Parabolic Equations with a Singular Absorption: Extinction Versus Blow-up

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Abstract. *We prove a local existence of weak solutions of semilinear parabolic equations with a strong singular absorption and a source. Moreover, we consider the qualitative behavior of solutions. We show that any solution exists globally and vanishes after a finite time if either the initial data or the source term is small enough. On the other hand, we point out some criteria such that solutions are explosive in a finite time.*

Keywords

Nonlinear parabolic equations, quenching phenomenon, singular absorption.

1. Introduction

In this paper, we are interested in nonnegative solutions of the following equation:

$$\begin{cases} \partial_t u - \Delta u + u^{-\beta}\chi_{\{u>0\}} = f(u) & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $0 < \beta < 1$, and $\chi_{\{u>0\}}$ denotes the characteristic function of the set of points (x,t) where $u(x,t) > 0$, i.e:

$$\chi_{\{u>0\}} = \begin{cases} 1, & \text{if } u > 0, \\ 0, & \text{if } u \leq 0. \end{cases}$$

Note that the absorption term $u^{-\beta}\chi_{\{u>0\}}$ becomes singular when u is near to 0, and we impose tactically $u^{-\beta}\chi_{\{u>0\}} = 0$ whenever $u = 0$.

Problem Eq. (1) can be considered as a limit of mathematical models describing enzymatic kinetics (see [1]), or the Langmuir-Hinshelwood model of the heterogeneous chemical catalyst (see, e.g. [19] p. 68, [22], [18]). This problem has been studied by the authors in [18], [29], [28], [13], [16], [9], [7], [8], [6], [20], and references therein. These authors have considered the existence and uniqueness, and the qualitative behavior of these solutions. For example, when $f = 0$, D. Phillips [18] proved the existence of solution for the Cauchy problem associating to Eq. (1). A partial uniqueness of solution of Eq. (1) was proved by J. Davila and M. Montenegro, [9] for a class of solutions with initial data $u_0(x) \geq C\mathrm{dist}(x,\partial\Omega)^\mu$, for $\mu \in (1, \frac{2}{1+\beta})$ (see also [5] the uniqueness in a different class of solutions). A beautiful result established by M. Winkler, [20], showed that the uniqueness of solution fails in general. One of the interesting behaviors of solutions of Eq. (1) is the extinction that any solution vanishes after a finite time even beginning with a positive initial data, see [18], [28] (see also [7] for a quasilinear equation of this type). It is known that this phenomenon occurs according to the presence of the nonlinear singular absorption $u^{-\beta}\chi_{\{u>0\}}$. One can see the same situation for the nonlinear absorption u^β, for $\beta \in (0,1)$, see [21] and references therein.

Equation (1) with source term $f(u)$ satisfying the sublinear condition, i.e: $f(u) \leq C(u + 1)$, was considered by J. Davila and M. Montenegro, [9]. The authors proved the existence of solution and showed that the measure of the set $\{(x, t) \in \Omega \times (0, \infty) : u(x, t) = 0\}$ is positive (see also a more general statement in [23]). In other words, the solution may exhibit the quenching behavior. Still in the sublinear case with source term $\lambda f(u)$, M. Montenegro [17] proved that there is a real number $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$, there is $t_0 > 0$ such that

$$u(x, t_0) = 0, \quad \forall x \in \Omega.$$

And it is called the complete quenching phenomenon.

From our knowledge, Eq. (1) with a general source term $f(u)$ has not been studied completely, such as $f(u) = u^2$. Thus, we would like to investigate existence, and the qualitative behavior of solutions of Eq. (1) for source term $f(u)$. It is well known that nonlinear parabolic equations with general source $f(u)$ may cause the finite time blow-up, i.e: there is a time $T_0 > 0$ such that $\lim_{t \to T_0} \|u(t)\|_\infty \to +\infty$. As mentioned above, the nonlinear absorption $u^{-\beta} \chi_{\{u>0\}}$ causes the complete quenching phenomenon. Thus, it is interesting to see when the complete quenching prevails the blow-up, and conversely. We also note that the similar questions were studied by [7], [8], [6] for the quasilinear parabolic equations of this type. To be simple, we consider $f(u) = u^p$, $p \geq 1$ through this paper, although our analysis can be applied to a general source $f(u)$, which is a locally Lipschitz function on $[0, \infty)$. Before discussing the behaviors of solutions of Eq. (1), it is necessary to introduce a notion of weak solution, and establish first a local existence of solutions of Eq. (1).

Definition 1. *Let $u_0 \in L^\infty(\Omega)$. A nonnegative function $u(x, t)$ is called a weak solution of Eq. (1) if $u^{-\beta} \chi_{\{u>0\}} \in L^1(\Omega \times (0, T))$, and $u \in L^p(0, T; W_0^{1,2}(\Omega)) \cap L^\infty(\Omega \times (0, T)) \cap \mathcal{C}([0, T); L^1(\Omega))$ satisfies Eq. (1) in the sense of distributions $\mathcal{D}'(\Omega \times (0, T))$, i.e:*

$$\int_0^T \int_\Omega \left(-u\phi_t + \nabla u.\nabla \phi + u^{-\beta} \chi_{\{u>0\}}\phi - f(u)\phi \right) dxdt = 0$$

(2)

for all $\phi \in \mathcal{C}_c^\infty(\Omega \times (0, T))$.

Our local existence result is as follows.

Theorem 1. *Let $u_0 \in L^\infty(\Omega)$, and $f(u) = u^p$. Then, there exists a finite time $T = T(u_0) > 0$ such that Eq. (1) has a maximal weak solution u in $\Omega \times (0, T)$, i.e: for any weak solution v in $\Omega \times (0, T)$, we have*

$$v \leq u, \quad in\ \Omega \times (0, T).$$

Moreover, there is a positive constant $C = C(\|u_0\|_\infty)$ such that

$$|\nabla u(x, \tau)|^2 \leq C u^{1-\beta} \left(\tau^{-1} + 1 \right), \quad (3)$$

for a.e $(x, \tau) \in \Omega \times (0, T)$.

Besides, if $\nabla(u_0^{\frac{1}{\gamma}}) \in L^\infty(\Omega)$, with $\gamma = \frac{2}{1+\beta}$, then there is a positive constant $C = C(\|u_0\|_\infty, \|\nabla(u_0^{\frac{1}{\gamma}})\|_\infty)$ such that

$$|\nabla u(x, \tau)|^2 \leq C u^{1-\beta}(x, \tau), \quad (4)$$

for a.e $(x, \tau) \in \Omega \times (0, T)$.

Remark 1. *The result of Theorem 1 implies that u is continuous up to the boundary. Furthermore, u is continuous up to $t = 0$ if provided $\nabla(u_0^{\frac{1}{\gamma}}) \in L^\infty(\Omega)$.*

Our next purpose is to study the global existence of solutions. Particularly, we are interested in the complete quenching phenomenon that any solution vanishes identically after a finite time under some circumstances. To state our global existence result and the complete quenching phenomenon, we consider Eq. (1) with source term λu^p, for any $\lambda > 0$. In some of our considerations, a crucial role is played by the first eigenvalue λ_1 of the Dirichlet problem:

$$\begin{cases} -\Delta \Phi = \lambda_1 \Phi & in\ \Omega, \\ \Phi(x) = 0, & on\ \partial\Omega \end{cases} \quad (5)$$

where Φ is the first normalized eigenfunction, $\int_\Omega \Phi(x)dx = 1$. It is known that λ_1 decreases when the measure of the spatial domain Ω increases. Then, we have a result of global existence of solution.

Theorem 2. *Let $u_0 \in L^\infty(\Omega)$, and $f(u) = \lambda u^p$, for $\lambda > 0$. Assume that there are an open*

bounded domain Ω_0, and a positive real number k_0 such that $\Omega \subset\subset \Omega_0$, and

$$\begin{cases} u_0(x) \leq k_0 \Phi_{\Omega_0}(x), \quad \text{for a.e } x \in \Omega, \\ \\ k_0 \lambda_{1,\Omega_0} \Phi_{\Omega_0}(x) + k_0^{-\beta} \Phi_{\Omega_0}^{-\beta}(x) \geq \lambda \left(k_0 \Phi_{\Omega_0}(x)\right)^p \\ \hfill \forall (x,t) \in \Omega \times (0,\infty), \end{cases}$$ (6)

where λ_{1,Ω_0} and Φ_{Ω_0} are the first eigenvalue and the first eigenfunction of problem Eq. (5) corresponding to Ω_0. Then, any solution v of Eq. (1) exists globally and satisfies

$$v(x,t) \leq k_0 \Phi_{\Omega_0}(x), \quad \text{in } \Omega \times (0,\infty). \quad (7)$$

Remark 2. *For a given $\lambda > 0$; if u_0 is small enough then Eq. (6) holds, and conversely. Thus, we obtain the global existence of solutions if provided either u_0 or λ is small enough.*

Next, we give the complete quenching results.

Theorem 3. *Let $u_0 \in L^\infty(\Omega)$, and $f(u) = \lambda u^p$. For a given $\lambda > 0$, then every solution of Eq. (1) is extinct after a finite time if provided that $\|u_0\|_\infty$ is small enough.*

Theorem 4. *For a given $u_0 \in L^\infty(\Omega)$, there is a real number $\lambda_0 > 0$ such that every solution of Eq. (1) quenches after a finite time if provided $\lambda \in (0, \lambda_0)$.*

Finally, we study the global nonexistence of solutions of Eq. (1), the so called finite time blow-up, see [15], [16], [12], [11], [25], [26], [30]. In this paper, we point out some criteria on initial data u_0 in order to guarantee the blow-up of solution in a finite time. Thus, it is convenient to introduce the energy functional

$$E(t) = \int_\Omega \left(\frac{1}{2} |\nabla u(t)|^2 + \frac{1}{1-\beta} u^{1-\beta}(t) \\ - \frac{1}{p+1} u^{p+1}(t) \right) dx.$$ (8)

Then we have a blowing up result as follows:

Theorem 5. *Let $u_0 \in L^\infty(\Omega) \cap H_0^1(\Omega)$. Suppose that $f(u) = u^p$, for $p > 1$, and $E(0) \leq 0$. Let u be a solution of Eq. (1). Then, u blows up in a finite time.*

The paper is organized as follows: In the next section, we prove some gradient estimates for the approximating solutions. In Sec. 3. we shall prove the local existence results. The global existence of solutions and the complete quenching phenomenon will be considered in Sec. 4. Section 5. is devoted to study the non-global existence of solution. In the final Section, we give some simulation in order to illustrate the interesting phenomenon: quenching versus blow-up.

Several notations which will be used through this paper are the following: we denote by C a general positive constant, possibly varying from line to line. Furthermore, the constants which depend on parameters will be emphasized by using parentheses. For example, $C = C(p, \beta, \tau)$ means that C depends on p, β, τ. And $Supp(f)$ is denoted as the support compact of f.

2. Gradient estimate for the approximate solutions

In this section, we consider the regularized problem $(P_{\varepsilon,\eta})$ of Eq. (1) as follows:

$$\begin{cases} \partial_t u - \Delta u + g_\varepsilon(u) = u^p & \text{in } \Omega \times (0,\infty), \\ u = \eta & \text{on } \partial\Omega \times (0,\infty), \\ u(0) = u_0 + \eta & \text{on } \Omega \end{cases}$$

for any $0 < \eta < \varepsilon$, with $g_\varepsilon(s) = \psi_\varepsilon(s) s^{-\beta}$, $\psi_\varepsilon(s) = \psi(\frac{s}{\varepsilon})$, and $\psi \in \mathcal{C}^\infty(\mathbb{R})$ is a nondecreasing function on $\mathbb{R}$ such that $\psi(s) = 0$ for $s \leq 1$, and $\psi(s) = 1$ for $s \geq 2$. Note that g_ε is a globally Lipschitz function for any $\varepsilon > 0$. We will show that solution $u_{\varepsilon,\eta}$ of equation $(P_{\varepsilon,\eta})$ tends to a solution of Eq. (1) as $\eta, \varepsilon \to 0$. In passing to the limit, we need to derive some gradient estimates for solution $u_{\varepsilon,\eta}$. Then, we have the following result:

Lemma 1. *Let $u_0 \in \mathcal{C}_c^\infty(\Omega)$, $u_0 \neq 0$. There exists a classical unique solution $u_{\varepsilon,\eta}$ of $(P_{\varepsilon,\eta})$ in $\Omega \times (0, T)$.*
i) Moreover, there is a constant $C > 0$ only depending on $\beta, T, f, \|u_0\|_\infty$ such that

$$|\nabla u_{\varepsilon,\eta}(x,\tau)|^2 \leq C u_{\varepsilon,\eta}^{1-\beta}(x,\tau)\left(\tau^{-1} + 1\right), \quad (9)$$

for any $(x, \tau) \in \Omega \times (0, T)$.

ii) If $\nabla(u_0^{\frac{1}{\gamma}}) \in L^\infty(\Omega)$, then we get

$$|\nabla u_{\varepsilon,\eta}(x,\tau)|^2 \le C u_{\varepsilon,\eta}^{1-\beta}(x,\tau), \qquad (10)$$

for any $(x, \tau) \in \Omega \times (0, T)$, with $C > 0$ merely depends on $\beta, T, \|u_0\|_\infty, \|\nabla(u_0^{\frac{1}{\gamma}})\|_\infty$.

Proof. The proof of this lemma is similar to the one in [20]. Thus, we skip it and refer to Lemma 3.1, [20]. □

Thanks to Lemma 1 and the classical regularity (see [14]), we can pass to the limit as $\eta \to 0$ in problem $(P_{\varepsilon,\eta})$. Then, problem (P_ε):

$$\begin{cases} \partial_t u - \Delta u + g_\varepsilon(u) = u^p & \text{in } \Omega \times (0, \infty), \\ u = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(0) = u_0 & \text{on } \Omega \end{cases}$$

has a unique solution u_ε. Note that u_ε also satisfies Lemma 1.

3. Local existence

In this section, we consider a local existence of solution of problem Eq. (1). We give the proof of Theorem 1.

Proof of Theorem 1. Let u_ε be a unique solution of problem (P_ε) in $\Omega \times (0, T)$. Then, we show that $\{u_\varepsilon\}_{\varepsilon > 0}$ is a non-decreasing sequence. Indeed, we have

$$g_{\varepsilon_1}(s) \ge g_{\varepsilon_2}(s), \quad \text{for any } 0 < \varepsilon_1 < \varepsilon_2.$$

This implies that u_{ε_1} is a sub-solution of the equation satisfied by u_{ε_2}. Therefore, the comparison principle yields

$$u_{\varepsilon_1} \le u_{\varepsilon_2}, \quad \text{in } \Omega \times (0, T), \quad \forall \varepsilon_1 < \varepsilon_2,$$

so the conclusion follows. Consequently, there is a nonnegative function u such that $u_\varepsilon \downarrow u$ as $\varepsilon \to 0^+$.

Obviously, we have from the comparison principle

$$u_\varepsilon \le \Gamma(t) = \left(\|u_0\|_\infty^{1-p} - (p-1)t\right)^{1-p},$$

for any $(x, t) \in \Omega \times (0, T_0)$, $T_0 = \frac{\|u_0\|_\infty^{1-p}}{p-1}$.

Integrating equation (P_ε) on $\Omega \times (0, T)$ yields

$$\int_\Omega u_\varepsilon(x, T)dx - \int_0^T \int_{\partial\Omega} \nabla u_\varepsilon \cdot \mathbf{n} d\sigma ds$$
$$+ \int_0^T \int_\Omega g_\varepsilon(u_\varepsilon)dxds = \int_\Omega u_\varepsilon(x, 0)dx$$
$$+ \int_0^T \int_\Omega u_\varepsilon^p dxds, \ \forall T \in (0, T_0),$$

where $\mathbf{n}$ is the unit outward normal vector of $\partial\Omega$.

Since $\nabla u_\varepsilon \cdot \mathbf{n} \le 0$, we obtain

$$\int_0^T \int_\Omega g_\varepsilon(u_\varepsilon)dxds \le \int_\Omega \left(u_0(x) + \varepsilon\right)dx$$
$$+ \int_0^T \int_\Omega u_\varepsilon^p dxds.$$

This inequality and the boundedness of u_ε above imply that $\|g_\varepsilon(u_\varepsilon)\|_{L^1(\Omega \times (0,T))}$ is bounded by a constant not depending on ε.

Thanks to Fatou's lemma, there is a function $\Upsilon \in L^1(\Omega \times (0, T))$ such that

$$\liminf_{\varepsilon \to 0} g_\varepsilon(u_\varepsilon) = \Upsilon, \quad \text{in } L^1(\Omega \times (0, T)). \quad (11)$$

Next, the monotonicity of $\{u_\varepsilon\}_{\varepsilon > 0}$ deduces

$$g_\varepsilon(u_\varepsilon)(x, t) \ge g_\varepsilon(u_\varepsilon)\chi_{\{u > 0\}}(x, t),$$

for $(x, t) \in \Omega \times (0, T)$, so

$$\liminf_{\varepsilon \to 0} g_\varepsilon(u_\varepsilon)(x, t) = \Upsilon(x, t) \ge u^{-\beta}\chi_{\{u > 0\}}(x, t),$$
$$(12)$$

for $(x, t) \in \Omega \times (0, T)$. Then, $u^{-\beta}\chi_{\{u > 0\}}$ is integrable on $\Omega \times (0, T)$.

Actually, we shall prove

$$\Upsilon = u^{-\beta}\chi_{\{u > 0\}}, \quad \text{in } L^1(\Omega \times (0, T)). \quad (13)$$

On the other hand, we use a result of gradient convergence of Boccardo et al., [3], [2] in order to obtain

$$\nabla u_\varepsilon \xrightarrow{\varepsilon \to 0} \nabla u, \quad \text{for a.e } (x, t) \in \Omega \times (0, T). \quad (14)$$

As a result, ∇u fulfills estimate Eq. (3) for a.e $(x, t) \in \Omega \times (0, T)$, and for any $\tau > 0$

$$\nabla u_\varepsilon \xrightarrow{\varepsilon \to 0} \nabla u, \quad \text{in } L^r(\Omega \times (\tau, T)), \quad \forall r \in [1, \infty). \quad (15)$$

Now, it suffices to demonstrate that u satisfies Eq. (1) in the sense of distribution.

For any $\eta > 0$ fixed, we use the test function $\psi_\eta(u_\varepsilon)\phi$, for any $\phi \in \mathcal{C}_c^\infty(\Omega \times (0,T))$, to the equation satisfied by u_ε. Then, using integration by parts yields

$$\int_{Supp(\phi)} \left(-\Psi_\eta(u_\varepsilon)\phi_t + \frac{1}{\eta}|\nabla u_\varepsilon|^2\psi'(\frac{u_\varepsilon}{\eta})\phi \right.$$
$$+\nabla u.\nabla\phi\psi_\eta(u_\varepsilon) + g_\varepsilon(u_\varepsilon)\psi_\eta(u_\varepsilon)\phi$$
$$\left. +u_\varepsilon^p\psi_\eta(u_\varepsilon)\phi \right)dxds = 0,$$

with $\Psi_\eta(u) = \int_0^u \psi_\eta(s)ds$. Note that $\psi_\eta(.)$ plays a role in avoiding the singularity of the term $u^{-\beta}\chi_{\{u>0\}}$ as u is near 0. Thus, there is no problem of going to the limit as $\varepsilon \to 0$ in the indicated equation in order to obtain

$$\int_{Supp(\phi)} \left(-\Psi_\eta(u)\phi_t + \frac{1}{\eta}|\nabla u|^2\psi'(\frac{u}{\eta})\phi \right.$$
$$+\nabla u.\nabla\phi\psi_\eta(u) + u^{-\beta}\psi_\eta(u)\phi$$
$$\left. +f(u,x,s)\psi_\eta(u)\phi \right)dxds = 0.$$

Next, we go to the limit as $\eta \to 0$ in the last equation.

By Eq. (14), Eq. (15), and the integration of $u^{-\beta}\chi_{\{u>0\}}$ in $\Omega \times (0,T)$, it is not difficult to verify

$$\begin{cases} \lim_{\eta\to 0}\int_{Supp(\phi)} \Psi_\eta(u)\phi_t dxds = \int_{Supp(\phi)} u\phi_t dxds, \\[2mm] \lim_{\eta\to 0}\int_{Supp(\phi)} \nabla u.\nabla\phi\psi_\eta(u)dxds = \int_{Supp(\phi)} \nabla u.\nabla\phi dxds, \\[2mm] \lim_{\eta\to 0}\int_{Supp(\phi)} u^{-\beta}\psi_\eta(u)\phi dxds = \int_{Supp(\phi)} u^{-\beta}\chi_{\{u>0\}}\phi, \\[2mm] \lim_{\eta\to 0}\int_{Supp(\phi)} u^p\psi_\eta(u)\phi dxds = \int_{Supp(\phi)} u^p\phi dxds. \end{cases}$$
$$(16)$$

Next, we show that

$$\lim_{\eta\to 0}\int_{Supp(\phi)} \frac{1}{\eta}|\nabla u|^2\psi'(\frac{u}{\eta})\phi \ dxds = 0. \quad (17)$$

In fact, since u satisfies estimate Eq. (3), we have

$$\frac{1}{\eta}\int_{Supp(\phi)} |\nabla u|^2|\psi'(\frac{u}{\eta})\phi|dxds$$
$$\leq C\frac{1}{\eta}\int_{Supp(\phi)\cap\{\eta<u<2\eta\}} u^{1-\beta}dxds$$
$$\leq 2C\int_{Supp(\phi)\cap\{\eta<u<2\eta\}} u^{-\beta}dxds,$$

where $Supp(\phi)$ means the support compact of ϕ, and the constant $C > 0$ is independent of η. Since $u^{-\beta}\chi_{\{u>0\}}$ is integrable on $\Omega \times (0,T)$, we obtain

$$\lim_{\eta\to 0}\int_{Supp(\phi)\cap\{\eta<u<2\eta\}} u^{-\beta}dxds = 0,$$

which implies the conclusion Eq. (17). A combination of Eq. (16) and Eq. (17) deduces

$$\int_{Supp(\phi)} \left(-u\phi_t + \nabla u.\nabla\phi \right.$$
$$\left. +u^{-\beta}\chi_{\{u>0\}}\phi + f(u,x,s)\phi \right)dxds = 0. \quad (18)$$

In other words, u satisfies Eq. (1) in $\mathcal{D}'(\Omega \times (0,T))$.

As mentioned above, we prove Eq. (13) now. From equation (P_ε), we have

$$\int_{Supp(\phi)} \left(-u_\varepsilon\phi_t + \nabla u_\varepsilon.\nabla\phi \right.$$
$$\left. +g_\varepsilon(u_\varepsilon)\phi + f(u_\varepsilon,x,s)\phi \right)dxds = 0,$$

for any $\phi \in \mathcal{C}_c^\infty(\Omega \times (0,T))$, $\phi \geq 0$.

Then, letting $\varepsilon \to 0$ deduces

$$\int_{Supp(\phi)} \left(-u\phi_t + \nabla u.\nabla\phi \right)dxds+$$
$$\lim_{\varepsilon\to 0}\int_{Supp(\phi)} g_\varepsilon(u_\varepsilon)\phi \ dxds \quad (19)$$
$$+\int_{Supp(\phi)} f(u,x,t)\phi \ dxds = 0.$$

By Eq. (18) and Eq. (19), we get

$$\lim_{\varepsilon\to 0}\int_{Supp(\phi)} g_\varepsilon(u_\varepsilon)\phi \ dxds$$
$$= \int_{Supp(\phi)} u^{-\beta}\chi_{\{u>0\}}\phi \ dxds. \quad (20)$$

According to Eq. (11), Eq. (20) and Fatou's lemma, we obtain

$$\int_{Supp(\phi)} u^{-\beta}\chi_{\{u>0\}}\phi dxds \geq \int_{Supp(\phi)} \Upsilon\phi dxds,$$

$\forall \phi \in \mathcal{C}_c^\infty(\Omega \times (0,T)),\ \phi \geq 0.$

The last inequality and Eq. (12) yield conclusion Eq. (13).

The conclusion $u \in \mathcal{C}([0,T];L^1(\Omega))$ is well known, so we skip its proof and refer to the compactness result of J. Simon, [33]. Thus, u is a weak solution of Eq. (1).

To complete the proof of Theorem 1, it remains to show that u is the maximal solution of Eq. (1).

Let v be any weak solution of Eq. (1) on $\Omega \times (0,T)$. Then, we have

$$v(x,t) \leq u(x,t), \quad \text{for a.e } (x,t) \in \Omega \times (0,T).$$

In fact, we observe that

$$g_\varepsilon(v) \leq v^{-\beta}\chi_{\{v>0\}}, \quad \forall \varepsilon > 0.$$

Thus,

$$\partial_t v - \Delta v + g_\varepsilon(v) \leq f(v,x,t), \quad \text{in } \mathcal{D}'(\Omega \times (0,T)),$$

which implies that v is a sub-solution of equation (P_ε).

By the comparison principle, we get

$$v(x,t) \leq u_\varepsilon(x,t), \quad \text{for a.e } (x,t) \in \Omega \times (0,T).$$

Letting $\varepsilon \to 0$ yields the result. $\qquad \square$

4. Global existence and complete quenching phenomenon

In this part, we study the global existence of solution and the complete quenching phenomenon through proving Theorem 2, and Theorem 3. Since u is the maximal solution, then it suffices to work on u.

Proof of Theorem 2. Let u be the maximal solution of Eq. (1) in $\Omega \times (0,T)$. To prove that u exists globally, we show that u is bounded by a constant not depending on t.

We first remind that $\inf_{x\in\Omega}\{\Phi_{\Omega_0}(x)\} > 0$, for $\Omega \subset\subset \Omega_0$.

Thus, for any $\varepsilon \in \left(0, \dfrac{1}{2}\inf_{x\in\Omega}\{k_0\Phi_{\Omega_0}(x)\}\right)$, we have $g_\varepsilon(k_0\Phi_{\Omega_0}) = k_0^{-\beta}\Phi_{\Omega_0}^{-\beta}$.

Put

$$\mathcal{L}_\varepsilon(v) := v_t - \Delta v + g_\varepsilon(v) - \lambda v^p.$$

Then,

$$\mathcal{L}_\varepsilon(k_0\Phi_{\Omega_0}) = k_0\lambda_1(\Omega_0)\Phi_{\Omega_0}+k_0^{-\beta}\Phi_{\Omega_0}^{-\beta}-\lambda(k_0\Phi_{\Omega_0})^p. \tag{21}$$

By Eq. (6), we observe that $\mathcal{L}_\varepsilon(k_0\Phi_{\Omega_0}) \geq 0$, in $\Omega \times (0,T)$. Therefore, $k_0\Phi_{\Omega_0}$ is a super-solution of equation (P_ε). The strong comparison theorem yields

$$u_\varepsilon(x,t) \leq k_0\Phi_{\Omega_0}(x), \quad \forall(x,t) \in \Omega \times (0,T).$$

Since u_ε is bounded on $\Omega \times (0,T)$, the standard argument allows us to extend the existence of u_ε on $\Omega \times (0, T+\delta_0)$, $\Omega \times (0, T+2\delta_0)$,..., for some $\delta_0 > 0$, thereby proves the global existence of solution u_ε. By the monotonicity of u_ε, u must exist globally, and the conclusion Eq. (7) follows immediately. $\qquad \square$

Next, we will show that for a given $\lambda > 0$, the maximal solution u must vanish identically after a finite time if $\|u_0\|_\infty$ is small enough.

Proof of Theorem 3. Since $\|u_0\|_\infty$ is small enough, we can choose a real number $k_0 > 0$ small as well, and an open bounded domain Ω_0 containing Ω, such that Eq. (6) holds. Thanks to Theorem 2, the maximal solution u exists globally, and it is bounded by $M = k_0\sup_{x\in\Omega}\{\Phi_{\Omega_0}(x)\}$.

Using the test function u to Eq. (1) gives us

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2(t)dx + \int_\Omega \left(|\nabla u(t)|^2 + u^{1-\beta}(t)\right) dx = \lambda\int_\Omega u^{p+1}dx, \quad \forall t > 0.$$

Since $u \leq M$, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2(t)dx + \int_\Omega \left(|\nabla u(t)|^2 + u^{1-\beta}(t)\right)dx$$
$$\leq \lambda M^{p+\beta}\int_\Omega u^{1-\beta}(t)dx, \quad \forall t > 0.$$

Or

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2(t)dx + \int_\Omega |\nabla u(t)|^2 dx + \qquad (22)$$
$$(1-c_M)\int_\Omega u^{1-\beta}(t)dx \leq 0,$$

where $c_M = \lambda M^{p+\beta}$. Note that M is as small as $\|u_0\|_\infty$. Thus, if $\|u_0\|_\infty$ is small enough then we get $(1-c_M) > c_0 > 0$.

It follows from Eq. (22) that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2(t)dx + c_0\int_\Omega \left(|\nabla u(t)|^2 dx + u^{1-\beta}(t)\right)dx \leq 0. \qquad (23)$$

Now, using Garliardo-Nirenberg's inequality [31] yields

$$\|u(t)\|_{L^2(\Omega)} \leq c\|\nabla u(t)\|^\theta_{L^2(\Omega)}\|u(t)\|^{1-\theta}_{L^1(\Omega)}$$
$$= c\left(\int_\Omega |\nabla u(t)|^2 dx\right)^{\frac{\theta}{2}}\left(\int_\Omega u(t)dx\right)^{1-\theta},$$

with $\theta = \frac{N+1}{N+2}$. Thus,

$$\|u(t)\|_{L^2(\Omega)} \leq c\left(\int_\Omega \left(|\nabla u(t)|^2 + u(t)\right)dx\right)^{\frac{\theta}{2}+1-\theta}$$
$$\leq c\left(\int_\Omega \left(|\nabla u(t)|^2 + M^\beta u^{1-\beta}(t)\right)dx\right)^{1-\theta/2}$$
$$\leq c_1\left(\int_\Omega \left(|\nabla u(t)|^2 + u^{1-\beta}(t)\right)dx\right)^{1-\theta/2},$$

with $c_1 = c_1(\beta, M) > 0$.

Then, we obtain

$$\left(\int_\Omega u^2(t)dx\right)^{\frac{N+2}{N+3}} \leq c_2\int_\Omega \left(|\nabla u(t)|^2 + u^{1-\beta}(t)\right)dx, \qquad (24)$$

with $c_2 = c_2(\beta, M, N) > 0$.

By Eq. (23) and Eq. (24), there is a positive constant $c_3 = c_3(\beta, M) > 0$ such that

$$y'(t) + c_3 y^{\frac{N+2}{N+3}}(t) \leq 0, \quad \text{for } t > 0, \qquad (25)$$

with $y(t) = \int_\Omega |u(x,t)|^2 dx$.

If we can show that there exists a time $t_0 \in [0,\infty)$ such that $y(t_0) = 0$, it follows then from Eq. (25) that $y(t) = 0$, for any $t \geq t_0$, thereby proves Theorem 3.

Indeed, we assume a contradiction that $y(t) > 0$, for any $t > 0$. Solving the ordinary differential inequality Eq. (25) yields

$$y^{\frac{1}{N+3}}(t) + c_4 t \leq y^{\frac{1}{N+3}}(0) = \|u_0\|^{\frac{2}{N+3}}_{L^2(\Omega)}, \quad \forall t > 0, \qquad (26)$$

with $c_4 = \frac{c_3}{N+3}$. This leads to a contradiction as t is sufficiently large. Thus, u must quench after a finite time. $\qquad \square$

Similarly, for a given initial data u_0, we also obtain the complete quenching result for the case λ small. Thus, Theorem 4 follows.

Remark 3. *Inequality Eq. (26) implies that the extinction time of u, denoted by $T^\star \leq \dfrac{\|u_0\|^{\frac{2}{N+2}}_{L^2(\Omega)}}{c_4}$.*

5. Non-global existence of solutions

In this section, we study the non-global existence of solutions of Eq. (1). We give the proof of Theorem 5.

Proof of Theorem 5. By multiplying u (resp. u_t) to Eq. (1), we have the following integral equations:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2(x,t)dx = -\int_\Omega \left(|\nabla u(x,t)|^2 + u^{1-\beta}(x,t) - u^{q+1}(x,t)\right)dx, \qquad (27)$$

and

$$\int_0^t\int_\Omega |u_t|^2 dxds + \int_\Omega \left(\frac{1}{2}|\nabla u(t)|^2 + \frac{1}{1-\beta}u^{1-\beta}(t)\right.$$
$$\left. -\frac{1}{q+1}u^{q+1}(t)\right)dx =$$
$$\int_\Omega \left(\frac{1}{2}|\nabla u_0|^2 + \frac{1}{1-\beta}u_0^{1-\beta} - \frac{1}{q+1}u_0^{q+1}\right)dx, \qquad (28)$$

see [30], [32]. By combining Eq. (27) and Eq. (28), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2(x,t)dx = -2E(t)$$
$$+\frac{1+\beta}{1-\beta}\int_\Omega u^{1-\beta}(x,t)dx + \frac{q-1}{q+1}\int_\Omega u^{q+1}(x,t)dx.$$

Since $E(0) \leq 0$, Eq. (28) implies $E(t) \leq 0$, for any $t > 0$. It follows then from the last inequality that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2(x,t)dx \geq \frac{q-1}{q+1}\int_\Omega u^{q+1}dx. \quad (29)$$

By Holder's inequality, we get

$$\int_\Omega u^2 dx \leq \left(\int_\Omega u^{q+1}dx\right)^{\frac{2}{q+1}}|\Omega|^{\frac{q-1}{q+1}}. \quad (30)$$

From Eq. (29) and Eq. (30), we obtain

$$y'(t) \geq Cy^{\frac{q+1}{2}}(t),$$

with $y(t) = \int_\Omega u^2(x,t)dx$, and $C = \frac{2(q-1)}{(q+1)|\Omega|^{\frac{q-1}{2}}}$.

This inequality implies that $y(t) \to +\infty$ as $t \to T_0^-$, with $T_0 = \dfrac{4\|u_0\|_{L^2(\Omega)}^{1-q}}{(q+1)|\Omega|^{\frac{q-1}{2}}}$. $\qquad\square$

6. Simulation results

In this part, we will illustrate our theoretical results with some numerical experiences. In the sequel, we consider Eq. (1) with $p = 2$, $\beta = 0.8$, $I = (0, L)$, and $u_0(x) = x(1 - x/L)$. Our numerical scheme mimics the one in the paper of [10]. Similarly, we use the linear finite elements with mass lumping in a uniform mess for the space variable to discretize our Eq. (1). The reader who is interested in detail can find in [10].

We fix $L = 2$.

With $\lambda = 14$, the maximal solution of Eq. (1) vanished after $t = 0.36$, see Fig. 1. While $\lambda = 15$, the maximal solution of Eq. (1) blows up at $t = 0.33$, see Fig. 2.
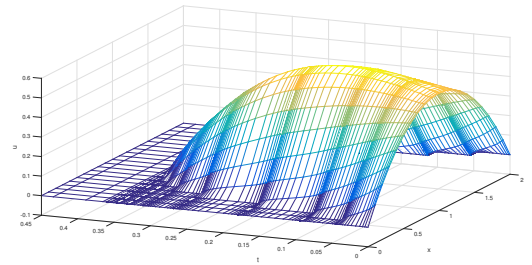

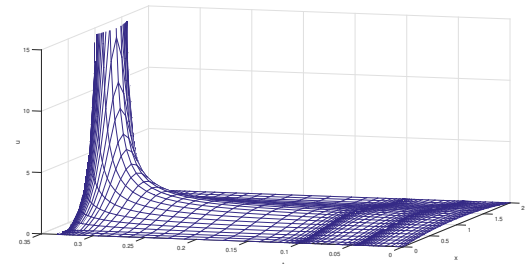**Fig. 1:** Evolution of the maximal solution of Eq. (1).

Fig. 2: Evolution of the maximal solution of Eq. (1).

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