\big)\right) x(\tau) d\tau > 0.$$

But the left hand of (3.1) $x(T_0 + \varepsilon)$ is negative, this is a contradiction.

Therefore, $0 \le x(t) < x_0$ $\forall t > 0$ and $x_0 > 0$. Based on assumption (iii), x(t) is strictly positive for this situation.

So, we draw a conclusion: for the initial value $|x_0|$ small enough, the non-zero solutions $|x(t)| < |x_0|$ that is, $x(t) > 0$ if $x_0 > 0$, $x(t) < 0$ if $x_0 < 0$.
Now for small $\delta > 0$, small initial value $|x_0| > 0$, and $(t, x) \in (0,+\infty) \times \{(-\delta, \delta)\backslash\{0\}\}$, one has $-Mx < f(x) < -mx$ if $x_0 > 0$, or, $-Mx > f(x) > -mx$ if $x_0 < 0$.

Now, consider following three equations:

$$D_*^\alpha x = -Mx, \qquad x(0) = x_0 \tag{8}$$
$$D_*^\alpha x = f(x), \qquad x(0) = x_0 \tag{9}$$

and

$$D_*^\alpha x - -mx \qquad x(0) - x_0 \tag{10}$$

These equations satisfy the conditions of Theorem 2.2, therefore
$x(t) < y(t) < z(t)$ if $x_0 > 0$ and $x(t) > y(t) > z(t)$ if $x_0 < 0$ for all $t > 0$, where x(t), y(t) and z(t) solve (9), (2) and (10), respectively.

It is evident that

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0.$$

Thus,

$$\lim_{t\to\infty} y(t) = 0.$$

This completes the proof.

4. CONCLUSIONS

In this paper, we first give Definitions about locally stable, local asymptotically stability, then we derive and prove a comparison theorem. At last, we aply the comparison theorem to give and prove a local asymptotical stability theorem for the nonlinear differential equation under some suitable conditions. These results are helpful to study fractional differential equations and establishing fractional models in science and engineering.

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TÓM TẮT

ĐỊNH LÝ SO SÁNH NGHIỆM CỦA PHƯƠNG TRÌNH VI PHÂN CẤP PHÂN SỐ PHI TUYẾN VÀ ỨNG DỤNG

Trong bài báo này, đầu tiên chúng tôi đưa ra và chứng minh định lý so sánh nghiệm cho phương trình vi phân cấp phân số phi tuyến, sau đó, chúng tôi áp dụng định lý này để thành lập và chứng minh định lí ổn định tiệm cận của phương trình vi phân cấp phân số phi tuyến dưới các điều kiện phù hợp.

Từ Khóa: Đạo hàm Caputo; Đạo hàm cấp phân số; Ổn định tiệm cận.

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