COLLEGE ALGEBRA

Enhanced with Graphing Utilities

Seventh Edition

SULLIVAN • SULLIVAN

BREAK THROUGH
To improving results

Available in MyMathLab® for Your College Algebra Course

Achieve Your Potential

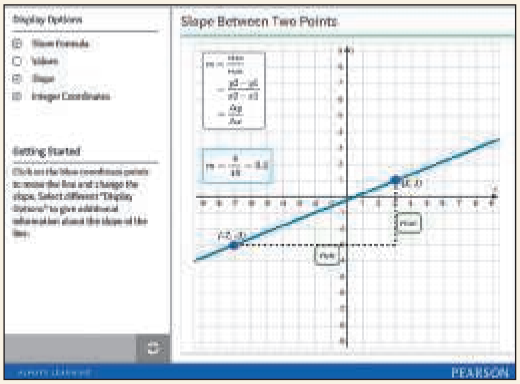
Success in math can make a difference in your life. MyMathLab is a learning experience with resources to help you achieve your potential in this course and beyond. MyMathLab will help you learn the new skills required, and also help you learn the concepts and make connections for future courses and careers.

Visualization and Conceptual Understanding

These MyMathLab resources will help you think visually and connect the concepts.

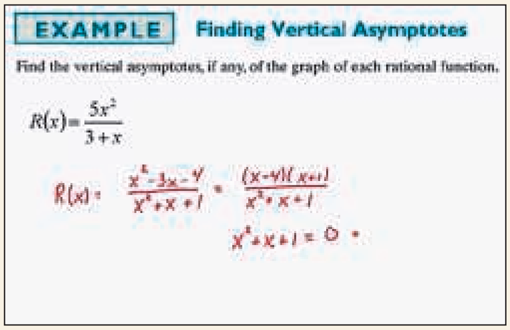
NEW! Guided Visualizations

These engaging interactive figures bring mathematical concepts to life, helping students visualize the concepts through directed explorations and purposeful manipulation. *Guided Visualizations* are assignable in MyMathLab and encourage active learning, critical thinking, and conceptual learning.



Getting Started

Click on the blue coordinate points to move the line and change the slope. Select different "Display Options" to give additional information about the slope of the line.



EXAMPLE Finding Vertical Asymptotes

Find the vertical asymptotes, if any, of the graph of each rational function.

$$R(x)=\frac{5x^2}{3+x}$$

$$R(x)=\frac{x^2-3x-4}{x^2+x+1}=\frac{(x-4)(x+1)}{x^2+x+1}$$

$$x^2+x+1=0$$

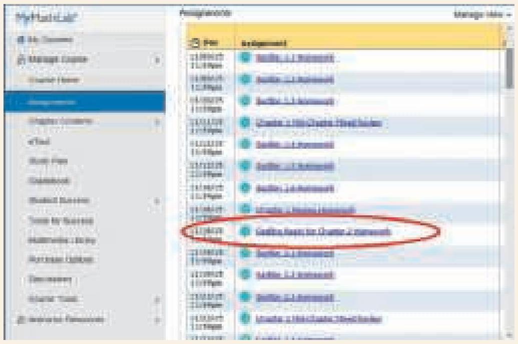
Video Assessment Exercises

Video assessment is tied to key Author in Action videos to check students' conceptual understanding of important math concepts. Students watch a video and work corresponding assessment questions.

www.mymathlab.com

Preparedness and Study Skills

MyMathLab® gives access to many learning resources that refresh knowledge of topics previously learned. *Getting Ready material, Retain Your Knowledge Exercises,* and *Note-Taking Guides* are some of the tools available.

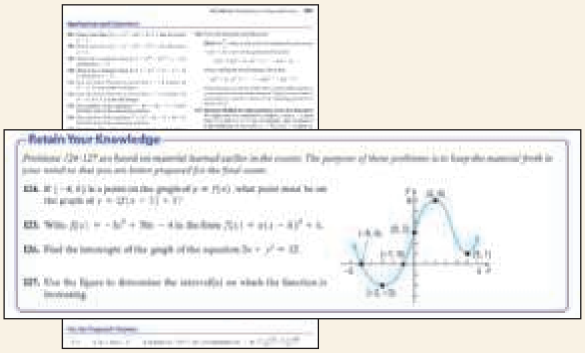


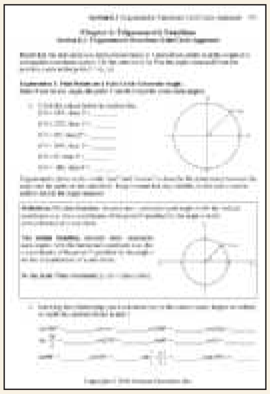
Getting Ready

Students refresh prerequisite topics through skill review quizzes and personalized homework integrated in MyMathLab. With *Getting Ready* content in MyMathLab students get just the help they need to be prepared to learn the new material.

Retain Your Knowledge Exercises

New! *Retain Your Knowledge Exercises* support ongoing review at the course level and help students maintain essential skills.





Guided Lecture Notes

Get help focusing on important concepts with the use of this structured organized note-taking tool. The *Guided Lecture Notes* are available in MyMathLab for download or as a printed student supplement.

For more information on how MyMathLab can help you Achieve Your Potential visit
http://www.pearsonhighered.com/achieve-your-potential/

BREAK THROUGH
To improving results

Get the most out of MyMathLab®

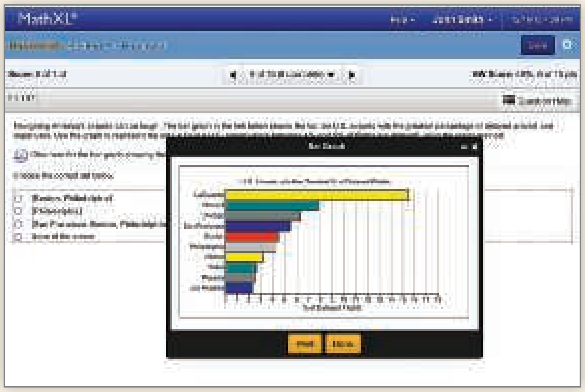
MyMathLab, Pearson's online learning management system, creates personalized experiences for students and provides powerful tools for instructors. With a wealth of tested and proven resources, each course can be tailored to fit your specific needs. Talk to your Pearson Representative about ways to integrate MyMathLab into your course for the best results.

Data-Driven Reporting for Instructors

- MyMathLab's comprehensive online gradebook automatically tracks students' results to tests, quizzes, homework, and work in the study plan.
- The Reporting Dashboard, found under More Gradebook Tools, makes it easier than ever to identify topics where students are struggling, or specific students who may need extra help.

Learning in Any Environment

- Because classroom formats and student needs continually change and evolve, MyMathLab has built-in flexibility to accommodate various course designs and formats.
- With a new, streamlined, mobile-friendly design, students and instructors can access courses from most mobile devices to work on exercises and review completed assignments.



Visit www.mymathlab.com and click Get Trained to make sure you're getting the most out of MyMathLab.

Prepare for Class "Read the Book"

Feature	Description	Benefit	Page(s)
Every Chapter Opener begins with …			
Chapter-Opening Topic & Project	Each chapter begins with a discussion of a topic of current interest and ends with a related project.	In the concluding project, you will apply what you have learned to solve a problem related to the topic.	407, 511
Internet-Based Projects	These projects allow for the integration of spreadsheet technology that you will need to be a productive member of the workforce.	The projects give you an opportunity to collaborate and use mathematics to deal with issues of current interest.	407, 511
Every Section begins with …			
Learning Objectives 2	Each section begins with a list of objectives. Individual objectives also appear in the text where they are covered.	These objectives focus your studying by emphasizing what's most important and where to find it.	428
Sections contain …			
PREPARING FOR THIS SECTION	Most sections begin with a list of key concepts to review, with page numbers.	Ever forget what you've learned? This feature highlights previously learned material to be used in this section. Review it, and you'll always be prepared to move forward.	428
Now Work the **'Are You Prepared?' Problems**	These problems assess whether you have the prerequisite knowledge for the upcoming section.	Not sure you need the Preparing for This Section review? Work the 'Are You Prepared?' problems. If you get one wrong, you'll know exactly what you need to review and where to review it!	428, 439
Now Work PROBLEMS	These follow most examples and direct you to a related exercise.	We learn best by doing. You'll solidify your understanding of examples if you try a similar problem right away, to be sure you understand what you've just read.	437
WARNING	Warnings are provided in the text.	These point out common mistakes and help you avoid them.	462
Explorations and **Seeing the Concept**	These graphing utility activities foreshadow a concept or reinforce a concept just presented.	You will obtain a deeper and more intuitive understanding of theorems and definitions.	377, 434
In Words	This feature provides alternative descriptions of select definitions and theorems.	Does math ever look foreign to you? This feature translates math into plain English.	430
Calculus	This symbol appears next to information essential for the study of calculus.	Pay attention–if you spend extra time now, you'll do better later!	236, 238, 373
SHOWCASE EXAMPLES	These examples provide "how to" instruction by offering a guided, step-by-step approach to solving a problem.	With each step presented on the left and the mathematics displayed on the right, you can immediately see how each step is employed.	342–343
Model It! Examples and Problems	These examples and problems require you to build a mathematical model from either a verbal description or data. The homework Model It! problems are marked by purple problem numbers.	It is rare for a problem to come in the form "*Solve the following equation.*" Rather, the equation must be developed based on an explanation of the problem. These problems require you to develop models that will enable you to describe the problem mathematically and suggest a solution to the problem.	453, 482

Practice "Work the Problems"

Feature	Description	Benefit	Page(s)
'Are You Prepared?' Problems	These problems assess your retention of the prerequisite material. Answers are given at the end of the section exercises. This feature is related to the Preparing for This Section feature.	Do you always remember what you've learned? Working these problems is the best way to find out. If you get one wrong, you'll know exactly what you need to review and where to review it!	428, 439
Concepts and Vocabulary	These short-answer questions, mainly fill-in-the-blank, multiple-choice, and true/false items, assess your understanding of key definitions and concepts in the current section.	It is difficult to learn math without knowing the language of mathematics. These problems test your understanding of the formulas and vocabulary.	440
Skill Building	Correlated with section examples, these problems provide straightforward practice.	It's important to dig in and develop your skills. These problems give you ample opportunity to do so.	440–442
Mixed Practice	These problems offer comprehensive assessment of the skills learned in the section by asking problems related to more than one concept or objective. These problems may also require you to utilize skills learned in previous sections.	Learning mathematics is a building process. Many concepts build on each other and are related. These problems help you see how mathematics builds on itself and how the concepts are linked together.	442
Applications and Extensions	These problems allow you to apply your skills to real-world problems. They also enable you to extend concepts learned in the section.	You will see that the material learned within the section has many uses in everyday life.	442–444
Explaining Concepts: Discussion and Writing	"Discussion and Writing" problems are colored red. They support class discussion, verbalization of mathematical ideas, and writing and research projects.	To verbalize an idea, or to describe it clearly in writing, shows real understanding. These problems nurture that understanding. Many are challenging, but you'll get out what you put in.	445
NEW! Retain Your Knowledge	These problems allow you to practice content learned earlier in the course.	Remembering how to solve all the different kinds of problems that you encounter throughout the course is difficult. This practice helps you remember previously learned skills.	445
Now Work PROBLEMS	Many examples refer you to a related homework problem. These related problems are marked by a pencil and orange numbers.	If you get stuck while working problems, look for the closest Now Work problem, and refer to the related example to see if it helps.	429, 437, 438, 441
Review Exercises	Every chapter concludes with a comprehensive list of exercises to practice. Use the list of objectives to determine what objective and examples correspond to each problem.	Work these problems to ensure that you understand all the skills and concepts employed in the chapter. Think of it as a comprehensive review of the chapter. All answers to Chapter Review problems appear in the back of the text.	506–509

Review "Study for Quizzes and Tests"

Feature	Description	Benefit	Page(s)
The Chapter Review at the end of each chapter contains …			
Things to Know	A detailed list of important theorems, formulas, and definitions from the chapter.	Review these and you'll know the most important material in the chapter!	504–505
You Should Be Able to …	A complete list of objectives by section and, for each, examples that illustrate the objective, and practice exercises that test your understanding of the objective.	Do the recommended exercises and you'll have mastered the key material. If you get something wrong, go back and work through the example listed, and try again.	505–506
Review Exercises	These provide comprehensive review and practice of key skills, matched to the Learning Objectives for each section.	Practice makes perfect. These problems combine exercises from all sections, giving you a comprehensive review in one place.	506–509
Chapter Test	About 15–20 problems that can be taken as a Chapter Test. Be sure to take the Chapter Test under test conditions—no notes!	Be prepared. Take the sample practice test under test conditions. This will get you ready for your instructor's test. If you get a problem wrong, you can watch the Chapter Test Prep Video.	509
Cumulative Review	These problem sets appear at the end of each chapter, beginning with Chapter 2. They combine problems from previous chapters, providing an ongoing cumulative review. When you use them in conjunction with the Retain Your Knowledge problems, you will be ready for the final exam.	These problem sets are really important. Completing them will ensure that you are not forgetting anything as you go. This will go a long way toward keeping you primed for the final exam.	510
Chapter Projects	The Chapter Projects apply to what you've learned in the chapter. Additional projects are available on the Instructor's Resource Center (IRC).	The Chapter Projects give you an opportunity to apply what you've learned in the chapter to the opening topic. If your instructor allows, these make excellent opportunities to work in a group, which is often the best way of learning math.	511
Internet-Based Projects	In selected chapters, a Web-based project is given.	These projects give you an opportunity to collaborate and use mathematics to deal with issues of current interest by using the Internet to research and collect data.	511

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Enhanced with Graphing Utilities

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In Memory of Mary…
Wife and Mother

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Three Distinct Series

Students have different goals, learning styles, and levels of preparation. Instructors have different teaching philosophies, styles, and techniques. Rather than write one series to fit all, the Sullivans have written three distinct series. All share the same goal—to develop a high level of mathematical understanding and an appreciation for the way mathematics can describe the world around us. The manner of reaching that goal, however, differs from series to series.

Enhanced with Graphing Utilities Series, Seventh Edition

This series provides a thorough integration of graphing utilities into topics, allowing students to explore mathematical concepts and encounter ideas usually studied in later courses. Using technology, the approach to solving certain problems differs from the Contemporary or Concepts through Functions Series, while the emphasis on understanding concepts and building strong skills does not: *College Algebra, Algebra & Trigonometry, Precalculus*.

Contemporary Series, Tenth Edition

The Contemporary Series is the most traditional in approach, yet modern in its treatment of precalculus mathematics. Graphing utility coverage is optional and can be included or excluded at the discretion of the instructor: *College Algebra, Algebra & Trigonometry, Trigonometry: A Unit Circle Approach, Precalculus.*

Concepts through Functions Series, Third Edition

This series differs from the others, utilizing a functions approach that serves as the organizing principle tying concepts together. Functions are introduced early in various formats. This approach supports the Rule of Four, which states that functions are represented symbolically, numerically, graphically, and verbally. Each chapter introduces a new type of function and then develops all concepts pertaining to that particular function. The solutions of equations and inequalities, instead of being developed as stand-alone topics, are developed in the context of the underlying functions. Graphing utility coverage is optional and can be included or excluded at the discretion of the instructor: *College Algebra; Precalculus, with a Unit Circle Approach to Trigonometry; Precalculus, with a Right Triangle Approach to Trigonometry*.

The Enhanced with Graphing Utilities Series

College Algebra

This text provides an approach to college algebra that completely integrates graphing technology without sacrificing mathematical analysis and conceptualization. The text has three chapters of review material preceding the chapters on functions. After completing this text, a student will be prepared for trigonometry, finite mathematics, and business calculus.

Algebra & Trigonometry

This text contains all the material in *College Algebra*, but it also develops the trigonometric functions using a right triangle approach and shows how that approach is related to the unit circle approach. Graphing techniques are emphasized, including a thorough discussion of polar coordinates, parametric equations, and conics using polar coordinates. Graphing calculator usage is integrated throughout. After completing this text, a student will be prepared for finite mathematics, business calculus, and engineering calculus.

Precalculus

This text contains one review chapter before covering the traditional precalculus topics of functions and their graphs, polynomial and rational functions, and exponential and logarithmic functions. The trigonometric functions are introduced using a unit circle approach and show how it is related to the right triangle approach. Graphing techniques are emphasized, including a thorough discussion of polar coordinates, parametric equations, and conics using polar coordinates. Graphing calculator usage is integrated throughout. The final chapter provides an introduction to calculus, with a discussion of the limit, the derivative, and the integral of a function. After completing this text, a student will be prepared for finite mathematics, business calculus, and engineering calculus.

Preface to the Instructor

As professors at an urban university and a community college, Michael Sullivan and Michael Sullivan III are aware of the varied needs of College Algebra students. Such students range from those who have little mathematical background and are fearful of mathematics courses, to those with a strong mathematical education and a high level of motivation. For some of your students, this will be their last course in mathematics, whereas others will further their mathematical education. We have written this text with both groups in mind.

As a teacher, and as an author of precalculus, engineering calculus, finite mathematics, and business calculus texts, Michael Sullivan understands what students must know if they are to be focused and successful in upper-level math courses. However, as a father of four, he also understands the realities of college life. As an author of a developmental mathematics series, Michael's son and co-author, Michael Sullivan III, understands the trepidations and skills that students bring to the College Algebra course. As the father of a current college student, Michael III realizes that today's college students demand a variety of media to support their education. This text addresses that demand by providing technology and video support that enhances understanding without sacrificing math skills. Together, both authors have taken great pains to ensure that the text offers solid, student-friendly examples and problems, as well as a clear and seamless writing style.

A tremendous benefit of authoring a successful series is the broad-based feedback we receive from teachers and students. We are sincerely grateful for their support. Virtually every change in this edition is the result of their thoughtful comments and suggestions. We are confident that, building on the success of the first six editions and incorporating many of these suggestions, we have made *College Algebra Enhanced with Graphing Utilities,* 7th Edition, an even better tool for learning and teaching. We continue to encourage you to share with us your experiences teaching from this text.

Features in the Seventh Edition

A descriptive list of the many special features of *College Algebra* can be found in the front of this text.

This list places the features in their proper context, as building blocks of an overall learning system that has been carefully crafted over the years to help students get the most out of the time they put into studying. Please take the time to review this and to discuss it with your students at the beginning of your course. When students utilize these features, they are more successful in the course.

New to the Seventh Edition

- **Retain Your Knowledge** This new category of problems in the exercise set is based on the article "To Retain New Learning, Do the Math" published in the *Edurati Review*. In this article, Kevin Washburn suggests that "the more students are required to recall new content or skills, the better their memory will be." It is frustrating when students cannot recall skills learned earlier in the course. To alleviate this recall problem, we have created "Retain Your Knowledge" problems. These are problems considered to be "final exam material" that students can use to maintain their skills. All the answers to these problems appear in the back of the text, and all are programmed in MyMathLab.
- **Guided Lecture Notes** Ideal for online, emporium/redesign courses, inverted classrooms, or traditional lecture classrooms. These lecture notes help students take thorough, organized, and understandable notes as they watch the Author in Action videos. They ask students to complete definitions, procedures, and examples based on the content of the videos and text. In addition, experience suggests that students learn by doing and understanding the why/how of the concept or property. Therefore, many sections have an exploration activity to motivate student learning. These explorations introduce the topic and/or connect it to either a real-world application or a previous section. For example, when the vertical-line test is discussed in Section 3.2, after the theorem statement, the notes ask the students to explain why the vertical-line test works by using the definition of a function. This challenge helps students process the information at a higher level of understanding.
- **Illustrations** Many of the figures now have captions to help connect the illustrations to the explanations in the body of the text.
- **TI Screen Shots** In this edition we have replaced all the screen shots from the sixth edition with screen shots using TI-84 Plus C. These updated screen shots help students visualize concepts clearly and help make stronger connections among equations, data, and graphs in full color.
- **Exercise Sets** All the exercises in the text have been reviewed and analyzed for this edition, some have been removed, and new ones have been added. All time-sensitive problems have been updated to the most recent information available. The problem sets remain classified according to purpose.

  The ***'Are You Prepared?'*** problems have been improved to better serve their purpose as a just-in-time review of concepts that the student will need to apply in the upcoming section.

  The ***Concepts and Vocabulary*** problems have been expanded and now include multiple-choice exercises. Together with the fill-in-the-blank and true/false problems, these exercises have been written to serve as reading quizzes.

Skill Building problems develop the student's computational skills with a large selection of exercises that are directly related to the objectives of the section. ***Mixed Practice*** problems offer a comprehensive assessment of skills that relate to more than one objective. Often these require skills learned earlier in the course.

Applications and Extensions problems have been updated. Further, many new application-type exercises have been added, especially ones involving information and data drawn from sources the student will recognize, to improve relevance and timeliness.

The ***Explaining Concepts: Discussion and Writing*** exercises have been improved and expanded to provide more opportunity for classroom discussion and group projects.

New to this edition, ***Retain Your Knowledge*** exercises consist of a collection of four problems in each exercise set that are based on material learned earlier in the course. They serve to keep information that has already been learned "fresh" in the mind of the student. Answers to all these problems appear in the Student Edition.

The ***Review Exercises*** in the Chapter Review have been streamlined, but they remain tied to the clearly expressed objectives of the chapter. Answers to all these problems appear in the Student Edition.

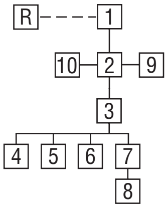
- **Annotated Instructor's Edition** As a guide, the author's suggestions for homework assignments are indicated by a blue underscore below the problem number. These problems are assignable in MyMathLab.

Content Changes in the Seventh Edition

- **Section 3.1** The objective Find the Difference Quotient of a Function has been added.
- **Section 5.2** The objective Use Descartes' Rule of Signs has been included.
- **Section 5.2** The theorem Bounds on the Zeros of a Polynomial Function is now based on the traditional method of using synthetic division.
- **Section 5.5** Content has been added that discusses the role of multiplicity of the zeros of the denominator of a rational function as it relates to the graph near a vertical asymptote.

Using the Seventh Edition Effectively with Your Syllabus

To meet the varied needs of diverse syllabi, this text contains more content than is likely to be covered in an College Algebra course. As the chart illustrates, this text has been organized with flexibility of use in mind. Within a given chapter, certain sections are optional (see the details that follow the accompanying figure) and can be omitted without loss of continuity.



Chapter R Review

This chapter consists of review material. It may be used as the first part of the course or later as a just-in-time review when the content is required. Specific references to this chapter occur throughout the text to assist in the review process.

Chapter 1 Equations and Inequalities

Primarily a review of intermediate algebra topics, this material is a prerequisite for later topics. The coverage of complex numbers and quadratic equations with a negative discriminant is optional and may be postponed or skipped entirely without loss of continuity.

Chapter 2 Graphs

This chapter lays the foundation for functions. Section 2.4 is optional.

Chapter 3 Functions and Their Graphs

This is perhaps the most important chapter. Section 3.6 is optional.

Chapter 4 Linear and Quadratic Functions

Topic selection depends on your syllabus. Sections 4.2 and 4.4 may be omitted without loss of continuity.

Chapter 5 Polynomial and Rational Functions

Topic selection depends on your syllabus.

Chapter 6 Exponential and Logarithmic Functions

Sections 6.1–6.6 follow in sequence. Sections 6.7, 6.8, and 6.9 are optional.

Chapter 7 Analytic Geometry

Sections 7.1–7.4 follow in sequence.

Chapter 8 Systems of Equations and Inequalities

Sections 8.2–8.7 may be covered in any order, but each requires Section 8.1. Section 8.8 requires Section 8.7.

Chapter 9 Sequences; Induction; The Binomial Theorem

There are three independent parts: Sections 9.1–9.3, Section 9.4, and Section 9.5.

Chapter 10 Counting and Probability

The sections follow in sequence.

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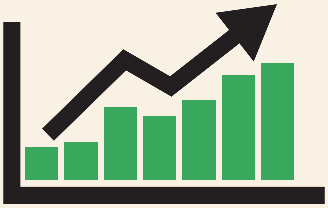

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To improving results

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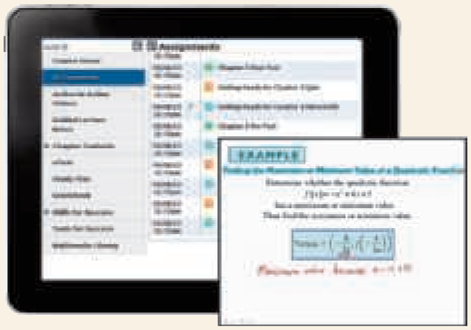

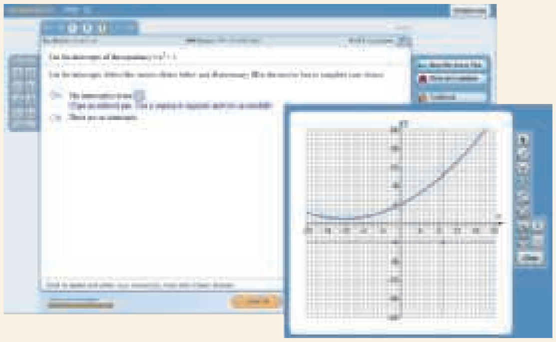

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Resources for Success

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Includes fully worked solutions to all exercises in the text.

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This guide includes additional examples and helpful teaching tips, by section.

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Applications Index

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To the Student

As you begin, you may feel anxious about the number of theorems, definitions, procedures, and equations you encounter. You may wonder if you can learn it all in time. Don't worry, your concerns are normal. This text was written with you in mind. If you attend class, work hard, and read and study effectively, you will build the knowledge and skills you need to be successful. Here's how you can use the text to your benefit.

Read Carefully

When you get busy, it's easy to skip reading and go right to the problems. Don't! The text provides a large number of examples and clear explanations to help you break down the mathematics into easy-to-understand steps. Reading will provide you with a clearer understanding, beyond simple memorization. Read before class (not after) so you can ask questions about anything you didn't understand. You'll be amazed at how much more you'll get out of class when you do this.

Use the Features

We use many different methods in the classroom to communicate. Those methods, when incorporated into the text, are called "features." The features serve many purposes, from supplying a timely review of material you learned before (just when you need it), to providing organized review sessions to help you prepare for quizzes and tests. Take advantage of the features and you will master the material.

To make this easier, we've provided a brief guide to getting the most from this book. Refer to the "Prepare for Class," "Practice," and "Review" guidelines on pages i–iii. Spend fifteen minutes reviewing the guide and familiarizing yourself with the features by flipping to the page numbers provided. Then, as you read, use them. This is the best way to make the most of your text.

Please do not hesitate to contact us, through Pearson Education, with any questions, comments, or suggestions about ways to improve this text. We look forward to hearing from you, and good luck with all of your studies.

Best Wishes!

Michael Sullivan
Michael Sullivan III

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R Review

A Look Ahead •••

Chapter R, as the title states, contains review material. Your instructor may choose to cover all or part of it as a regular chapter at the beginning of your course or later as a just-in-time review when the content is required. Regardless, when information in this chapter is needed, a specific reference to this chapter will be made so you can review.

Outline

PREPARING FOR THIS TEXT *Before getting started, read "To the Student" at the front of this text.*

OBJECTIVES
1 Work with Sets (p. 2)
2 Classify Numbers (p. 4)
3 Evaluate Numerical Expressions (p. 8)
4 Work with Properties of Real Numbers (p. 10)

1 Work with Sets

A **set** is a well-defined collection of distinct objects. The objects of a set are called its **elements**. By **well-defined**, we mean that there is a rule that enables us to determine whether a given object is an element of the set. If a set has no elements, it is called the **empty set**, or **null set**, and is denoted by the symbol $\varnothing$.

For example, the set of **digits** consists of the collection of numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. If we use the symbol D to denote the set of digits, then we can write

$$D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

In this notation, the braces $\{\quad\}$ are used to enclose the objects, or **elements**, in the set. This method of denoting a set is called the **roster method**. A second way to denote a set is to use **set-builder notation**, where the set D of digits is written as

$$D = \underbrace{\overset{}{\{} \qquad \overset{}{x} \qquad \overset{}{|}}_{\uparrow\qquad\uparrow\qquad\uparrow} \quad x \text{ is a digit}\}$$

Read as "*D* is the set of all *x* such that *x* is a digit."

EXAMPLE 1 Using Set-builder Notation and the Roster Method

(a) $E = \{x|x \text{ is an even digit}\} = \{0, 2, 4, 6, 8\}$
(b) $O = \{x|x \text{ is an odd digit}\} = \{1, 3, 5, 7, 9\}$

Because the elements of a set are distinct, we never repeat elements. For example, we would never write $\{1, 2, 3, 2\}$; the correct listing is $\{1, 2, 3\}$. Because a set is a collection, the order in which the elements are listed is immaterial. $\{1, 2, 3\}$, $\{1, 3, 2\}$, $\{2, 1, 3\}$, and so on, all represent the same set.

If every element of a set A is also an element of a set B, then A is a **subset** of B, which is denoted $A \subseteq B$. If two sets A and B have the same elements, then A **equals** B, which is denoted $A = B$.

For example, $\{1, 2, 3\} \subseteq \{1, 2, 3, 4, 5\}$ and $\{1, 2, 3\} = \{2, 3, 1\}$.

DEFINITION

If A and B are sets, the **intersection** of A with B, denoted $A \cap B$, is the set consisting of elements that belong to both A and B. The **union** of A with B, denoted $A \cup B$, is the set consisting of elements that belong to either A or B, or both.

EXAMPLE 2 Finding the Intersection and Union of Sets

Let $A = \{1, 3, 5, 8\}$, $B = \{3, 5, 7\}$, and $C = \{2, 4, 6, 8\}$. Find:

(a) $A \cap B$ (b) $A \cup B$ (c) $B \cap (A \cup C)$

Solution

(a) $A \cap B = \{1, 3, 5, 8\} \cap \{3, 5, 7\} = \{3, 5\}$

(b) $A \cup B = \{1, 3, 5, 8\} \cup \{3, 5, 7\} = \{1, 3, 5, 7, 8\}$

(c) $B \cap (A \cup C) = \{3, 5, 7\} \cap [\{1, 3, 5, 8\} \cup \{2, 4, 6, 8\}]$
$= \{3, 5, 7\} \cap \{1, 2, 3, 4, 5, 6, 8\} = \{3, 5\}$ ■

Now Work PROBLEM 15

Usually, in working with sets, we designate a **universal set** U, the set consisting of all the elements that we wish to consider. Once a universal set has been designated, we can consider elements of the universal set not found in a given set.

DEFINITION

If A is a set, the **complement** of A, denoted $\overline{A}$, is the set consisting of all the elements in the universal set that are not in A.* ■

EXAMPLE 3 Finding the Complement of a Set

If the universal set is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and if $A = \{1, 3, 5, 7, 9\}$, then $\overline{A} = \{2, 4, 6, 8\}$. ■

It follows from the definition of complement that $A \cup \overline{A} = U$ and $A \cap \overline{A} = \varnothing$. Do you see why?

Now Work PROBLEM 19

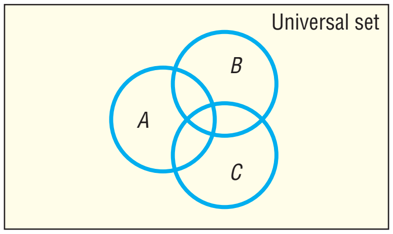

Figure 1 Venn diagram

It is often helpful to draw pictures of sets. Such pictures, called **Venn diagrams**, represent sets as circles enclosed in a rectangle, which represents the universal set. Such diagrams often help us to visualize various relationships among sets. See Figure 1.

If we know that $A \subseteq B$, we might use the Venn diagram in Figure 2(a). If we know that A and B have no elements in common—that is, if $A \cap B = \varnothing$—we might use the Venn diagram in Figure 2(b). The sets A and B in Figure 2(b) are said to be **disjoint**.

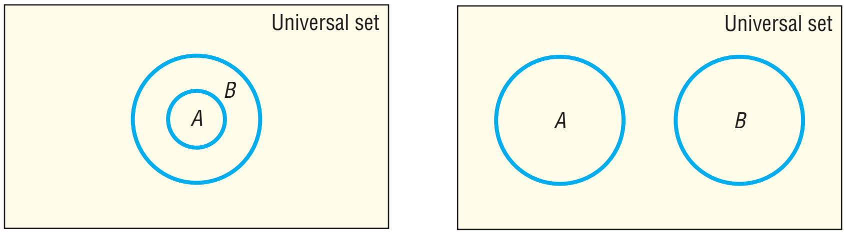

Figure 2 (a) $A \subseteq B$ subset (b) $A \cap B = \varnothing$ disjoint sets

Figures 3(a), 3(b), and 3(c) use Venn diagrams to illustrate the definitions of intersection, union, and complement, respectively.

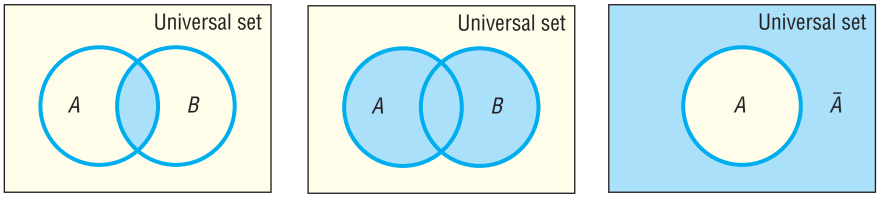

Figure 3 (a) $A \cap B$ intersection (b) $A \cup B$ union (c) $\overline{A}$ complement

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*Some texts use the notation A' for the complement of A.

2 Classify Numbers

It is helpful to classify the various kinds of numbers that we deal with as sets. The **counting numbers**, or **natural numbers**, are the numbers in the set $\{1, 2, 3, 4, \dots\}$. (The three dots, called an **ellipsis**, indicate that the pattern continues indefinitely.) As their name implies, these numbers are often used to count things. For example, there are 26 letters in our alphabet; there are 100 cents in a dollar. The **whole numbers** are the numbers in the set $\{0, 1, 2, 3, \dots\}$—that is, the counting numbers together with 0. The set of counting numbers is a subset of the set of whole numbers.

DEFINITION

The **integers** are the set of numbers $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$.

These numbers are useful in many situations. For example, if your checking account has \$10 in it and you write a check for \$15, you can represent the current balance as −\$5.

Each time we expand a number system, such as from the whole numbers to the integers, we do so in order to be able to handle new, and usually more complicated, problems. The integers enable us to solve problems requiring both positive and negative counting numbers, such as profit/loss, height above/below sea level, temperature above/below 0°F, and so on.

But integers alone are not sufficient for *all* problems. For example, they do not answer the question "What part of a dollar is 38 cents?" To answer such a question, we enlarge our number system to include *rational numbers*. For example, $\frac{38}{100}$ answers the question "What part of a dollar is 38 cents?"

DEFINITION

A **rational number** is a number that can be expressed as a quotient $\frac{a}{b}$ of two integers. The integer a is called the **numerator**, and the integer b, which cannot be 0, is called the **denominator**. The rational numbers are the numbers in the set $\left\{x \middle| x = \frac{a}{b}, \text{ where } a, b \text{ are integers and } b \neq 0\right\}$.

Examples of rational numbers are $\frac{3}{4}, \frac{5}{2}, \frac{0}{4}, -\frac{2}{3}$, and $\frac{100}{3}$. Since $\frac{a}{1} = a$ for any integer a, it follows that the set of integers is a subset of the set of rational numbers.

Rational numbers may be represented as **decimals**. For example, the rational numbers $\frac{3}{4}, \frac{5}{2}, -\frac{2}{3}$, and $\frac{7}{66}$ may be represented as decimals by merely carrying out the indicated division:

$$\frac{3}{4} = 0.75 \qquad \frac{5}{2} = 2.5 \qquad -\frac{2}{3} = -0.666\ldots = -0.\overline{6} \qquad \frac{7}{66} = 0.1060606\ldots = 0.1\overline{06}$$

Notice that the decimal representations of $\frac{3}{4}$ and $\frac{5}{2}$ terminate, or end. The decimal representations of $-\frac{2}{3}$ and $\frac{7}{66}$ do not terminate, but they do exhibit a pattern of repetition. For $-\frac{2}{3}$, the 6 repeats indefinitely, as indicated by the bar over the 6; for $\frac{7}{66}$, the block 06 repeats indefinitely, as indicated by the bar over the 06. It can be shown that every rational number may be represented by a decimal that either terminates or is nonterminating with a repeating block of digits, and vice versa.

On the other hand, some decimals do not fit into either of these categories. Such decimals represent **irrational numbers**. Every irrational number may be represented by a decimal that neither repeats nor terminates. In other words, irrational numbers cannot be written in the form $\frac{a}{b}$, where a, b are integers and $b \neq 0$.

Irrational numbers occur naturally. For example, consider the isosceles right triangle whose legs are each of length 1. See Figure 4. The length of the hypotenuse is $\sqrt{2}$, an irrational number.

Also, the number that equals the ratio of the circumference C to the diameter d of any circle, denoted by the symbol π (the Greek letter pi), is an irrational number. See Figure 5.

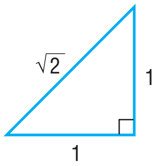


Figure 4

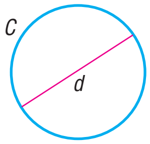


Figure 5 $\pi = \dfrac{C}{d}$

DEFINITION

The set of **real numbers** is the union of the set of rational numbers with the set of irrational numbers.

Figure 6 shows the relationship of various types of numbers.*

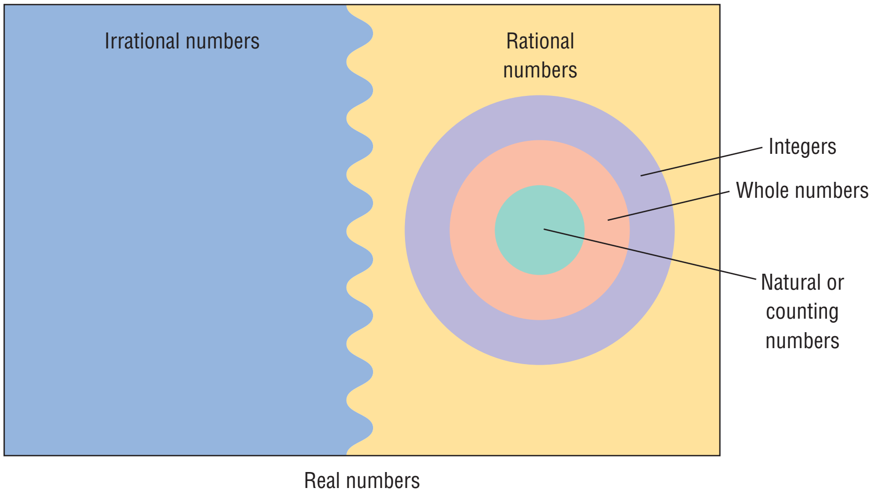


Figure 6

EXAMPLE 4 Classifying the Numbers in a Set

List the numbers in the set

$$\left\{-3, \frac{4}{3}, 0.12, \sqrt{2}, \pi, 10, 2.151515\ldots\text{(where the block 15 repeats)}\right\}$$

that are

(a) Natural numbers (b) Integers (c) Rational numbers
(d) Irrational numbers (e) Real numbers

Solution

(a) 10 is the only natural number.
(b) -3 and 10 are integers.
(c) $-3, 10, \frac{4}{3}, 0.12$, and $2.151515\ldots$ are rational numbers.
(d) $\sqrt{2}$ and π are irrational numbers.
(e) All the numbers listed are real numbers.

Now Work PROBLEM 25

---

*The set of real numbers is a subset of the set of complex numbers. We discuss complex numbers in Chapter 1, Section 1.4.

Approximations

Every decimal may be represented by a real number (either rational or irrational), and every real number may be represented by a decimal.

In practice, the decimal representation of an irrational number is given as an approximation. For example, using the symbol ≈ (read as "approximately equal to"), we can write

$$\sqrt{2} \approx 1.4142 \qquad \pi \approx 3.1416$$

In approximating decimals, we either *round off* or *truncate* to a given number of decimal places.* The number of places establishes the location of the *final digit* in the decimal approximation.

> **Truncation:** Drop all of the digits that follow the specified final digit in the decimal.
>
> **Rounding:** Identify the specified final digit in the decimal. If the next digit is 5 or more, add 1 to the final digit; if the next digit is 4 or less, leave the final digit as it is. Then truncate following the final digit.

EXAMPLE 5 Approximating a Decimal to Two Places

Approximate 20.98752 to two decimal places by

(a) Truncating
(b) Rounding

Solution For 20.98752, the final digit is 8, since it is two decimal places from the decimal point.

(a) To truncate, we remove all digits following the final digit 8. The truncation of 20.98752 to two decimal places is 20.98.

(b) The digit following the final digit 8 is the digit 7. Since 7 is 5 or more, we add 1 to the final digit 8 and truncate. The rounded form of 20.98752 to two decimal places is 20.99. ■

EXAMPLE 6 Approximating a Decimal to Two and Four Places

Number	Rounded to Two Decimal Places	Rounded to Four Decimal Places	Truncated to Two Decimal Places	Truncated to Four Decimal Places
(a) 3.14159	3.14	3.1416	3.14	3.1415
(b) 0.056128	0.06	0.0561	0.05	0.0561
(c) 893.46125	893.46	893.4613	893.46	893.4612

■

Now Work PROBLEM 29

Significant Digits

There are two types of numbers—*exact* and *approximate*. **Exact numbers** are numbers whose value is known with 100% certainty and accuracy. For example, there are 12 donuts in a dozen donuts, or there are 50 states in the United States.

---

*Sometimes we say "correct to a given number of decimal places" instead of "truncated."

Approximate numbers are numbers whose value is not known with 100% certainty or whose measurement is inexact. When values are determined from measurements they are typically approximate numbers because the exact measurement is limited by the accuracy of the measuring device and the skill of the individual obtaining the measurement. The **number of significant digits** in a number represents the level of accuracy of the measurement.

The following rules are used to determine the number of significant digits in approximate numbers.

The Number of Significant Digits

- Leading zeros are not significant. For example, 0.0034 has two significant digits.
- Embedded zeros are significant. For example, 208 has three significant digits.
- Trailing zeros are significant only if the decimal point is specified. For example, 2800 has two significant digits. However, if we specify the measurement is accurate to the ones digit, then 2800 has four significant digits.

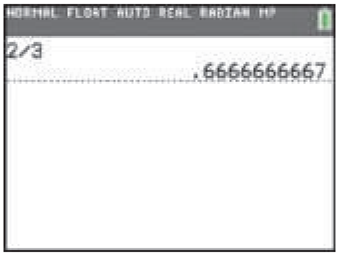
When performing computations with approximate numbers, it is important not to report the result with more accuracy than the measurements used in the computation.

When performing computations using significant digits, proceed with the computation as you normally would, then round the final answer to the number of significant digits as the least accurately known number. For example, suppose we want to find the area of a rectangle whose width is 1.94 inches (three significant digits) and whose length is 2.7 inches (two significant digits). Because the length has two significant digits, we report the area to two significant digits. The area, (1.94 inches) (2.7 inches) $=$ 5.238 square inches, can only be written to two significant digits and is reported as 5.2 square inches.

Calculators

Calculators are incapable of displaying decimals that contain a large number of digits. For example, some calculators are capable of displaying only eight digits. When a number requires more than eight digits, the calculator either truncates or rounds. To see how your calculator handles decimals, divide 2 by 3. How many digits do you see? Is the last digit a 6 or a 7? If it is a 6, your calculator truncates; if it is a 7, your calculator rounds.

There are different kinds of calculators. An **arithmetic** calculator can only add, subtract, multiply, and divide numbers; therefore, this type is not adequate for this course. **Scientific** calculators have all the capabilities of arithmetic calculators and contain **function keys** labeled ln, log, sin, cos, tan, x^y, inv, and so on. **Graphing** calculators have all the capabilities of scientific calculators and contain a screen on which graphs can be displayed. As you proceed through this text, you will discover how to use many of the function keys.



```
2/3
                .6666666667
```

Figure 7

Figure 7 shows $\frac{2}{3}$ on a TI-84 Plus C graphing calculator. How many digits are displayed? Does a TI-84 Plus C round or truncate? What does your calculator do?

Operations

In algebra, we use letters such as x, y, a, b, and c to represent numbers. The symbols used in algebra for the operations of addition, subtraction, multiplication, and division are $+, -, \cdot$, and $/$. The words used to describe the results of these operations are **sum**, **difference, product**, and **quotient**. Table 1 summarizes these ideas.

Table 1

Operation	Symbol	Words
Addition	$a + b$	Sum: a plus b
Subtraction	$a - b$	Difference: a minus b
Multiplication	$a \cdot b$, $(a) \cdot b$, $a \cdot (b)$, $(a) \cdot (b)$, ab, $(a)b$, $a(b)$, $(a)(b)$	Product: a times b
Division	a/b or $\frac{a}{b}$	Quotient: a divided by b

In algebra, we generally avoid using the multiplication sign × and the division sign ÷ so familiar in arithmetic. Notice also that when two expressions are placed next to each other without an operation symbol, as in ab, or in parentheses, as in $(a)(b)$, it is understood that the expressions, called **factors**, are to be multiplied.

We also prefer not to use mixed numbers in algebra. When mixed numbers are used, addition is understood; for example, $2\frac{3}{4}$ means $2 + \frac{3}{4}$. In algebra, use of a mixed number may be confusing because the absence of an operation symbol between two terms is generally taken to mean multiplication. The expression $2\frac{3}{4}$ is therefore written instead as 2.75 or as $\frac{11}{4}$.

The symbol $=$, called an **equal sign** and read as "equals" or "is," is used to express the idea that the number or expression on the left of the equal sign is equivalent to the number or expression on the right.

EXAMPLE 7 **Writing Statements Using Symbols**

(a) The sum of 2 and 7 equals 9. In symbols, this statement is written as $2 + 7 = 9$.

(b) The product of 3 and 5 is 15. In symbols, this statement is written as $3 \cdot 5 = 15$.

■

Now Work PROBLEM 41

3 Evaluate Numerical Expressions

Consider the expression $2 + 3 \cdot 6$. It is not clear whether we should add 2 and 3 to get 5, and then multiply by 6 to get 30; or first multiply 3 and 6 to get 18, and then add 2 to get 20. To avoid this ambiguity, we have the following agreement.

In Words
Multiply first, then add.

We agree that whenever the two operations of addition and multiplication separate three numbers, the multiplication operation will always be performed first, followed by the addition operation.

For $2 + 3 \cdot 6$, then, we have

$$2 + 3 \cdot 6 = 2 + 18 = 20$$

EXAMPLE 8 **Finding the Value of an Expression**

Evaluate each expression.

(a) $3 + 4 \cdot 5$ (b) $8 \cdot 2 + 1$ (c) $2 + 2 \cdot 2$

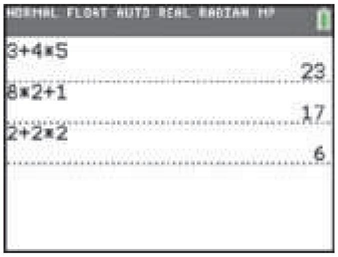
Solution (a) $3 + 4\cdot 5 = 3 + 20 = 23$

↑ Multiply first.

(b) $8\cdot 2 + 1 = 16 + 1 = 17$

↑ Multiply first.

(c) $2 + 2\cdot 2 = 2 + 4 = 6$ ■



```
NORMAL FLOAT AUTO REAL RADIAN MP
3+4*5
                    23
8*2+1
                    17
2+2*2
                     6
```

Figure 8

Figure 8 shows the solution to Example 8 using a TI-84 Plus C graphing calculator. Notice that the calculator follows the agreed order of operations.

Now Work PROBLEM 53

When we want to indicate adding 3 and 4 and then multiplying the result by 5, we use parentheses and write $(3 + 4)\cdot 5$. Whenever parentheses appear in an expression, it means "perform the operations within the parentheses first!"

EXAMPLE 9 Finding the Value of an Expression

(a) $(5 + 3)\cdot 4 = 8\cdot 4 = 32$

(b) $(4 + 5)\cdot(8 - 2) = 9\cdot 6 = 54$ ■

When we divide two expressions, as in

$$\frac{2 + 3}{4 + 8}$$

it is understood that the division bar acts like parentheses; that is,

$$\frac{2 + 3}{4 + 8} = \frac{(2 + 3)}{(4 + 8)}$$

Rules for the Order of Operations

1. Begin with the innermost parentheses and work outward. Remember that in dividing two expressions, we treat the numerator and denominator as if they were enclosed in parentheses.
2. Perform multiplications and divisions, working from left to right.
3. Perform additions and subtractions, working from left to right.

EXAMPLE 10 Finding the Value of an Expression

Evaluate each expression.

(a) $8\cdot 2 + 3$

(b) $5\cdot(3 + 4) + 2$

(c) $\dfrac{2 + 5}{2 + 4\cdot 7}$

(d) $2 + [4 + 2\cdot(10 + 6)]$

Solution (a) $8\cdot 2 + 3 = 16 + 3 = 19$

↑ Multiply first.

(b) $5\cdot(3 + 4) + 2 = 5\cdot 7 + 2 = 35 + 2 = 37$

↑ Parentheses first ↑ Multiply before adding.

(c) $\dfrac{2+5}{2+4\cdot 7} = \dfrac{2+5}{2+28} = \dfrac{7}{30}$

(d) $2 + [4 + 2\cdot(10+6)] = 2 + [4 + 2\cdot(16)]$

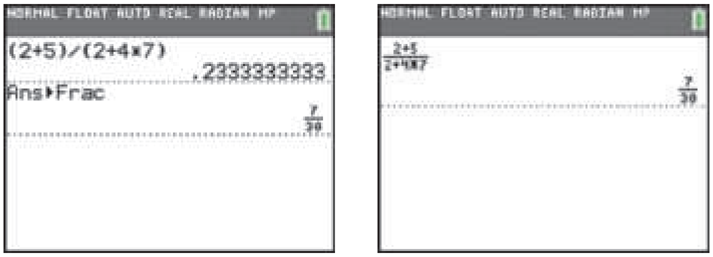
$= 2 + [4 + 32] = 2 + [36] = 38$ ■

Be careful if you use a calculator. For Example 10(c), you need to use parentheses. See Figure 9(a).* If you don't, the calculator will compute the expression

$$2 + \frac{5}{2} + 4\cdot 7 = 2 + 2.5 + 28 = 32.5$$

giving a wrong answer.

Another option, when using a TI-84 Plus C graphing calculator, is to use the fraction template. See Figure 9(b).



```
NORMAL FLOAT AUTO REAL RADIAN MP
(2+5)/(2+4*7)
                     .2333333333
Ans▸Frac
                               7
                              30
```

```
NORMAL FLOAT AUTO REAL RADIAN MP
 2+5
2+4*7
                               7
                              30
```

Figure 9 (a) (b)

Now Work PROBLEMS 59 AND 67

4 Work with Properties of Real Numbers

The equal sign is used to mean that one expression is equivalent to another. Four important properties of equality are listed next. In this list, *a*, *b*, and *c* represent real numbers.

1. The **reflexive property** states that a number equals itself; that is, $a = a$.
2. The **symmetric property** states that if $a = b$, then $b = a$.
3. The **transitive property** states that if $a = b$ and $b = c$, then $a = c$.
4. The **principle of substitution** states that if $a = b$, then we may substitute b for a in any expression containing a.

Now, let's consider some other properties of real numbers.

EXAMPLE 11 Commutative Properties

(a) $3 + 5 = 8$
$5 + 3 = 8$
$3 + 5 = 5 + 3$

(b) $2\cdot 3 = 6$
$3\cdot 2 = 6$
$2\cdot 3 = 3\cdot 2$ ■

This example illustrates the **commutative property** of real numbers, which states that the order in which addition or multiplication takes place does not affect the final result.

---

*Notice that we converted the decimal into its fraction form in Figure 9(a). Consult your manual to see how to enter such expressions on your calculator.

Commutative Properties

$$a + b = b + a \tag{1a}$$

$$a \cdot b = b \cdot a \tag{1b}$$

Here, and in the properties listed next and on pages 12–14, *a*, *b*, and *c* represent real numbers.

EXAMPLE 12 Associative Properties

(a) $2 + (3 + 4) = 2 + 7 = 9$
$(2 + 3) + 4 = 5 + 4 = 9$
$2 + (3 + 4) = (2 + 3) + 4$

(b) $2 \cdot (3 \cdot 4) = 2 \cdot 12 = 24$
$(2 \cdot 3) \cdot 4 = 6 \cdot 4 = 24$
$2 \cdot (3 \cdot 4) = (2 \cdot 3) \cdot 4$ ■

The way we add or multiply three real numbers does not affect the final result. Expressions such as $2 + 3 + 4$ and $3 \cdot 4 \cdot 5$ present no ambiguity, even though addition and multiplication are performed on one pair of numbers at a time. This property is called the **associative property**.

Associative Properties

$$a + (b + c) = (a + b) + c = a + b + c \tag{2a}$$

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c = a \cdot b \cdot c \tag{2b}$$

Distributive Property

$$a \cdot (b + c) = a \cdot b + a \cdot c \tag{3a}$$

$$(a + b) \cdot c = a \cdot c + b \cdot c \tag{3b}$$

The **distributive property** may be used in two different ways.

EXAMPLE 13 Distributive Property

(a) $2 \cdot (x + 3) = 2 \cdot x + 2 \cdot 3 = 2x + 6$ **Use to remove parentheses.**

(b) $3x + 5x = (3 + 5)x = 8x$ **Use to combine two expressions.**

(c) $(x + 2)(x + 3) = x(x + 3) + 2(x + 3) = (x^2 + 3x) + (2x + 6)$
$= x^2 + (3x + 2x) + 6 = x^2 + 5x + 6$ ■

Now Work PROBLEM 89

The real numbers 0 and 1 have unique properties called the *identity properties.*

EXAMPLE 14 Identity Properties

(a) $4 + 0 = 0 + 4 = 4$ (b) $3 \cdot 1 = 1 \cdot 3 = 3$ ■

Identity Properties

$$0 + a = a + 0 = a \qquad \textbf{(4a)}$$

$$a \cdot 1 = 1 \cdot a = a \qquad \textbf{(4b)}$$

We call 0 the **additive identity** and 1 the **multiplicative identity**.

For each real number a, there is a real number $-a$, called the **additive inverse** of a, having the following property:

Additive Inverse Property

$$a + (-a) = -a + a = 0 \qquad \textbf{(5a)}$$

EXAMPLE 15 Finding an Additive Inverse

(a) The additive inverse of 6 is -6, because $6 + (-6) = 0$.

(b) The additive inverse of -8 is $-(-8) = 8$, because $-8 + 8 = 0$. ■

The additive inverse of a, that is, $-a$, is often called the *negative* of a or the *opposite* of a. The use of such terms can be dangerous, because they suggest that the additive inverse is a negative number, which may not be the case. For example, the additive inverse of -3, or $-(-3)$, equals 3, a positive number.

For each *nonzero* real number a, there is a real number $\frac{1}{a}$, called the **multiplicative inverse** of a, having the following property:

Multiplicative Inverse Property

$$a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1 \qquad \text{if } a \neq 0 \qquad \textbf{(5b)}$$

The multiplicative inverse $\frac{1}{a}$ of a nonzero real number a is also referred to as the **reciprocal** of a.

EXAMPLE 16 Finding a Reciprocal

(a) The reciprocal of 6 is $\frac{1}{6}$, because $6 \cdot \frac{1}{6} = 1$.

(b) The reciprocal of -3 is $\frac{1}{-3}$, because $-3 \cdot \frac{1}{-3} = 1$.

(c) The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$, because $\frac{2}{3} \cdot \frac{3}{2} = 1$. ■

With these properties for adding and multiplying real numbers, we can define the operations of subtraction and division as follows:

DEFINITION

The **difference** $a - b$, also read "a less b" or "a minus b," is defined as

$$a - b = a + (-b) \qquad \textbf{(6)}$$

■

To subtract b from a, add the opposite of b to a.

DEFINITION

If b is a nonzero real number, the **quotient** $\frac{a}{b}$, also read as "a divided by b" or "the ratio of a to b," is defined as

$$\frac{a}{b} = a \cdot \frac{1}{b} \qquad \text{if } b \neq 0 \qquad \textbf{(7)}$$

EXAMPLE 17 Working with Differences and Quotients

(a) $8 - 5 = 8 + (-5) = 3$

(b) $4 - 9 = 4 + (-9) = -5$

(c) $\frac{5}{8} = 5 \cdot \frac{1}{8}$

For any number a, the product of a times 0 is always 0; that is,

In Words
The result of multiplying by zero is zero.

Multiplication by Zero

$$a \cdot 0 = 0 \qquad \textbf{(8)}$$

For a nonzero number a,

Division Properties

$$\frac{0}{a} = 0 \qquad \frac{a}{a} = 1 \qquad \text{if } a \neq 0 \qquad \textbf{(9)}$$

Note: Division by 0 is not defined. One reason is to avoid the following difficulty: $\frac{2}{0} = x$ means to find x such that $0 \cdot x = 2$. But $0 \cdot x$ equals 0 for all x, so there is no unique number x such that $\frac{2}{0} = x$. ■

Rules of Signs

$$a(-b) = -(ab) \qquad (-a)b = -(ab) \qquad (-a)(-b) = ab$$
$$-(-a) = a \qquad \frac{a}{-b} = \frac{-a}{b} = -\frac{a}{b} \qquad \frac{-a}{-b} = \frac{a}{b} \qquad \textbf{(10)}$$

EXAMPLE 18 Applying the Rules of Signs

(a) $2(-3) = -(2 \cdot 3) = -6$

(b) $(-3)(-5) = 3 \cdot 5 = 15$

(c) $\frac{3}{-2} = \frac{-3}{2} = -\frac{3}{2}$

(d) $\frac{-4}{-9} = \frac{4}{9}$

(e) $\frac{x}{-2} = \frac{1}{-2} \cdot x = -\frac{1}{2}x$

Reduction Properties

$$ac = bc \quad \text{implies} \quad a = b \quad \text{if } c \neq 0$$

$$\frac{ac}{bc} = \frac{a}{b} \qquad \text{if } b \neq 0, c \neq 0 \qquad \textbf{(11)}$$

EXAMPLE 19 **Using the Reduction Properties**

(a) If $2x = 6$, then

$$\begin{aligned} 2x &= 6 \\ 2x &= 2 \cdot 3 && \textbf{Factor 6.} \\ x &= 3 && \textbf{Divide out the 2's.} \end{aligned}$$

(b) $\dfrac{18}{12} = \dfrac{3 \cdot \cancel{6}}{2 \cdot \cancel{6}} = \dfrac{3}{2}$

↑ **Divide out the 6's.** ■

Note: We follow the common practice of using slash marks to indicate factors dividing out. ■

In Words

If a product equals 0, then one or both of the factors is 0.

Zero-Product Property

$$\text{If } ab = 0, \text{ then } a = 0, \text{ or } b = 0, \text{ or both.} \qquad \textbf{(12)}$$

EXAMPLE 20 **Using the Zero-Product Property**

If $2x = 0$, then either $2 = 0$ or $x = 0$. Since $2 \neq 0$, it follows that $x = 0$. ■

Arithmetic of Quotients

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd} \qquad \text{if } b \neq 0, d \neq 0 \qquad \textbf{(13)}$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \qquad \text{if } b \neq 0, d \neq 0 \qquad \textbf{(14)}$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc} \qquad \text{if } b \neq 0, c \neq 0, d \neq 0 \qquad \textbf{(15)}$$

EXAMPLE 21 **Adding, Subtracting, Multiplying, and Dividing Quotients**

(a) $\dfrac{2}{3} + \dfrac{5}{2} = \dfrac{2 \cdot 2}{3 \cdot 2} + \dfrac{3 \cdot 5}{3 \cdot 2} = \dfrac{2 \cdot 2 + 3 \cdot 5}{3 \cdot 2} = \dfrac{4 + 15}{6} = \dfrac{19}{6}$

↑ **By equation (13)**

(b) $\dfrac{3}{5} - \dfrac{2}{3} = \dfrac{3}{5} + \left(-\dfrac{2}{3}\right) = \dfrac{3}{5} + \dfrac{-2}{3}$

↑ **By equation (6)** ↑ **By equation (10)**

$= \dfrac{3 \cdot 3 + 5 \cdot (-2)}{5 \cdot 3} = \dfrac{9 + (-10)}{15} = \dfrac{-1}{15} = -\dfrac{1}{15}$

↑ **By equation (13)**

Note: Slanting the slash marks in different directions for different factors, as shown here, is a good practice to follow, since it will help in checking for errors.

(c) $$\frac{8}{3}\cdot\frac{15}{4} \underset{\substack{\uparrow\\ \text{By equation (14)}}}{=} \frac{8\cdot 15}{3\cdot 4} = \frac{2\cdot\cancel{4}\cdot\cancel{3}\cdot 5}{\cancel{3}\cdot\cancel{4}\cdot 1} \underset{\substack{\uparrow\\ \text{By equation (11)}}}{=} \frac{2\cdot 5}{1} = 10$$

(d) $$\frac{\frac{3}{5}}{\frac{7}{9}} \underset{\substack{\uparrow\\ \text{By equation (15)}}}{=} \frac{3}{5}\cdot\frac{9}{7} \underset{\substack{\uparrow\\ \text{By equation (14)}}}{=} \frac{3\cdot 9}{5\cdot 7} = \frac{27}{35}$$

Note: In writing quotients, we shall follow the usual convention and write the quotient in lowest terms. That is, we write it so that any common factors of the numerator and the denominator have been removed using the Reduction Properties, equation (11). As examples,

$$\frac{90}{24} = \frac{15\cdot\cancel{6}}{4\cdot\cancel{6}} = \frac{15}{4}$$

$$\frac{24x^2}{18x} = \frac{4\cdot\cancel{6}\cdot x\cdot\cancel{x}}{3\cdot\cancel{6}\cdot\cancel{x}} = \frac{4x}{3} \qquad x \neq 0$$

Now Work PROBLEMS 69, 73, AND 83

Sometimes it is easier to add two fractions using *least common multiples* (LCM). The LCM of two numbers is the smallest number that each has as a common multiple.

EXAMPLE 22 Finding the Least Common Multiple of Two Numbers

Find the least common multiple of 15 and 12.

Solution To find the LCM of 15 and 12, we look at multiples of 15 and 12.

15, 30, 45, 60, 75, 90, 105, 120, . . .

12, 24, 36, 48, 60, 72, 84, 96, 108, 120, . . .

The *common* multiples are in blue. The *least* common multiple is 60.

EXAMPLE 23 Using the Least Common Multiple to Add Two Fractions

Find: $\frac{8}{15} + \frac{5}{12}$

Solution We use the LCM of the denominators of the fractions and rewrite each fraction using the LCM as a common denominator. The LCM of the denominators (12 and 15) is 60. Rewrite each fraction using 60 as the denominator.

$$\begin{aligned}\frac{8}{15} + \frac{5}{12} &= \frac{8}{15}\cdot\frac{4}{4} + \frac{5}{12}\cdot\frac{5}{5}\\ &= \frac{32}{60} + \frac{25}{60}\\ &= \frac{32 + 25}{60}\\ &= \frac{57}{60}\\ &= \frac{19}{20}\end{aligned}$$

Now Work PROBLEM 77

Historical Feature

The real number system has a history that stretches back at least to the ancient Babylonians (1800 BC). It is remarkable how much the ancient Babylonian attitudes resemble our own. As we stated in the text, the fundamental difficulty with irrational numbers is that they cannot be written as quotients of integers or, equivalently, as repeating or terminating decimals. The Babylonians wrote their numbers in a system based on 60 in the same way that we write ours based on 10. They would carry as many places for π as the accuracy of the problem demanded, just as we now use

$$\pi \approx 3\frac{1}{7} \quad \text{or} \quad \pi \approx 3.1416 \quad \text{or} \quad \pi \approx 3.14159$$
$$\text{or} \quad \pi \approx 3.14159265358979$$

depending on how accurate we need to be.

Things were very different for the Greeks, whose number system allowed only rational numbers. When it was discovered that $\sqrt{2}$ was not a rational number, this was regarded as a fundamental flaw in the number concept. So serious was the matter that the Pythagorean Brotherhood (an early mathematical society) is said to have drowned one of its members for revealing this terrible secret. Greek mathematicians then turned away from the number concept, expressing facts about whole numbers in terms of line segments.

In astronomy, however, Babylonian methods, including the Babylonian number system, continued to be used. Simon Stevin (1548–1620), probably using the Babylonian system as a model, invented the decimal system, complete with rules of calculation, in 1585. [Others, for example, al-Kashi of Samarkand (d. 1429), had made some progress in the same direction.] The decimal system so effectively conceals the difficulties that the need for more logical precision began to be felt only in the early 1800s. Around 1880, Georg Cantor (1845–1918) and Richard Dedekind (1831–1916) gave precise definitions of real numbers. Cantor's definition, although more abstract and precise, has its roots in the decimal (and hence Babylonian) numerical system.

Sets and set theory were a spin-off of the research that went into clarifying the foundations of the real number system. Set theory has developed into a large discipline of its own, and many mathematicians regard it as the foundation upon which modern mathematics is built. Cantor's discoveries that infinite sets can also be counted and that there are different sizes of infinite sets are among the most astounding results of modern mathematics.

R.1 Assess Your Understanding

Concepts and Vocabulary

1. The numbers in the set $\left\{x \middle| x = \frac{a}{b}, \text{ where } a, b \text{ are integers and } b \neq 0\right\}$ are called ________ numbers.

2. The value of the expression $4 + 5 \cdot 6 - 3$ is ___.

3. The fact that $2x + 3x = (2 + 3)x$ is a consequence of the __________ Property.

4. "The product of 5 and $x + 3$ equals 6" may be written as ____________.

5. The intersection of sets A and B is denoted by which of the following?

(a) $A \cap B$ (b) $A \cup B$ (c) $A \subseteq B$ (d) $A \varnothing B$

6. Choose the correct name for the set of numbers $\{0, 1, 2, 3, \ldots\}$.

(a) Counting numbers (b) Whole numbers
(c) Integers (d) Irrational numbers

7. ***True or False*** Rational numbers have decimals that either terminate or are nonterminating with a repeating block of digits.

8. ***True or False*** The Zero-Product Property states that the product of any number and zero equals zero.

9. ***True or False*** The least common multiple of 12 and 18 is 6.

10. ***True or False*** No real number is both rational and irrational.

Skill Building

In Problems 11–22, use U = universal set = $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 3, 4, 5, 9\}$, $B = \{2, 4, 6, 7, 8\}$, *and* $C = \{1, 3, 4, 6\}$ *to find each set.*

11. $A \cup B$
12. $A \cup C$
13. $A \cap B$
14. $A \cap C$
15. $(A \cup B) \cap C$
16. $(A \cap B) \cup C$
17. $\overline{A}$
18. $\overline{C}$
19. $\overline{A \cap B}$
20. $\overline{B \cup C}$
21. $\overline{A} \cup \overline{B}$
22. $\overline{B} \cap \overline{C}$

In Problems 23–28, list the numbers in each set that are (a) Natural numbers, (b) Integers, (c) Rational numbers, (d) Irrational numbers, (e) Real numbers.

23. $A = \left\{-6, \frac{1}{2}, -1.333\ldots \text{(the 3's repeat)}, \pi, 2, 5\right\}$

24. $B = \left\{-\frac{5}{3}, 2.060606\ldots \text{(the block 06 repeats)}, 1.25, 0, 1, \sqrt{5}\right\}$

25. $C = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right\}$

26. $D = \{-1, -1.1, -1.2, -1.3\}$

27. $E = \left\{\sqrt{2}, \pi, \sqrt{2} + 1, \pi + \frac{1}{2}\right\}$

28. $F = \left\{-\sqrt{2}, \pi + \sqrt{2}, \frac{1}{2} + 10.3\right\}$

In Problems 29–40, approximate each number (a) rounded and (b) truncated to three decimal places.

29. 18.9526 **30.** 25.86134 **31.** 28.65319 **32.** 99.05249 **33.** 0.06291 **34.** 0.05388

35. 9.9985 **36.** 1.0006 **37.** $\frac{3}{7}$ **38.** $\frac{5}{9}$ **39.** $\frac{521}{15}$ **40.** $\frac{81}{5}$

In Problems 41–50, write each statement using symbols.

41. The sum of 3 and 2 equals 5.

42. The product of 5 and 2 equals 10.

43. The sum of x and 2 is the product of 3 and 4.

44. The sum of 3 and y is the sum of 2 and 2.

45. The product of 3 and y is the sum of 1 and 2.

46. The product of 2 and x is the product of 4 and 6.

47. The difference x less 2 equals 6.

48. The difference 2 less y equals 6.

49. The quotient x divided by 2 is 6.

50. The quotient 2 divided by x is 6.

In Problems 51–88, evaluate each expression.

51. $9 - 4 + 2$ **52.** $6 - 4 + 3$ **53.** $-6 + 4 \cdot 3$ **54.** $8 - 4 \cdot 2$

55. $4 + 5 - 8$ **56.** $8 - 3 - 4$ **57.** $4 + \frac{1}{3}$ **58.** $2 - \frac{1}{2}$

59. $6 - [3 \cdot 5 + 2 \cdot (3 - 2)]$ **60.** $2 \cdot [8 - 3(4 + 2)] - 3$ **61.** $2 \cdot (3 - 5) + 8 \cdot 2 - 1$ **62.** $1 - (4 \cdot 3 - 2 + 2)$

63. $10 - [6 - 2 \cdot 2 + (8 - 3)] \cdot 2$ **64.** $2 - 5 \cdot 4 - [6 \cdot (3 - 4)]$

65. $(5 - 3)\frac{1}{2}$ **66.** $(5 + 4)\frac{1}{3}$ **67.** $\frac{4 + 8}{5 - 3}$ **68.** $\frac{2 - 4}{5 - 3}$

69. $\frac{3}{5} \cdot \frac{10}{21}$ **70.** $\frac{5}{9} \cdot \frac{3}{10}$ **71.** $\frac{6}{25} \cdot \frac{10}{27}$ **72.** $\frac{21}{25} \cdot \frac{100}{3}$

73. $\frac{3}{4} + \frac{2}{5}$ **74.** $\frac{4}{3} + \frac{1}{2}$ **75.** $\frac{5}{6} + \frac{9}{5}$ **76.** $\frac{8}{9} + \frac{15}{2}$

77. $\frac{5}{18} + \frac{1}{12}$ **78.** $\frac{2}{15} + \frac{8}{9}$ **79.** $\frac{1}{30} - \frac{7}{18}$ **80.** $\frac{3}{14} - \frac{2}{21}$

81. $\frac{3}{20} - \frac{2}{15}$ **82.** $\frac{6}{35} - \frac{3}{14}$ **83.** $\frac{\frac{5}{18}}{\frac{11}{27}}$ **84.** $\frac{\frac{5}{21}}{\frac{2}{35}}$

85. $\frac{1}{2} \cdot \frac{3}{5} + \frac{7}{10}$ **86.** $\frac{2}{3} + \frac{4}{5} \cdot \frac{1}{6}$ **87.** $2 \cdot \frac{3}{4} + \frac{3}{8}$ **88.** $3 \cdot \frac{5}{6} - \frac{1}{2}$

In Problems 89–100, use the Distributive Property to remove the parentheses.

89. $6(x + 4)$ **90.** $4(2x - 1)$ **91.** $x(x - 4)$ **92.** $4x(x + 3)$

93. $2\left(\frac{3}{4}x - \frac{1}{2}\right)$ **94.** $3\left(\frac{2}{3}x + \frac{1}{6}\right)$ **95.** $(x + 2)(x + 4)$ **96.** $(x + 5)(x + 1)$

97. $(x - 2)(x + 1)$ **98.** $(x - 4)(x + 1)$ **99.** $(x - 8)(x - 2)$ **100.** $(x - 4)(x - 2)$

Explaining Concepts: Discussion and Writing

101. Explain to a friend how the Distributive Property is used to justify the fact that $2x + 3x = 5x$.

102. Explain to a friend why $2 + 3 \cdot 4 = 14$, whereas $(2 + 3) \cdot 4 = 20$.

103. Explain why $2(3 \cdot 4)$ is not equal to $(2 \cdot 3) \cdot (2 \cdot 4)$.

104. Explain why $\frac{4 + 3}{2 + 5}$ is not equal to $\frac{4}{2} + \frac{3}{5}$.

105. Is subtraction commutative? Support your conclusion with an example.

106. Is subtraction associative? Support your conclusion with an example.

107. Is division commutative? Support your conclusion with an example.

108. Is division associative? Support your conclusion with an example.

109. If $2 = x$, why does $x = 2$?

110. If $x = 5$, why does $x^2 + x = 30$?

111. Are there any real numbers that are both rational and irrational? Are there any real numbers that are neither? Explain your reasoning.

112. Explain why the sum of a rational number and an irrational number must be irrational.

113. A rational number is defined as the quotient of two integers. When written as a decimal, the decimal will either repeat or terminate. By looking at the denominator of the rational number, there is a way to tell in advance whether its decimal representation will repeat or terminate. Make a list of rational numbers and their decimals. See if you can discover the pattern. Confirm your conclusion by consulting books on number theory at the library. Write a brief essay on your findings.

114. The current time is 12 noon CST. What time (CST) will it be 12,997 hours from now?

115. Both $\frac{a}{0}\,(a \neq 0)$ and $\frac{0}{0}$ are undefined, but for different reasons. Write a paragraph or two explaining the different reasons.

R.2 Algebra Essentials

OBJECTIVES
1 Graph Inequalities (p. 19)
2 Find Distance on the Real Number Line (p. 20)
3 Evaluate Algebraic Expressions (p. 21)
4 Determine the Domain of a Variable (p. 22)
5 Use the Laws of Exponents (p. 22)
6 Evaluate Square Roots (p. 24)
7 Use a Calculator to Evaluate Exponents (p. 25)
8 Use Scientific Notation (p. 25)

The Real Number Line

Real numbers can be represented by points on a line called the **real number line**. There is a one-to-one correspondence between real numbers and points on a line. That is, every real number corresponds to a point on the line, and each point on the line has a unique real number associated with it.

Pick a point on a line somewhere in the center, and label it O. This point, called the **origin**, corresponds to the real number 0. See Figure 10. The point 1 unit to the right of O corresponds to the number 1. The distance between 0 and 1 determines the **scale** of the number line. For example, the point associated with the number 2 is twice as far from O as 1. Notice that an arrowhead on the right end of the line indicates the direction in which the numbers increase. Points to the left of the origin correspond to the real numbers $-1, -2$, and so on. Figure 10 also shows the points associated with the rational numbers $-\frac{1}{2}$ and $\frac{1}{2}$ and with the irrational numbers $\sqrt{2}$ and π.

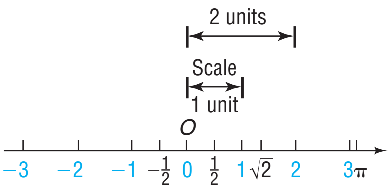


Figure 10 Real number line

DEFINITION

The real number associated with a point P is called the **coordinate** of P, and the line whose points have been assigned coordinates is called the **real number line**.

Now Work PROBLEM 13

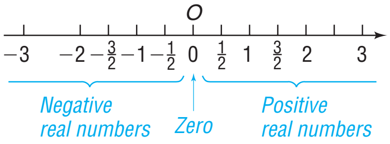


Figure 11

The real number line consists of three classes of real numbers, as shown in Figure 11.

1. The **negative real numbers** are the coordinates of points to the left of the origin O.
2. The real number **zero** is the coordinate of the origin O.
3. The **positive real numbers** are the coordinates of points to the right of the origin O.

Multiplication Properties of Positive and Negative Numbers

1. The product of two positive numbers is a positive number.
2. The product of two negative numbers is a positive number.
3. The product of a positive number and a negative number is a negative number.

1 Graph Inequalities

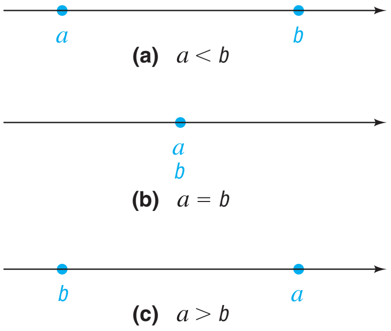


Figure 12

An important property of the real number line follows from the fact that, given two numbers (points) a and b, either a is to the left of b, or a is at the same location as b, or a is to the right of b. See Figure 12.

If a is to the left of b, then "a is less than b," which is written $a < b$. If a is to the right of b, then "a is greater than b," which is written $a > b$. If a is at the same location as b, then $a = b$. If a is either less than or equal to b, then $a \leq b$. Similarly, $a \geq b$ means that a is either greater than or equal to b. Collectively, the symbols $<, >, \leq$, and $\geq$ are called **inequality symbols**.

Note that $a < b$ and $b > a$ mean the same thing. It does not matter whether we write $2 < 3$ or $3 > 2$.

Furthermore, if $a < b$ or if $b > a$, then the difference $b - a$ is positive. Do you see why?

EXAMPLE 1 **Using Inequality Symbols**

(a) $3 < 7$ (b) $-8 > -16$ (c) $-6 < 0$

(d) $-8 < -4$ (e) $4 > -1$ (f) $8 > 0$ ■

In Example 1(a), we conclude that $3 < 7$ either because 3 is to the left of 7 on the real number line or because the difference, $7 - 3 = 4$, is a positive real number.

Similarly, we conclude in Example 1(b) that $-8 > -16$ either because -8 lies to the right of -16 on the real number line or because the difference, $-8 - (-16) = -8 + 16 = 8$, is a positive real number.

Look again at Example 1. Note that the inequality symbol always points in the direction of the smaller number.

An **inequality** is a statement in which two expressions are related by an inequality symbol. The expressions are referred to as the **sides** of the inequality. Inequalities of the form $a < b$ or $b > a$ are called **strict inequalities**, whereas inequalities of the form $a \leq b$ or $b \geq a$ are called **nonstrict inequalities**.

Based on the discussion so far, we conclude that

$a > 0$ is equivalent to a is positive

$a < 0$ is equivalent to a is negative

We sometimes read $a > 0$ by saying that "a is positive." If $a \geq 0$, then either $a > 0$ or $a = 0$, and we may read this as "a is nonnegative."

Now Work PROBLEMS 17 AND 27

EXAMPLE 2 Graphing Inequalities

(a) On the real number line, graph all numbers x for which $x > 4$.
(b) On the real number line, graph all numbers x for which $x \leq 5$.

Solution (a) See Figure 13. Notice that we use a left parenthesis to indicate that the number 4 is *not* part of the graph.

(b) See Figure 14. Notice that we use a right bracket to indicate that the number 5 *is* part of the graph. ■

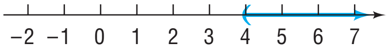


Figure 13 $x > 4$

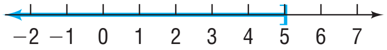


Figure 14 $x \leq 5$

Now Work PROBLEM 33

2 Find Distance on the Real Number Line

The *absolute value* of a number a is the distance from 0 to a on the number line. For example, -4 is 4 units from 0, and 3 is 3 units from 0. See Figure 15. That is, the absolute value of -4 is 4, and the absolute value of 3 is 3.

A more formal definition of absolute value is given next.

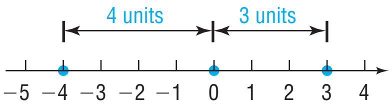


Figure 15

DEFINITION

The **absolute value** of a real number a, denoted by the symbol $|a|$, is defined by the rules

$$\boxed{|a| = a \quad \text{if } a \geq 0 \qquad \text{and} \qquad |a| = -a \quad \text{if } a < 0}$$

For example, because $-4 < 0$, the second rule must be used to get $|-4| = -(-4) = 4$.

EXAMPLE 3 Computing Absolute Value

(a) $|8| = 8$ (b) $|0| = 0$ (c) $|-15| = -(-15) = 15$ ■

Look again at Figure 15. The distance from -4 to 3 is 7 units. This distance is the difference $3 - (-4)$, obtained by subtracting the smaller coordinate from the larger. However, since $|3 - (-4)| = |7| = 7$ and $|-4 - 3| = |-7| = 7$, we can use absolute value to calculate the distance between two points without being concerned about which is smaller.

DEFINITION

If P and Q are two points on a real number line with coordinates a and b, respectively, the **distance between P and Q**, denoted by $d(P, Q)$, is

$$\boxed{d(P, Q) = |b - a|}$$

Since $|b - a| = |a - b|$, it follows that $d(P, Q) = d(Q, P)$.

EXAMPLE 4

Finding Distance on a Number Line

Let P, Q, and R be points on a real number line with coordinates -5, 7, and -3, respectively. Find the distance

(a) between P and Q (b) between Q and R

Solution See Figure 16.

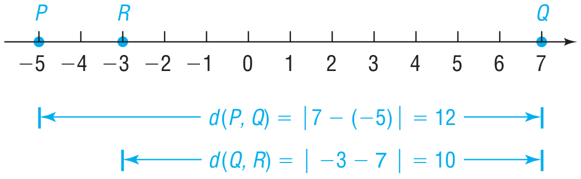


Figure 16

(a) $d(P, Q) = |7 - (-5)| = |12| = 12$
(b) $d(Q, R) = |-3 - 7| = |-10| = 10$ ■

Now Work PROBLEM 39

3 Evaluate Algebraic Expressions

Remember, in algebra we use letters such as x, y, a, b, and c to represent numbers. If the letter used is to represent *any* number from a given set of numbers, it is called a **variable**. A **constant** is either a fixed number, such as 5 or $\sqrt{3}$, or a letter that represents a fixed (possibly unspecified) number.

Constants and variables are combined using the operations of addition, subtraction, multiplication, and division to form *algebraic expressions*. Examples of algebraic expressions include

$$x + 3 \qquad \frac{3}{1 - t} \qquad 7x - 2y$$

To evaluate an algebraic expression, substitute a numerical value for each variable.

EXAMPLE 5

Evaluating an Algebraic Expression

Evaluate each expression if $x = 3$ and $y = -1$.

(a) $x + 3y$ (b) $5xy$ (c) $\dfrac{3y}{2 - 2x}$ (d) $|-4x + y|$

Solution (a) Substitute 3 for x and -1 for y in the expression $x + 3y$.

$$x + 3y \underset{\substack{\uparrow \\ \mathbf{x = 3,\, y = -1}}}{=} 3 + 3(-1) = 3 + (-3) = 0$$

(b) If $x = 3$ and $y = -1$, then

$$5xy = 5(3)(-1) = -15$$

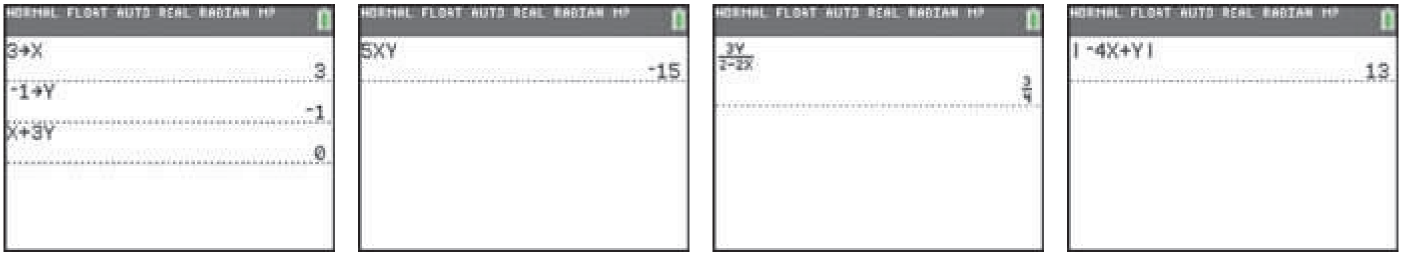
(c) If $x = 3$ and $y = -1$, then

$$\frac{3y}{2 - 2x} = \frac{3(-1)}{2 - 2(3)} = \frac{-3}{2 - 6} = \frac{-3}{-4} = \frac{3}{4}$$

(d) If $x = 3$ and $y = -1$, then

$$|-4x + y| = |-4(3) + (-1)| = |-12 + (-1)| = |-13| = 13$$ ■

A graphing calculator can be used to evaluate an algebraic expression. Figure 17 shows the results of Example 5 using a TI-84 Plus C.



```
NORMAL FLOAT AUTO REAL RADIAN MP
3→X
                               3
-1→Y
                              -1
X+3Y
                               0
```

(a)

```
NORMAL FLOAT AUTO REAL RADIAN MP
5XY
                             -15
```

(b)

```
NORMAL FLOAT AUTO REAL RADIAN MP
3Y/(2-2X)
                             3/4
```

(c)

```
NORMAL FLOAT AUTO REAL RADIAN MP
|-4X+Y|
                              13
```

(d)

Figure 17

Now Work PROBLEMS 41 AND 49

4 Determine the Domain of a Variable

In working with expressions or formulas involving variables, the variables may be allowed to take on values from only a certain set of numbers. For example, in the formula for the area A of a circle of radius r, $A = \pi r^2$, the variable r is necessarily restricted to the positive real numbers. In the expression $\frac{1}{x}$, the variable x cannot take on the value 0, since division by 0 is not defined.

DEFINITION

The set of values that a variable may assume is called the **domain of the variable**.

EXAMPLE 6 Finding the Domain of a Variable

The domain of the variable x in the expression

$$\frac{5}{x-2}$$

is $\{x \mid x \neq 2\}$ since, if $x = 2$, the denominator becomes 0, which is not defined.

EXAMPLE 7 Circumference of a Circle

In the formula for the circumference C of a circle of radius r,

$$C = 2\pi r$$

the domain of the variable r, representing the radius of the circle, is the set of positive real numbers, $\{r \mid r > 0\}$. The domain of the variable C, representing the circumference of the circle, is also the set of positive real numbers, $\{C \mid C > 0\}$.

In describing the domain of a variable, we may use either set notation or words, whichever is more convenient.

Now Work PROBLEM 59

5 Use the Laws of Exponents

Integer exponents provide a shorthand notation for representing repeated multiplications of a real number. For example,

$$3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$$

Additionally, many formulas have exponents. For example,

- The formula for the horsepower rating H of an engine is

$$H = \frac{D^2 N}{2.5}$$

where D is the diameter of a cylinder and N is the number of cylinders.

- A formula for the resistance R of blood flowing in a blood vessel is

$$R = C\frac{L}{r^4}$$

where L is the length of the blood vessel, r is the radius, and C is a positive constant.

DEFINITION

If a is a real number and n is a positive integer, then the symbol $\boldsymbol{a^n}$ represents the product of n factors of a. That is,

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}} \tag{1}$$

From this definition $a^1 = a$, $a^2 = a \cdot a$, $a^3 = a \cdot a \cdot a$, and so on. In the expression a^n, a is called the **base** and n is called the **exponent**, or **power**. We read a^n as "a raised to the power n" or as "a to the nth power." We usually read a^2 as "a squared" and a^3 as "a cubed."

In working with exponents, the operation of *raising to a power* is performed before any other operation. As examples,

$$4 \cdot 3^2 = 4 \cdot 9 = 36 \qquad 2^2 + 3^2 = 4 + 9 = 13$$

$$-2^4 = -16 \qquad 5 \cdot 3^2 + 2 \cdot 4 = 5 \cdot 9 + 2 \cdot 4 = 45 + 8 = 53$$

WARNING Be careful with minus signs and exponents.

$$-2^4 = -1 \cdot 2^4 = -16$$

whereas

$$(-2)^4 = (-2)(-2)(-2)(-2) = 16$$

Parentheses are used to indicate operations to be performed first. For example,

$$(-2)^4 = (-2)(-2)(-2)(-2) = 16 \qquad (2 + 3)^2 = 5^2 = 25$$

DEFINITION

If $a \neq 0$, then

$$\boxed{a^0 = 1}$$

DEFINITION

If $a \neq 0$ and if n is a positive integer, then

$$\boxed{a^{-n} = \frac{1}{a^n}}$$

Whenever you encounter a negative exponent, think "reciprocal."

EXAMPLE 8 Evaluating Expressions Containing Negative Exponents

(a) $2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$ (b) $x^{-4} = \dfrac{1}{x^4}$ (c) $\left(\dfrac{1}{5}\right)^{-2} = \dfrac{1}{\left(\dfrac{1}{5}\right)^2} = \dfrac{1}{\dfrac{1}{25}} = 25$

Now Work PROBLEMS 77 AND 97

The following properties, called the **Laws of Exponents**, can be proved using the preceding definitions. In the list, a and b are real numbers, and m and n are integers.

THEOREM

Laws of Exponents

$$a^m a^n = a^{m+n} \qquad (a^m)^n = a^{mn} \qquad (ab)^n = a^n b^n$$

$$\frac{a^m}{a^n} = a^{m-n} = \frac{1}{a^{n-m}} \quad \text{if } a \neq 0 \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \quad \text{if } b \neq 0$$

EXAMPLE 9 Using the Laws of Exponents

(a) $x^{-3} \cdot x^5 = x^{-3+5} = x^2 \quad x \neq 0$

(b) $(x^{-3})^2 = x^{-3\cdot 2} = x^{-6} = \dfrac{1}{x^6} \quad x \neq 0$

(c) $(2x)^3 = 2^3 \cdot x^3 = 8x^3$

(d) $\left(\dfrac{2}{3}\right)^4 = \dfrac{2^4}{3^4} = \dfrac{16}{81}$

(e) $\dfrac{x^{-2}}{x^{-5}} = x^{-2-(-5)} = x^3 \quad x \neq 0$ ■

Note: Always write the final answer using positive exponents. ■

Now Work PROBLEM 79

EXAMPLE 10 Using the Laws of Exponents

Write each expression so that all exponents are positive.

(a) $\dfrac{x^5 y^{-2}}{x^3 y} \quad x \neq 0, \quad y \neq 0$

(b) $\left(\dfrac{x^{-3}}{3y^{-1}}\right)^{-2} \quad x \neq 0, \quad y \neq 0$

Solution

(a) $\dfrac{x^5 y^{-2}}{x^3 y} = \dfrac{x^5}{x^3} \cdot \dfrac{y^{-2}}{y} = x^{5-3} \cdot y^{-2-1} = x^2 y^{-3} = x^2 \cdot \dfrac{1}{y^3} = \dfrac{x^2}{y^3}$

(b) $\left(\dfrac{x^{-3}}{3y^{-1}}\right)^{-2} = \dfrac{(x^{-3})^{-2}}{(3y^{-1})^{-2}} = \dfrac{x^6}{3^{-2}(y^{-1})^{-2}} = \dfrac{x^6}{\frac{1}{9}y^2} = \dfrac{9x^6}{y^2}$ ■

Now Work PROBLEM 89

6 Evaluate Square Roots

In Words
$\sqrt{36}$ means "give me the nonnegative number whose square is 36."

A real number is squared when it is raised to the power 2. The inverse of squaring is finding a **square root**. For example, since $6^2 = 36$ and $(-6)^2 = 36$, the numbers 6 and -6 are square roots of 36.

The symbol $\sqrt{\ \ }$, called a **radical sign**, is used to denote the **principal**, or nonnegative, square root. For example, $\sqrt{36} = 6$.

DEFINITION

> If a is a nonnegative real number, the nonnegative number b such that $b^2 = a$ is the **principal square root** of a, and is denoted by $b = \sqrt{a}$.

The following comments are noteworthy:

1. Negative numbers do not have square roots (in the real number system), because the square of any real number is *nonnegative*. For example, $\sqrt{-4}$ is not a real number, because there is no real number whose square is -4.
2. The principal square root of 0 is 0, since $0^2 = 0$. That is, $\sqrt{0} = 0$.
3. The principal square root of a positive number is positive.
4. If $c \geq 0$, then $(\sqrt{c})^2 = c$. For example, $(\sqrt{2})^2 = 2$ and $(\sqrt{3})^2 = 3$.

EXAMPLE 11 Evaluating Square Roots

(a) $\sqrt{64} = 8$ (b) $\sqrt{\dfrac{1}{16}} = \dfrac{1}{4}$ (c) $(\sqrt{1.4})^2 = 1.4$ ■

Examples 11(a) and (b) are examples of square roots of perfect squares, since $64 = 8^2$ and $\dfrac{1}{16} = \left(\dfrac{1}{4}\right)^2$.

Consider the expression $\sqrt{a^2}$. Since $a^2 \geq 0$, the principal square root of a^2 is defined whether $a > 0$ or $a < 0$. However, since the principal square root is nonnegative, we need an absolute value to ensure the nonnegative result. That is,

$$\sqrt{a^2} = |a| \quad a \text{ any real number} \qquad \textbf{(2)}$$

EXAMPLE 12

Using Equation (2)

(a) $\sqrt{(2.3)^2} = |2.3| = 2.3$ (b) $\sqrt{(-2.3)^2} = |-2.3| = 2.3$

(c) $\sqrt{x^2} = |x|$ ■

Now Work PROBLEM 85

7 Use a Calculator to Evaluate Exponents

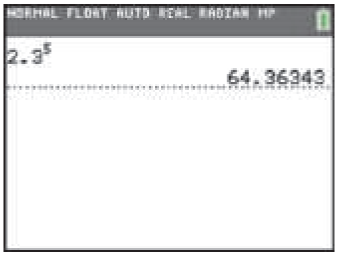
Your calculator has either a caret key, $\boxed{\wedge}$, or an $\boxed{x^y}$ key, that is used for computations involving exponents.

EXAMPLE 13

Exponents on a Graphing Calculator

Evaluate: $(2.3)^5$

Solution Figure 18 shows the result using a TI-84 Plus C graphing calculator. ■



```
2.3⁵
                  64.36343
```

Figure 18

Now Work PROBLEM 115

8 Use Scientific Notation

Measurements of physical quantities can range from very small to very large. For example, the mass of a proton is approximately 0.00000000000000000000000000167 kilogram and the mass of Earth is about 5,980,000,000,000,000,000,000,000 kilograms. These numbers obviously are tedious to write down and difficult to read, so we use exponents to rewrite them.

DEFINITION

When a number has been written as the product of a number x, where $1 \leq x < 10$, times a power of 10, it is said to be written in **scientific notation**.

In scientific notation,

$$\text{Mass of a proton} = 1.67 \times 10^{-27} \text{ kilogram}$$

$$\text{Mass of Earth} = 5.98 \times 10^{24} \text{ kilograms}$$

Converting a Decimal into Scientific Notation

To change a positive number into scientific notation:

1. Count the number N of places that the decimal point must be moved to arrive at a number x, where $1 \leq x < 10$.
2. If the original number is greater than or equal to 1, the scientific notation is $x \times 10^N$. If the original number is between 0 and 1, the scientific notation is $x \times 10^{-N}$.

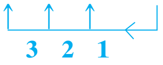
EXAMPLE 14 Using Scientific Notation

Write each number in scientific notation.

(a) 9582 (b) 1.245 (c) 0.285 (d) 0.000561

Solution (a) The decimal point in 9582 follows the 2. Count left from the decimal point

$$9\quad 5\quad 8\quad 2\quad .$$



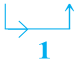
stopping after three moves, because 9.582 is a number between 1 and 10. Since 9582 is greater than 1, we write

$$9582 = 9.582 \times 10^3$$

(b) The decimal point in 1.245 is between the 1 and 2. Since the number is already between 1 and 10, the scientific notation for it is $1.245 \times 10^0 = 1.245$.

(c) The decimal point in 0.285 is between the 0 and the 2. We count

$$0\quad .\quad 2\quad 8\quad 5$$

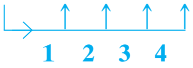


stopping after one move, because 2.85 is a number between 1 and 10. Since 0.285 is between 0 and 1, we write

$$0.285 = 2.85 \times 10^{-1}$$

(d) The decimal point in 0.000561 is moved as follows:

$$0\quad .\quad 0\quad 0\quad 0\quad 5\quad 6\quad 1$$



As a result,

$$0.000561 = 5.61 \times 10^{-4}$$ ■

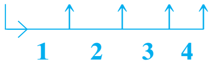
Now Work PROBLEM 121

EXAMPLE 15 Changing from Scientific Notation to Decimals

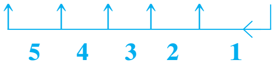
Write each number as a decimal.

(a) 2.1×10^4 (b) 3.26×10^{-5} (c) 1×10^{-2}

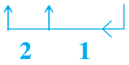
Solution (a) $2.1 \times 10^4 = 2\quad .\quad 1\quad 0\quad 0\quad 0\quad \times 10^4 = 21{,}000$



(b) $3.26 \times 10^{-5} = 0\quad 0\quad 0\quad 0\quad 0\quad 3\quad .\quad 2\quad 6\quad \times 10^{-5} = 0.0000326$



(c) $1 \times 10^{-2} = 0\quad 0\quad 1\quad .\quad 0 \times 10^{-2} = 0.01$



■

Now Work PROBLEM 129

EXAMPLE 16 **Using Scientific Notation**

(a) The diameter of the smallest living cell is only about 0.00001 centimeter (cm).* Express this number in scientific notation.

(b) The surface area of Earth is about 1.97×10^8 square miles.† Express the surface area as a whole number.

Solution (a) $0.00001 \text{ cm} = 1 \times 10^{-5}$ cm because the decimal point is moved five places and the number is less than 1.

(b) 1.97×10^8 square miles = 197,000,000 square miles. ■

Now Work PROBLEM 155

COMMENT On a calculator, a number such as 3.615×10^{12} is usually displayed as $\boxed{\text{3.615E12.}}$ ■

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* *Powers of Ten,* Philip and Phylis Morrison.

† *2011 Information Please Almanac*.

Historical Feature

The word *algebra* is derived from the Arabic word *al-jabr*. This word is a part of the title of a ninth-century work, "Hisâb al-jabr w'al-muqâbalah," written by Mohammed ibn Músâ al-Khwârizmî. The word *al-jabr* means "a restoration," a reference to the fact that if a number is added to one side of an equation, then it must also be added to the other side in order to "restore" the equality. The title of the work, freely translated, is "The Science of Reduction and Cancellation." Of course, today, algebra has come to mean a great deal more.

R.2 Assess Your Understanding

Concepts and Vocabulary

1. A(n) ________ is a letter used in algebra to represent any number from a given set of numbers.

2. On the real number line, the real number zero is the coordinate of the ______.

3. An inequality of the form $a > b$ is called a(n) ______ inequality.

4. In the expression 2^4, the number 2 is called the _____ and 4 is called the ________________.

5. In scientific notation, 1234.5678 = ______________.

6. ***True or False*** The product of two negative real numbers is always greater than zero.

7. ***True or False*** The distance between two distinct points on the real number line is always greater than zero.

8. ***True or False*** The absolute value of a real number is always greater than zero.

9. ***True or False*** When a number is expressed in scientific notation, it is expressed as the product of a number x, $0 \le x < 1$, and a power of 10.

10. ***True or False*** To multiply two expressions having the same base, retain the base and multiply the exponents.

11. If a is a nonnegative real number, then which inequality statement best describes a?

(a) $a < 0$ (b) $a > 0$ (c) $a \le 0$ (d) $a \ge 0$

12. Let a and b be non-zero real numbers and m and n be integers. Which of the following is not a law of exponents?

(a) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ (b) $(a^m)^n = a^{m+n}$

(c) $\frac{a^m}{a^n} = a^{m-n}$ (d) $(ab)^n = a^n b^n$

Skill Building

13. On the real number line, label the points with coordinates $0, 1, -1, \frac{5}{2}, -2.5, \frac{3}{4}$, and 0.25.

14. Repeat Problem 13 for the coordinates $0, -2, 2, -1.5, \frac{3}{2}, \frac{1}{3}$, and $\frac{2}{3}$.

In Problems 15–24, replace the question mark by <, >, or =, whichever is correct.

15. $\frac{1}{2}$? 0

16. 5 ? 6

17. -1 ? -2

18. -3 ? $-\frac{5}{2}$

19. π ? 3.14

20. $\sqrt{2}$? 1.41

21. $\frac{1}{2}$? 0.5

22. $\frac{1}{3}$? 0.33

23. $\frac{2}{3}$? 0.67

24. $\frac{1}{4}$? 0.25

In Problems 25–30, write each statement as an inequality.

25. x is positive

26. z is negative

27. x is less than 2

28. y is greater than -5

29. x is less than or equal to 1

30. x is greater than or equal to 2

In Problems 31–34, graph the numbers x on the real number line.

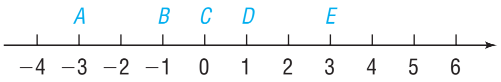
31. $x \geq -2$

32. $x < 4$

33. $x > -1$

34. $x \leq 7$

In Problems 35–40, use the given real number line to compute each distance.



35. $d(C, D)$

36. $d(C, A)$

37. $d(D, E)$

38. $d(C, E)$

39. $d(A, E)$

40. $d(D, B)$

In Problems 41–48, evaluate each expression if $x = -2$ and $y = 3$.

41. $x + 2y$

42. $3x + y$

43. $5xy + 2$

44. $-2x + xy$

45. $\dfrac{2x}{x - y}$

46. $\dfrac{x + y}{x - y}$

47. $\dfrac{3x + 2y}{2 + y}$

48. $\dfrac{2x - 3}{y}$

In Problems 49–58, find the value of each expression if $x = 3$ and $y = -2$.

49. $|x + y|$

50. $|x - y|$

51. $|x| + |y|$

52. $|x| - |y|$

53. $\dfrac{|x|}{x}$

54. $\dfrac{|y|}{y}$

55. $|4x - 5y|$

56. $|3x + 2y|$

57. $||4x| - |5y||$

58. $3|x| + 2|y|$

In Problems 59–66, determine which of the values (a) through (d), if any, must be excluded from the domain of the variable in each expression.

(a) $x = 3$ (b) $x = 1$ (c) $x = 0$ (d) $x = -1$

59. $\dfrac{x^2 - 1}{x}$

60. $\dfrac{x^2 + 1}{x}$

61. $\dfrac{x}{x^2 - 9}$

62. $\dfrac{x}{x^2 + 9}$

63. $\dfrac{x^2}{x^2 + 1}$

64. $\dfrac{x^3}{x^2 - 1}$

65. $\dfrac{x^2 + 5x - 10}{x^3 - x}$

66. $\dfrac{-9x^2 - x + 1}{x^3 + x}$

In Problems 67–70, determine the domain of the variable x in each expression.

67. $\dfrac{4}{x - 5}$

68. $\dfrac{-6}{x + 4}$

69. $\dfrac{x}{x + 4}$

70. $\dfrac{x - 2}{x - 6}$

In Problems 71–74, use the formula $C = \dfrac{5}{9}(F - 32)$ for converting degrees Fahrenheit into degrees Celsius to find the Celsius measure of each Fahrenheit temperature.

71. $F = 32°$

72. $F = 212°$

73. $F = 77°$

74. $F = -4°$

In Problems 75–86, simplify each expression.

75. $(-4)^2$

76. -4^2

77. 4^{-2}

78. -4^{-2}

79. $3^{-6} \cdot 3^4$

80. $4^{-2} \cdot 4^3$

81. $(3^{-2})^{-1}$

82. $(2^{-1})^{-3}$

83. $\sqrt{25}$

84. $\sqrt{36}$

85. $\sqrt{(-4)^2}$

86. $\sqrt{(-3)^2}$

In Problems 87–96, simplify each expression. Express the answer so that all exponents are positive. Whenever an exponent is 0 or negative, we assume that the base is not 0.

87. $(8x^3)^2$

88. $(-4x^2)^{-1}$

89. $(x^2y^{-1})^2$

90. $(x^{-1}y)^3$

91. $\dfrac{x^2y^3}{xy^4}$

92. $\dfrac{x^{-2}y}{xy^2}$

93. $\dfrac{(-2)^3x^4(yz)^2}{3^2xy^3z}$

94. $\dfrac{4x^{-2}(yz)^{-1}}{2^3x^4y}$

95. $\left(\dfrac{3x^{-1}}{4y^{-1}}\right)^{-2}$

96. $\left(\dfrac{5x^{-2}}{6y^{-2}}\right)^{-3}$

In Problems 97–108, find the value of each expression if $x = 2$ and $y = -1$.

97. $2xy^{-1}$ **98.** $-3x^{-1}y$ **99.** $x^2 + y^2$ **100.** x^2y^2

101. $(xy)^2$ **102.** $(x + y)^2$ **103.** $\sqrt{x^2}$ **104.** $(\sqrt{x})^2$

105. $\sqrt{x^2 + y^2}$ **106.** $\sqrt{x^2} + \sqrt{y^2}$ **107.** x^y **108.** y^x

109. Find the value of the expression $2x^3 - 3x^2 + 5x - 4$ if $x = 2$. What is the value if $x = 1$?

110. Find the value of the expression $4x^3 + 3x^2 - x + 2$ if $x = 1$. What is the value if $x = 2$?

111. What is the value of $\dfrac{(666)^4}{(222)^4}$? **112.** What is the value of $(0.1)^3(20)^3$?

In Problems 113–120, use a calculator to evaluate each expression. Round your answer to three decimal places.

113. $(8.2)^6$ **114.** $(3.7)^5$ **115.** $(6.1)^{-3}$ **116.** $(2.2)^{-5}$

117. $(-2.8)^6$ **118.** $-(2.8)^6$ **119.** $(-8.11)^{-4}$ **120.** $-(8.11)^{-4}$

In Problems 121–128, write each number in scientific notation.

121. 454.2 **122.** 32.14 **123.** 0.013 **124.** 0.00421

125. 32,155 **126.** 21,210 **127.** 0.000423 **128.** 0.0514

In Problems 129–136, write each number as a decimal.

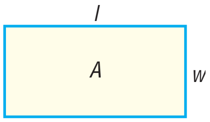
129. 6.15×10^4 **130.** 9.7×10^3 **131.** 1.214×10^{-3} **132.** 9.88×10^{-4}

133. 1.1×10^8 **134.** 4.112×10^2 **135.** 8.1×10^{-2} **136.** 6.453×10^{-1}

Applications and Extensions

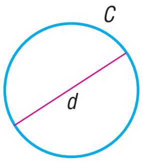
In Problems 137–146, express each statement as an equation involving the indicated variables.

137. Area of a Rectangle The area A of a rectangle is the product of its length l and its width w.

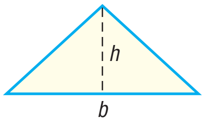


138. Perimeter of a Rectangle The perimeter P of a rectangle is twice the sum of its length l and its width w.

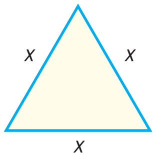
139. Circumference of a Circle The circumference C of a circle is the product of π and its diameter d.



140. Area of a Triangle The area A of a triangle is one-half the product of its base b and its height h.

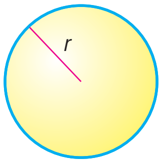


141. Area of an Equilateral Triangle The area A of an equilateral triangle is $\dfrac{\sqrt{3}}{4}$ times the square of the length x of one side.



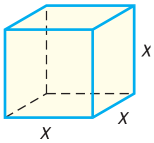
142. Perimeter of an Equilateral Triangle The perimeter P of an equilateral triangle is 3 times the length x of one side.

143. Volume of a Sphere The volume V of a sphere is $\dfrac{4}{3}$ times π times the cube of the radius r.



144. Surface Area of a Sphere The surface area S of a sphere is 4 times π times the square of the radius r.

145. Volume of a Cube The volume V of a cube is the cube of the length x of a side.



146. Surface Area of a Cube The surface area S of a cube is 6 times the square of the length x of a side.

147. Manufacturing Cost The weekly production cost C of manufacturing x watches is given by the formula $C = 4000 + 2x$, where the variable C is in dollars.

(a) What is the cost of producing 1000 watches?

(b) What is the cost of producing 2000 watches?

148. Balancing a Checkbook At the beginning of the month, Mike had a balance of \$210 in his checking account. During the next month, he deposited \$80, wrote a check for \$120, made another deposit of \$25, and wrote two checks: one for \$60 and the other for \$32. He was also assessed a monthly service charge of \$5. What was his balance at the end of the month?

In Problems 149 and 150, write an inequality using an absolute value to describe each statement.

149. x is at least 6 units from 4.

150. x is more than 5 units from 2.

151. U.S. Voltage In the United States, normal household voltage is 110 volts. It is acceptable for the actual voltage x to differ from normal by at most 5 volts. A formula that describes this is

$$|x - 110| \leq 5$$

(a) Show that a voltage of 108 volts is acceptable.

(b) Show that a voltage of 104 volts is not acceptable.

152. Foreign Voltage In other countries, normal household voltage is 220 volts. It is acceptable for the actual voltage x to differ from normal by at most 8 volts. A formula that describes this is

$$|x - 220| \leq 8$$

(a) Show that a voltage of 214 volts is acceptable.

(b) Show that a voltage of 209 volts is not acceptable.

153. Making Precision Ball Bearings The FireBall Company manufactures ball bearings for precision equipment. One of its products is a ball bearing with a stated radius of 3 centimeters (cm). Only ball bearings with a radius within 0.01 cm of this stated radius are acceptable. If x is the radius of a ball bearing, a formula describing this situation is

$$|x - 3| \leq 0.01$$

(a) Is a ball bearing of radius $x = 2.999$ acceptable?

(b) Is a ball bearing of radius $x = 2.89$ acceptable?

154. Body Temperature Normal human body temperature is 98.6°F. A temperature x that differs from normal by at least 1.5°F is considered unhealthy. A formula that describes this is

$$|x - 98.6| \geq 1.5$$

(a) Show that a temperature of 97°F is unhealthy.

(b) Show that a temperature of 100°F is not unhealthy.

155. Distance from Earth to Its Moon The distance from Earth to the Moon is about 4×10^8 meters.* Express this distance as a whole number.

156. Height of Mt. Everest The height of Mt. Everest is 8848 meters.* Express this height in scientific notation.

157. Wavelength of Visible Light The wavelength of visible light is about 5×10^{-7} meter.* Express this wavelength as a decimal.

158. Diameter of an Atom The diameter of an atom is about 1×10^{-10} meter.* Express this diameter as a decimal.

159. Diameter of Copper Wire The smallest commercial copper wire is about 0.0005 inch in diameter.† Express this diameter using scientific notation.

160. Smallest Motor The smallest motor ever made is less than 0.05 centimeter wide.† Express this width using scientific notation.

161. Astronomy One light-year is defined by astronomers to be the distance that a beam of light will travel in 1 year (365 days). If the speed of light is 186,000 miles per second, how many miles are in a light-year? Express your answer in scientific notation.

162. Astronomy How long does it take a beam of light to reach Earth from the Sun when the Sun is 93,000,000 miles from Earth? Express your answer in seconds, using scientific notation.

163. Does $\frac{1}{3}$ equal 0.333? If not, which is larger? By how much?

164. Does $\frac{2}{3}$ equal 0.666? If not, which is larger? By how much?

Explaining Concepts: Discussion and Writing

165. Is there a positive real number "closest" to 0?

166. Number game I'm thinking of a number! It lies between 1 and 10; its square is rational and lies between 1 and 10. The number is larger than π. Correct to two decimal places (that is, truncated to two decimal places), name the number. Now think of your own number, describe it, and challenge a fellow student to name it.

167. Write a brief paragraph that illustrates the similarities and differences between "less than" $(<)$ and "less than or equal to" $(\leq)$.

168. Give a reason why the statement $5 < 8$ is true.

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* *Powers of Ten,* Philip and Phylis Morrison.

† *2011 Information Please Almanac.*

R.3 Geometry Essentials

OBJECTIVES
1 Use the Pythagorean Theorem and Its Converse (p. 31)
2 Know Geometry Formulas (p. 32)
3 Understand Congruent Triangles and Similar Triangles (p. 33)

1 Use the Pythagorean Theorem and Its Converse

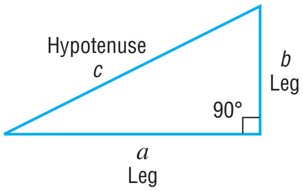


Figure 19 A right triangle

The *Pythagorean Theorem* is a statement about *right triangles*. A **right triangle** is one that contains a **right angle**—that is, an angle of 90°. The side of the triangle opposite the 90° angle is called the **hypotenuse**; the remaining two sides are called **legs**. In Figure 19 we have used c to represent the length of the hypotenuse and a and b to represent the lengths of the legs. Notice the use of the symbol ⊾ to show the 90° angle. We now state the Pythagorean Theorem.

PYTHAGOREAN THEOREM

In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs. That is, in the right triangle shown in Figure 19,

$$\boxed{c^2 = a^2 + b^2 \qquad (1)}$$

A proof of the Pythagorean Theorem is given at the end of this section.

EXAMPLE 1 **Finding the Hypotenuse of a Right Triangle**

In a right triangle, one leg has length 4 and the other has length 3. What is the length of the hypotenuse?

Solution Since the triangle is a right triangle, we use the Pythagorean Theorem with $a = 4$ and $b = 3$ to find the length c of the hypotenuse. From equation (1),

$$c^2 = a^2 + b^2$$
$$c^2 = 4^2 + 3^2 = 16 + 9 = 25$$
$$c = \sqrt{25} = 5$$

Now Work PROBLEM 15

The converse of the Pythagorean Theorem is also true.

CONVERSE OF THE PYTHAGOREAN THEOREM

In a triangle, if the square of the length of one side equals the sum of the squares of the lengths of the other two sides, the triangle is a right triangle. The 90° angle is opposite the longest side.

A proof is given at the end of this section.

EXAMPLE 2 **Verifying That a Triangle Is a Right Triangle**

Show that a triangle whose sides are of lengths 5, 12, and 13 is a right triangle. Identify the hypotenuse.

Solution Square the lengths of the sides.

$$5^2 = 25 \qquad 12^2 = 144 \qquad 13^2 = 169$$

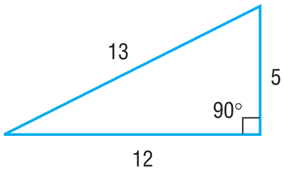


Figure 20

Notice that the sum of the first two squares (25 and 144) equals the third square (169). That is, because $5^2 + 12^2 = 13^2$, the triangle is a right triangle. The longest side, 13, is the hypotenuse. See Figure 20. ■

Now Work PROBLEM 23

EXAMPLE 3 Applying the Pythagorean Theorem

The tallest building in the world is Burj Khalifa in Dubai, United Arab Emirates, at 2717 feet and 163 floors. The observation deck is 1483 feet above ground level. How far can a person standing on the observation deck see (with the aid of a telescope)? Use 3960 miles for the radius of Earth.

Source: Council on Tall Buildings and Urban Habitat

Solution From the center of Earth, draw two radii: one through Burj Khalifa and the other to the farthest point a person can see from the observation deck. See Figure 21. Apply the Pythagorean Theorem to the right triangle.

Since 1 mile = 5280 feet, $1483 \text{ feet} = \frac{1483}{5280}$ mile. Then

$$d^2 + (3960)^2 = \left(3960 + \frac{1483}{5280}\right)^2$$

$$d^2 = \left(3960 + \frac{1483}{5280}\right)^2 - (3960)^2 \approx 2224.58$$

$$d \approx 47.17$$

A person can see more than 47 miles from the observation deck.

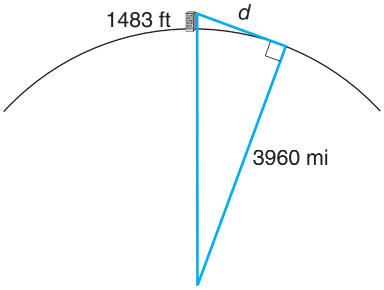


Figure 21

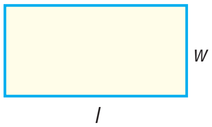
■

Now Work PROBLEM 55

2 Know Geometry Formulas

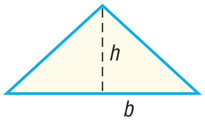
Certain formulas from geometry are useful in solving algebra problems.

For a rectangle of length l and width w,

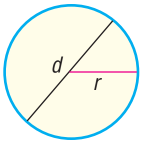


$$\text{Area} = lw \qquad \text{Perimeter} = 2l + 2w$$

For a triangle with base b and altitude h,

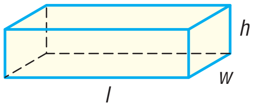


$$\text{Area} = \frac{1}{2}bh$$



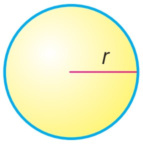
For a circle of radius r (diameter $d = 2r$),

$$\text{Area} = \pi r^2 \qquad \text{Circumference} = 2\pi r = \pi d$$



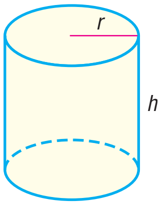
For a closed rectangular box of length l, width w, and height h,

$$\text{Volume} = lwh \qquad \text{Surface area} = 2lh + 2wh + 2lw$$



For a sphere of radius r,

$$\text{Volume} = \frac{4}{3}\pi r^3 \qquad \text{Surface area} = 4\pi r^2$$



For a closed right circular cylinder of height h and radius r,

$$\text{Volume} = \pi r^2 h \qquad \text{Surface area} = 2\pi r^2 + 2\pi rh$$

Now Work PROBLEM 31

EXAMPLE 4

Using Geometry Formulas

A Christmas tree ornament is in the shape of a semicircle on top of a triangle. How many square centimeters (cm^2) of copper is required to make the ornament if the height of the triangle is 6 cm and the base is 4 cm?

Solution See Figure 22. The amount of copper required equals the shaded area. This area is the sum of the areas of the triangle and the semicircle. The triangle has height $h = 6$ and base $b = 4$. The semicircle has diameter $d = 4$, so its radius is $r = 2$.

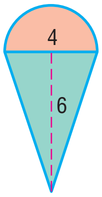


Figure 22

$$\begin{aligned}\text{Total area} &= \text{Area of triangle} + \text{Area of semicircle}\\ &= \frac{1}{2}bh + \frac{1}{2}\pi r^2 = \frac{1}{2}(4)(6) + \frac{1}{2}\pi \cdot 2^2 \quad b = 4;\ h = 6;\ r = 2\\ &= 12 + 2\pi \approx 18.28\ \text{cm}^2\end{aligned}$$

About 18.28 cm^2 of copper is required.

Now Work PROBLEM 49

3 Understand Congruent Triangles and Similar Triangles

Throughout the text we will make reference to triangles. We begin with a discussion of *congruent* triangles. According to dictionary.com, the word **congruent** means "coinciding exactly when superimposed." For example, two angles are congruent if they have the same measure, and two line segments are congruent if they have the same length.

DEFINITION

Two triangles are **congruent** if each pair of corresponding angles have the same measure and each pair of corresponding sides are the same length.

In Words
Two triangles are congruent if they have the same size and shape.

In Figure 23 on the next page, corresponding angles are equal and the corresponding sides are equal in length: $a = d$, $b = e$, and $c = f$. As a result, these triangles are congruent.

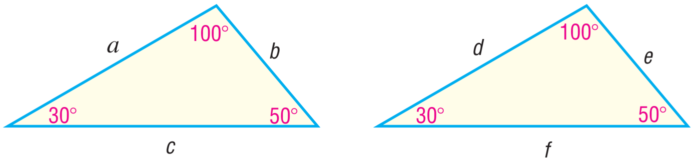


Figure 23 Congruent triangles

Actually, it is not necessary to verify that all three angles and all three sides are the same measure to determine whether two triangles are congruent.

Determining Congruent Triangles

1. **Angle–Side–Angle Case** Two triangles are congruent if two of the angles are equal and the lengths of the corresponding sides between the two angles are equal.

   For example, in Figure 24(a), the two triangles are congruent because two angles and the included side are equal.

2. **Side–Side–Side Case** Two triangles are congruent if the lengths of the corresponding sides of the triangles are equal.

   For example, in Figure 24(b), the two triangles are congruent because the three corresponding sides are all equal.

3. **Side–Angle–Side Case** Two triangles are congruent if the lengths of two corresponding sides are equal and the angles between the two sides are the same.

   For example, in Figure 24(c), the two triangles are congruent because two sides and the included angle are equal.

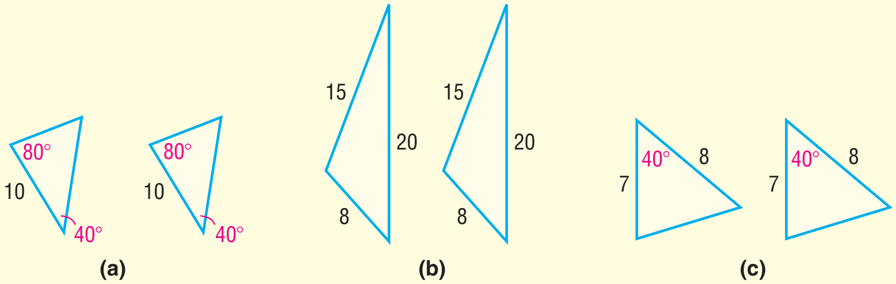


Figure 24

We contrast congruent triangles with *similar* triangles.

DEFINITION Two triangles are **similar** if the corresponding angles are equal and the lengths of the corresponding sides are proportional.

In Words
Two triangles are similar if they have the same shape, but (possibly) different sizes.

For example, the triangles in Figure 25 are similar because the corresponding angles are equal. In addition, the lengths of the corresponding sides are proportional because each side in the triangle on the right is twice as long as each corresponding side in the triangle on the left. That is, the ratio of the corresponding sides is a constant: $\frac{d}{a} = \frac{e}{b} = \frac{f}{c} = 2$.

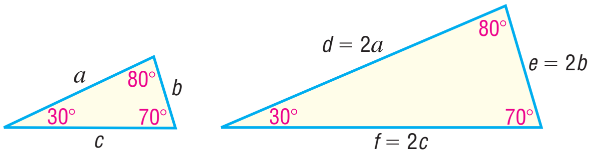


Figure 25 Similar triangles

It is not necessary to verify that all three angles are equal and all three sides are proportional to determine whether two triangles are similar.

Determining Similar Triangles

1. **Angle–Angle Case** Two triangles are similar if two of the corresponding angles are equal.
   For example, in Figure 26(a), the two triangles are similar because two angles are equal.
2. **Side–Side–Side Case** Two triangles are similar if the lengths of all three sides of each triangle are proportional.
   For example, in Figure 26(b), the two triangles are similar because
$$\frac{10}{30} = \frac{5}{15} = \frac{6}{18} = \frac{1}{3}$$
3. **Side–Angle–Side Case** Two triangles are similar if two corresponding sides are proportional and the angles between the two sides are equal.

   For example, in Figure 26(c), the two triangles are similar because $\frac{4}{6} = \frac{12}{18} = \frac{2}{3}$ and the angles between the sides are equal.

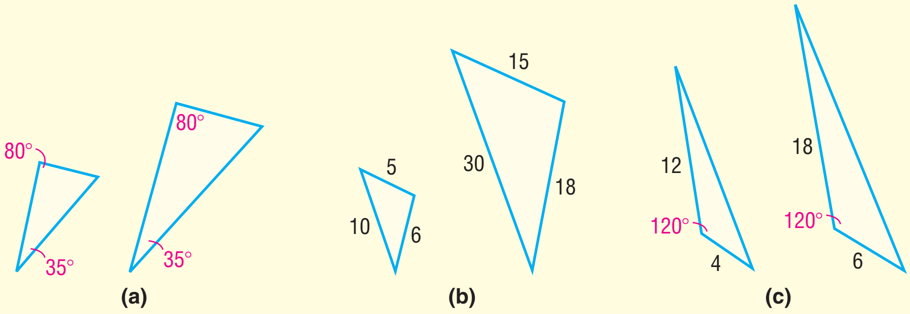


Figure 26

EXAMPLE 5

Using Similar Triangles

Given that the triangles in Figure 27 are similar, find the missing length x and the angles A, B, and C.

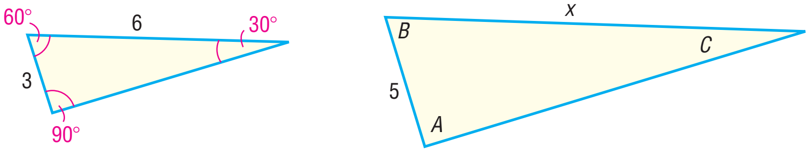


Figure 27

Solution Because the triangles are similar, corresponding angles are equal. So $A = 90°$, $B = 60°$, and $C = 30°$. Also, the corresponding sides are proportional. That is, $\frac{3}{5} = \frac{6}{x}$. We solve this equation for x.

$$\frac{3}{5} = \frac{6}{x}$$

$$5x \cdot \frac{3}{5} = 5x \cdot \frac{6}{x} \qquad \text{Multiply both sides by } 5x.$$

$$3x = 30 \qquad \text{Simplify.}$$

$$x = 10 \qquad \text{Divide both sides by 3.}$$

The missing length is 10 units. ■

Now Work PROBLEM 43

Proof of the Pythagorean Theorem Begin with a square, each side of length $a + b$. In this square, form four right triangles, each having legs equal in length to a and b. See Figure 28. All these triangles are congruent (two sides and their included angle are equal). As a result, the hypotenuse of each is the same, say c, and the pink shading in Figure 28 indicates a square with an area equal to c^2.

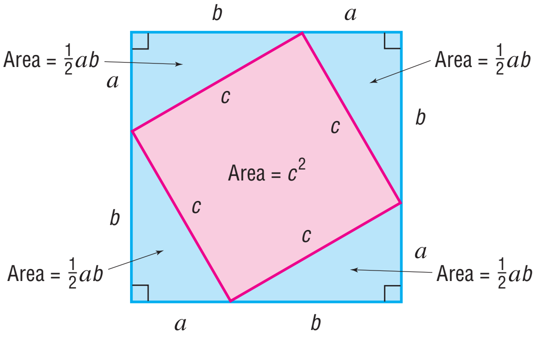


Figure 28

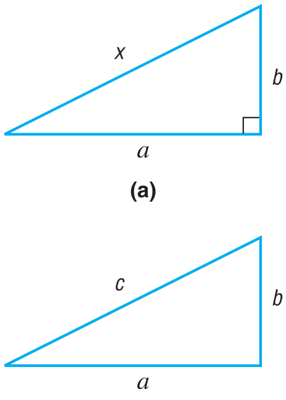
The area of the original square with sides $a + b$ equals the sum of the areas of the four triangles (each of area $\frac{1}{2}ab$) plus the area of the square with side c. That is,

$$(a + b)^2 = \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab + c^2$$

$$a^2 + 2ab + b^2 = 2ab + c^2$$

$$a^2 + b^2 = c^2$$

The proof is complete. ■



(a)

(b) $c^2 = a^2 + b^2$

Figure 29

Proof of the Converse of the Pythagorean Theorem Begin with two triangles: one a right triangle with legs a and b and the other a triangle with sides a, b, and c for which $c^2 = a^2 + b^2$. See Figure 29. By the Pythagorean Theorem, the length x of the third side of the first triangle is

$$x^2 = a^2 + b^2$$

But $c^2 = a^2 + b^2$. Then,

$$x^2 = c^2$$

$$x = c$$

The two triangles have the same sides and are therefore congruent. This means corresponding angles are equal, so the angle opposite side c of the second triangle equals 90°.

The proof is complete. ■

R.3 Assess Your Understanding

Concepts and Vocabulary

1. A(n) ____ triangle is one that contains an angle of 90 degrees. The longest side is called the __________.

2. For a triangle with base b and altitude h, a formula for the area A is ________.

3. The formula for the circumference C of a circle of radius r is ________.

4. Two triangles are _______ if corresponding angles are equal and the lengths of the corresponding sides are proportional.

5. Which of the following is not a case for determining congruent triangles?

(a) Angle–Side–Angle (b) Side–Angle–Side

(c) Angle–Angle–Angle (d) Side-Side-Side

6. Choose the formula for the volume of a sphere of radius r.

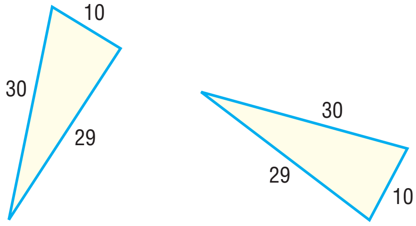
(a) $\frac{4}{3}\pi r^2$ (b) $\frac{4}{3}\pi r^3$ (c) $4\pi r^3$ (d) $4\pi r^2$

7. ***True or False*** In a right triangle, the square of the length of the longest side equals the sum of the squares of the lengths of the other two sides.

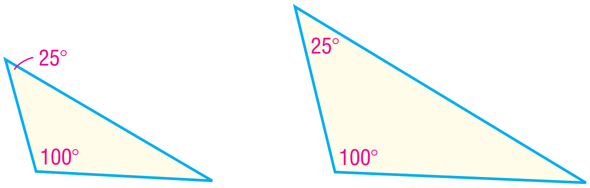
8. ***True or False*** The triangle with sides of lengths 6, 8, and 10 is a right triangle.

9. ***True or False*** The surface area of a sphere of radius r is $\frac{4}{3}\pi r^2$.

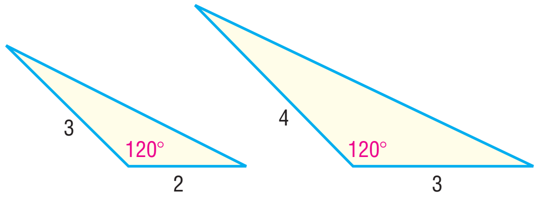
10. ***True or False*** The triangles shown are congruent.



11. ***True or False*** The triangles shown are similar.



12. ***True or False*** The triangles shown are similar.



Skill Building

In Problems 13–18, the lengths of the legs of a right triangle are given. Find the hypotenuse.

13. $a = 5, \quad b = 12$

14. $a = 6, \quad b = 8$

15. $a = 10, \quad b = 24$

16. $a = 4, \quad b = 3$

17. $a = 7, \quad b = 24$

18. $a = 14, \quad b = 48$

In Problems 19–26, the lengths of the sides of a triangle are given. Determine which are right triangles. For those that are, identify the hypotenuse.

19. 3, 4, 5

20. 6, 8, 10

21. 4, 5, 6

22. 2, 2, 3

23. 7, 24, 25

24. 10, 24, 26

25. 6, 4, 3

26. 5, 4, 7

27. Find the area A of a rectangle with length 4 inches and width 2 inches.

28. Find the area A of a rectangle with length 9 centimeters and width 4 centimeters.

29. Find the area A of a triangle with height 4 inches and base 2 inches.

30. Find the area A of a triangle with height 9 centimeters and base 4 centimeters.

31. Find the area A and circumference C of a circle of radius 5 meters.

32. Find the area A and circumference C of a circle of radius 2 feet.

33. Find the volume V and surface area S of a closed rectangular box with length 8 feet, width 4 feet, and height 7 feet.

34. Find the volume V and surface area S of a closed rectangular box with length 9 inches, width 4 inches, and height 8 inches.

35. Find the volume V and surface area S of a sphere of radius 4 centimeters.

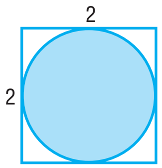
36. Find the volume V and surface area S of a sphere of radius 3 feet.

37. Find the volume V and surface area S of a closed right circular cylinder with radius 9 inches and height 8 inches.

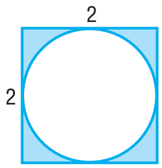
38. Find the volume V and surface area S of a closed right circular cylinder with radius 8 inches and height 9 inches.

In Problems 39–42, find the area of the shaded region.

39.



40.



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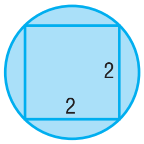

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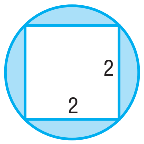
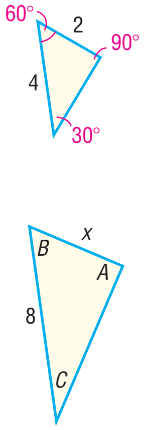


In Problems 43–46, the triangles in each pair are similar. Find the missing length x and the missing angles A, B, and C.

43.



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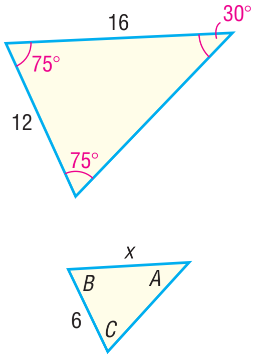

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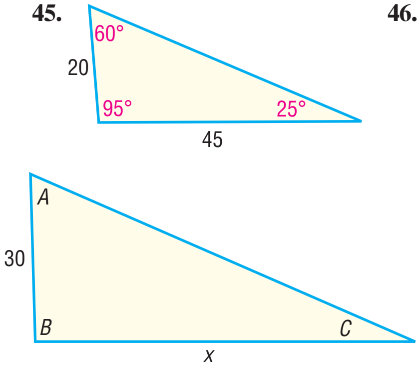

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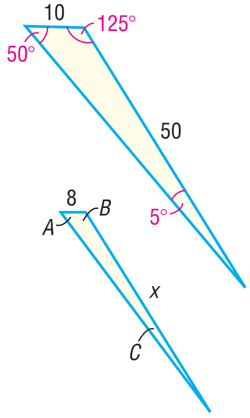
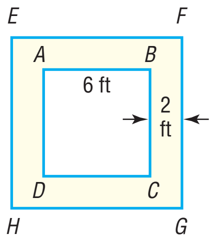


Applications and Extensions

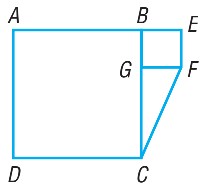
47. How many feet has a wheel with a diameter of 16 inches traveled after four revolutions?

48. How many revolutions will a circular disk with a diameter of 4 feet have completed after it has rolled 20 feet?

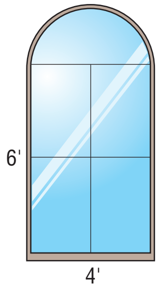
49. In the figure shown, *ABCD* is a square, with each side of length 6 feet. The width of the border (shaded portion) between the outer square *EFGH* and *ABCD* is 2 feet. Find the area of the border.



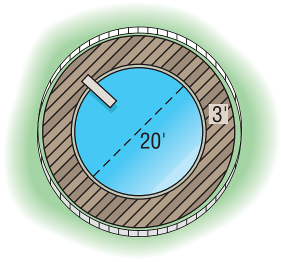
50. Refer to the figure. Square *ABCD* has an area of 100 square feet; square *BEFG* has an area of 16 square feet. What is the area of the triangle *CGF*?



51. Architecture A Norman window consists of a rectangle surmounted by a semicircle. Find the area of the Norman window shown in the illustration. How much wood frame is needed to enclose the window?

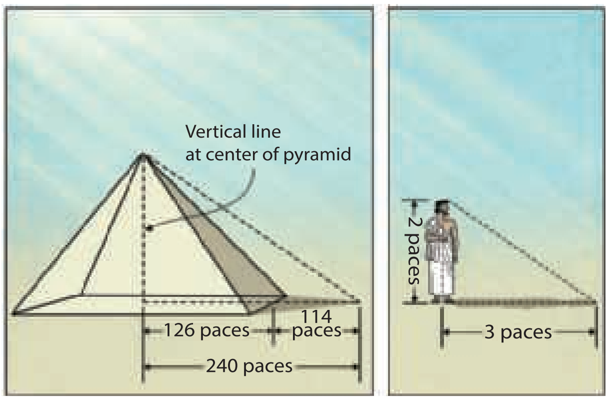


52. Construction A circular swimming pool that is 20 feet in diameter is enclosed by a wooden deck that is 3 feet wide. What is the area of the deck? How much fence is required to enclose the deck?



53. How Tall Is the Great Pyramid? The ancient Greek philosopher Thales of Miletus is reported on one occasion to have visited Egypt and calculated the height of the Great Pyramid of Cheops by means of shadow reckoning. Thales knew that each side of the base of the pyramid was 252 paces and that his own height was 2 paces. He measured the length of the pyramid's shadow to be 114 paces and determined the length of his shadow to be 3 paces. See the illustration. Using similar triangles, determine the height of the Great Pyramid in terms of the number of paces.

Source: Diggins, Julie E, *String Straightedge and Shadow: The Story of Geometry,* 2003, Whole Spirit Press, http://wholespiritpress.com.



54. The Bermuda Triangle Karen is doing research on the Bermuda Triangle, which she defines roughly by Hamilton, Bermuda; San Juan, Puerto Rico; and Fort Lauderdale, Florida. On her atlas Karen measures the straight-line distances from Hamilton to Fort Lauderdale, Fort Lauderdale to San Juan, and San Juan to Hamilton to be approximately 57 millimeters (mm), 58 mm, and 53.5 mm respectively. If the actual distance from Fort Lauderdale to San Juan is 1046 miles, approximate the actual distances from San Juan to Hamilton and from Hamilton to Fort Lauderdale.

In Problems 55–57, use the facts that the radius of Earth is 3960 miles and 1 mile = 5280 feet.

55. How Far Can You See? The conning tower of the U.S.S. *Silversides,* a World War II submarine now permanently stationed in Muskegon, Michigan, is approximately 20 feet above sea level. How far can you see from the conning tower?

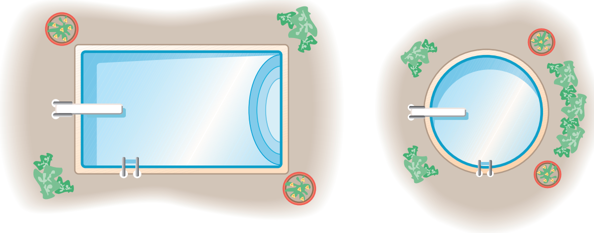
56. How Far Can You See? A person who is 6 feet tall is standing on the beach in Fort Lauderdale, Florida, and looks out onto the Atlantic Ocean. Suddenly, a ship appears on the horizon. How far is the ship from shore?

57. How Far Can You See? The deck of a destroyer is 100 feet above sea level. How far can a person see from the deck? How far can a person see from the bridge, which is 150 feet above sea level?

58. Suppose that m and n are positive integers with $m > n$. If $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$, show that a, b, and c are the lengths of the sides of a right triangle. (This formula can be used to find the sides of a right triangle that are integers, such as 3, 4, 5; 5, 12, 13; and so on. Such triplets of integers are called **Pythagorean triples**.)

Explaining Concepts: Discussion and Writing

59. You have 1000 feet of flexible pool siding and intend to construct a swimming pool. Experiment with rectangular-shaped pools with perimeters of 1000 feet. How do their areas vary? What is the shape of the rectangle with the largest area? Now compute the area enclosed by a circular pool with a perimeter (circumference) of 1000 feet. What would be your choice of shape for the pool? If rectangular, what is your preference for dimensions? Justify your choice. If your only consideration is to have a pool that encloses the most area, what shape should you use?



60. The Gibb's Hill Lighthouse, Southampton, Bermuda, in operation since 1846, stands 117 feet high on a hill 245 feet high, so its beam of light is 362 feet above sea level. A brochure states that the light itself can be seen on the horizon about 26 miles distant. Verify the accuracy of this information. The brochure further states that ships 40 miles away can see the light and that planes flying at 10,000 feet can see it 120 miles away. Verify the accuracy of these statements. What assumption did the brochure make about the height of the ship?

OBJECTIVES
1 Recognize Monomials (p. 40)
2 Recognize Polynomials (p. 41)
3 Add and Subtract Polynomials (p. 42)
4 Multiply Polynomials (p. 43)
5 Know Formulas for Special Products (p. 44)
6 Divide Polynomials Using Long Division (p. 45)
7 Work with Polynomials in Two Variables (p. 48)

We have described algebra as a generalization of arithmetic in which letters are used to represent real numbers. From now on, we shall use the letters at the end of the alphabet, such as x, y, and z, to represent variables and use the letters at the beginning of the alphabet, such as a, b, and c, to represent constants. In the expressions $3x + 5$ and $ax + b$, it is understood that x is a variable and that a and b are constants, even though the constants a and b are unspecified. As you will find out, the context usually makes the intended meaning clear.

1 Recognize Monomials

DEFINITION

A **monomial** in one variable is the product of a constant and a variable raised to a nonnegative integer power. A monomial is of the form

$$\boxed{ax^k}$$

where a is a constant, x is a variable, and $k \geq 0$ is an integer. The constant a is called the **coefficient** of the monomial. If $a \neq 0$, then k is called the **degree** of the monomial. ■

Note: The nonnegative integers are the integers 0, 1, 2, 3,.... ■

EXAMPLE 1 Examples of Monomials

Monomial	**Coefficient**	**Degree**	
(a) $6x^2$	6	2	
(b) $-\sqrt{2}x^3$	$-\sqrt{2}$	3	
(c) 3	3	0	Since $3 = 3 \cdot 1 = 3x^0, x \neq 0$
(d) $-5x$	-5	1	Since $-5x = -5x^1$
(e) x^4	1	4	Since $x^4 = 1 \cdot x^4$

■

EXAMPLE 2 Examples of Nonmonomial Expressions

(a) $3x^{1/2}$ is not a monomial, since the exponent of the variable x is $\frac{1}{2}$, and $\frac{1}{2}$ is not a nonnegative integer.

(b) $4x^{-3}$ is not a monomial, since the exponent of the variable x is -3, and -3 is not a nonnegative integer. ■

Now Work PROBLEM 9

2 Recognize Polynomials

Two monomials with the same variable raised to the same power are called **like terms**. For example, $2x^4$ and $-5x^4$ are like terms. In contrast, the monomials $2x^3$ and $2x^5$ are not like terms.

We can add or subtract like terms using the Distributive Property. For example,

$$2x^2 + 5x^2 = (2 + 5)x^2 = 7x^2 \quad \text{and} \quad 8x^3 - 5x^3 = (8 - 5)x^3 = 3x^3$$

The sum or difference of two monomials having different degrees is called a **binomial**. The sum or difference of three monomials with three different degrees is called a **trinomial**. For example,

$x^2 - 2$ is a binomial.
$x^3 - 3x + 5$ is a trinomial.
$2x^2 + 5x^2 + 2 = 7x^2 + 2$ is a binomial.

DEFINITION

A **polynomial** in one variable is an algebraic expression of the form

$$\boxed{a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \qquad \textbf{(1)}}$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants,* called the **coefficients** of the polynomial, $n \geq 0$ is an integer, and x is a variable. If $a_n \neq 0$, it is called the **leading coefficient**, a_nx^n is called the **leading term**, and n is the **degree** of the polynomial. ■

In Words
A polynomial is a sum of monomials.

The monomials that make up a polynomial are called its **terms**. If all of the coefficients are 0, the polynomial is called the **zero polynomial**, which has no degree.

Polynomials are usually written in **standard form**, beginning with the nonzero term of highest degree and continuing with terms in descending order according to degree. If a power of x is missing, it is because its coefficient is zero.

EXAMPLE 3 Examples of Polynomials

Polynomial	**Coefficients**	**Degree**
$-8x^3 + 4x^2 - 6x + 2$	$-8, 4, -6, 2$	3
$3x^2 - 5 = 3x^2 + 0 \cdot x + (-5)$	$3, 0, -5$	2
$8 - 2x + x^2 = 1 \cdot x^2 + (-2)x + 8$	$1, -2, 8$	2
$5x + \sqrt{2} = 5x^1 + \sqrt{2}$	$5, \sqrt{2}$	1
$3 = 3 \cdot 1 = 3 \cdot x^0$	3	0
0	0	No degree ■

Although we have been using x to represent the variable, letters such as y and z are also commonly used.

$3x^4 - x^2 + 2$ is a polynomial (in x) of degree 4.
$9y^3 - 2y^2 + y - 3$ is a polynomial (in y) of degree 3.
$z^5 + \pi$ is a polynomial (in z) of degree 5.

Algebraic expressions such as

$$\frac{1}{x} \quad \text{and} \quad \frac{x^2 + 1}{x + 5}$$

---

* The notation a_n is read as "a sub n." The number n is called a **subscript** and should not be confused with an exponent. We use subscripts to distinguish one constant from another when a large or undetermined number of constants are required.

are not polynomials. The first is not a polynomial because $\frac{1}{x} = x^{-1}$ has an exponent that is not a nonnegative integer. Although the second expression is the quotient of two polynomials, the polynomial in the denominator has degree greater than 0, so the expression cannot be a polynomial.

Now Work PROBLEM 19

3 Add and Subtract Polynomials

Polynomials are added and subtracted by combining like terms.

EXAMPLE 4 Adding Polynomials

Find the sum of the polynomials:

$$8x^3 - 2x^2 + 6x - 2 \quad \text{and} \quad 3x^4 - 2x^3 + x^2 + x$$

Solution We shall find the sum in two ways.

Horizontal Addition: The idea here is to group the like terms and then combine them.

$$\begin{aligned} &(8x^3 - 2x^2 + 6x - 2) + (3x^4 - 2x^3 + x^2 + x) \\ &= 3x^4 + (8x^3 - 2x^3) + (-2x^2 + x^2) + (6x + x) - 2 \\ &= 3x^4 + 6x^3 - x^2 + 7x - 2 \end{aligned}$$

Vertical Addition: The idea here is to vertically line up the like terms in each polynomial and then add the coefficients.

$$\begin{array}{rrrrrr} & x^4 & x^3 & x^2 & x^1 & x^0 \\ & & 8x^3 & -\ 2x^2 & +\ 6x & -\ 2 \\ + & 3x^4 & -\ 2x^3 & +\ x^2 & +\ x & \\ \hline & 3x^4 & +\ 6x^3 & -\ x^2 & +\ 7x & -\ 2 \end{array}$$

■

We can subtract two polynomials horizontally or vertically as well.

EXAMPLE 5 Subtracting Polynomials

Find the difference: $(3x^4 - 4x^3 + 6x^2 - 1) - (2x^4 - 8x^2 - 6x + 5)$

Solution ***Horizontal Subtraction:***

$$\begin{aligned} &(3x^4 - 4x^3 + 6x^2 - 1) - (2x^4 - 8x^2 - 6x + 5) \\ &= 3x^4 - 4x^3 + 6x^2 - 1 + \underbrace{(-2x^4 + 8x^2 + 6x - 5)}_{\text{Be sure to change the sign of each term in the second polynomial.}} \\ &\underset{\substack{\uparrow \\ \text{Group like terms.}}}{=} (3x^4 - 2x^4) + (-4x^3) + (6x^2 + 8x^2) + 6x + (-1 - 5) \\ &= x^4 - 4x^3 + 14x^2 + 6x - 6 \end{aligned}$$

COMMENT Vertical subtraction will be used when we divide polynomials. ■

Vertical Subtraction: We line up like terms, change the sign of each coefficient of the second polynomial, and add.

$$\begin{array}{crrrrr} & x^4 & x^3 & x^2 & x^1 & x^0 \\ & 3x^4 & -\,4x^3 & +\,6x^2 & & -\,1 \\ - & [\underline{2x^4} & & \underline{-\,8x^2} & \underline{-\,6x} & \underline{+\,5}\,] \end{array} = \begin{array}{crrrrr} & x^4 & x^3 & x^2 & x^1 & x^0 \\ & 3x^4 & -\,4x^3 & +\;\;6x^2 & & -\,1 \\ + & -2x^4 & & +\;\;8x^2 & +\,6x & -\,5 \\ \hline & x^4 & -\,4x^3 & +\,14x^2 & +\,6x & -\,6 \end{array}$$

■

Which method to use for adding and subtracting polynomials is up to you. To save space, we shall most often use the horizontal format.

Now Work PROBLEM 31

4 Multiply Polynomials

Two monomials may be multiplied using the Laws of Exponents and the Commutative and Associative Properties. For example,

$$(2x^3)\cdot(5x^4) = (2\cdot 5)\cdot(x^3\cdot x^4) = 10x^{3+4} = 10x^7$$

Products of polynomials are found by repeated use of the Distributive Property and the Laws of Exponents. Again, you have a choice of horizontal or vertical format.

EXAMPLE 6 Multiplying Polynomials

Find the product: $(2x+5)(x^2-x+2)$

Solution ***Horizontal Multiplication:***

$$\begin{aligned} (2x+5)(x^2-x+2) &\underset{\substack{\uparrow \\ \text{Distributive Property}}}{=} 2x(x^2-x+2)+5(x^2-x+2) \\ &\underset{\substack{\uparrow \\ \text{Distributive Property}}}{=} (2x\cdot x^2 - 2x\cdot x + 2x\cdot 2) + (5\cdot x^2 - 5\cdot x + 5\cdot 2) \\ &\underset{\substack{\uparrow \\ \text{Law of Exponents}}}{=} (2x^3 - 2x^2 + 4x) + (5x^2 - 5x + 10) \\ &\underset{\substack{\uparrow \\ \text{Combine like terms.}}}{=} 2x^3 + 3x^2 - x + 10 \end{aligned}$$

Vertical Multiplication: The idea here is very much like multiplying a two-digit number by a three-digit number.

$$\begin{array}{rrrrrl} & & x^2 & -\;\;x & +\;\;2 & \\ & & & 2x & +\;\;5 & \\ \hline & 2x^3 & -\,2x^2 & +\,4x & & \text{This line is } 2x(x^2-x+2). \\ (+) & & 5x^2 & -\,5x & +\,10 & \text{This line is } 5(x^2-x+2). \\ \hline & 2x^3 & +\,3x^2 & -\;\;x & +\,10 & \text{Sum of the above two lines} \end{array}$$

■

Now Work PROBLEM 47

5 Know Formulas for Special Products

Certain products, which we call **special products**, occur frequently in algebra. We can calculate them easily using the **FOIL** (*F*irst, *O*uter, *I*nner, *L*ast) method of multiplying two binomials.

Outer

First

$$(ax+b)(cx+d) = ax(cx+d) + b(cx+d)$$

Inner

Last

$$= \overbrace{ax\cdot cx}^{\text{First}} + \overbrace{ax\cdot d}^{\text{Outer}} + \overbrace{b\cdot cx}^{\text{Inner}} + \overbrace{b\cdot d}^{\text{Last}}$$
$$= acx^2 + adx + bcx + bd$$
$$= acx^2 + (ad+bc)x + bd$$

EXAMPLE 7

Using FOIL

(a) $(x-3)(x+3) = \underset{\text{F}}{x^2} + \underset{\text{O}}{3x} - \underset{\text{I}}{3x} - \underset{\text{L}}{9} = x^2 - 9$

(b) $(x+2)^2 = (x+2)(x+2) = x^2 + 2x + 2x + 4 = x^2 + 4x + 4$

(c) $(x-3)^2 = (x-3)(x-3) = x^2 - 3x - 3x + 9 = x^2 - 6x + 9$

(d) $(x+3)(x+1) = x^2 + x + 3x + 3 = x^2 + 4x + 3$

(e) $(2x+1)(3x+4) = 6x^2 + 8x + 3x + 4 = 6x^2 + 11x + 4$ ■

Now Work PROBLEMS 49 AND 57

Some products have been given special names because of their form. The following special products are based on Examples 7(a), (b), and (c).

Difference of Two Squares

$$(x-a)(x+a) = x^2 - a^2 \qquad \textbf{(2)}$$

Squares of Binomials, or Perfect Squares

$$(x+a)^2 = x^2 + 2ax + a^2 \qquad \textbf{(3a)}$$
$$(x-a)^2 = x^2 - 2ax + a^2 \qquad \textbf{(3b)}$$

EXAMPLE 8

Using Special Product Formulas

(a) $(x-5)(x+5) = x^2 - 5^2 = x^2 - 25$ — Difference of two squares

(b) $(x+7)^2 = x^2 + 2\cdot 7\cdot x + 7^2 = x^2 + 14x + 49$ — Square of a binomial

(c) $(2x+1)^2 = (2x)^2 + 2\cdot 1\cdot 2x + 1^2 = 4x^2 + 4x + 1$ — Notice that we used $2x$ in place of x in formula (3a).

(d) $(3x-4)^2 = (3x)^2 - 2\cdot 4\cdot 3x + 4^2 = 9x^2 - 24x + 16$ — Replace x by $3x$ in formula (3b). ■

Now Work PROBLEMS 67, 69, AND 71

Let's look at some more examples that lead to general formulas.

EXAMPLE 9 Cubing a Binomial

(a) $$\begin{aligned}(x+2)^3 = (x+2)(x+2)^2 &= (x+2)(x^2+4x+4) \quad \textbf{Formula (3a)}\\ &= (x^3+4x^2+4x)+(2x^2+8x+8)\\ &= x^3+6x^2+12x+8\end{aligned}$$

(b) $$\begin{aligned}(x-1)^3 = (x-1)(x-1)^2 &= (x-1)(x^2-2x+1) \quad \textbf{Formula (3b)}\\ &= (x^3-2x^2+x)-(x^2-2x+1)\\ &= x^3-3x^2+3x-1\end{aligned}$$

■

Cubes of Binomials, or Perfect Cubes

$$(x+a)^3 = x^3+3ax^2+3a^2x+a^3 \qquad \textbf{(4a)}$$

$$(x-a)^3 = x^3-3ax^2+3a^2x-a^3 \qquad \textbf{(4b)}$$

Now Work PROBLEM 87

EXAMPLE 10 Forming the Difference of Two Cubes

$$\begin{aligned}(x-1)(x^2+x+1) &= x(x^2+x+1)-1(x^2+x+1)\\ &= x^3+x^2+x-x^2-x-1\\ &= x^3-1\end{aligned}$$

■

EXAMPLE 11 Forming the Sum of Two Cubes

$$\begin{aligned}(x+2)(x^2-2x+4) &= x(x^2-2x+4)+2(x^2-2x+4)\\ &= x^3-2x^2+4x+2x^2-4x+8\\ &= x^3+8\end{aligned}$$

■

Examples 10 and 11 lead to two more special products.

Difference of Two Cubes

$$(x-a)(x^2+ax+a^2) = x^3-a^3 \qquad \textbf{(5)}$$

Sum of Two Cubes

$$(x+a)(x^2-ax+a^2) = x^3+a^3 \qquad \textbf{(6)}$$

6 Divide Polynomials Using Long Division

The procedure for dividing two polynomials is similar to the procedure for dividing two integers.

EXAMPLE 12 Dividing Two Integers

Divide 842 by 15.

Solution

$$\begin{array}{rrl} & 56 & \leftarrow \textbf{Quotient} \\ \textbf{Divisor} \rightarrow & 15\overline{)842} & \leftarrow \textbf{Dividend} \\ & \underline{75} & \leftarrow \textbf{5} \cdot \textbf{15 (subtract)} \\ & 92 & \leftarrow \textbf{Bring down the 2.} \\ & \underline{90} & \leftarrow \textbf{6} \cdot \textbf{15 (subtract)} \\ & 2 & \leftarrow \textbf{Remainder} \end{array}$$

So, $\frac{842}{15} = 56 + \frac{2}{15}$. ■

In the long-division process detailed in Example 12, the number 15 is called the **divisor**, the number 842 is called the **dividend**, the number 56 is called the **quotient**, and the number 2 is called the **remainder**.

To check the answer obtained in a division problem, multiply the quotient by the divisor and add the remainder. The answer should be the dividend.

$$\boxed{(\text{Quotient})(\text{Divisor}) + \text{Remainder} = \text{Dividend}}$$

For example, we can check the results obtained in Example 12 as follows:

$$(56)(15) + 2 = 840 + 2 = 842$$

To divide two polynomials, we first must write each polynomial in standard form. The process then follows a pattern similar to that of Example 12. The next example illustrates the procedure.

EXAMPLE 13 Dividing Two Polynomials

Find the quotient and the remainder when

$$3x^3 + 4x^2 + x + 7 \quad \text{is divided by} \quad x^2 + 1$$

Solution Each polynomial is in standard form. The dividend is $3x^3 + 4x^2 + x + 7$, and the divisor is $x^2 + 1$.

Note: Remember, a polynomial is in standard form when its terms are written in descending powers of x. ■

STEP 1: Divide the leading term of the dividend, $3x^3$, by the leading term of the divisor, x^2. Enter the result, $3x$, over the term $3x^3$, as follows:

$$\begin{array}{rl} & 3x \\ x^2 + 1 & \overline{)3x^3 + 4x^2 + x + 7} \end{array}$$

STEP 2: Multiply $3x$ by $x^2 + 1$, and enter the result below the dividend.

$$\begin{array}{rll} & 3x & \\ x^2 + 1 & \overline{)3x^3 + 4x^2 + x + 7} & \\ & \underline{3x^3 + 3x } & \leftarrow \mathbf{3x \cdot (x^2 + 1) = 3x^3 + 3x} \\ & \uparrow & \end{array}$$

Align the 3*x* term under the *x* to make the next step easier.

STEP 3: Subtract and bring down the remaining terms.

$$\begin{array}{rll} & 3x & \\ x^2 + 1 & \overline{)3x^3 + 4x^2 + x + 7} & \\ & \underline{3x^3 + 3x } & \leftarrow \textbf{Subtract (change the signs and add).} \\ & 4x^2 - 2x + 7 & \leftarrow \textbf{Bring down the } \mathbf{4x^2} \textbf{ and the 7.} \end{array}$$

Step 4: Repeat Steps 1–3 using $4x^2 - 2x + 7$ as the dividend.

$$
\begin{array}{rl}
 & 3x \quad + 4 \\
x^2 + 1 & \overline{)3x^3 + 4x^2 + x + 7} \\
 & \underline{3x^3 \qquad\quad + 3x\phantom{{}+7}} \\
 & \qquad\quad 4x^2 - 2x + 7 \quad \leftarrow \textbf{Divide } 4x^2 \textbf{ by } x^2 \textbf{ to get 4.} \\
 & \qquad\quad \underline{4x^2 \qquad\; + 4} \quad \leftarrow \textbf{Multiply } x^2 + 1 \textbf{ by 4; subtract.} \\
 & \qquad\qquad\quad -2x + 3
\end{array}
$$

COMMENT If the degree of the divisor is greater than the degree of the dividend, then the process ends. ■

Since x^2 does not divide $-2x$ evenly (that is, the result is not a monomial), the process ends. The quotient is $3x + 4$, and the remainder is $-2x + 3$.

✓ **Check:** (Quotient)(Divisor) + Remainder

$$
\begin{aligned}
&= (3x + 4)(x^2 + 1) + (-2x + 3) \\
&= 3x^3 + 3x + 4x^2 + 4 + (-2x + 3) \\
&= 3x^3 + 4x^2 + x + 7 = \text{Dividend}
\end{aligned}
$$

Then

$$\frac{3x^3 + 4x^2 + x + 7}{x^2 + 1} = 3x + 4 + \frac{-2x + 3}{x^2 + 1}$$

■

The next example combines the steps involved in long division.

EXAMPLE 14 Dividing Two Polynomials

Find the quotient and the remainder when

$$x^4 - 3x^3 + 2x - 5 \quad \text{is divided by} \quad x^2 - x + 1$$

Solution In setting up this division problem, it is necessary to leave a space for the missing x^2 term in the dividend.

$$
\begin{array}{rrll}
 & & x^2 - 2x \;\; - 3 & \leftarrow \textbf{Quotient} \\
\textbf{Divisor} \rightarrow & x^2 - x + 1 & \overline{)x^4 - 3x^3 \qquad\qquad + 2x - 5} & \leftarrow \textbf{Dividend} \\
\textbf{Subtract} \rightarrow & & \underline{x^4 - x^3 + x^2} & \\
 & & \qquad -2x^3 - x^2 + 2x - 5 & \\
\textbf{Subtract} \rightarrow & & \qquad \underline{-2x^3 + 2x^2 - 2x} & \\
 & & \qquad\qquad\qquad -3x^2 + 4x - 5 & \\
\textbf{Subtract} \rightarrow & & \qquad\qquad\qquad \underline{-3x^2 + 3x - 3} & \\
 & & \qquad\qquad\qquad\qquad\qquad x - 2 & \leftarrow \textbf{Remainder}
\end{array}
$$

✓ **Check:** (Quotient)(Divisor) + Remainder

$$
\begin{aligned}
&= (x^2 - 2x - 3)(x^2 - x + 1) + (x - 2) \\
&= x^4 - x^3 + x^2 - 2x^3 + 2x^2 - 2x - 3x^2 + 3x - 3 + x - 2 \\
&= x^4 - 3x^3 + 2x - 5 = \text{Dividend}
\end{aligned}
$$

As a result,

$$\frac{x^4 - 3x^3 + 2x - 5}{x^2 - x + 1} = x^2 - 2x - 3 + \frac{x - 2}{x^2 - x + 1}$$

■

The process of dividing two polynomials leads to the following result:

THEOREM

Let Q be a polynomial of positive degree, and let P be a polynomial whose degree is greater than or equal to the degree of Q. The remainder after dividing P by Q is either the zero polynomial or a polynomial whose degree is less than the degree of the divisor Q. ■

Now Work PROBLEM 95

7 Work with Polynomials in Two Variables

A **monomial in two variables** x and y has the form $ax^n y^m$, where a is a constant, x and y are variables, and n and m are nonnegative integers. The **degree** of a monomial is the sum of the powers of the variables.

For example,

$$2xy^3, \quad x^2y^2, \quad \text{and} \quad x^3y$$

are all monomials that have degree 4.

A **polynomial in two variables** x and y is the sum of one or more monomials in two variables. The **degree of a polynomial** in two variables is the highest degree of all the monomials with nonzero coefficients.

EXAMPLE 15 **Examples of Polynomials in Two Variables**

$3x^2 + 2x^3y + 5$	$\pi x^3 - y^2$	$x^4 + 4x^3y - xy^3 + y^4$
Two variables, degree is 4.	Two variables, degree is 3.	Two variables, degree is 4.

■

Multiplying polynomials in two variables is handled in the same way as multiplying polynomials in one variable.

EXAMPLE 16 **Using a Special Product Formula**

To multiply $(2x - y)^2$, use the Squares of Binomials formula (3b) with $2x$ instead of x and with y instead of a.

$$\begin{aligned}(2x - y)^2 &= (2x)^2 - 2 \cdot y \cdot 2x + y^2 \\ &= 4x^2 - 4xy + y^2\end{aligned}$$

■

Now Work PROBLEM 81

R.4 Assess Your Understanding

Concepts and Vocabulary

1. The polynomial $3x^4 - 2x^3 + 13x^2 - 5$ is of degree __. The leading coefficient is __.

2. $(x^2 - 4)(x^2 + 4) =$ ________.

3. $(x - 2)(x^2 + 2x + 4) =$ _______.

4. The monomials that make up a polynomial are called which of the following?

(a) terms (b) variables (c) factors (d) coefficients

5. Choose the degree of the monomial $3x^4y^2$.

(a) 3 (b) 8 (c) 6 (d) 2

6. ***True or False*** $4x^{-2}$ is a monomial of degree -2.

7. ***True or False*** The degree of the product of two nonzero polynomials equals the sum of their degrees.

8. ***True or False*** $(x + a)(x^2 + ax + a) = x^3 + a^3$.

Skill Building

In Problems 9–18, tell whether the expression is a monomial. If it is, name the variable(s) and the coefficient, and give the degree of the monomial. If it is not a monomial, state why not.

9. $2x^3$
10. $-4x^2$
11. $\frac{8}{x}$
12. $-2x^{-3}$
13. $-2xy^2$
14. $5x^2y^3$
15. $\frac{8x}{y}$
16. $-\frac{2x^2}{y^3}$
17. $x^2 + y^2$
18. $3x^2 + 4$

In Problems 19–28, tell whether the expression is a polynomial. If it is, give its degree. If it is not, state why not.

19. $3x^2 - 5$
20. $1 - 4x$
21. 5
22. $-\pi$
23. $3x^2 - \frac{5}{x}$
24. $\frac{3}{x} + 2$
25. $2y^3 - \sqrt{2}$
26. $10z^2 + z$
27. $\frac{x^2 + 5}{x^3 - 1}$
28. $\frac{3x^3 + 2x - 1}{x^2 + x + 1}$

In Problems 29–48, add, subtract, or multiply, as indicated. Express your answer as a single polynomial in standard form.

29. $(x^2 + 4x + 5) + (3x - 3)$
30. $(x^3 + 3x^2 + 2) + (x^2 - 4x + 4)$
31. $(x^3 - 2x^2 + 5x + 10) - (2x^2 - 4x + 3)$
32. $(x^2 - 3x - 4) - (x^3 - 3x^2 + x + 5)$
33. $(6x^5 + x^3 + x) + (5x^4 - x^3 + 3x^2)$
34. $(10x^5 - 8x^2) + (3x^3 - 2x^2 + 6)$
35. $(x^2 - 3x + 1) + 2(3x^2 + x - 4)$
36. $-2(x^2 + x + 1) + (-5x^2 - x + 2)$
37. $6(x^3 + x^2 - 3) - 4(2x^3 - 3x^2)$
38. $8(4x^3 - 3x^2 - 1) - 6(4x^3 + 8x - 2)$
39. $(x^2 - x + 2) + (2x^2 - 3x + 5) - (x^2 + 1)$
40. $(x^2 + 1) - (4x^2 + 5) + (x^2 + x - 2)$
41. $9(y^2 - 3y + 4) - 6(1 - y^2)$
42. $8(1 - y^3) + 4(1 + y + y^2 + y^3)$
43. $x(x^2 + x - 4)$
44. $4x^2(x^3 - x + 2)$
45. $-2x^2(4x^3 + 5)$
46. $5x^3(3x - 4)$
47. $(x + 1)(x^2 + 2x - 4)$
48. $(2x - 3)(x^2 + x + 1)$

In Problems 49–66, multiply the polynomials using the FOIL method. Express your answer as a single polynomial in standard form.

49. $(x + 2)(x + 4)$
50. $(x + 3)(x + 5)$
51. $(2x + 5)(x + 2)$
52. $(3x + 1)(2x + 1)$
53. $(x - 4)(x + 2)$
54. $(x + 4)(x - 2)$
55. $(x - 3)(x - 2)$
56. $(x - 5)(x - 1)$
57. $(2x + 3)(x - 2)$
58. $(2x - 4)(3x + 1)$
59. $(-2x + 3)(x - 4)$
60. $(-3x - 1)(x + 1)$
61. $(-x - 2)(-2x - 4)$
62. $(-2x - 3)(3 - x)$
63. $(x - 2y)(x + y)$
64. $(2x + 3y)(x - y)$
65. $(-2x - 3y)(3x + 2y)$
66. $(x - 3y)(-2x + y)$

In Problems 67–90, multiply the polynomials using the special product formulas. Express your answer as a single polynomial in standard form.

67. $(x - 7)(x + 7)$
68. $(x - 1)(x + 1)$
69. $(2x + 3)(2x - 3)$
70. $(3x + 2)(3x - 2)$
71. $(x + 4)^2$
72. $(x + 5)^2$
73. $(x - 4)^2$
74. $(x - 5)^2$
75. $(3x + 4)(3x - 4)$
76. $(5x - 3)(5x + 3)$
77. $(2x - 3)^2$
78. $(3x - 4)^2$

79. $(x+y)(x-y)$

80. $(x+3y)(x-3y)$

81. $(3x+y)(3x-y)$

82. $(3x+4y)(3x-4y)$

83. $(x+y)^2$

84. $(x-y)^2$

85. $(x-2y)^2$

86. $(2x+3y)^2$

87. $(x-2)^3$

88. $(x+1)^3$

89. $(2x+1)^3$

90. $(3x-2)^3$

In Problems 91–106, find the quotient and the remainder. Check your work by verifying that

$$(\textit{Quotient})(\textit{Divisor}) + \textit{Remainder} = \textit{Dividend}$$

91. $4x^3 - 3x^2 + x + 1$ divided by $x + 2$

92. $3x^3 - x^2 + x - 2$ divided by $x + 2$

93. $4x^3 - 3x^2 + x + 1$ divided by x^2

94. $3x^3 - x^2 + x - 2$ divided by x^2

95. $5x^4 - 3x^2 + x + 1$ divided by $x^2 + 2$

96. $5x^4 - x^2 + x - 2$ divided by $x^2 + 2$

97. $4x^5 - 3x^2 + x + 1$ divided by $2x^3 - 1$

98. $3x^5 - x^2 + x - 2$ divided by $3x^3 - 1$

99. $2x^4 - 3x^3 + x + 1$ divided by $2x^2 + x + 1$

100. $3x^4 - x^3 + x - 2$ divided by $3x^2 + x + 1$

101. $-4x^3 + x^2 - 4$ divided by $x - 1$

102. $-3x^4 - 2x - 1$ divided by $x - 1$

103. $1 - x^2 + x^4$ divided by $x^2 + x + 1$

104. $1 - x^2 + x^4$ divided by $x^2 - x + 1$

105. $x^3 - a^3$ divided by $x - a$

106. $x^5 - a^5$ divided by $x - a$

Explaining Concepts: Discussion and Writing

107. Explain why the degree of the product of two nonzero polynomials equals the sum of their degrees.

108. Explain why the degree of the sum of two polynomials of different degrees equals the larger of their degrees.

109. Give a careful statement about the degree of the sum of two polynomials of the same degree.

110. Do you prefer adding two polynomials using the horizontal method or the vertical method? Write a brief position paper defending your choice.

111. Do you prefer to memorize the rule for the square of a binomial $(x+a)^2$ or to use FOIL to obtain the product? Write a brief position paper defending your choice.

R.5 Factoring Polynomials

OBJECTIVES
1. Factor the Difference of Two Squares and the Sum and Difference of Two Cubes (p. 51)
2. Factor Perfect Squares (p. 52)
3. Factor a Second-Degree Polynomial: $x^2 + Bx + C$ (p. 53)
4. Factor by Grouping (p. 54)
5. Factor a Second-Degree Polynomial: $Ax^2 + Bx + C,\ A \neq 1$ (p. 55)
6. Complete the Square (p. 57)

Consider the following product:

$$(2x+3)(x-4) = 2x^2 - 5x - 12$$

The two polynomials on the left side are called **factors** of the polynomial on the right side. Expressing a given polynomial as a product of other polynomials—that is, finding the factors of a polynomial—is called **factoring**.

We shall restrict our discussion here to factoring polynomials in one variable into products of polynomials in one variable, where all coefficients are integers. We call this **factoring over the integers**.

Any polynomial can be written as the product of 1 times itself or as -1 times its additive inverse. If a polynomial cannot be written as the product of two other polynomials (excluding 1 and -1), then the polynomial is **prime**. When a polynomial has been written as a product consisting only of prime factors, it is **factored completely**. Examples of prime polynomials (over the integers) are

$$2, \quad 3, \quad 5, \quad x, \quad x+1, \quad x-1, \quad 3x+4, \quad x^2+4$$

COMMENT Over the real numbers, $3x + 4$ factors into $3\left(x + \frac{4}{3}\right)$. It is the noninteger $\frac{4}{3}$ that causes $3x + 4$ to be prime over the integers. In most instances, we will be factoring over the integers. ■

The first factor to look for in a factoring problem is a common monomial factor present in each term of the polynomial. If one is present, use the Distributive Property to factor it out. Continue factoring out monomial factors until none are left.

EXAMPLE 1

Identifying Common Monomial Factors

Polynomial	Common Monomial Factor	Remaining Factor	Factored Form
$2x + 4$	2	$x + 2$	$2x + 4 = 2(x + 2)$
$3x - 6$	3	$x - 2$	$3x - 6 = 3(x - 2)$
$2x^2 - 4x + 8$	2	$x^2 - 2x + 4$	$2x^2 - 4x + 8 = 2(x^2 - 2x + 4)$
$8x - 12$	4	$2x - 3$	$8x - 12 = 4(2x - 3)$
$x^2 + x$	x	$x + 1$	$x^2 + x = x(x + 1)$
$x^3 - 3x^2$	x^2	$x - 3$	$x^3 - 3x^2 = x^2(x - 3)$
$6x^2 + 9x$	$3x$	$2x + 3$	$6x^2 + 9x = 3x(2x + 3)$ ■

Notice that once all common monomial factors have been removed from a polynomial, the remaining factor is either a prime polynomial of degree 1 or a polynomial of degree 2 or higher. (Do you see why?)

Now Work PROBLEM 9

1 Factor the Difference of Two Squares and the Sum and Difference of Two Cubes

When you factor a polynomial, first check for common monomial factors. Then see whether you can use one of the special formulas discussed in the previous section.

Difference of Two Squares $\quad x^2 - a^2 = (x - a)(x + a)$

Perfect Squares $\quad x^2 + 2ax + a^2 = (x + a)^2$

$\quad x^2 - 2ax + a^2 = (x - a)^2$

Sum of Two Cubes $\quad x^3 + a^3 = (x + a)(x^2 - ax + a^2)$

Difference of Two Cubes $\quad x^3 - a^3 = (x - a)(x^2 + ax + a^2)$

EXAMPLE 2

Factoring the Difference of Two Squares

Factor completely: $x^2 - 4$

Solution Note that $x^2 - 4$ is the difference of two squares, x^2 and 2^2.

$$x^2 - 4 = (x - 2)(x + 2)$$ ■

EXAMPLE 3 **Factoring the Difference of Two Cubes**

Factor completely: $x^3 - 1$

Solution Because $x^3 - 1$ is the difference of two cubes, x^3 and 1^3,

$$x^3 - 1 = (x - 1)(x^2 + x + 1)$$ ■

EXAMPLE 4 **Factoring the Sum of Two Cubes**

Factor completely: $x^3 + 8$

Solution Because $x^3 + 8$ is the sum of two cubes, x^3 and 2^3,

$$x^3 + 8 = (x + 2)(x^2 - 2x + 4)$$ ■

EXAMPLE 5 **Factoring the Difference of Two Squares**

Factor completely: $x^4 - 16$

Solution Because $x^4 - 16$ is the difference of two squares, $x^4 = (x^2)^2$ and $16 = 4^2$,

$$x^4 - 16 = (x^2 - 4)(x^2 + 4)$$

But $x^2 - 4$ is also the difference of two squares. Then,

$$x^4 - 16 = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4)$$ ■

Now Work PROBLEMS 19 AND 37

2 Factor Perfect Squares

When the first term and third term of a trinomial are both positive and are perfect squares, such as x^2, $9x^2$, 1, and 4, check to see whether the trinomial is a perfect square.

EXAMPLE 6 **Factoring a Perfect Square**

Factor completely: $x^2 + 6x + 9$

Solution The first term, x^2, and the third term, $9 = 3^2$, are perfect squares. Because the middle term, $6x$, is twice the product of x and 3, we have a perfect square.

$$x^2 + 6x + 9 = (x + 3)^2$$ ■

EXAMPLE 7 **Factoring a Perfect Square**

Factor completely: $9x^2 - 6x + 1$

Solution The first term, $9x^2 = (3x)^2$, and the third term, $1 = 1^2$, are perfect squares. Because the middle term, $-6x$, is -2 times the product of $3x$ and 1, we have a perfect square.

$$9x^2 - 6x + 1 = (3x - 1)^2$$ ■

EXAMPLE 8 **Factoring a Perfect Square**

Factor completely: $25x^2 + 30x + 9$

Solution The first term, $25x^2 = (5x)^2$, and the third term, $9 = 3^2$, are perfect squares. Because the middle term, $30x$, is twice the product of $5x$ and 3, we have a perfect square.

$$25x^2 + 30x + 9 = (5x + 3)^2$$ ■

Now Work PROBLEMS 29 AND 103

If a trinomial is not a perfect square, it may be possible to factor it using the technique discussed next.

3 Factor a Second-Degree Polynomial: $x^2 + Bx + C$

The idea behind factoring a second-degree polynomial like $x^2 + Bx + C$ is to see whether it can be made equal to the product of two (possibly equal) first-degree polynomials.

For example, consider

$$(x+3)(x+4) = x^2 + 7x + 12$$

The factors of $x^2 + 7x + 12$ are $x + 3$ and $x + 4$. Notice the following:

$$x^2 + 7x + 12 = (x+3)(x+4)$$

12 is the product of 3 and 4.

7 is the sum of 3 and 4.

In general, if $x^2 + Bx + C = (x+a)(x+b) = x^2 + (a+b)x + ab$, then $ab = C$ and $a + b = B$.

> To factor a second-degree polynomial $x^2 + Bx + C$, find integers whose product is C and whose sum is B. That is, if there are numbers a, b, where $ab = C$ and $a + b = B$, then
>
> $$\boxed{x^2 + Bx + C = (x+a)(x+b)}$$

EXAMPLE 9 Factoring a Trinomial

Factor completely: $x^2 + 7x + 10$

Solution First determine all pairs of integers whose product is 10, and then compute their sums.

Integers whose product is 10	1, 10	−1, −10	2, 5	−2, −5
Sum	11	−11	7	−7

The integers 2 and 5 have a product of 10 and add up to 7, the coefficient of the middle term. As a result,

$$x^2 + 7x + 10 = (x+2)(x+5)$$

■

EXAMPLE 10 Factoring a Trinomial

Factor completely: $x^2 - 6x + 8$

Solution First determine all pairs of integers whose product is 8, and then compute each sum.

Integers whose product is 8	1, 8	−1, −8	2, 4	−2, −4
Sum	9	−9	6	−6

Since −6 is the coefficient of the middle term,

$$x^2 - 6x + 8 = (x-2)(x-4)$$

■

EXAMPLE 11 **Factoring a Trinomial**

Factor completely: $x^2 - x - 12$

Solution First determine all pairs of integers whose product is -12, and then compute each sum.

Integers whose product is -12	$1, -12$	$-1, 12$	$2, -6$	$-2, 6$	$3, -4$	$-3, 4$
Sum	-11	11	-4	4	-1	1

Since -1 is the coefficient of the middle term,

$$x^2 - x - 12 = (x + 3)(x - 4)$$ ■

EXAMPLE 12 **Factoring a Trinomial**

Factor completely: $x^2 + 4x - 12$

Solution The integers -2 and 6 have a product of -12 and have the sum 4. So,

$$x^2 + 4x - 12 = (x - 2)(x + 6)$$ ■

To avoid errors in factoring, always check your answer by multiplying it out to see whether the result equals the original expression.

When none of the possibilities works, the polynomial is prime.

EXAMPLE 13 **Identifying a Prime Polynomial**

Show that $x^2 + 9$ is prime.

Solution First list the pairs of integers whose product is 9, and then compute their sums.

Integers whose product is 9	$1, 9$	$-1, -9$	$3, 3$	$-3, -3$
Sum	10	-10	6	-6

Since the coefficient of the middle term in $x^2 + 9 = x^2 + 0x + 9$ is 0 and none of the sums equals 0, we conclude that $x^2 + 9$ is prime. ■

Example 13 demonstrates a more general result:

THEOREM

Any polynomial of the form $x^2 + a^2$, a real, is prime. ■

Now Work PROBLEMS 43 AND 87

4 Factor by Grouping

Sometimes a common factor does not occur in every term of the polynomial but does occur in each of several groups of terms that together make up the polynomial. When this happens, the common factor can be factored out of each group by means of the Distributive Property. This technique is called **factoring by grouping**.

EXAMPLE 14 **Factoring by Grouping**

Factor completely by grouping: $(x^2 + 2)x + (x^2 + 2) \cdot 3$

Solution Notice the common factor $x^2 + 2$. Applying the Distributive Property yields

$$(x^2 + 2)x + (x^2 + 2) \cdot 3 = (x^2 + 2)(x + 3)$$

Since $x^2 + 2$ and $x + 3$ are prime, the factorization is complete. ■

The next example shows a factoring problem that occurs in calculus.

EXAMPLE 15 Factoring by Grouping

Factor completely by grouping: $3(x-1)^2(x+2)^4 + 4(x-1)^3(x+2)^3$

Solution Here, $(x-1)^2(x+2)^3$ is a common factor of both $3(x-1)^2(x+2)^4$ and $4(x-1)^3(x+2)^3$. As a result,

$$\begin{aligned} 3(x-1)^2(x+2)^4 + 4(x-1)^3(x+2)^3 &= (x-1)^2(x+2)^3[3(x+2) + 4(x-1)] \\ &= (x-1)^2(x+2)^3[3x+6+4x-4] \\ &= (x-1)^2(x+2)^3(7x+2) \end{aligned}$$

■

EXAMPLE 16 Factoring by Grouping

Factor completely by grouping: $x^3 - 4x^2 + 2x - 8$

Solution To see whether factoring by grouping will work, group the first two terms and the last two terms. Then look for a common factor in each group. In this example, factor x^2 from $x^3 - 4x^2$ and 2 from $2x - 8$. The remaining factor in each case is the same, $x - 4$. This means that factoring by grouping will work, as follows:

$$\begin{aligned} x^3 - 4x^2 + 2x - 8 &= (x^3 - 4x^2) + (2x - 8) \\ &= x^2(x-4) + 2(x-4) \\ &= (x-4)(x^2+2) \end{aligned}$$

Since $x^2 + 2$ and $x - 4$ are prime, the factorization is complete. ■

Now Work PROBLEMS **55** AND **131**

5 Factor a Second-Degree Polynomial: $Ax^2 + Bx + C, A \neq 1$

To factor a second-degree polynomial $Ax^2 + Bx + C$, when $A \neq 1$ and A, B, and C have no common factors, follow these steps:

Steps for Factoring $Ax^2 + Bx + C$, When $A \neq 1$ and A, B, and C Have No Common Factors

STEP 1: Find the value of AC.

STEP 2: Find a pair of integers whose product is AC and that add up to B. That is, find a and b such that $ab = AC$ and $a + b = B$.

STEP 3: Write $Ax^2 + Bx + C = Ax^2 + ax + bx + C$.

STEP 4: Factor this last expression by grouping.

EXAMPLE 17 Factoring a Trinomial

Factor completely: $2x^2 + 5x + 3$

Solution Comparing $2x^2 + 5x + 3$ to $Ax^2 + Bx + C$, we find that $A = 2$, $B = 5$, and $C = 3$.

STEP 1: The value of AC is $2 \cdot 3 = 6$.

STEP 2: Determine the pairs of integers whose product is $AC = 6$ and compute their sums.

Integers whose product is 6	1, 6	−1, −6	2, 3	−2, −3
Sum	7	−7	5	−5

Step 3: The integers whose product is 6 that add up to $B = 5$ are 2 and 3.

$$2x^2 + 5x + 3 = 2x^2 + 2x + 3x + 3$$

Step 4: Factor by grouping.

$$\begin{aligned} 2x^2 + 2x + 3x + 3 &= (2x^2 + 2x) + (3x + 3) \\ &= 2x(x + 1) + 3(x + 1) \\ &= (x + 1)(2x + 3) \end{aligned}$$

As a result,

$$2x^2 + 5x + 3 = (x + 1)(2x + 3)$$

■

EXAMPLE 18 Factoring a Trinomial

Factor completely: $2x^2 - x - 6$

Solution Comparing $2x^2 - x - 6$ to $Ax^2 + Bx + C$, we find that $A = 2, B = -1$, and $C = -6$.

Step 1: The value of AC is $2 \cdot (-6) = -12$.

Step 2: Determine the pairs of integers whose product is $AC = -12$ and compute their sums.

Integers whose product is -12	$1, -12$	$-1, 12$	$2, -6$	$-2, 6$	$3, -4$	$-3, 4$
Sum	-11	11	-4	4	-1	1

Step 3: The integers whose product is -12 that add up to $B = -1$ are -4 and 3.

$$2x^2 - x - 6 = 2x^2 - 4x + 3x - 6$$

Step 4: Factor by grouping.

$$\begin{aligned} 2x^2 - 4x + 3x - 6 &= (2x^2 - 4x) + (3x - 6) \\ &= 2x(x - 2) + 3(x - 2) \\ &= (x - 2)(2x + 3) \end{aligned}$$

As a result,

$$2x^2 - x - 6 = (x - 2)(2x + 3)$$

■

Now Work PROBLEM 61

SUMMARY

Type of Polynomial	Method	Example
Any polynomial	Look for common monomial factors. (Always do this first!)	$6x^2 + 9x = 3x(2x + 3)$
Binomials of degree 2 or higher	Check for a special product:	
	Difference of two squares, $x^2 - a^2$	$x^2 - 16 = (x - 4)(x + 4)$
	Difference of two cubes, $x^3 - a^3$	$x^3 - 64 = (x - 4)(x^2 + 4x + 16)$
	Sum of two cubes, $x^3 + a^3$	$x^3 + 27 = (x + 3)(x^2 - 3x + 9)$
Trinomials of degree 2	Check for a perfect square, $(x \pm a)^2$	$x^2 + 8x + 16 = (x + 4)^2$ $x^2 - 10x + 25 = (x - 5)^2$
	Factoring $x^2 + Bx + C$ (p. 53)	$x^2 - x - 2 = (x - 2)(x + 1)$
	Factoring $Ax^2 + Bx + C$ (p. 55)	$6x^2 + x - 1 = (2x + 1)(3x - 1)$
Four or more terms	Grouping	$2x^3 - 3x^2 + 4x - 6 = (2x - 3)(x^2 + 2)$

6 Complete the Square

The idea behind completing the square in one variable is to "adjust" an expression of the form $x^2 + bx$ to make it a perfect square. Perfect squares are trinomials of the form

$$x^2 + 2ax + a^2 = (x + a)^2 \text{ or } x^2 - 2ax + a^2 = (x - a)^2$$

For example, $x^2 + 6x + 9$ is a perfect square because $x^2 + 6x + 9 = (x + 3)^2$. And $p^2 - 12p + 36$ is a perfect square because $p^2 - 12p + 36 = (p - 6)^2$.

So how do we "adjust" $x^2 + bx$ to make it a perfect square? We do it by adding a number. For example, to make $x^2 + 6x$ a perfect square, add 9. But how do we know to add 9? If we divide the coefficient of the first-degree term, 6, by 2, and then square the result, we obtain 9. This approach works in general.

WARNING To use $\left(\frac{1}{2}b\right)^2$ to complete the square, the coefficient of the x^2 term must be 1. ■

Completing the Square of $x^2 + bx$

Identify the coefficient of the first-degree term. Multiply this coefficient by $\frac{1}{2}$ and then square the result. That is, determine the value of b in $x^2 + bx$ and compute $\left(\frac{1}{2}b\right)^2$.

EXAMPLE 19

Completing the Square

Determine the number that must be added to each expression to complete the square. Then factor the expression.

Start	Add	Result	Factored Form
$y^2 + 8y$	$\left(\frac{1}{2}\cdot 8\right)^2 = 16$	$y^2 + 8y + 16$	$(y + 4)^2$
$x^2 + 12x$	$\left(\frac{1}{2}\cdot 12\right)^2 = 36$	$x^2 + 12x + 36$	$(x + 6)^2$
$a^2 - 20a$	$\left(\frac{1}{2}\cdot (-20)\right)^2 = 100$	$a^2 - 20a + 100$	$(a - 10)^2$
$p^2 - 5p$	$\left(\frac{1}{2}\cdot (-5)\right)^2 = \frac{25}{4}$	$p^2 - 5p + \frac{25}{4}$	$\left(p - \frac{5}{2}\right)^2$

■

Notice that the factored form of a perfect square is either

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2 \quad \text{or} \quad x^2 - bx + \left(\frac{b}{2}\right)^2 = \left(x - \frac{b}{2}\right)^2$$

Now Work PROBLEM 73

Are you wondering why we refer to making an expression a perfect square as "completing the square"? Look at the square in Figure 30. Its area is $(y + 4)^2$. The yellow area is y^2 and each orange area is $4y$ (for a total area of $8y$). The sum of these areas is $y^2 + 8y$. To complete the square, we need to add the area of the green region: $4 \cdot 4 = 16$. As a result, $y^2 + 8y + 16 = (y + 4)^2$.

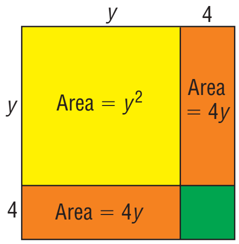


Figure 30

R.5 Assess Your Understanding

Concepts and Vocabulary

1. If factored completely, $3x^3 - 12x =$ ______________.

2. If a polynomial cannot be written as the product of two other polynomials (excluding 1 and -1), then the polynomial is said to be ______.

3. For $x^2 + Bx + C = (x + a)(x + b)$, which of the following must be true?
(a) $ab = B$ and $a + b = C$
(b) $a + b = C$ and $a - b = B$
(c) $ab = C$ and $a + b = B$
(d) $ab = B$ and $a - b = C$

4. Choose the best description of $x^2 - 64$.
(a) Prime (b) Difference of two squares
(c) Difference of two cubes (d) Perfect Square

5. Choose the complete factorization of $4x^2 - 8x - 60$.
(a) $2(x + 3)(x - 5)$ (b) $4(x^2 - 2x - 15)$
(c) $(2x + 6)(2x - 10)$ (d) $4(x + 3)(x - 5)$

6. To complete the square of $x^2 + bx$, use which of the following?
(a) $(2b)^2$ (b) $2b^2$ (c) $\left(\frac{1}{2}b\right)^2$ (d) $\frac{1}{2}b^2$

7. ***True or False*** The polynomial $x^2 + 4$ is prime.

8. ***True or False*** $3x^3 - 2x^2 - 6x + 4 = (3x - 2)(x^2 + 2)$.

Skill Building

In Problems 9–18, factor each polynomial by removing the common monomial factor.

9. $3x + 6$
10. $7x - 14$
11. $ax^2 + a$
12. $ax - a$
13. $x^3 + x^2 + x$
14. $x^3 - x^2 + x$
15. $2x^2 - 2x$
16. $3x^2 - 3x$
17. $3x^2y - 6xy^2 + 12xy$
18. $60x^2y - 48xy^2 + 72x^3y$

In Problems 19–26, factor the difference of two squares.

19. $x^2 - 1$
20. $x^2 - 4$
21. $4x^2 - 1$
22. $9x^2 - 1$
23. $x^2 - 16$
24. $x^2 - 25$
25. $25x^2 - 4$
26. $36x^2 - 9$

In Problems 27–36, factor the perfect squares.

27. $x^2 + 2x + 1$
28. $x^2 - 4x + 4$
29. $x^2 + 4x + 4$
30. $x^2 - 2x + 1$
31. $x^2 - 10x + 25$
32. $x^2 + 10x + 25$
33. $4x^2 + 4x + 1$
34. $9x^2 + 6x + 1$
35. $16x^2 + 8x + 1$
36. $25x^2 + 10x + 1$

In Problems 37–42, factor the sum or difference of two cubes.

37. $x^3 - 27$
38. $x^3 + 125$
39. $x^3 + 27$
40. $27 - 8x^3$
41. $8x^3 + 27$
42. $64 - 27x^3$

In Problems 43–54, factor each polynomial.

43. $x^2 + 5x + 6$
44. $x^2 + 6x + 8$
45. $x^2 + 7x + 6$
46. $x^2 + 9x + 8$
47. $x^2 + 7x + 10$
48. $x^2 + 11x + 10$
49. $x^2 - 10x + 16$
50. $x^2 - 17x + 16$
51. $x^2 - 7x - 8$
52. $x^2 - 2x - 8$
53. $x^2 + 7x - 8$
54. $x^2 + 2x - 8$

In Problems 55–60, factor by grouping.

55. $2x^2 + 4x + 3x + 6$
56. $3x^2 - 3x + 2x - 2$
57. $2x^2 - 4x + x - 2$
58. $3x^2 + 6x - x - 2$
59. $18x^2 + 27x + 12x + 18$
60. $45x^3 - 30x^2 + 15x^2 - 10x$

In Problems 61–72, factor each polynomial.

61. $3x^2 + 4x + 1$
62. $2x^2 + 3x + 1$
63. $2z^2 + 5z + 3$
64. $6z^2 + 5z + 1$
65. $3x^2 + 2x - 8$
66. $3x^2 + 10x + 8$
67. $3x^2 - 2x - 8$
68. $3x^2 - 10x + 8$
69. $12x^4 + 56x^3 + 32x^2$
70. $21x^2 - 98x + 56$
71. $3x^2 + 10x - 8$
72. $3x^2 - 10x - 8$

In Problems 73–78, determine what number should be added to complete the square of each expression. Then factor each expression.

73. $x^2 + 10x$
74. $p^2 + 14p$
75. $y^2 - 6y$
76. $x^2 - 4x$
77. $x^2 - \frac{1}{2}x$
78. $x^2 + \frac{1}{3}x$

Mixed Practice

In Problems 79–126, factor each polynomial completely. If the polynomial cannot be factored, say it is prime.

79. $x^2 - 36$
80. $x^2 - 9$
81. $2 - 8x^2$
82. $3 - 27x^2$
83. $8x^2 + 88x + 80$
84. $10x^3 + 50x^2 + 40x$
85. $x^2 - 10x + 21$
86. $x^2 - 6x + 8$
87. $4x^2 - 8x + 32$
88. $3x^2 - 12x + 15$
89. $x^2 + 4x + 16$
90. $x^2 + 12x + 36$
91. $15 + 2x - x^2$
92. $14 + 6x - x^2$
93. $3x^2 - 12x - 36$
94. $x^3 + 8x^2 - 20x$
95. $y^4 + 11y^3 + 30y^2$
96. $3y^3 - 18y^2 - 48y$
97. $8x^5 + 24x^4 + 18x^3$
98. $36x^6 - 48x^5 + 16x^4$
99. $6x^2 + 8x + 2$
100. $8x^2 + 6x - 2$
101. $x^4 - 81$
102. $x^4 - 1$
103. $x^6 - 2x^3 + 1$
104. $x^6 + 2x^3 + 1$
105. $x^7 - x^5$
106. $x^8 - x^5$
107. $16x^2 + 24x + 9$
108. $9x^2 - 24x + 16$
109. $5 + 16x - 16x^2$
110. $5 + 11x - 16x^2$
111. $4y^2 - 16y + 15$
112. $9y^2 + 9y - 4$
113. $1 - 8x^2 - 9x^4$
114. $4 - 14x^2 - 8x^4$
115. $x(x + 3) - 6(x + 3)$
116. $5(3x - 7) + x(3x - 7)$
117. $(x + 2)^2 - 5(x + 2)$
118. $(x - 1)^2 - 2(x - 1)$
119. $(3x - 2)^3 - 27$
120. $(5x + 1)^3 - 1$
121. $3(x^2 + 10x + 25) - 4(x + 5)$
122. $7(x^2 - 6x + 9) + 5(x - 3)$
123. $x^3 + 2x^2 - x - 2$
124. $x^3 - 3x^2 - x + 3$
125. $x^4 - x^3 + x - 1$
126. $x^4 + x^3 + x + 1$

Applications and Extensions

In Problems 127–136, expressions that occur in calculus are given. Factor each expression completely.

127. $2(3x + 4)^2 + (2x + 3) \cdot 2(3x + 4) \cdot 3$
128. $5(2x + 1)^2 + (5x - 6) \cdot 2(2x + 1) \cdot 2$
129. $2x(2x + 5) + x^2 \cdot 2$
130. $3x^2(8x - 3) + x^3 \cdot 8$
131. $2(x + 3)(x - 2)^3 + (x + 3)^2 \cdot 3(x - 2)^2$
132. $4(x + 5)^3(x - 1)^2 + (x + 5)^4 \cdot 2(x - 1)$
133. $(4x - 3)^2 + x \cdot 2(4x - 3) \cdot 4$
134. $3x^2(3x + 4)^2 + x^3 \cdot 2(3x + 4) \cdot 3$
135. $2(3x - 5) \cdot 3(2x + 1)^3 + (3x - 5)^2 \cdot 3(2x + 1)^2 \cdot 2$
136. $3(4x + 5)^2 \cdot 4(5x + 1)^2 + (4x + 5)^3 \cdot 2(5x + 1) \cdot 5$
137. Show that $x^2 + 4$ is prime.
138. Show that $x^2 + x + 1$ is prime.

Explaining Concepts: Discussion and Writing

139. Make up a polynomial that factors into a perfect square.

140. Explain to a fellow student what you look for first when presented with a factoring problem. What do you do next?

R.6 Synthetic Division

OBJECTIVE 1 Divide Polynomials Using Synthetic Division (p. 59)

1 Divide Polynomials Using Synthetic Division

To find the quotient as well as the remainder when a polynomial of degree 1 or higher is divided by $x - c$, a shortened version of long division, called **synthetic division**, makes the task simpler.

To see how synthetic division works, first consider long division for dividing the polynomial $2x^3 - x^2 + 3$ by $x - 3$.

$$
\begin{array}{r|rrrrl}
 & 2x^2 & +\,5x & +\,15 & & \leftarrow \textbf{Quotient} \\
\cline{2-5}
x - 3 & 2x^3 & -\,x^2 & & +\,3 & \\
 & 2x^3 & -\,6x^2 & & & \\
\cline{2-3}
 & & 5x^2 & & & \\
 & & 5x^2 & -\,15x & & \\
\cline{3-4}
 & & & 15x & +\,3 & \\
 & & & 15x & -\,45 & \\
\cline{4-5}
 & & & & 48 & \leftarrow \textbf{Remainder}
\end{array}
$$

✓**Check:** (Divisor) · (Quotient) + Remainder

$$
\begin{aligned}
&= (x - 3)(2x^2 + 5x + 15) + 48 \\
&= 2x^3 + 5x^2 + 15x - 6x^2 - 15x - 45 + 48 \\
&= 2x^3 - x^2 + 3
\end{aligned}
$$

The process of synthetic division arises from rewriting the long division in a more compact form, using simpler notation. For example, in the long division above, the terms in blue are not really necessary because they are identical to the terms directly above them. With these terms removed, we have

$$
\begin{array}{r|rrrr}
 & 2x^2 & +\,5x & +\,15 & \\
\cline{2-5}
x - 3 & 2x^3 & -\,x^2 & & +\,3 \\
 & & -\,6x^2 & & \\
\cline{2-3}
 & & 5x^2 & & \\
 & & & -\,15x & \\
\cline{3-5}
 & & & 15x & \\
 & & & & -\,45 \\
\cline{4-5}
 & & & & 48
\end{array}
$$

Most of the x's that appear in this process can also be removed, provided that we are careful about positioning each coefficient. In this regard, we will need to use 0 as the coefficient of x in the dividend, because that power of x is missing. Now we have

$$
\begin{array}{r|rrrr}
 & 2x^2 & +\,5x & +\,15 & \\
\cline{2-5}
x - 3 & 2 & -\,1 & 0 & 3 \\
 & & -\,6 & & \\
\cline{2-3}
 & & 5 & & \\
 & & & -\,15 & \\
\cline{3-5}
 & & & 15 & \\
 & & & & -\,45 \\
\cline{4-5}
 & & & & 48
\end{array}
$$

We can make this display more compact by moving the lines up until the numbers in blue align horizontally.

$$
\begin{array}{r|rrrrl}
 & 2x^2 & +\,5x & +\,15 & & \textbf{Row 1} \\
\cline{2-5}
x - 3 & 2 & -\,1 & 0 & 3 & \textbf{Row 2} \\
 & & -\,6 & -\,15 & -\,45 & \textbf{Row 3} \\
\cline{2-5}
 & \bigcirc & 5 & 15 & 48 & \textbf{Row 4}
\end{array}
$$

Because the leading coefficient of the divisor is always 1, the leading coefficient of the dividend will also be the leading coefficient of the quotient. So we place the leading coefficient of the quotient, 2, in the circled position. Now, the first three numbers in row 4 are precisely the coefficients of the quotient, and the last number in row 4 is the remainder. Since row 1 is not really needed, we can compress the process to three rows, where the bottom row contains both the coefficients of the quotient and the remainder.

$$\begin{array}{r|rrrrl}
\cline{2-5}
x-3 & 2 & -1 & 0 & 3 & \textbf{Row 1}\\
 & & -6 & -15 & -45 & \textbf{Row 2 (subtract)}\\
\cline{2-5}
 & 2 & 5 & 15 & 48 & \textbf{Row 3}
\end{array}$$

Recall that the entries in row 3 are obtained by subtracting the entries in row 2 from those in row 1. Rather than subtracting the entries in row 2, we can change the sign of each entry and add. With this modification, our display will look like this:

$$\begin{array}{r|rrrrl}
\cline{2-5}
x-3 & 2 & -1 & 0 & 3 & \textbf{Row 1}\\
 & & 6 & 15 & 45 & \textbf{Row 2 (add)}\\
\cline{2-5}
 & 2\,{+}\overset{3}{\nearrow} & 5\,{+}\overset{3}{\nearrow} & 15\,{+}\overset{3}{\nearrow} & 48 & \textbf{Row 3}
\end{array}$$

Notice that the entries in row 2 are three times the prior entries in row 3. Our last modification to the display replaces the $x - 3$ by 3. The entries in row 3 give the quotient and the remainder, as shown next.

$$\begin{array}{r|rrrrl}
\cline{2-5}
3 & 2 & -1 & 0 & 3 & \textbf{Row 1}\\
 & & 6 & 15 & 45 & \textbf{Row 2 (add)}\\
\cline{2-5}
 & 2 & 5 & 15 & 48 & \textbf{Row 3}
\end{array}$$

$$\underbrace{2 \qquad 5 \qquad 15}_{\textbf{Quotient}} \qquad 48 \nwarrow$$

$$\overbrace{2x^2 + 5x + 15} \qquad 48 \swarrow \quad \textbf{Remainder}$$

Let's go through an example step by step.

EXAMPLE 1 Using Synthetic Division to Find the Quotient and Remainder

Use synthetic division to find the quotient and remainder when

$$x^3 - 4x^2 - 5 \quad \text{is divided by} \quad x - 3$$

Solution

Step 1: Write the dividend in descending powers of x. Then copy the coefficients, remembering to insert a 0 for any missing powers of x.

$$1 \quad -4 \quad 0 \quad -5 \qquad \textbf{Row 1}$$

Step 2: Insert the usual division symbol. In synthetic division, the divisor is of the form $x - c$, and c is the number placed to the left of the division symbol. Here, since the divisor is $x - 3$, insert 3 to the left of the division symbol.

$$\begin{array}{r|rrrrl}
\cline{2-5}
3 & 1 & -4 & 0 & -5 & \textbf{Row 1}
\end{array}$$

Step 3: Bring the 1 down two rows, and enter it in row 3.

$$\begin{array}{r|rrrrl}
\cline{2-5}
3 & 1 & -4 & 0 & -5 & \textbf{Row 1}\\
 & \downarrow & & & & \textbf{Row 2}\\
\cline{2-5}
 & 1 & & & & \textbf{Row 3}
\end{array}$$

Step 4: Multiply the latest entry in row 3 by 3, and place the result in row 2, one column over to the right.

$$\begin{array}{r|rrrrl}
\cline{2-5}
3 & 1 & -4 & 0 & -5 & \textbf{Row 1}\\
 & & 3 & & & \textbf{Row 2}\\
\cline{2-5}
 & 1\,{+}\overset{3}{\nearrow} & & & & \textbf{Row 3}
\end{array}$$

Step 5: Add the entry in row 2 to the entry above it in row 1, and enter the sum in row 3.

$$\begin{array}{r|rrrrl}
\cline{2-5}
3 & 1 & -4 & 0 & -5 & \textbf{Row 1}\\
 & & 3 & & & \textbf{Row 2}\\
\cline{2-5}
 & 1\,{+}\overset{3}{\nearrow} & -1 & & & \textbf{Row 3}
\end{array}$$

STEP 6: Repeat Steps 4 and 5 until no more entries are available in row 1.

$$\begin{array}{r|rrrrl} 3 & 1 & -4 & 0 & -5 & \text{Row 1} \\ & & 3 & -3 & -9 & \text{Row 2} \\ \hline & 1 \overset{\times 3}{\nearrow} & -1 \overset{\times 3}{\nearrow} & -3 \overset{\times 3}{\nearrow} & -14 & \text{Row 3} \end{array}$$

STEP 7: The final entry in row 3, the -14, is the remainder; the other entries in row 3, the 1, -1, and -3, are the coefficients (in descending order) of a polynomial whose degree is 1 less than that of the dividend. This is the quotient. That is,

$$\text{Quotient} = x^2 - x - 3 \qquad \text{Remainder} = -14$$

✓ **Check:** (Divisor) (Quotient) + Remainder

$$\begin{aligned} &= (x-3)(x^2 - x - 3) + (-14) \\ &= (x^3 - x^2 - 3x - 3x^2 + 3x + 9) + (-14) \\ &= x^3 - 4x^2 - 5 = \text{Dividend} \end{aligned}$$

■

Let's do an example in which all seven steps are combined.

EXAMPLE 2 Using Synthetic Division to Verify a Factor

Use synthetic division to show that $x + 3$ is a factor of

$$2x^5 + 5x^4 - 2x^3 + 2x^2 - 2x + 3$$

Solution The divisor is $x + 3 = x - (-3)$, so place -3 to the left of the division symbol. Then the row 3 entries will be multiplied by -3, entered in row 2, and added to row 1.

$$\begin{array}{r|rrrrrrl} -3 & 2 & 5 & -2 & 2 & -2 & 3 & \text{Row 1} \\ & & -6 & 3 & -3 & 3 & -3 & \text{Row 2} \\ \hline & 2 & -1 & 1 & -1 & 1 & 0 & \text{Row 3} \end{array}$$

Because the remainder is 0, we have

$$\begin{aligned} &(\text{Divisor})(\text{Quotient}) + \text{Remainder} \\ &= (x+3)(2x^4 - x^3 + x^2 - x + 1) = 2x^5 + 5x^4 - 2x^3 + 2x^2 - 2x + 3 \end{aligned}$$

As we see, $x + 3$ is a factor of $2x^5 + 5x^4 - 2x^3 + 2x^2 - 2x + 3$. ■

As Example 2 illustrates, the remainder after division gives information about whether the divisor is, or is not, a factor. We shall have more to say about this in Chapter 5.

Now Work PROBLEMS 9 AND 19

R.6 Assess Your Understanding

Concepts and Vocabulary

1. To check division, the expression that is being divided, the dividend, should equal the product of the ________ and the ________ plus the ________.

2. To divide $2x^3 - 5x + 1$ by $x + 3$ using synthetic division, the first step is to write ____ $\overline{)\qquad\qquad}$.

3. Choose the division problem that cannot be done using synthetic division.

(a) $2x^3 - 4x^2 + 6x - 8$ is divided by $x - 8$
(b) $x^4 - 3$ is divided by $x + 1$
(c) $x^5 + 3x^2 - 9x + 2$ is divided by $x + 10$
(d) $x^4 - 5x^3 + 3x^2 - 9x + 13$ is divided by $x^2 + 5$

4. Choose the correct conclusion based on the following synthetic division:

$$\begin{array}{r|rrrr} -5 & 2 & 3 & -38 & -15 \\ & & -10 & 35 & 15 \\ \hline & 2 & -7 & -3 & 0 \end{array}$$

(a) $x + 5$ is a factor of $2x^3 + 3x^2 - 38x - 15$
(b) $x - 5$ is a factor of $2x^3 + 3x^2 - 38x - 15$
(c) $x + 5$ is not a factor of $2x^3 + 3x^2 - 38x - 15$
(d) $x - 5$ is not a factor of $2x^3 + 3x^2 - 38x - 15$

5. ***True or False*** In using synthetic division, the divisor is always a polynomial of degree 1, whose leading coefficient is 1.

6. ***True or False*** $\begin{array}{r|rrrr} -2 & 5 & 3 & 2 & 1 \\ & & -10 & 14 & -32 \\ \hline & 5 & -7 & 16 & -31 \end{array}$ means $\dfrac{5x^3 + 3x^2 + 2x + 1}{x + 2} = 5x^2 - 7x + 16 + \dfrac{-31}{x + 2}$.

Skill Building

In Problems 7–18, use synthetic division to find the quotient and remainder when:

7. $x^3 - x^2 + 2x + 4$ is divided by $x - 2$

8. $x^3 + 2x^2 - 3x + 1$ is divided by $x + 1$

9. $3x^3 + 2x^2 - x + 3$ is divided by $x - 3$

10. $-4x^3 + 2x^2 - x + 1$ is divided by $x + 2$

11. $x^5 - 4x^3 + x$ is divided by $x + 3$

12. $x^4 + x^2 + 2$ is divided by $x - 2$

13. $4x^6 - 3x^4 + x^2 + 5$ is divided by $x - 1$

14. $x^5 + 5x^3 - 10$ is divided by $x + 1$

15. $0.1x^3 + 0.2x$ is divided by $x + 1.1$

16. $0.1x^2 - 0.2$ is divided by $x + 2.1$

17. $x^5 - 1$ is divided by $x - 1$

18. $x^5 + 1$ is divided by $x + 1$

In Problems 19–28, use synthetic division to determine whether $x - c$ is a factor of the given polynomial.

19. $4x^3 - 3x^2 - 8x + 4; \quad x - 2$

20. $-4x^3 + 5x^2 + 8; \quad x + 3$

21. $3x^4 - 6x^3 - 5x + 10; \quad x - 2$

22. $4x^4 - 15x^2 - 4; \quad x - 2$

23. $3x^6 + 82x^3 + 27; \quad x + 3$

24. $2x^6 - 18x^4 + x^2 - 9; \quad x + 3$

25. $4x^6 - 64x^4 + x^2 - 15; \quad x + 4$

26. $x^6 - 16x^4 + x^2 - 16; \quad x + 4$

27. $2x^4 - x^3 + 2x - 1; \quad x - \frac{1}{2}$

28. $3x^4 + x^3 - 3x + 1; \quad x + \frac{1}{3}$

Applications and Extensions

29. Find the sum of a, b, c, and d if

$$\frac{x^3 - 2x^2 + 3x + 5}{x + 2} = ax^2 + bx + c + \frac{d}{x + 2}$$

Explaining Concepts: Discussion and Writing

30. When dividing a polynomial by $x - c$, do you prefer to use long division or synthetic division? Does the value of c make a difference to you in choosing? Give reasons.

R.7 Rational Expressions

OBJECTIVES
1 Reduce a Rational Expression to Lowest Terms (p. 63)
2 Multiply and Divide Rational Expressions (p. 64)
3 Add and Subtract Rational Expressions (p. 65)
4 Use the Least Common Multiple Method (p. 67)
5 Simplify Complex Rational Expressions (p. 69)

1 Reduce a Rational Expression to Lowest Terms

If we form the quotient of two polynomials, the result is called a **rational expression**. Some examples of rational expressions are

$$\text{(a)}\ \frac{x^3 + 1}{x} \qquad \text{(b)}\ \frac{3x^2 + x - 2}{x^2 + 5} \qquad \text{(c)}\ \frac{x}{x^2 - 1} \qquad \text{(d)}\ \frac{xy^2}{(x - y)^2}$$

Expressions (a), (b), and (c) are rational expressions in one variable, x, whereas (d) is a rational expression in two variables, x and y.

Rational expressions are described in the same manner as rational numbers. In expression (a), the polynomial $x^3 + 1$ is the **numerator**, and x is the **denominator**. When the numerator and denominator of a rational expression contain no common factors (except 1 and -1), we say that the rational expression is **reduced to lowest terms**, or **simplified**.

The polynomial in the denominator of a rational expression cannot be equal to 0 because division by 0 is not defined. For example, for the expression $\frac{x^3 + 1}{x}$, x cannot take on the value 0. The domain of the variable x is $\{x \mid x \neq 0\}$.

A rational expression is reduced to lowest terms by factoring the numerator and the denominator completely and dividing out any common factors using the Reduction Property:

$$\frac{a\cancel{c}}{b\cancel{c}} = \frac{a}{b} \qquad \text{if } b \neq 0, c \neq 0 \qquad \textbf{(1)}$$

EXAMPLE 1 **Reducing a Rational Expression to Lowest Terms**

Reduce to lowest terms: $\dfrac{x^2 + 4x + 4}{x^2 + 3x + 2}$

Solution Begin by factoring the numerator and the denominator.

$$x^2 + 4x + 4 = (x + 2)(x + 2)$$

$$x^2 + 3x + 2 = (x + 2)(x + 1)$$

WARNING Apply the Reduction Property only to rational expressions written in factored form. Be sure to divide out only common factors, not common terms! ■

Since a common factor, $x + 2$, appears, the original expression is not in lowest terms. To reduce it to lowest terms, use the Reduction Property:

$$\frac{x^2 + 4x + 4}{x^2 + 3x + 2} = \frac{\cancel{(x + 2)}(x + 2)}{\cancel{(x + 2)}(x + 1)} = \frac{x + 2}{x + 1} \qquad x \neq -2, -1$$ ■

EXAMPLE 2 **Reducing Rational Expressions to Lowest Terms**

Reduce each rational expression to lowest terms.

(a) $\dfrac{x^3 - 8}{x^3 - 2x^2}$ (b) $\dfrac{8 - 2x}{x^2 - x - 12}$

Solution

(a) $$\frac{x^3 - 8}{x^3 - 2x^2} = \frac{\cancel{(x - 2)}(x^2 + 2x + 4)}{x^2\cancel{(x - 2)}} = \frac{x^2 + 2x + 4}{x^2} \qquad x \neq 0, 2$$

(b) $$\frac{8 - 2x}{x^2 - x - 12} = \frac{2(4 - x)}{(x - 4)(x + 3)} = \frac{2(-1)\cancel{(x - 4)}}{\cancel{(x - 4)}(x + 3)} = \frac{-2}{x + 3} \qquad x \neq -3, 4$$ ■

Now Work PROBLEM 7

2 Multiply and Divide Rational Expressions

The rules for multiplying and dividing rational expressions are the same as the rules for multiplying and dividing rational numbers. If $\frac{a}{b}$ and $\frac{c}{d}$, $b \neq 0$, $d \neq 0$, are two rational expressions, then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \qquad \text{if } b \neq 0, d \neq 0 \qquad \textbf{(2)}$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc} \qquad \text{if } b \neq 0, c \neq 0, d \neq 0 \qquad \textbf{(3)}$$

In using equations (2) and (3) with rational expressions, be sure first to factor each polynomial completely so that common factors can be divided out. Leave your answer in factored form.

EXAMPLE 3

Multiplying and Dividing Rational Expressions

Perform the indicated operation and simplify the result. Leave your answer in factored form.

(a) $\dfrac{x^2-2x+1}{x^3+x}\cdot\dfrac{4x^2+4}{x^2+x-2}$ (b) $\dfrac{\dfrac{x+3}{x^2-4}}{\dfrac{x^2-x-12}{x^3-8}}$

Solution

(a)
$$\begin{aligned}\frac{x^2-2x+1}{x^3+x}\cdot\frac{4x^2+4}{x^2+x-2}&=\frac{(x-1)^2}{x(x^2+1)}\cdot\frac{4(x^2+1)}{(x+2)(x-1)}\\&=\frac{(x-1)^{\cancel{2}}(4)\cancel{(x^2+1)}}{x\cancel{(x^2+1)}(x+2)\cancel{(x-1)}}\\&=\frac{4(x-1)}{x(x+2)}\qquad x\neq-2,0,1\end{aligned}$$

(b)
$$\begin{aligned}\frac{\dfrac{x+3}{x^2-4}}{\dfrac{x^2-x-12}{x^3-8}}&=\frac{x+3}{x^2-4}\cdot\frac{x^3-8}{x^2-x-12}\\&=\frac{x+3}{(x-2)(x+2)}\cdot\frac{(x-2)(x^2+2x+4)}{(x-4)(x+3)}\\&=\frac{\cancel{(x+3)}\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}(x+2)(x-4)\cancel{(x+3)}}\\&=\frac{x^2+2x+4}{(x+2)(x-4)}\qquad x\neq-3,-2,2,4\end{aligned}$$

■

Now Work PROBLEMS 19 AND 27

3 Add and Subtract Rational Expressions

The rules for adding and subtracting rational expressions are the same as the rules for adding and subtracting rational numbers. If the denominators of two rational expressions to be added (or subtracted) are equal, then add (or subtract) the numerators and keep the common denominator.

In Words

To add (or subtract) two rational expressions with the same denominator, keep the common denominator and add (or subtract) the numerators.

If $\dfrac{a}{b}$ and $\dfrac{c}{b}$ are two rational expressions, then

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}\qquad\frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}\qquad\text{if } b\neq 0\tag{4}$$

EXAMPLE 4

Adding and Subtracting Rational Expressions with Equal Denominators

Perform the indicated operation and simplify the result. Leave your answer in factored form.

(a) $\dfrac{2x^2-4}{2x+5}+\dfrac{x+3}{2x+5}\quad x\neq-\dfrac{5}{2}$ (b) $\dfrac{x}{x-3}-\dfrac{3x+2}{x-3}\quad x\neq 3$

Solution

(a)
$$\begin{aligned}\frac{2x^2-4}{2x+5}+\frac{x+3}{2x+5}&=\frac{(2x^2-4)+(x+3)}{2x+5}\\&=\frac{2x^2+x-1}{2x+5}=\frac{(2x-1)(x+1)}{2x+5}\end{aligned}$$

$$\text{(b)}\quad \frac{x}{x-3}-\frac{3x+2}{x-3}=\frac{x-(3x+2)}{x-3}=\frac{x-3x-2}{x-3}$$
$$=\frac{-2x-2}{x-3}=\frac{-2(x+1)}{x-3}$$

■

EXAMPLE 5

Adding Rational Expressions Whose Denominators Are Additive Inverses of Each Other

Perform the indicated operation and simplify the result. Leave your answer in factored form.

$$\frac{2x}{x-3}+\frac{5}{3-x}\qquad x\neq 3$$

Solution Notice that the denominators of the two rational expressions are different. However, the denominator of the second expression is the additive inverse of the denominator of the first. That is,

$$3-x=-x+3=-1\cdot(x-3)=-(x-3)$$

Then

$$\frac{2x}{x-3}+\frac{5}{3-x}\underset{\substack{\uparrow\\ 3-x=-(x-3)}}{=}\frac{2x}{x-3}+\frac{5}{-(x-3)}\underset{\substack{\uparrow\\ \frac{a}{-b}=\frac{-a}{b}}}{=}\frac{2x}{x-3}+\frac{-5}{x-3}$$
$$=\frac{2x+(-5)}{x-3}=\frac{2x-5}{x-3}$$

■

Now Work PROBLEMS 39 AND 45

If the denominators of two rational expressions to be added or subtracted are not equal, we can use the general formulas for adding and subtracting rational expressions.

$$\frac{a}{b}+\frac{c}{d}=\frac{a\cdot d}{b\cdot d}+\frac{b\cdot c}{b\cdot d}=\frac{ad+bc}{bd}\qquad \text{if } b\neq 0,\, d\neq 0 \qquad \textbf{(5a)}$$
$$\frac{a}{b}-\frac{c}{d}=\frac{a\cdot d}{b\cdot d}-\frac{b\cdot c}{b\cdot d}=\frac{ad-bc}{bd}\qquad \text{if } b\neq 0,\, d\neq 0 \qquad \textbf{(5b)}$$

EXAMPLE 6

Adding and Subtracting Rational Expressions with Unequal Denominators

Perform the indicated operation and simplify the result. Leave your answer in factored form.

$$\text{(a)}\quad \frac{x-3}{x+4}+\frac{x}{x-2}\qquad x\neq -4, 2 \qquad\qquad \text{(b)}\quad \frac{x^2}{x^2-4}-\frac{1}{x}\qquad x\neq -2, 0, 2$$

Solution

$$\text{(a)}\quad \frac{x-3}{x+4}+\frac{x}{x-2}\underset{\substack{\uparrow\\ \textbf{(5a)}}}{=}\frac{x-3}{x+4}\cdot\frac{x-2}{x-2}+\frac{x+4}{x+4}\cdot\frac{x}{x-2}$$
$$=\frac{(x-3)(x-2)+(x+4)(x)}{(x+4)(x-2)}$$
$$=\frac{x^2-5x+6+x^2+4x}{(x+4)(x-2)}=\frac{2x^2-x+6}{(x+4)(x-2)}$$

$$\text{(b)}\quad \frac{x^2}{x^2-4} - \frac{1}{x} \underset{\overset{\uparrow}{\text{(5b)}}}{=} \frac{x^2}{x^2-4}\cdot\frac{x}{x} - \frac{x^2-4}{x^2-4}\cdot\frac{1}{x} = \frac{x^2(x) - (x^2-4)(1)}{(x^2-4)(x)}$$

$$= \frac{x^3 - x^2 + 4}{(x-2)(x+2)(x)} \qquad ■$$

Now Work PROBLEM 49

4 Use the Least Common Multiple Method

If the denominators of two rational expressions to be added (or subtracted) have common factors, we usually do not use the general rules given by equations (5a) and (5b). Just as with fractions, we apply the **least common multiple (LCM) method**. The LCM method uses the polynomial of least degree that has each denominator polynomial as a factor.

The LCM Method for Adding or Subtracting Rational Expressions

The Least Common Multiple (LCM) Method requires four steps:

STEP 1: Factor completely the polynomial in the denominator of each rational expression.

STEP 2: The LCM of the denominators is the product of each of these factors raised to a power equal to the greatest number of times that the factor occurs in the polynomials.

STEP 3: Write each rational expression using the LCM as the common denominator.

STEP 4: Add or subtract the rational expressions using equation (4).

We begin with an example that requires only Steps 1 and 2.

EXAMPLE 7 Finding the Least Common Multiple

Find the least common multiple of the following pair of polynomials:

$$x(x-1)^2(x+1) \quad \text{and} \quad 4(x-1)(x+1)^3$$

Solution

STEP 1: The polynomials are already factored completely as

$$x(x-1)^2(x+1) \quad \text{and} \quad 4(x-1)(x+1)^3$$

STEP 2: Start by writing the factors of the left-hand polynomial. (Or you could start with the one on the right.)

$$x(x-1)^2(x+1)$$

Now look at the right-hand polynomial. Its first factor, 4, does not appear in our list, so we insert it.

$$4x(x-1)^2(x+1)$$

The next factor, $x - 1$, is already in our list, so no change is necessary. The final factor is $(x+1)^3$. Since our list has $x + 1$ to the first power only, we replace $x + 1$ in the list by $(x+1)^3$. The LCM is

$$4x(x-1)^2(x+1)^3$$

Notice that the LCM is, in fact, the polynomial of least degree that contains $x(x-1)^2(x+1)$ and $4(x-1)(x+1)^3$ as factors. ■

Now Work PROBLEM 55

EXAMPLE 8 Using the Least Common Multiple to Add Rational Expressions

Perform the indicated operation and simplify the result. Leave your answer in factored form.

$$\frac{x}{x^2+3x+2}+\frac{2x-3}{x^2-1} \qquad x \neq -2, -1, 1$$

Solution **STEP 1:** Factor completely the polynomials in the denominators.

$$\begin{aligned} x^2+3x+2 &= (x+2)(x+1) \\ x^2-1 &= (x-1)(x+1) \end{aligned}$$

STEP 2: The LCM is $(x+2)(x+1)(x-1)$. Do you see why?
STEP 3: Write each rational expression using the LCM as the denominator.

$$\frac{x}{x^2+3x+2}=\frac{x}{(x+2)(x+1)}=\frac{x}{(x+2)(x+1)}\cdot\frac{x-1}{x-1}=\frac{x(x-1)}{(x+2)(x+1)(x-1)}$$

↑ Multiply numerator and denominator by $x - 1$ to get the LCM in the denominator.

$$\frac{2x-3}{x^2-1}=\frac{2x-3}{(x-1)(x+1)}=\frac{2x-3}{(x-1)(x+1)}\cdot\frac{x+2}{x+2}=\frac{(2x-3)(x+2)}{(x-1)(x+1)(x+2)}$$

↑ Multiply numerator and denominator by $x + 2$ to get the LCM in the denominator.

STEP 4: Now add by using equation (4).

$$\begin{aligned} \frac{x}{x^2+3x+2}+\frac{2x-3}{x^2-1} &= \frac{x(x-1)}{(x+2)(x+1)(x-1)}+\frac{(2x-3)(x+2)}{(x+2)(x+1)(x-1)} \\ &= \frac{(x^2-x)+(2x^2+x-6)}{(x+2)(x+1)(x-1)} \\ &= \frac{3x^2-6}{(x+2)(x+1)(x-1)}=\frac{3(x^2-2)}{(x+2)(x+1)(x-1)} \end{aligned}$$

■

EXAMPLE 9 Using the Least Common Multiple to Subtract Rational Expressions

Perform the indicated operation and simplify the result. Leave your answer in factored form.

$$\frac{3}{x^2+x}-\frac{x+4}{x^2+2x+1} \qquad x \neq -1, 0$$

Solution **STEP 1:** Factor completely the polynomials in the denominators.

$$\begin{aligned} x^2+x &= x(x+1) \\ x^2+2x+1 &= (x+1)^2 \end{aligned}$$

STEP 2: The LCM is $x(x+1)^2$.

STEP 3: Write each rational expression using the LCM as the denominator.

$$\frac{3}{x^2+x} = \frac{3}{x(x+1)} = \frac{3}{x(x+1)} \cdot \frac{x+1}{x+1} = \frac{3(x+1)}{x(x+1)^2}$$

$$\frac{x+4}{x^2+2x+1} = \frac{x+4}{(x+1)^2} = \frac{x+4}{(x+1)^2} \cdot \frac{x}{x} = \frac{x(x+4)}{x(x+1)^2}$$

STEP 4: Subtract, using equation (4).

$$\begin{aligned} \frac{3}{x^2+x} - \frac{x+4}{x^2+2x+1} &= \frac{3(x+1)}{x(x+1)^2} - \frac{x(x+4)}{x(x+1)^2} \\ &= \frac{3(x+1) - x(x+4)}{x(x+1)^2} \\ &= \frac{3x+3-x^2-4x}{x(x+1)^2} \\ &= \frac{-x^2-x+3}{x(x+1)^2} \end{aligned}$$

■

Now Work PROBLEM 65

5 Simplify Complex Rational Expressions

When sums and/or differences of rational expressions appear as the numerator and/or denominator of a quotient, the quotient is called a **complex rational expression**.* For example,

$$\frac{1+\dfrac{1}{x}}{1-\dfrac{1}{x}} \quad \text{and} \quad \frac{\dfrac{x^2}{x^2-4}-3}{\dfrac{x-3}{x+2}-1}$$

are complex rational expressions. To **simplify** a complex rational expression means to write it as a rational expression reduced to lowest terms. This can be accomplished in either of two ways.

Simplifying a Complex Rational Expression

Option 1: Treat the numerator and denominator of the complex rational expression separately, performing whatever operations are indicated and simplifying the results. Follow this by simplifying the resulting rational expression.

Option 2: Find the LCM of the denominators of all rational expressions that appear in the complex rational expression. Multiply the numerator and denominator of the complex rational expression by the LCM and simplify the result.

We use both options in the next example. By carefully studying each option, you can discover situations in which one may be easier to use than the other.

EXAMPLE 10 Simplifying a Complex Rational Expression

Simplify: $\dfrac{\dfrac{1}{2}+\dfrac{3}{x}}{\dfrac{x+3}{4}} \qquad x \neq -3, 0$

---

* Some texts use the term **complex fraction**.

Solution ***Option 1:*** First, we perform the indicated operation in the numerator, and then we divide.

$$\frac{\frac{1}{2}+\frac{3}{x}}{\frac{x+3}{4}} \underset{\uparrow}{=} \frac{\frac{1\cdot x+2\cdot 3}{2\cdot x}}{\frac{x+3}{4}} = \frac{\frac{x+6}{2x}}{\frac{x+3}{4}} \underset{\uparrow}{=} \frac{x+6}{2x}\cdot\frac{4}{x+3}$$

Rule for adding quotients Rule for dividing quotients

$$\underset{\uparrow}{=} \frac{(x+6)\cdot 4}{2\cdot x\cdot(x+3)} = \frac{\cancel{2}\cdot 2\cdot(x+6)}{\cancel{2}\cdot x\cdot(x+3)} = \frac{2(x+6)}{x(x+3)}$$

Rule for multiplying quotients

Option 2: The rational expressions that appear in the complex rational expression are

$$\frac{1}{2},\quad \frac{3}{x},\quad \frac{x+3}{4}$$

The LCM of their denominators is $4x$. We multiply the numerator and denominator of the complex rational expression by $4x$ and then simplify.

$$\frac{\frac{1}{2}+\frac{3}{x}}{\frac{x+3}{4}} \underset{\uparrow}{=} \frac{4x\cdot\left(\frac{1}{2}+\frac{3}{x}\right)}{4x\cdot\left(\frac{x+3}{4}\right)} \underset{\uparrow}{=} \frac{4x\cdot\frac{1}{2}+4x\cdot\frac{3}{x}}{\frac{4x\cdot(x+3)}{4}}$$

Multiply the numerator and denominator by $4x$. Use the Distributive Property in the numerator.

$$= \frac{\cancel{2}\cdot 2x\cdot\frac{1}{\cancel{2}}+4\cancel{x}\cdot\frac{3}{\cancel{x}}}{\frac{\cancel{4}x\cdot(x+3)}{\cancel{4}}} \underset{\uparrow}{=} \frac{2x+12}{x(x+3)} \underset{\uparrow}{=} \frac{2(x+6)}{x(x+3)}$$

Simplify. Factor. ■

EXAMPLE 11 Simplifying a Complex Rational Expression

Simplify: $\dfrac{\frac{x^2}{x-4}+2}{\frac{2x-2}{x}-1} \qquad x \neq 0, 2, 4$

Solution We will use Option 1.

$$\frac{\frac{x^2}{x-4}+2}{\frac{2x-2}{x}-1} = \frac{\frac{x^2}{x-4}+\frac{2(x-4)}{x-4}}{\frac{2x-2}{x}-\frac{x}{x}} = \frac{\frac{x^2+2x-8}{x-4}}{\frac{2x-2-x}{x}}$$

$$= \frac{\frac{(x+4)(x-2)}{x-4}}{\frac{x-2}{x}} = \frac{(x+4)\cancel{(x-2)}}{x-4}\cdot\frac{x}{\cancel{x-2}}$$

$$= \frac{(x+4)\cdot x}{x-4}$$

■

Now Work PROBLEM 75

Application

EXAMPLE 12 **Solving an Application in Electricity**

An electrical circuit contains two resistors connected in parallel, as shown in Figure 31. If these two resistors provide resistance of R_1 and R_2 ohms, respectively, their combined resistance R is given by the formula

$$R = \frac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2}}$$

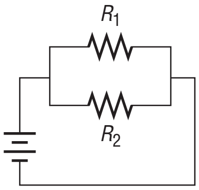


Figure 31

Express R as a rational expression; that is, simplify the right-hand side of this formula. Evaluate the rational expression if $R_1 = 6$ ohms and $R_2 = 10$ ohms.

Solution We will use Option 2. If we consider 1 as the fraction $\frac{1}{1}$, the rational expressions in the complex rational expression are

$$\frac{1}{1}, \quad \frac{1}{R_1}, \quad \frac{1}{R_2}$$

The LCM of the denominators is $R_1 R_2$. We multiply the numerator and denominator of the complex rational expression by $R_1 R_2$ and simplify.

$$\frac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2}} = \frac{1 \cdot R_1 R_2}{\left(\dfrac{1}{R_1} + \dfrac{1}{R_2}\right) \cdot R_1 R_2} = \frac{R_1 R_2}{\dfrac{1}{R_1} \cdot R_1 R_2 + \dfrac{1}{R_2} \cdot R_1 R_2} = \frac{R_1 R_2}{R_2 + R_1}$$

So,

$$R = \frac{R_1 R_2}{R_2 + R_1}$$

If $R_1 = 6$ and $R_2 = 10$, then

$$R = \frac{6 \cdot 10}{10 + 6} = \frac{60}{16} = \frac{15}{4} \quad \text{ohms}$$ ■

R.7 Assess Your Understanding

Concepts and Vocabulary

1. When the numerator and denominator of a rational expression contain no common factors (except 1 and -1), the rational expression is in ______ ______.

2. LCM is an abbreviation for ______ ________ ________.

3. Choose the statement that is not true. Assume $b \neq 0, c \neq 0$, and $d \neq 0$ as necessary.

(a) $\frac{ac}{bc} = \frac{a}{b}$ (b) $\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$

(c) $\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$ (d) $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \frac{ac}{bd}$

4. Choose the rational expression that simplifies to -1.

(a) $\frac{a - b}{b - a}$ (b) $\frac{a - b}{a - b}$

(c) $\frac{a + b}{a - b}$ (d) $\frac{b - a}{b + a}$

5. ***True or False*** The rational expression $\frac{2x^3 - 4x}{x - 2}$ is reduced to lowest terms.

6. ***True or False*** The LCM of $2x^3 + 6x^2$ and $6x^4 + 4x^3$ is $4x^3(x + 1)$.

Skill Building

In Problems 7–18, reduce each rational expression to lowest terms.

7. $\frac{3x + 9}{x^2 - 9}$ **8.** $\frac{4x^2 + 8x}{12x + 24}$ **9.** $\frac{x^2 - 2x}{3x - 6}$ **10.** $\frac{15x^2 + 24x}{3x^2}$

11. $\frac{24x^2}{12x^2 - 6x}$ **12.** $\frac{x^2 + 4x + 4}{x^2 - 4}$ **13.** $\frac{y^2 - 25}{2y^2 - 8y - 10}$ **14.** $\frac{3y^2 - y - 2}{3y^2 + 5y + 2}$

15. $\frac{x^2 + 4x - 5}{x^2 - 2x + 1}$ **16.** $\frac{x - x^2}{x^2 + x - 2}$ **17.** $\frac{x^2 + 5x - 14}{2 - x}$ **18.** $\frac{2x^2 + 5x - 3}{1 - 2x}$

In Problems 19–36, perform the indicated operation and simplify the result. Leave your answer in factored form.

19. $\dfrac{3x+6}{5x^2}\cdot\dfrac{x}{x^2-4}$

20. $\dfrac{3}{2x}\cdot\dfrac{x^2}{6x+10}$

21. $\dfrac{4x^2}{x^2-16}\cdot\dfrac{x^3-64}{2x}$

22. $\dfrac{12}{x^2+x}\cdot\dfrac{x^3+1}{4x-2}$

23. $\dfrac{4x-8}{-3x}\cdot\dfrac{12}{12-6x}$

24. $\dfrac{6x-27}{5x}\cdot\dfrac{2}{4x-18}$

25. $\dfrac{x^2-3x-10}{x^2+2x-35}\cdot\dfrac{x^2+4x-21}{x^2+9x+14}$

26. $\dfrac{x^2+x-6}{x^2+4x-5}\cdot\dfrac{x^2-25}{x^2+2x-15}$

27. $\dfrac{\dfrac{6x}{x^2-4}}{\dfrac{3x-9}{2x+4}}$

28. $\dfrac{\dfrac{12x}{5x+20}}{\dfrac{4x^2}{x^2-16}}$

29. $\dfrac{\dfrac{8x}{x^2-1}}{\dfrac{10x}{x+1}}$

30. $\dfrac{\dfrac{x-2}{4x}}{\dfrac{x^2-4x+4}{12x}}$

31. $\dfrac{\dfrac{4-x}{4+x}}{\dfrac{4x}{x^2-16}}$

32. $\dfrac{\dfrac{3+x}{3-x}}{\dfrac{x^2-9}{9x^3}}$

33. $\dfrac{\dfrac{x^2+7x+12}{x^2-7x+12}}{\dfrac{x^2+x-12}{x^2-x-12}}$

34. $\dfrac{\dfrac{x^2+7x+6}{x^2+x-6}}{\dfrac{x^2+5x-6}{x^2+5x+6}}$

35. $\dfrac{\dfrac{2x^2-x-28}{3x^2-x-2}}{\dfrac{4x^2+16x+7}{3x^2+11x+6}}$

36. $\dfrac{\dfrac{9x^2+3x-2}{12x^2+5x-2}}{\dfrac{9x^2-6x+1}{8x^2-10x-3}}$

In Problems 37–54, perform the indicated operation and simplify the result. Leave your answer in factored form.

37. $\dfrac{x}{2}+\dfrac{5}{2}$

38. $\dfrac{3}{x}-\dfrac{6}{x}$

39. $\dfrac{x^2}{2x-3}-\dfrac{4}{2x-3}$

40. $\dfrac{3x^2}{2x-1}-\dfrac{9}{2x-1}$

41. $\dfrac{x+1}{x-3}+\dfrac{2x-3}{x-3}$

42. $\dfrac{2x-5}{3x+2}+\dfrac{x+4}{3x+2}$

43. $\dfrac{3x+5}{2x-1}-\dfrac{2x-4}{2x-1}$

44. $\dfrac{5x-4}{3x+4}-\dfrac{x+1}{3x+4}$

45. $\dfrac{4}{x-2}+\dfrac{x}{2-x}$

46. $\dfrac{6}{x-1}-\dfrac{x}{1-x}$

47. $\dfrac{4}{x-1}-\dfrac{2}{x+2}$

48. $\dfrac{2}{x+5}-\dfrac{5}{x-5}$

49. $\dfrac{x}{x+1}+\dfrac{2x-3}{x-1}$

50. $\dfrac{3x}{x-4}+\dfrac{2x}{x+3}$

51. $\dfrac{x-3}{x+2}-\dfrac{x+4}{x-2}$

52. $\dfrac{2x-3}{x-1}-\dfrac{2x+1}{x+1}$

53. $\dfrac{x}{x^2-4}+\dfrac{1}{x}$

54. $\dfrac{x-1}{x^3}+\dfrac{x}{x^2+1}$

In Problems 55–62, find the LCM of the given polynomials.

55. $x^2-4,\quad x^2-x-2$

56. $x^2-x-12,\quad x^2-8x+16$

57. $x^3-x,\quad x^2-x$

58. $3x^2-27,\quad 2x^2-x-15$

59. $4x^3-4x^2+x,\quad 2x^3-x^2,\quad x^3$

60. $x-3,\quad x^2+3x,\quad x^3-9x$

61. $x^3-x,\quad x^3-2x^2+x,\quad x^3-1$

62. $x^2+4x+4,\quad x^3+2x^2,\quad (x+2)^3$

In Problems 63–74, perform the indicated operations and simplify the result. Leave your answer in factored form.

63. $\dfrac{x}{x^2-7x+6}-\dfrac{x}{x^2-2x-24}$

64. $\dfrac{x}{x-3}-\dfrac{x+1}{x^2+5x-24}$

65. $\dfrac{4x}{x^2-4}-\dfrac{2}{x^2+x-6}$

66. $\dfrac{3x}{x-1}-\dfrac{x-4}{x^2-2x+1}$

67. $\dfrac{3}{(x-1)^2(x+1)}+\dfrac{2}{(x-1)(x+1)^2}$

68. $\dfrac{2}{(x+2)^2(x-1)}-\dfrac{6}{(x+2)(x-1)^2}$

69. $\dfrac{x+4}{x^2-x-2}-\dfrac{2x+3}{x^2+2x-8}$

70. $\dfrac{2x-3}{x^2+8x+7}-\dfrac{x-2}{(x+1)^2}$

71. $\dfrac{1}{x}-\dfrac{2}{x^2+x}+\dfrac{3}{x^3-x^2}$

72. $\dfrac{x}{(x-1)^2}+\dfrac{2}{x}-\dfrac{x+1}{x^3-x^2}$

73. $\dfrac{1}{h}\left(\dfrac{1}{x+h}-\dfrac{1}{x}\right)$

74. $\dfrac{1}{h}\left[\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}\right]$

In Problems 75–86, perform the indicated operations and simplify the result. Leave your answer in factored form.

75. $\dfrac{1+\dfrac{1}{x}}{1-\dfrac{1}{x}}$

76. $\dfrac{4+\dfrac{1}{x^2}}{3-\dfrac{1}{x^2}}$

77. $\dfrac{2-\dfrac{x+1}{x}}{3+\dfrac{x-1}{x+1}}$

78. $\dfrac{1-\dfrac{x}{x+1}}{2-\dfrac{x-1}{x}}$

79. $\dfrac{\dfrac{x+4}{x-2}-\dfrac{x-3}{x+1}}{x+1}$

80. $\dfrac{\dfrac{x-2}{x+1}-\dfrac{x}{x-2}}{x+3}$

81. $\dfrac{\dfrac{x-2}{x+2}+\dfrac{x-1}{x+1}}{\dfrac{x}{x+1}-\dfrac{2x-3}{x}}$

82. $\dfrac{\dfrac{2x+5}{x}-\dfrac{x}{x-3}}{\dfrac{x^2}{x-3}-\dfrac{(x+1)^2}{x+3}}$

83. $1-\dfrac{1}{1-\dfrac{1}{x}}$

84. $1-\dfrac{1}{1-\dfrac{1}{1-x}}$

85. $\dfrac{2(x-1)^{-1}+3}{3(x-1)^{-1}+2}$

86. $\dfrac{4(x+2)^{-1}-3}{3(x+2)^{-1}-1}$

In Problems 87–94, expressions that occur in calculus are given. Reduce each expression to lowest terms.

87. $\dfrac{(2x+3)\cdot 3-(3x-5)\cdot 2}{(3x-5)^2}$

88. $\dfrac{(4x+1)\cdot 5-(5x-2)\cdot 4}{(5x-2)^2}$

89. $\dfrac{x\cdot 2x-(x^2+1)\cdot 1}{(x^2+1)^2}$

90. $\dfrac{x\cdot 2x-(x^2-4)\cdot 1}{(x^2-4)^2}$

91. $\dfrac{(3x+1)\cdot 2x-x^2\cdot 3}{(3x+1)^2}$

92. $\dfrac{(2x-5)\cdot 3x^2-x^3\cdot 2}{(2x-5)^2}$

93. $\dfrac{(x^2+1)\cdot 3-(3x+4)\cdot 2x}{(x^2+1)^2}$

94. $\dfrac{(x^2+9)\cdot 2-(2x-5)\cdot 2x}{(x^2+9)^2}$

Applications and Extensions

95. The Lensmaker's Equation The focal length f of a lens with index of refraction n is

$$\frac{1}{f}=(n-1)\left[\frac{1}{R_1}+\frac{1}{R_2}\right]$$

where R_1 and R_2 are the radii of curvature of the front and back surfaces of the lens. Express f as a rational expression. Evaluate the rational expression for $n = 1.5$, $R_1 = 0.1$ meter, and $R_2 = 0.2$ meter.

96. Electrical Circuits An electrical circuit contains three resistors connected in parallel. If these three resistors provide resistance of R_1, R_2, and R_3 ohms, respectively, their combined resistance R is given by the formula

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}$$

Express R as a rational expression. Evaluate R for $R_1 = 5$ ohms, $R_2 = 4$ ohms, and $R_3 = 10$ ohms.

Explaining Concepts: Discussion and Writing

97. The following expressions are called **continued fractions**:

$$1+\frac{1}{x},\quad 1+\cfrac{1}{1+\cfrac{1}{x}},\quad 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}},\quad 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}}},\ldots$$

Each simplifies to an expression of the form

$$\frac{ax+b}{bx+c}$$

Trace the successive values of a, b, and c as you "continue" the fraction. Can you discover the patterns that these values follow? Go to the library and research Fibonacci numbers. Write a report on your findings.

98. Explain to a fellow student when you would use the LCM method to add two rational expressions. Give two examples of adding two rational expressions: one in which you use the LCM and the other in which you do not.

99. Which of the two options given in the text for simplifying complex rational expressions do you prefer? Write a brief paragraph stating the reasons for your choice.

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Exponents, Square Roots (Section R.2, pp. 22–25)

Now Work the 'Are You Prepared?' problems on page 79.

OBJECTIVES
1 Work with nth Roots (p. 74)
2 Simplify Radicals (p. 75)
3 Rationalize Denominators (p. 76)
4 Simplify Expressions with Rational Exponents (p. 77)

1 Work with *n*th Roots

DEFINITION

The **principal nth root of a real number a**, $n \geq 2$ an integer, symbolized by $\sqrt[n]{a}$, is defined as follows:

$$\boxed{\sqrt[n]{a} = b \quad \text{means} \quad a = b^n}$$

where $a \geq 0$ and $b \geq 0$ if n is even and a, b are any real numbers if n is odd. ▪

In Words
The symbol $\sqrt[n]{a}$ means "give me the number that, when raised to the power n, equals a."

Notice that if a is negative and n is even, then $\sqrt[n]{a}$ is not defined. When it is defined, the principal nth root of a number is unique.

The symbol $\sqrt[n]{a}$ for the principal nth root of a is called a **radical**; the integer n is called the **index**, and a is called the **radicand**. If the index of a radical is 2, we call $\sqrt[2]{a}$ the **square root** of a and omit the index 2 by simply writing $\sqrt{a}$. If the index is 3, we call $\sqrt[3]{a}$ the **cube root** of a.

EXAMPLE 1

Simplifying Principal *n*th Roots

(a) $\sqrt[3]{8} = \sqrt[3]{2^3} = 2$

(b) $\sqrt[3]{-64} = \sqrt[3]{(-4)^3} = -4$

(c) $\sqrt[4]{\frac{1}{16}} = \sqrt[4]{\left(\frac{1}{2}\right)^4} = \frac{1}{2}$

(d) $\sqrt[6]{(-2)^6} = |-2| = 2$ ▪

These are examples of **perfect roots**, since each simplifies to a rational number. Notice the absolute value in Example 1(d). If n is even, then the principal nth root must be nonnegative.

In general, if $n \geq 2$ is an integer and a is a real number, we have

$$\sqrt[n]{a^n} = a \qquad \text{if } n \geq 3 \text{ is odd} \qquad \textbf{(1a)}$$

$$\sqrt[n]{a^n} = |a| \qquad \text{if } n \geq 2 \text{ is even} \qquad \textbf{(1b)}$$

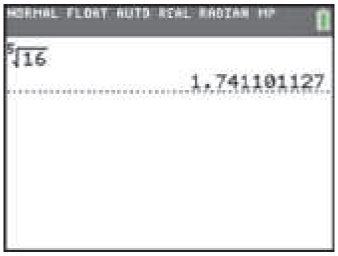
Now Work PROBLEM 11

Radicals provide a way of representing many irrational real numbers. For example, it can be shown that there is no rational number whose square is 2. Using radicals, we can say that $\sqrt{2}$ is the positive number whose square is 2.

EXAMPLE 2 Using a Calculator to Approximate Roots

Use a calculator to approximate $\sqrt[5]{16}$.

Solution Figure 32 shows the result using a TI-84 Plus C graphing calculator. ■



```
5√16
          1.741101127
```

Figure 32

Now Work PROBLEM 119

2 Simplify Radicals

Let $n \ge 2$ and $m \ge 2$ denote integers, and let a and b represent real numbers. Assuming that all radicals are defined, we have the following properties:

Properties of Radicals

$$\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b} \qquad \textbf{(2a)}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \qquad b \neq 0 \qquad \textbf{(2b)}$$

$$\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m \qquad \textbf{(2c)}$$

When used in reference to radicals, the direction to "simplify" will mean to remove from the radicals any perfect roots that occur as factors.

EXAMPLE 3 Simplifying Radicals

(a) $\sqrt{32} \underset{\uparrow}{=} \sqrt{16\cdot 2} \underset{\uparrow}{=} \sqrt{16}\cdot\sqrt{2} = 4\sqrt{2}$

Factor out 16, a perfect square. **(2a)**

(b) $\sqrt[3]{16} \underset{\uparrow}{=} \sqrt[3]{8\cdot 2} \underset{\uparrow}{=} \sqrt[3]{8}\cdot\sqrt[3]{2} = \sqrt[3]{2^3}\cdot\sqrt[3]{2} = 2\sqrt[3]{2}$

Factor out 8, a perfect cube. **(2a)**

(c) $\sqrt[3]{-16x^4} \underset{\uparrow}{=} \sqrt[3]{-8\cdot 2\cdot x^3\cdot x} \underset{\uparrow}{=} \sqrt[3]{(-8x^3)(2x)}$

Factor perfect cubes inside radical. **Group perfect cubes.**

$$= \sqrt[3]{(-2x)^3\cdot 2x} \underset{\uparrow}{=} \sqrt[3]{(-2x)^3}\cdot\sqrt[3]{2x} = -2x\sqrt[3]{2x}$$

(2a)

(d) $\sqrt[4]{\dfrac{16x^5}{81}} = \sqrt[4]{\dfrac{2^4x^4x}{3^4}} = \sqrt[4]{\left(\dfrac{2x}{3}\right)^4\cdot x} = \sqrt[4]{\left(\dfrac{2x}{3}\right)^4}\cdot\sqrt[4]{x} = \left|\dfrac{2x}{3}\right|\sqrt[4]{x}$ ■

Now Work PROBLEMS 15 AND 27

Two or more radicals can be combined, provided that they have the same index and the same radicand. Such radicals are called **like radicals**.

EXAMPLE 4 Combining Like Radicals

(a) $-8\sqrt{12} + \sqrt{3} = -8\sqrt{4\cdot 3} + \sqrt{3}$

$$= -8\cdot\sqrt{4}\,\sqrt{3} + \sqrt{3}$$

$$= -16\sqrt{3} + \sqrt{3} = -15\sqrt{3}$$

(b) $$\begin{aligned}\sqrt[3]{8x^4} + \sqrt[3]{-x} + 4\sqrt[3]{27x} &= \sqrt[3]{2^3x^3x} + \sqrt[3]{-1\cdot x} + 4\sqrt[3]{3^3x}\\ &= \sqrt[3]{(2x)^3}\cdot\sqrt[3]{x} + \sqrt[3]{-1}\cdot\sqrt[3]{x} + 4\sqrt[3]{3^3}\cdot\sqrt[3]{x}\\ &= 2x\sqrt[3]{x} - 1\cdot\sqrt[3]{x} + 12\sqrt[3]{x}\\ &= (2x+11)\sqrt[3]{x}\end{aligned}$$ ■

Now Work PROBLEM 45

3 Rationalize Denominators

When radicals occur in quotients, it is customary to rewrite the quotient so that the new denominator contains no radicals. This process is referred to as **rationalizing the denominator**.

The idea is to multiply by an appropriate expression so that the new denominator contains no radicals. For example:

If a Denominator Contains the Factor	Multiply by	To Obtain a Denominator Free of Radicals
$\sqrt{3}$	$\sqrt{3}$	$(\sqrt{3})^2 = 3$
$\sqrt{3}+1$	$\sqrt{3}-1$	$(\sqrt{3})^2 - 1^2 = 3 - 1 = 2$
$\sqrt{2}-3$	$\sqrt{2}+3$	$(\sqrt{2})^2 - 3^2 = 2 - 9 = -7$
$\sqrt{5}-\sqrt{3}$	$\sqrt{5}+\sqrt{3}$	$(\sqrt{5})^2 - (\sqrt{3})^2 = 5 - 3 = 2$
$\sqrt[3]{4}$	$\sqrt[3]{2}$	$\sqrt[3]{4}\cdot\sqrt[3]{2} = \sqrt[3]{8} = 2$

In rationalizing the denominator of a quotient, be sure to multiply both the numerator and the denominator by the expression.

EXAMPLE 5 **Rationalizing Denominators**

Rationalize the denominator of each expression:

(a) $\dfrac{1}{\sqrt{3}}$ (b) $\dfrac{5}{4\sqrt{2}}$ (c) $\dfrac{\sqrt{2}}{\sqrt{3}-3\sqrt{2}}$

Solution (a) The denominator contains the factor $\sqrt{3}$, so we multiply the numerator and denominator by $\sqrt{3}$ to obtain

$$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{(\sqrt{3})^2} = \frac{\sqrt{3}}{3}$$

(b) The denominator contains the factor $\sqrt{2}$, so we multiply the numerator and denominator by $\sqrt{2}$ to obtain

$$\frac{5}{4\sqrt{2}} = \frac{5}{4\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{2}}{4(\sqrt{2})^2} = \frac{5\sqrt{2}}{4\cdot 2} = \frac{5\sqrt{2}}{8}$$

(c) The denominator contains the factor $\sqrt{3}-3\sqrt{2}$, so we multiply the numerator and denominator by $\sqrt{3}+3\sqrt{2}$ to obtain

$$\begin{aligned}\frac{\sqrt{2}}{\sqrt{3}-3\sqrt{2}} &= \frac{\sqrt{2}}{\sqrt{3}-3\sqrt{2}}\cdot\frac{\sqrt{3}+3\sqrt{2}}{\sqrt{3}+3\sqrt{2}} = \frac{\sqrt{2}(\sqrt{3}+3\sqrt{2})}{(\sqrt{3})^2-(3\sqrt{2})^2}\\ &= \frac{\sqrt{2}\sqrt{3}+3(\sqrt{2})^2}{3-18} = \frac{\sqrt{6}+6}{-15} = -\frac{6+\sqrt{6}}{15}\end{aligned}$$ ■

Now Work PROBLEM 59

4 Simplify Expressions with Rational Exponents

Radicals are used to define rational exponents.

DEFINITION

If a is a real number and $n \geq 2$ is an integer, then

$$a^{1/n} = \sqrt[n]{a} \tag{3}$$

provided that $\sqrt[n]{a}$ exists.

Note that if n is even and $a < 0$, then $\sqrt[n]{a}$ and $a^{1/n}$ do not exist.

EXAMPLE 6 Writing Expressions Containing Fractional Exponents as Radicals

(a) $4^{1/2} = \sqrt{4} = 2$ (b) $8^{1/2} = \sqrt{8} = 2\sqrt{2}$

(c) $(-27)^{1/3} = \sqrt[3]{-27} = -3$ (d) $16^{1/3} = \sqrt[3]{16} = 2\sqrt[3]{2}$

DEFINITION

If a is a real number and m and n are integers containing no common factors, with $n \geq 2$, then

$$a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m \tag{4}$$

provided that $\sqrt[n]{a}$ exists.

We have two comments about equation (4):

1. The exponent $\dfrac{m}{n}$ must be in lowest terms, and $n \geq 2$ must be positive.
2. In simplifying the rational expression $a^{m/n}$, either $\sqrt[n]{a^m}$ or $(\sqrt[n]{a})^m$ may be used, the choice depending on which is easier to simplify. Generally, taking the root first, as in $(\sqrt[n]{a})^m$, is easier.

EXAMPLE 7 Using Equation (4)

(a) $4^{3/2} = (\sqrt{4})^3 = 2^3 = 8$ (b) $(-8)^{4/3} = (\sqrt[3]{-8})^4 = (-2)^4 = 16$

(c) $(32)^{-2/5} = (\sqrt[5]{32})^{-2} = 2^{-2} = \dfrac{1}{4}$ (d) $25^{6/4} = 25^{3/2} = (\sqrt{25})^3 = 5^3 = 125$

Now Work PROBLEM 69

It can be shown that the Laws of Exponents hold for rational exponents. The next example illustrates using the Laws of Exponents to simplify.

EXAMPLE 8 Simplifying Expressions Containing Rational Exponents

Simplify each expression. Express your answer so that only positive exponents occur. Assume that the variables are positive.

(a) $(x^{2/3}y)(x^{-2}y)^{1/2}$ (b) $\left(\dfrac{2x^{1/3}}{y^{2/3}}\right)^{-3}$ (c) $\left(\dfrac{9x^2y^{1/3}}{x^{1/3}y}\right)^{1/2}$

Solution (a)

$$\begin{aligned}(x^{2/3}y)(x^{-2}y)^{1/2} &= (x^{2/3}y)[(x^{-2})^{1/2}y^{1/2}]\\ &= x^{2/3}yx^{-1}y^{1/2}\\ &= (x^{2/3}\cdot x^{-1})(y\cdot y^{1/2})\\ &= x^{-1/3}y^{3/2}\\ &= \frac{y^{3/2}}{x^{1/3}}\end{aligned}$$

(b)

$$\left(\frac{2x^{1/3}}{y^{2/3}}\right)^{-3} = \left(\frac{y^{2/3}}{2x^{1/3}}\right)^{3} = \frac{(y^{2/3})^3}{(2x^{1/3})^3} = \frac{y^2}{2^3(x^{1/3})^3} = \frac{y^2}{8x}$$

(c)

$$\left(\frac{9x^2y^{1/3}}{x^{1/3}y}\right)^{1/2} = \left(\frac{9x^{2-(1/3)}}{y^{1-(1/3)}}\right)^{1/2} = \left(\frac{9x^{5/3}}{y^{2/3}}\right)^{1/2} = \frac{9^{1/2}(x^{5/3})^{1/2}}{(y^{2/3})^{1/2}} = \frac{3x^{5/6}}{y^{1/3}}$$

■

Now Work PROBLEM 89

The next two examples illustrate some algebra that you will need to know for certain calculus problems.

EXAMPLE 9 Writing an Expression as a Single Quotient

Write the following expression as a single quotient in which only positive exponents appear.

$$(x^2+1)^{1/2} + x\cdot\frac{1}{2}(x^2+1)^{-1/2}\cdot 2x$$

Solution

$$\begin{aligned}(x^2+1)^{1/2} + x\cdot\frac{1}{\cancel{2}}(x^2+1)^{-1/2}\cdot\cancel{2}x &= (x^2+1)^{1/2} + \frac{x^2}{(x^2+1)^{1/2}}\\ &= \frac{(x^2+1)^{1/2}(x^2+1)^{1/2}+x^2}{(x^2+1)^{1/2}}\\ &= \frac{(x^2+1)+x^2}{(x^2+1)^{1/2}}\\ &= \frac{2x^2+1}{(x^2+1)^{1/2}}\end{aligned}$$

■

Now Work PROBLEM 95

EXAMPLE 10 Factoring an Expression Containing Rational Exponents

Factor and simplify: $\frac{4}{3}x^{1/3}(2x+1) + 2x^{4/3}$

Solution Begin by writing $2x^{4/3}$ as a fraction with 3 as the denominator.

$$\frac{4}{3}x^{1/3}(2x+1) + 2x^{4/3} = \frac{4x^{1/3}(2x+1)}{3} + \frac{6x^{4/3}}{3} \underset{\substack{\uparrow\\ \text{Add the two fractions}}}{=} \frac{4x^{1/3}(2x+1)+6x^{4/3}}{3}$$

$$\underset{\substack{\uparrow\\ 2 \text{ and } x^{1/3} \text{ are common factors}}}{=} \frac{2x^{1/3}[2(2x+1)+3x]}{3} \underset{\substack{\uparrow\\ \text{Simplify}}}{=} \frac{2x^{1/3}(7x+2)}{3}$$

■

Now Work PROBLEM 107

Historical Note

The radical sign, $\surd$, was first used in print by Christoff Rudolff in 1525. It is thought to be the manuscript form of the letter *r* (for the Latin word *radix* = *root*), although this has not been quite conclusively confirmed. It took a long time for $\surd$ to become the standard symbol for a square root and much longer to standardize $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, $\sqrt[5]{\ }$, and so on. The indexes of the root were placed in every conceivable position, with

$$\surd\overset{3}{8}, \quad \surd\textcircled{3}8, \quad \text{and} \quad \underset{3}{\surd}8$$

all being variants for $\sqrt[3]{8}$. The notation $\surd\sqrt{16}$ was popular for $\sqrt[4]{16}$. By the 1700s, the index had settled where we now put it.

The bar on top of the present radical symbol, as follows,

$$\sqrt{a^2 + 2ab + b^2}$$

is the last survivor of the **vinculum**, a bar placed atop an expression to indicate what we would now indicate with parentheses. For example,

$$a\overline{b + c} = a(b + c)$$

■

R.8 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages in red.*

1. $(-3)^2 =$ __; $-3^2 =$ ____ (pp. 22–25)

2. $\sqrt{16} =$ __; $\sqrt{(-4)^2} =$ __ (pp. 22–25)

Concepts and Vocabulary

3. In the symbol $\sqrt[n]{a}$, the integer n is called the _____.

4. We call $\sqrt[3]{a}$ the _____ _____ of a.

5. Let $n \geq 2$ and $m \geq 2$ be integers, and let a and b be real numbers. Which of the following is not a property of radicals? Assume all radicals are defined.

(a) $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$ (b) $\sqrt[n]{a + b} = \sqrt[n]{a} + \sqrt[n]{b}$

(c) $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ (d) $\sqrt[n]{a^m} = (\sqrt[n]{a})^m$

6. If a is a real number and $n \geq 2$ is an integer, then which of the following expressions is equivalent to $\sqrt[n]{a}$, provided that it exists?

(a) a^{-n} (b) a^n (c) $\dfrac{1}{a^n}$ (d) $a^{1/n}$

7. Which of the following phrases best defines like radicals?
(a) Radical expressions that have the same index
(b) Radical expressions that have the same radicand
(c) Radical expressions that have the same index and the same radicand
(d) Radical expressions that have the same variable

8. To rationalize the denominator of the expression $\dfrac{\sqrt{2}}{1 - \sqrt{3}}$, multiply both the numerator and the denominator by which of the following?

(a) $\sqrt{3}$ (b) $\sqrt{2}$ (c) $1 + \sqrt{3}$ (d) $1 - \sqrt{3}$

9. ***True or False*** $\sqrt[5]{-32} = -2$

10. ***True or False*** $\sqrt[4]{(-3)^4} = -3$

Skill Building

In Problems 11–54, simplify each expression. Assume that all variables are positive when they appear.

11. $\sqrt[3]{27}$
12. $\sqrt[4]{16}$
13. $\sqrt[3]{-8}$
14. $\sqrt[3]{-1}$
15. $\sqrt{8}$
16. $\sqrt{75}$
17. $\sqrt{700}$
18. $\sqrt{45x^3}$
19. $\sqrt[3]{32}$
20. $\sqrt[3]{54}$
21. $\sqrt[3]{-8x^4}$
22. $\sqrt[3]{192x^5}$
23. $\sqrt[4]{243}$
24. $\sqrt[4]{48x^5}$
25. $\sqrt[4]{x^{12}y^8}$
26. $\sqrt[5]{x^{10}y^5}$
27. $\sqrt[4]{\dfrac{x^9y^7}{xy^3}}$
28. $\sqrt[3]{\dfrac{3xy^2}{81x^4y^2}}$
29. $\sqrt{36x}$
30. $\sqrt{9x^5}$
31. $\sqrt[4]{162x^9y^{12}}$
32. $\sqrt[3]{-40x^{14}y^{10}}$
33. $\sqrt{3x^2}\sqrt{12x}$
34. $\sqrt{5x}\sqrt{20x^3}$
35. $(\sqrt{5}\sqrt[3]{9})^2$
36. $(\sqrt[3]{3}\sqrt{10})^4$
37. $(3\sqrt{6})(2\sqrt{2})$
38. $(5\sqrt{8})(-3\sqrt{3})$
39. $3\sqrt{2} + 4\sqrt{2}$
40. $6\sqrt{5} - 4\sqrt{5}$
41. $-\sqrt{18} + 2\sqrt{8}$
42. $2\sqrt{12} - 3\sqrt{27}$
43. $(\sqrt{3} + 3)(\sqrt{3} - 1)$
44. $(\sqrt{5} - 2)(\sqrt{5} + 3)$
45. $5\sqrt[3]{2} - 2\sqrt[3]{54}$
46. $9\sqrt[3]{24} - \sqrt[3]{81}$
47. $(\sqrt{x} - 1)^2$
48. $(\sqrt{x} + \sqrt{5})^2$
49. $\sqrt[3]{16x^4} - \sqrt[3]{2x}$
50. $\sqrt[4]{32x} + \sqrt[4]{2x^5}$
51. $\sqrt{8x^3} - 3\sqrt{50x}$
52. $3x\sqrt{9y} + 4\sqrt{25y}$
53. $\sqrt[3]{16x^4y} - 3x\sqrt[3]{2xy} + 5\sqrt[3]{-2xy^4}$
54. $8xy - \sqrt{25x^2y^2} + \sqrt[3]{8x^3y^3}$

In Problems 55–68, rationalize the denominator of each expression. Assume that all variables are positive when they appear.

55. $\dfrac{1}{\sqrt{2}}$

56. $\dfrac{2}{\sqrt{3}}$

57. $\dfrac{-\sqrt{3}}{\sqrt{5}}$

58. $\dfrac{-\sqrt{3}}{\sqrt{8}}$

59. $\dfrac{\sqrt{3}}{5-\sqrt{2}}$

60. $\dfrac{\sqrt{2}}{\sqrt{7}+2}$

61. $\dfrac{2-\sqrt{5}}{2+3\sqrt{5}}$

62. $\dfrac{\sqrt{3}-1}{2\sqrt{3}+3}$

63. $\dfrac{5}{\sqrt{2}-1}$

64. $\dfrac{-3}{\sqrt{5}+4}$

65. $\dfrac{5}{\sqrt[3]{2}}$

66. $\dfrac{-2}{\sqrt[3]{9}}$

67. $\dfrac{\sqrt{x+h}-\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$

68. $\dfrac{\sqrt{x+h}+\sqrt{x-h}}{\sqrt{x+h}-\sqrt{x-h}}$

In Problems 69–84, simplify each expression.

69. $8^{2/3}$

70. $4^{3/2}$

71. $(-27)^{1/3}$

72. $16^{3/4}$

73. $16^{3/2}$

74. $25^{3/2}$

75. $9^{-3/2}$

76. $16^{-3/2}$

77. $\left(\dfrac{9}{8}\right)^{3/2}$

78. $\left(\dfrac{27}{8}\right)^{2/3}$

79. $\left(\dfrac{8}{9}\right)^{-3/2}$

80. $\left(\dfrac{8}{27}\right)^{-2/3}$

81. $(-1000)^{-1/3}$

82. $-25^{-1/2}$

83. $\left(-\dfrac{64}{125}\right)^{-2/3}$

84. $-81^{-3/4}$

In Problems 85–92, simplify each expression. Express your answer so that only positive exponents occur. Assume that the variables are positive.

85. $x^{3/4}x^{1/3}x^{-1/2}$

86. $x^{2/3}x^{1/2}x^{-1/4}$

87. $(x^3y^6)^{1/3}$

88. $(x^4y^8)^{3/4}$

89. $\dfrac{(x^2y)^{1/3}(xy^2)^{2/3}}{x^{2/3}y^{2/3}}$

90. $\dfrac{(xy)^{1/4}(x^2y^2)^{1/2}}{(x^2y)^{3/4}}$

91. $\dfrac{(16x^2y^{-1/3})^{3/4}}{(xy^2)^{1/4}}$

92. $\dfrac{(4x^{-1}y^{1/3})^{3/2}}{(xy)^{3/2}}$

Applications and Extensions

In Problems 93–106, expressions that occur in calculus are given. Write each expression as a single quotient in which only positive exponents and/or radicals appear.

93. $\dfrac{x}{(1+x)^{1/2}}+2(1+x)^{1/2} \quad x>-1$

94. $\dfrac{1+x}{2x^{1/2}}+x^{1/2} \quad x>0$

95. $2x(x^2+1)^{1/2}+x^2\cdot\dfrac{1}{2}(x^2+1)^{-1/2}\cdot 2x$

96. $(x+1)^{1/3}+x\cdot\dfrac{1}{3}(x+1)^{-2/3} \quad x\neq -1$

97. $\sqrt{4x+3}\cdot\dfrac{1}{2\sqrt{x-5}}+\sqrt{x-5}\cdot\dfrac{1}{5\sqrt{4x+3}} \quad x>5$

98. $\dfrac{\sqrt[3]{8x+1}}{3\sqrt[3]{(x-2)^2}}+\dfrac{\sqrt[3]{x-2}}{24\sqrt[3]{(8x+1)^2}} \quad x\neq 2, x\neq -\dfrac{1}{8}$

99. $\dfrac{\sqrt{1+x}-x\cdot\dfrac{1}{2\sqrt{1+x}}}{1+x} \quad x>-1$

100. $\dfrac{\sqrt{x^2+1}-x\cdot\dfrac{2x}{2\sqrt{x^2+1}}}{x^2+1}$

101. $\dfrac{(x+4)^{1/2}-2x(x+4)^{-1/2}}{x+4} \quad x>-4$

102. $\dfrac{(9-x^2)^{1/2}+x^2(9-x^2)^{-1/2}}{9-x^2} \quad -3<x<3$

103. $\dfrac{\dfrac{x^2}{(x^2-1)^{1/2}}-(x^2-1)^{1/2}}{x^2} \quad x<-1 \text{ or } x>1$

104. $\dfrac{(x^2+4)^{1/2}-x^2(x^2+4)^{-1/2}}{x^2+4}$

105. $\dfrac{\dfrac{1+x^2}{2\sqrt{x}}-2x\sqrt{x}}{(1+x^2)^2} \quad x>0$

106. $\dfrac{2x(1-x^2)^{1/3}+\dfrac{2}{3}x^3(1-x^2)^{-2/3}}{(1-x^2)^{2/3}} \quad x\neq -1, x\neq 1$

In Problems 107–116, expressions that occur in calculus are given. Factor each expression. Express your answer so that only positive exponents occur.

107. $(x+1)^{3/2} + x \cdot \frac{3}{2}(x+1)^{1/2} \quad x \geq -1$

108. $(x^2+4)^{4/3} + x \cdot \frac{4}{3}(x^2+4)^{1/3} \cdot 2x$

109. $6x^{1/2}(x^2+x) - 8x^{3/2} - 8x^{1/2} \quad x \geq 0$

110. $6x^{1/2}(2x+3) + x^{3/2} \cdot 8 \quad x \geq 0$

111. $3(x^2+4)^{4/3} + x \cdot 4(x^2+4)^{1/3} \cdot 2x$

112. $2x(3x+4)^{4/3} + x^2 \cdot 4(3x+4)^{1/3}$

113. $4(3x+5)^{1/3}(2x+3)^{3/2} + 3(3x+5)^{4/3}(2x+3)^{1/2} \quad x \geq -\frac{3}{2}$

114. $6(6x+1)^{1/3}(4x-3)^{3/2} + 6(6x+1)^{4/3}(4x-3)^{1/2} \quad x \geq \frac{3}{4}$

115. $3x^{-1/2} + \frac{3}{2}x^{1/2} \quad x > 0$

116. $8x^{1/3} - 4x^{-2/3} \quad x \neq 0$

In Problems 117–124, use a calculator to approximate each radical. Round your answer to two decimal places.

117. $\sqrt{2}$

118. $\sqrt{7}$

119. $\sqrt[3]{4}$

120. $\sqrt[3]{-5}$

121. $\dfrac{2+\sqrt{3}}{3-\sqrt{5}}$

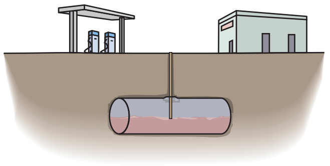
122. $\dfrac{\sqrt{5}-2}{\sqrt{2}+4}$

123. $\dfrac{3\sqrt[3]{5}-\sqrt{2}}{\sqrt{3}}$

124. $\dfrac{2\sqrt{3}-\sqrt[3]{4}}{\sqrt{2}}$

125. Calculating the Amount of Gasoline in a Tank A Shell station stores its gasoline in underground tanks that are right circular cylinders lying on their sides. See the illustration. The volume V of gasoline in the tank (in gallons) is given by the formula

$$V = 40h^2\sqrt{\frac{96}{h} - 0.608}$$



where h is the height of the gasoline (in inches) as measured on a depth stick.

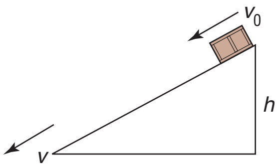
(a) If $h = 12$ inches, how many gallons of gasoline are in the tank?
(b) If $h = 1$ inch, how many gallons of gasoline are in the tank?

126. Inclined Planes The final velocity v of an object in feet per second (ft/sec) after it slides down a frictionless inclined plane of height h feet is

$$v = \sqrt{64h + v_0^2}$$

where v_0 is the initial velocity (in ft/sec) of the object.

(a) What is the final velocity v of an object that slides down a frictionless inclined plane of height 4 feet? Assume that the initial velocity is 0.
(b) What is the final velocity v of an object that slides down a frictionless inclined plane of height 16 feet? Assume that the initial velocity is 0.
(c) What is the final velocity v of an object that slides down a frictionless inclined plane of height 2 feet with an initial velocity of 4 ft/sec?



Problems 127 and 128 require the following information.

Period of a Pendulum *The period T, in seconds, of a pendulum of length l, in feet, may be approximated using the formula*

$$T = 2\pi\sqrt{\frac{l}{32}}$$

In Problems 127 and 128, express your answer both as a square root and as a decimal.

127. Find the period T of a pendulum whose length is 64 feet.

128. Find the period T of a pendulum whose length is 16 feet.

Explaining Concepts: Discussion and Writing

129. Give an example to show that $\sqrt{a^2}$ is not equal to a. Use it to explain why $\sqrt{a^2} = |a|$.

‘Are You Prepared?’ Answers

1. 9; −9 **2.** 4; 4

1 Graphs, Equations, and Inequalities

Financing a Purchase

Whenever we make a major purchase, such as an automobile or a house, we often need to finance the purchase by borrowing money from a lending institution, such as a bank. Have you ever wondered how the bank determines the monthly payment? How much total interest will be paid over the course of the loan? What roles do the rate of interest and the length of the loan play?

—See the Internet-based Chapter Project I—

Outline

A Look Ahead •••

In this chapter, we use the rectangular coordinate system to graph equations. Then, we algebraically solve equations and use the rectangular coordinate system to visualize the solution. The idea of using a system of rectangular coordinates dates back to ancient times, when such a system was used for surveying and city planning. Apollonius of Perga, in 200 BC, used a form of rectangular coordinates in his work on conics, although this use does not stand out as clearly as it does in modern treatments. Sporadic use of rectangular coordinates continued until the 1600s. By that time, algebra had developed sufficiently so that René Descartes (1596–1650) and Pierre de Fermat (1601–1665) could take the crucial step, which was the use of rectangular coordinates to translate geometry problems into algebra problems, and vice versa. This step allowed both geometers and algebraists to gain new insights into their subjects, which previously had been regarded as separate, but now were seen to be connected in many important ways.

With the advent of modern technology, in particular, graphing utilities, not only are we able to visualize the dual roles of algebra and geometry, but we are also able to solve many problems that required advanced methods before this technology.

1.1 The Distance and Midpoint Formulas; Graphing Utilities; Introduction to Graphing Equations

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Algebra Essentials (Section R.2, pp. 18–27)
- Geometry Essentials (Section R.3, pp. 31–36)

Now Work the **'Are You Prepared?'** problems on page 94.

OBJECTIVES
1 Use the Distance Formula (p. 85)
2 Use the Midpoint Formula (p. 87)
3 Graph Equations by Plotting Points (p. 88)
4 Graph Equations Using a Graphing Utility (p. 90)
5 Use a Graphing Utility to Create Tables (p. 92)
6 Find Intercepts from a Graph (p. 93)
7 Use a Graphing Utility to Approximate Intercepts (p. 93)

Rectangular Coordinates

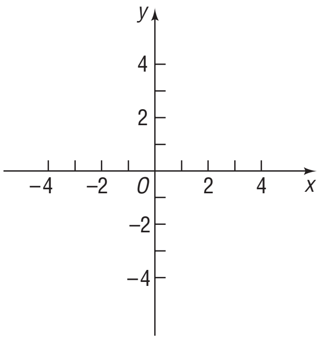


Figure 1 *xy*-Plane

We locate a point on the real number line by assigning it a single real number, called the *coordinate of the point*. For work in a two-dimensional plane, points are located by using two numbers.

Begin with two real number lines located in the same plane: one horizontal and the other vertical. The horizontal line is called the ***x*-axis**, the vertical line the ***y*-axis**, and the point of intersection the **origin *O***. See Figure 1. Assign coordinates to every point on these number lines using a convenient scale. Recall that the scale of a number line is the distance between 0 and 1. In mathematics, we usually use the same scale on each axis, but in applications, different scales appropriate to the application may be used.

The origin O has a value of 0 on both the x-axis and y-axis. Points on the x-axis to the right of O are associated with positive real numbers, and those to the left of O are associated with negative real numbers. Points on the y-axis above O are associated with positive real numbers, and those below O are associated with negative real numbers. In Figure 1, the x-axis and y-axis are labeled as x and y, respectively, and we have used an arrow at the end of each axis to denote the positive direction.

The coordinate system described here is called a **rectangular** or **Cartesian*** **coordinate system**. The plane formed by the x-axis and y-axis is sometimes called the ***xy*-plane**, and the x-axis and y-axis are referred to as the **coordinate axes**.

Any point P in the xy-plane can be located by using an **ordered pair** (x, y) of real numbers. Let x denote the signed distance of P from the y-axis (*signed* means that, if P is to the right of the y-axis, then $x > 0$, and if P is to the left of the y-axis, then $x < 0$); and let y denote the signed distance of P from the x-axis. The ordered pair (x, y), also called the **coordinates** of P, gives us enough information to locate the point P in the plane.

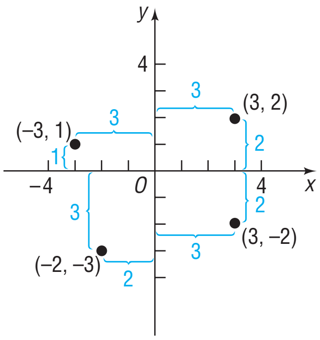


Figure 2

For example, to locate the point whose coordinates are $(-3, 1)$, go 3 units along the x-axis to the left of O and then go straight up 1 unit. We **plot** this point by placing a dot at this location. See Figure 2, in which the points with coordinates $(-3, 1)$, $(-2, -3)$, $(3, -2)$, and $(3, 2)$ are plotted.

The origin has coordinates $(0, 0)$. Any point on the x-axis has coordinates of the form $(x, 0)$, and any point on the y-axis has coordinates of the form $(0, y)$.

If (x, y) are the coordinates of a point P, then x is called the ***x*-coordinate**, or **abscissa**, of P and y is the ***y*-coordinate**, or **ordinate**, of P. We identify the point P using its coordinates (x, y) by writing $P = (x, y)$. Usually, we will simply say "the point (x, y)" rather than "the point whose coordinates are (x, y)."

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*Named after René Descartes (1596–1650), a French mathematician, philosopher, and theologian.

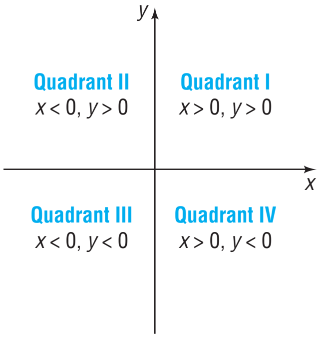


Figure 3

The coordinate axes divide the xy-plane into four sections called **quadrants**, as shown in Figure 3. In quadrant I, both the x-coordinate and the y-coordinate of all points are positive; in quadrant II, x is negative and y is positive; in quadrant III, both x and y are negative; and in quadrant IV, x is positive and y is negative. Points on the coordinate axes belong to no quadrant.

Now Work PROBLEM 15

Graphing Utilities

All graphing utilities (graphing calculators and computer software graphing packages) graph equations by plotting points on a screen. The screen itself actually consists of small rectangles, called **pixels**. The more pixels the screen has, the better the resolution. Most graphing calculators have 50 to 100 pixels per square inch; most smartphones have 300 to 450 pixels per square inch. When a point to be plotted lies inside a pixel, the pixel is turned on (lights up). The graph of an equation is a collection of lighted pixels. Figure 4 shows how the graph of $y = 2x$ looks on a TI-84 Plus C graphing calculator.

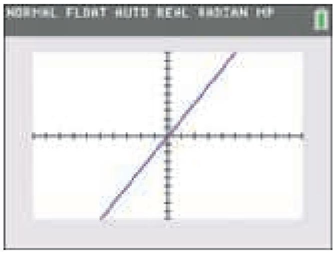


Figure 4 $Y_1 = 2x$

The screen of a graphing utility will display the coordinate axes of a rectangular coordinate system. However, the scale must be set on each axis. The smallest and largest values of x and y to be included in the graph must also be set. This is called **setting the viewing rectangle** or **viewing window**. Figure 5 illustrates a typical viewing window.

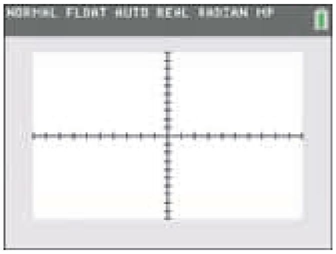


Figure 5 Viewing window

To set the viewing window, values must be given to the following expressions:

*X*min: the smallest value of x shown on the viewing window
*X*max: the largest value of x shown on the viewing window
*X*scl: the number of units per tick mark on the x-axis
*Y*min: the smallest value of y shown on the viewing window
*Y*max: the largest value of y shown on the viewing window
*Y*scl: the number of units per tick mark on the y-axis

Figure 6 illustrates these settings and their relation to the Cartesian coordinate system.

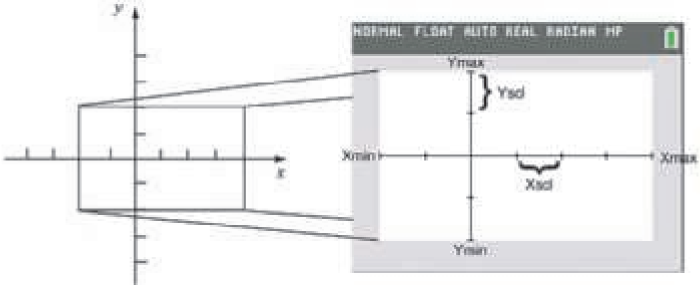


Figure 6

If the scale used on each axis is known, the minimum and maximum values of x and y shown on the screen can be determined by counting the tick marks. Look again at Figure 5. For a scale of 1 on each axis, the minimum and maximum values of x are -10 and 10, respectively; the minimum and maximum values of y are also -10 and 10. If the scale is 2 on each axis, then the minimum and maximum values of x are -20 and 20, respectively; and the minimum and maximum values of y are -20 and 20, respectively.

Conversely, if the minimum and maximum values of x and y are known, the scales can be determined by counting the tick marks displayed. This text follows the practice of showing the minimum and maximum values of x and y in illustrations

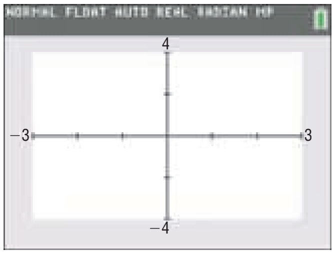


Figure 7

so that the reader will know how the viewing window was set. See Figure 7, from which the following window settings can be determined:

$$Xmin = -3 \qquad Ymin = -4$$
$$Xmax = 3 \qquad Ymax = 4$$
$$Xscl = 1 \qquad Yscl = 2$$

Now Work PROBLEMS 19 AND 29

1 Use the Distance Formula

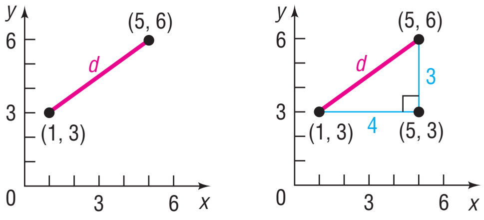
If the same units of measurement (such as inches, centimeters, and so on) are used for both the *x*-axis and *y*-axis, then all distances in the *xy*-plane can be measured using this unit of measurement.

EXAMPLE 1 Finding the Distance between Two Points

Find the distance d between the points $(1, 3)$ and $(5, 6)$.

Solution First plot the points $(1, 3)$ and $(5, 6)$ and connect them with a straight line. See Figure 8(a). To find the length d, begin by drawing a horizontal line from $(1, 3)$ to $(5, 3)$ and a vertical line from $(5, 3)$ to $(5, 6)$, forming a right triangle, as in Figure 8(b). One leg of the triangle is of length 4 (since $|5 - 1| = 4$) and the other is of length 3 (since $|6 - 3| = 3$). By the Pythagorean Theorem, the square of the distance d that we seek is

$$d^2 = 4^2 + 3^2 = 16 + 9 = 25$$
$$d = \sqrt{25} = 5$$



(a) (b)

Figure 8

■

The **distance formula** provides a straightforward method for computing the distance between two points.

THEOREM

Distance Formula

The distance between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, denoted by $d(P_1, P_2)$, is

$$\boxed{d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad \textbf{(1)}}$$

■

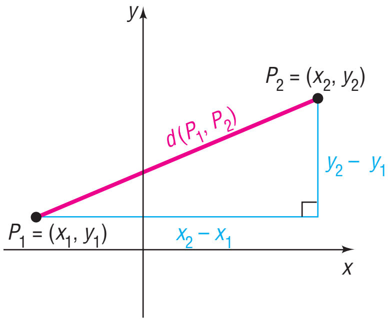


Figure 9 Illustration of the Distance Formula

Figure 9 illustrates the theorem.

Proof of the Distance Formula Let (x_1, y_1) denote the coordinates of point P_1, and let (x_2, y_2) denote the coordinates of point P_2. Assume that the line joining P_1 and P_2 is neither horizontal nor vertical. Refer to Figure 10(a). The coordinates of P_3 are (x_2, y_1). The horizontal distance from P_1 to P_3 is the absolute value of

In Words

To compute the distance between two points, find the difference of the *x*-coordinates, square it, and add this to the square of the difference of the *y*-coordinates. The square root of this sum is the distance.

the difference of the x-coordinates, $|x_2 - x_1|$. The vertical distance from P_3 to P_2 is the absolute value of the difference of the y-coordinates, $|y_2 - y_1|$. See Figure 10(b). The distance $d(P_1, P_2)$ is the length of the hypotenuse of the right triangle, so, by the Pythagorean Theorem, it follows that

$$\begin{aligned}[d(P_1, P_2)]^2 &= |x_2 - x_1|^2 + |y_2 - y_1|^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ d(P_1, P_2) &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\end{aligned}$$

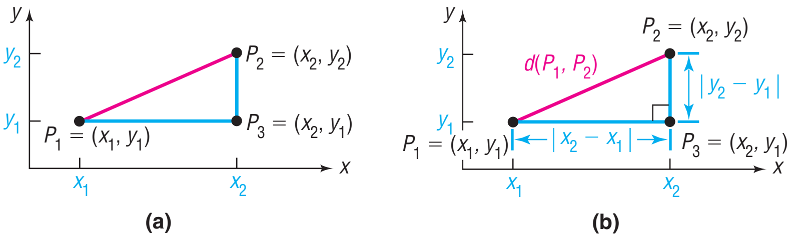


Figure 10

Now, if the line joining P_1 and P_2 is horizontal, then the y-coordinate of P_1 equals the y-coordinate of P_2; that is, $y_1 = y_2$. Refer to Figure 11(a). In this case, the distance formula (1) still works, because, for $y_1 = y_2$, it reduces to

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + 0^2} = \sqrt{(x_2 - x_1)^2} = |x_2 - x_1|$$

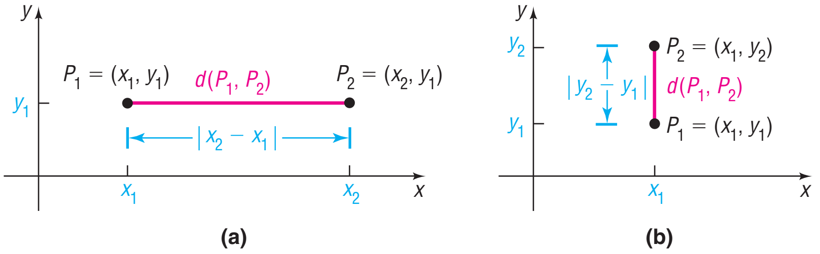


Figure 11

A similar argument holds if the line joining P_1 and P_2 is vertical. See Figure 11(b). ■

EXAMPLE 2 Finding the Length of a Line Segment

Find the length of the line segment shown in Figure 12.

Solution The length of the line segment is the distance between the points $P_1 = (x_1, y_1) = (-4, 5)$ and $P_2 = (x_2, y_2) = (3, 2)$. Using the distance formula (1) with $x_1 = -4$, $y_1 = 5$, $x_2 = 3$, and $y_2 = 2$, the length d is

$$\begin{aligned}d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{[3 - (-4)]^2 + (2 - 5)^2} \\ &= \sqrt{7^2 + (-3)^2} = \sqrt{49 + 9} = \sqrt{58} \approx 7.62\end{aligned}$$

■

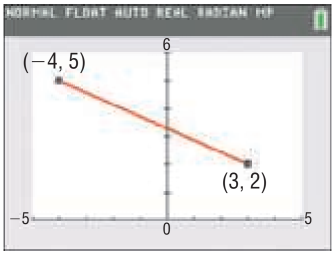


Figure 12

Now Work PROBLEM 35

The distance between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is never a negative number. Also, the distance between two points is 0 only when the points are identical—that is, when $x_1 = x_2$ and $y_1 = y_2$. Also, because $(x_2 - x_1)^2 = (x_1 - x_2)^2$ and $(y_2 - y_1)^2 = (y_1 - y_2)^2$, it makes no difference whether the distance is computed from P_1 to P_2 or from P_2 to P_1; that is, $d(P_1, P_2) = d(P_2, P_1)$.

The introduction to this chapter mentioned that rectangular coordinates enable us to translate geometry problems into algebra problems, and vice versa. The next example shows how algebra (the distance formula) can be used to solve geometry problems.

EXAMPLE 3 Using Algebra to Solve Geometry Problems

Consider the three points $A = (-2, 1)$, $B = (2, 3)$, and $C = (3, 1)$.

(a) Plot each point and form the triangle ABC.
(b) Find the length of each side of the triangle.
(c) Verify that the triangle is a right triangle.
(d) Find the area of the triangle.

Solution (a) Figure 13 shows the points A, B, C and the triangle ABC.

(b) To find the length of each side of the triangle, use the distance formula (1).

$$\begin{aligned} d(A,B) &= \sqrt{[2-(-2)]^2 + (3-1)^2} = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5} \\ d(B,C) &= \sqrt{(3-2)^2 + (1-3)^2} = \sqrt{1+4} = \sqrt{5} \\ d(A,C) &= \sqrt{[3-(-2)]^2 + (1-1)^2} = \sqrt{25+0} = 5 \end{aligned}$$

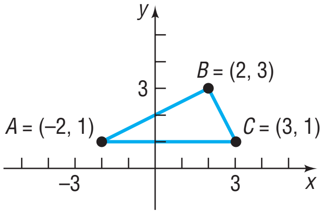


Figure 13

(c) If the sum of the squares of the lengths of two of the sides equals the square of the length of the third side, the triangle is a right triangle. (Why is this sufficient?) Looking at Figure 13, it seems reasonable to conjecture that the angle at vertex B might be a right angle. We shall check to see whether

$$[d(A,B)]^2 + [d(B,C)]^2 = [d(A,C)]^2$$

Using the results from part (b),

$$\begin{aligned} [d(A,B)]^2 + [d(B,C)]^2 &= \left(2\sqrt{5}\right)^2 + \left(\sqrt{5}\right)^2 \\ &= 20 + 5 = 25 = [d(A,C)]^2 \end{aligned}$$

It follows from the converse of the Pythagorean Theorem that triangle ABC is a right triangle.

(d) Because the right angle is at vertex B, the sides AB and BC form the base and height of the triangle. Its area is

$$\text{Area} = \frac{1}{2}(\text{Base})(\text{Height}) = \frac{1}{2}\left(2\sqrt{5}\right)\left(\sqrt{5}\right) = 5 \text{ square units}$$ ■

Now Work PROBLEM 51

2 Use the Midpoint Formula

We now derive a formula for the coordinates of the **midpoint of a line segment**. Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be the endpoints of a line segment, and let $M = (x, y)$ be the point on the line segment that is the same distance from P_1 as it is from P_2. See Figure 14. The triangles P_1AM and MBP_2 are congruent. [Do you see why? Angle $AP_1M =$ angle BMP_2,* angle $P_1MA =$ angle MP_2B, and $d(P_1, M) = d(M, P_2)$ is given. So, we have angle–side–angle.] Hence, corresponding sides are equal in length. That is,

$$\begin{aligned} x - x_1 &= x_2 - x & \quad \text{and} \quad y - y_1 &= y_2 - y \\ 2x &= x_1 + x_2 & 2y &= y_1 + y_2 \\ x &= \frac{x_1 + x_2}{2} & y &= \frac{y_1 + y_2}{2} \end{aligned}$$

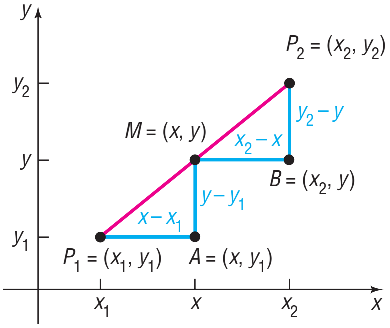


Figure 14 Illustration of midpoint

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*A postulate from geometry states that the transversal $\overline{P_1P_2}$ forms congruent corresponding angles with the parallel line segments $\overline{P_1A}$ and $\overline{MB}$.

THEOREM

Midpoint Formula

The midpoint $M = (x, y)$ of the line segment from $P_1 = (x_1, y_1)$ to $P_2 = (x_2, y_2)$ is

$$\boxed{M = (x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \qquad \textbf{(2)}}$$

In Words

To find the midpoint of a line segment, average the *x*-coordinates of the endpoints, and average the *y*-coordinates of the endpoints.

EXAMPLE 4

Finding the Midpoint of a Line Segment

Find the midpoint of a line segment from $P_1 = (-5, 5)$ to $P_2 = (3, 1)$. Plot the points P_1 and P_2 and their midpoint.

Solution

Apply the midpoint formula (2) using $x_1 = -5$, $y_1 = 5$, $x_2 = 3$, and $y_2 = 1$. Then the coordinates (x, y) of the midpoint M are

$$x = \frac{x_1 + x_2}{2} = \frac{-5 + 3}{2} = -1 \quad \text{and} \quad y = \frac{y_1 + y_2}{2} = \frac{5 + 1}{2} = 3$$

That is, $M = (-1, 3)$. See Figure 15. ■

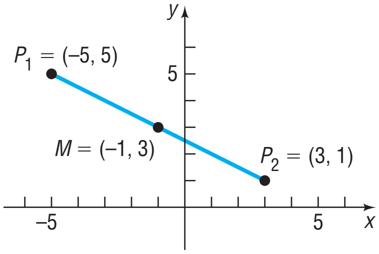


Figure 15

Now Work PROBLEM 57

3 Graph Equations by Plotting Points

An **equation in two variables**, say x and y, is a statement in which two expressions involving x and y are equal. The expressions are called the **sides** of the equation. Since an equation is a statement, it may be true or false, depending on the value of the variables. Any values of x and y that result in a true statement are said to **satisfy** the equation.

For example, the following are all equations in two variables x and y:

$$x^2 + y^2 = 5 \qquad 2x - y = 6 \qquad y = 2x + 5 \qquad x^2 = y$$

The first of these, $x^2 + y^2 = 5$, is satisfied for $x = 1, y = 2$, since $1^2 + 2^2 = 5$. Other choices of x and y, such as $x = -1, y = -2$, also satisfy this equation. It is not satisfied for $x = 2$ and $y = 3$, since $2^2 + 3^2 = 4 + 9 = 13 \neq 5$.

The **graph of an equation in two variables** x and y consists of the set of points in the xy-plane whose coordinates (x, y) satisfy the equation.

Graphs play an important role in helping us to visualize the relationships that exist between two variables or quantities. Figure 16 shows the relation between the

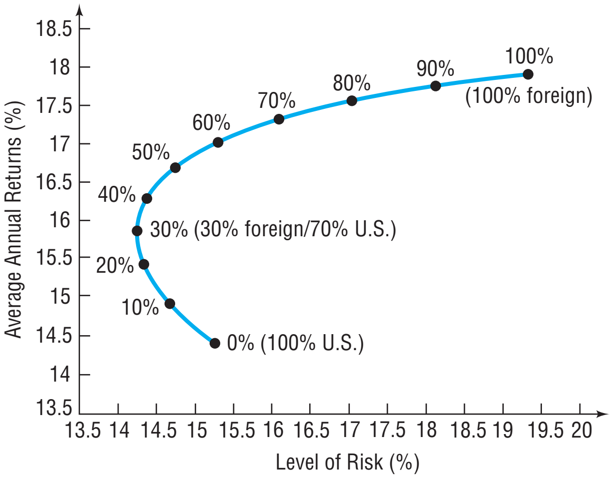


Figure 16
Source: *T. Rowe Price*

level of risk in a stock portfolio and the average annual rate of return. The graph shows that, when 30% of a portfolio of stocks is invested in foreign companies, risk is minimized.

EXAMPLE 5 Determining Whether a Point Is on the Graph of an Equation

Determine whether each of the following points is on the graph of the equation $2x - y = 6$.

(a) $(2, 3)$ (b) $(2, -2)$

Solution (a) For the point $(2, 3)$, check to see whether $x = 2$, $y = 3$ satisfies the equation $2x - y = 6$.

$$2x - y = 2(2) - 3 = 4 - 3 = 1 \neq 6$$

The equation is not satisfied, so the point $(2, 3)$ is not on the graph of $2x - y = 6$.

(b) For the point $(2, -2)$,

$$2x - y = 2(2) - (-2) = 4 + 2 = 6$$

The equation is satisfied, so the point $(2, -2)$ is on the graph of $2x - y = 6$. ■

Now Work PROBLEM 65

EXAMPLE 6 Graphing an Equation by Plotting Points

Graph the equation: $y = -2x + 3$

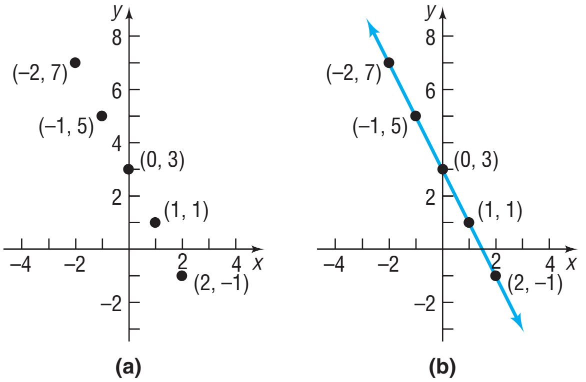
Step-by-Step Solution

Step 1: Find some points (x, y) that satisfy the equation. To determine these points, choose values of x and use the equation to find the corresponding values for y. See Table 1.

Table 1

x	$y = -2x + 3$	(x, y)
−2	$-2(-2) + 3 = 7$	$(-2, 7)$
−1	$-2(-1) + 3 = 5$	$(-1, 5)$
0	$-2(0) + 3 = 3$	$(0, 3)$
1	$-2(1) + 3 = 1$	$(1, 1)$
2	$-2(2) + 3 = -1$	$(2, -1)$

Step 2: Plot the points listed in the table as shown in Figure 17(a). Now connect the points to obtain the graph of the equation (a line), as shown in Figure 17(b).



(a) (b)

Figure 17 $y = -2x + 3$ ■

EXAMPLE 7 Graphing an Equation by Plotting Points

Graph the equation: $y = x^2$

Solution Table 2 on page 90 provides several points on the graph. Plotting these points and connecting them with a smooth curve gives the graph (a *parabola*) shown in Figure 18.

Table 2

x	$y = x^2$	(x, y)
−4	16	(−4, 16)
−3	9	(−3, 9)
−2	4	(−2, 4)
−1	1	(−1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	4	(2, 4)
3	9	(3, 9)
4	16	(4, 16)

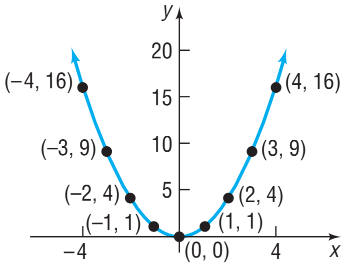


Figure 18 $y = x^2$

The graphs of the equations shown in Figures 17(b) and 18 do not show all the points that are on the graph. For example, in Figure 17(b), the point $(20, -37)$ is a part of the graph of $y = -2x + 3$, but it is not shown. Since the graph of $y = -2x + 3$ could be extended out as far as we please, we use arrows to indicate that the pattern shown continues. It is important when illustrating a graph to present enough of the graph so that any viewer of the illustration will "see" the rest of it as an obvious continuation of what is actually there. This is referred to as a **complete graph**.

One way to obtain a complete graph of an equation is to continue plotting points on the graph until a pattern becomes evident. Then these points are connected with a smooth curve following the suggested pattern. But how many points are sufficient? Sometimes knowledge about the equation tells us. For example, we will learn in the next chapter that if an equation is of the form $y = mx + b$, then its graph is a line. In this case, two points would suffice to obtain the graph.

One purpose of this text is to investigate the properties of equations in order to decide whether a graph is complete. Sometimes we shall graph equations by plotting points on the graph until a pattern becomes evident and then connect these points with a smooth curve, following the suggested pattern. (Shortly, we shall investigate various techniques that will enable us to graph an equation without plotting so many points.) Other times we shall graph equations using a graphing utility.

4 Graph Equations Using a Graphing Utility

From Examples 6 and 7, we see that a graph can be obtained by plotting points in a rectangular coordinate system and connecting them. Graphing utilities perform these same steps when graphing an equation. For example, the TI-84 Plus C determines 265 evenly spaced input values (starting at Xmin and ending at Xmax),* uses the equation to determine the output values, plots these points on the screen, and finally (if in the connected mode) draws a line between consecutive points.

To graph an equation in two variables x and y using a graphing utility requires that the equation be written in the form $y = \{\text{expression in } x\}$. If the original equation is not in this form, rewrite it using equivalent equations until the form $y = \{\text{expression in } x\}$ is obtained. In general, there are four ways to obtain equivalent equations.

Procedures That Result in Equivalent Equations

1. Interchange the two sides of the equation:

$$3x + 5 = y \quad \text{is equivalent to} \quad y = 3x + 5$$

2. Simplify the sides of the equation by combining like terms, eliminating parentheses, and so on:

$$2y + 2 + 6 = 2x + 5(x + 1) \quad \text{is equivalent to} \quad 2y + 8 = 7x + 5$$

---

*These input values depend on the values of Xmin and Xmax. For example, if $X\text{min} = -10$ and $X\text{max} = 10$, then the first input value will be -10 and the next input value will be $-10 + (10 - (-10))/264 = -9.9242$, and so on.

3. Add or subtract the same expression on both sides of the equation:

$$y + 3x - 5 = 4 \quad \text{is equivalent to} \quad y + 3x - 5 + 5 = 4 + 5$$

4. Multiply or divide both sides of the equation by the same nonzero expression:

$$3y = 6 - 2x \quad \text{is equivalent to} \quad \frac{1}{3} \cdot 3y = \frac{1}{3}(6 - 2x)$$

EXAMPLE 8 Expressing an Equation in the Form $y = \{\text{expression in } x\}$

Solve for y: $2y + 3x - 5 = 4$

Solution Replace the original equation by a succession of equivalent equations.

$$2y + 3x - 5 = 4$$

$$2y + 3x - 5 + 5 = 4 + 5 \qquad \textbf{Add 5 to both sides.}$$

$$2y + 3x = 9 \qquad \textbf{Simplify.}$$

$$2y + 3x - 3x = 9 - 3x \qquad \textbf{Subtract } \boldsymbol{3x} \textbf{ from both sides.}$$

$$2y = 9 - 3x \qquad \textbf{Simplify.}$$

$$\frac{2y}{2} = \frac{9 - 3x}{2} \qquad \textbf{Divide both sides by 2.}$$

$$y = \frac{9 - 3x}{2} \qquad \textbf{Simplify.}$$

■

WARNING Be careful when entering the expression $\frac{9 - 3x}{2}$. Use a fraction template $\left(\frac{\square}{\square}\right)$ or use parentheses as follows: $(9 - 3x)/2$. ■

We are now ready to graph equations using a graphing utility.

EXAMPLE 9 Graphing an Equation Using a Graphing Utility

Use a graphing utility to graph the equation: $6x^2 + 3y = 36$

Step-by-Step Solution

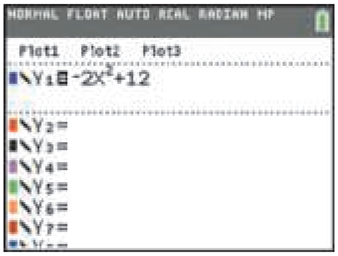
Step 1: Solve the equation for y in terms of x.

$$6x^2 + 3y = 36$$

$$3y = -6x^2 + 36 \qquad \textbf{Subtract } \boldsymbol{6x^2} \textbf{ from both sides.}$$

$$y = -2x^2 + 12 \qquad \textbf{Divide both sides by 3 and simplify.}$$

Step 2: Enter the equation to be graphed into your graphing utility. Figure 19 shows the equation to be graphed entered on a TI-84 Plus C.



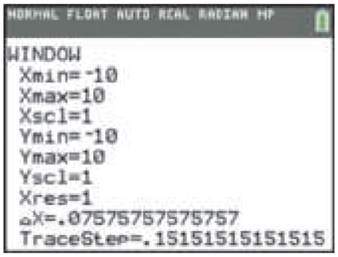
```
NORMAL FLOAT AUTO REAL RADIAN MP
 Plot1  Plot2  Plot3
\Y1=-2X²+12
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

Figure 19

Step 3: Choose an initial viewing window. Without any knowledge about the behavior of the graph, it is common to choose the standard viewing window as the initial viewing window. The standard viewing window is

$X\text{min} = -10$ $Y\text{min} = -10$
$X\text{max} = 10$ $Y\text{max} = 10$
$X\text{scl} = 1$ $Y\text{scl} = 1$

See Figure 20.



```
NORMAL FLOAT AUTO REAL RADIAN MP
WINDOW
 Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-10
 Ymax=10
 Yscl=1
 Xres=1
 ΔX=.07575757575757
 TraceStep=.15151515151515
```

Figure 20 Standard viewing window

Step 4: Graph the equation. See Figure 21.

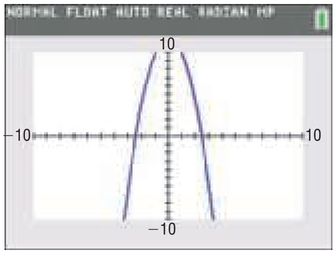


Figure 21 $Y_1 = -2x^2 + 12$

Step 5: Adjust the viewing window until a complete graph is obtained.

The graph of $y = -2x^2 + 12$ is not complete. The value of Ymax must be increased so that the top portion of the graph is visible. After increasing the value of Ymax to 12, we obtain the graph in Figure 22. The graph is now complete.

Note: Some graphing utilities have a ZOOM-STANDARD feature that automatically sets the viewing window to the standard viewing window. In addition, some graphing utilities have a ZOOM-FIT feature that determines the appropriate *Y*min and *Y*max for a given *X*min and *X*max. Consult your owner's manual for the appropriate keystrokes.

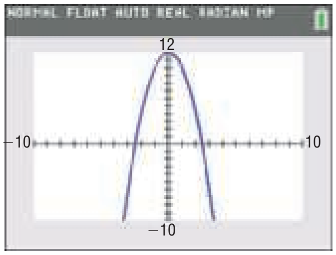


Figure 22 $Y_1 = -2x^2 + 12$

Now Work PROBLEM 83

Look again at Figure 22. Although a complete graph is shown, the graph might be improved by adjusting the values of Xmin and Xmax. Figure 23 shows the graph of $y = -2x^2 + 12$ using $X\text{min} = -4$ and $X\text{max} = 4$. Do you think this is a better choice for the viewing window?

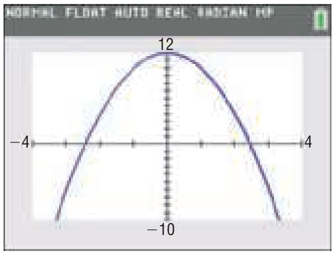


Figure 23 $Y_1 = -2x^2 + 12$

5 Use a Graphing Utility to Create Tables

In addition to graphing equations, graphing utilities can also be used to create a table of values that satisfy the equation. This feature is especially useful in determining an appropriate viewing window when graphing an equation.

EXAMPLE 10 **Creating a Table Using a Graphing Utility**

Create a table that displays the points on the graph of $6x^2 + 3y = 36$ for $x = -3, -2, -1, 0, 1, 2,$ and 3.

Step-by-Step Solution

Step 1: Solve the equation for *y* in terms of *x*.

We solved the equation for y in terms of x in Example 9 and obtained $y = -2x^2 + 12$.

Step 2: Enter the expression in *x* following the *Y* = prompt of the graphing utility.

See Figure 19 on page 91.

Step 3: Set up the table. Graphing utilities typically have two modes for creating tables. In the AUTO mode, the user determines a starting point for the table (TblStart) and ΔTbl (pronounced "delta table"). The ΔTbl feature determines the increment for *x* in the table. The ASK mode requires the user to enter values of *x*, and then the utility determines the corresponding value of *y*.

Create a table using AUTO mode. The table we wish to create starts at −3, so TblStart = −3. The increment in x is 1, so $\Delta\text{Tbl} = 1$. See Figure 24.

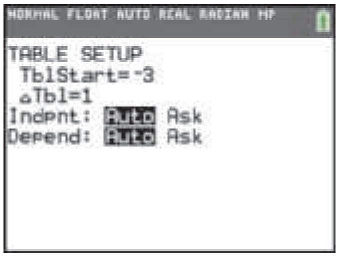
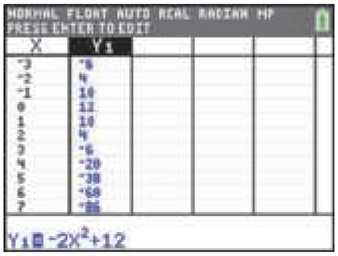

NORMAL FLOAT AUTO REAL RADIAN MP

TABLE SETUP
TblStart=-3
ΔTbl=1
Indpnt: Auto Ask
Depend: Auto Ask

Figure 24 Table setup

Step 4: Create the table. See Table 3.

Table 3



NORMAL FLOAT AUTO REAL RADIAN MP
PRESS ENTER TO EDIT

X	Y_1
-3	-6
-2	4
-1	10
0	12
1	10
2	4
3	-6
4	-20
5	-38
6	-60
7	-86

$Y_1 = -2X^2+12$

In AUTO mode, the user can scroll forward or backward within the table to find additional values. ■

In looking at Table 3, notice that $y = 12$ when $x = 0$. This information could have been used to help to create the initial viewing window by letting us know that Ymax needs to be at least 12 in order to get a complete graph.

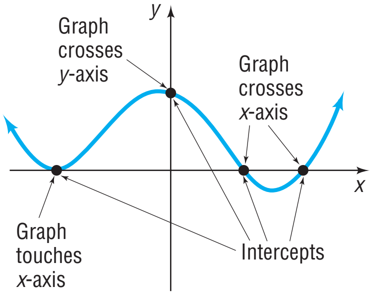


Figure 25 Intercepts

6 Find Intercepts from a Graph

The points, if any, at which a graph crosses or touches the coordinate axes are called the **intercepts**. See Figure 25. The x-coordinate of a point at which the graph crosses or touches the x-axis is an ***x*-intercept**, and the y-coordinate of a point at which the graph crosses or touches the y-axis is a ***y*-intercept**. For a graph to be complete, all its intercepts must be displayed.

EXAMPLE 11 Finding Intercepts from a Graph

Find the intercepts of the graph in Figure 26. What are its x-intercepts? What are its y-intercepts?

Solution The intercepts of the graph are the points

$$(-3, 0),\quad (0, 3),\quad \left(\frac{3}{2}, 0\right),\quad \left(0, -\frac{4}{3}\right),\quad (0, -3.5),\quad (4.5, 0)$$

The x-intercepts are -3, $\frac{3}{2}$, and 4.5; the y-intercepts are -3.5, $-\frac{4}{3}$, and 3. ■

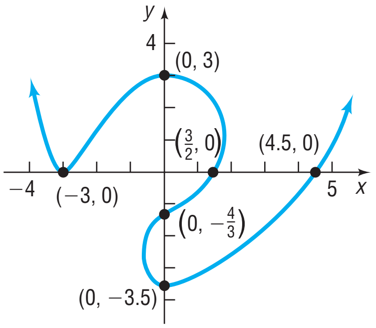


Figure 26

In Example 11 notice the following usage: If the type of intercept is not specified (x- versus y-), then report the intercept as an ordered pair. However, if the type of intercept is specified, then report the coordinate of the specified intercept. For x-intercepts, report the x-coordinate of the intercept; for y-intercepts, report the y-coordinate of the intercept.

Now Work PROBLEM 71

7 Use a Graphing Utility to Approximate Intercepts

We can use a graphing utility to approximate the intercepts of the graph of an equation.

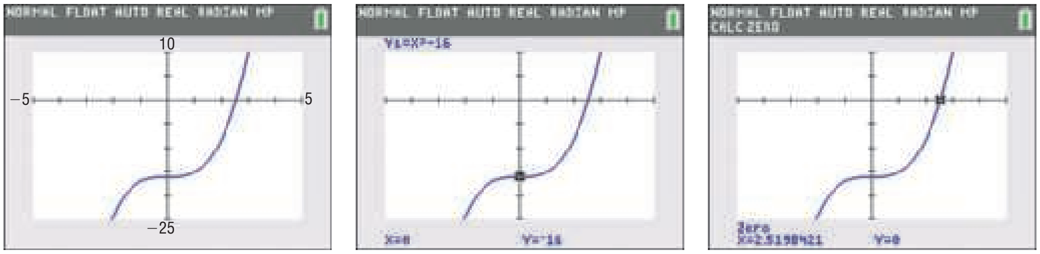
EXAMPLE 12 Approximating Intercepts Using a Graphing Utility

Use a graphing utility to approximate the intercepts of the equation $y = x^3 - 16$.

Solution Figure 27(a) on page 94 shows the graph of $y = x^3 - 16$.

The eVALUEate feature of a TI-84 Plus C graphing calculator accepts as input a value of x and determines the value of y. If we let $x = 0$, the y-intercept is found to be -16. See Figure 27(b).

The ZERO feature of a TI-84 Plus C is used to find the x-intercept(s). See Figure 27(c). Rounded to two decimal places, the x-intercept is 2.52.



(a) (b) (c)

Figure 27

Now Work PROBLEM 93

To find the intercepts algebraically requires the ability to solve equations, the subject of the following four sections.

1.1 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. On a real number line the origin is assigned the number ______. (p. 18)
2. If −3 and 5 are the coordinates of two points on the real number line, the distance between these points is ______. (pp. 20–21)
3. If 3 and 4 are the legs of a right triangle, the hypotenuse is ______. (p. 31)
4. Use the converse of the Pythagorean Theorem to show that a triangle whose sides are of lengths 11, 60, and 61 is a right triangle. (p. 32)
5. The area of a triangle whose base is b and whose altitude is h is $A =$ ______. (p. 32)
6. ***True or False*** Two triangles are congruent if two angles and the included side of one equals two angles and the included side of the other. (pp. 33–34)

Concepts and Vocabulary

7. If (x, y) are the coordinates of a point P in the xy-plane, then x is called the ____________ of P and y is the ____________ of P.
8. The coordinate axes divide the xy-plane into four sections called __________.
9. If three distinct points P, Q, and R all lie on a line and if $d(P, Q) = d(Q, R)$, then Q is called the __________ of the line segment from P to R.
10. ***True or False*** The distance between two points is sometimes a negative number.
11. ***True or False*** The point $(-1, 4)$ lies in quadrant IV of the Cartesian plane.
12. ***True or False*** The midpoint of a line segment is found by averaging the x-coordinates and averaging the y-coordinates of the endpoints.
13. Given that the intercepts of a graph are $(-4, 0)$ and $(0, 5)$, choose the statement that is true.
    (a) The y-intercept is −4 and the x-intercept is 5.
    (b) The y-intercepts are −4 and 5.
    (c) The x-intercepts are −4 and 5.
    (d) The x-intercept is −4 and the y-intercept is 5.
14. Which of the following points does not satisfy the equation $2x^2 - 5y = 20$?
    (a) $(0, -4)$ (b) $(5, 6)$ (c) $(-5, -14)$ (d) $(10, 36)$

Skill Building

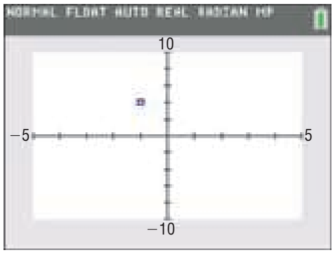
In Problems 15 and 16, plot each point in the xy-plane. Tell in which quadrant or on what coordinate axis each point lies.

15. (a) $A = (-3, 2)$
    (b) $B = (6, 0)$
    (c) $C = (-2, -2)$
    (d) $D = (6, 5)$
    (e) $E = (0, -3)$
    (f) $F = (6, -3)$
16. (a) $A = (1, 4)$
    (b) $B = (-3, -4)$
    (c) $C = (-3, 4)$
    (d) $D = (4, 1)$
    (e) $E = (0, 1)$
    (f) $F = (-3, 0)$

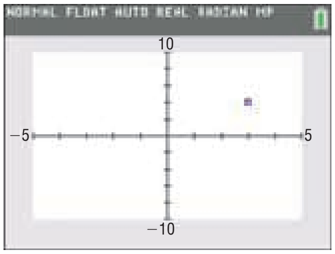
17. Plot the points $(2, 0)$, $(2, -3)$, $(2, 4)$, $(2, 1)$, and $(2, -1)$. Describe the set of all points of the form $(2, y)$, where y is a real number.
18. Plot the points $(0, 3)$, $(1, 3)$, $(-2, 3)$, $(5, 3)$, and $(-4, 3)$. Describe the set of all points of the form $(x, 3)$, where x is a real number.

In Problems 19–22, determine the coordinates of the points shown. Tell in which quadrant each point lies. Assume the coordinates are integers.

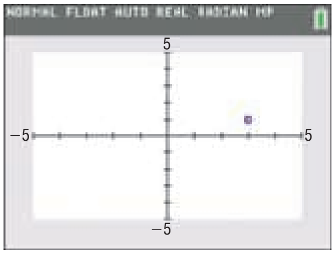
19.



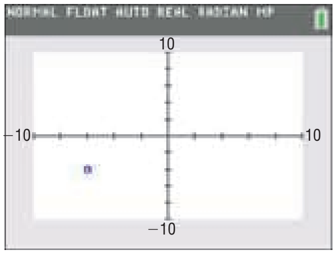
20.



21.



22.



In Problems 23–28, select a setting so that each given point will lie within the viewing window.

23. $(-10, 5), (3, -2), (4, -1)$

24. $(5, 0), (6, 8), (-2, -3)$

25. $(40, 20), (-20, -80), (10, 40)$

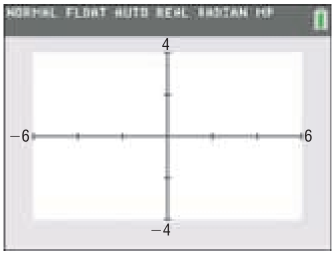
26. $(-80, 60), (20, -30), (-20, -40)$

27. $(0, 0), (100, 5), (5, 150)$

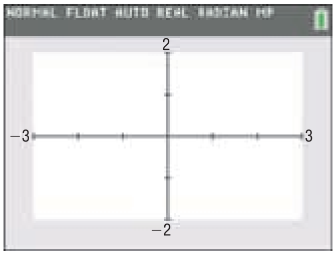
28. $(0, -1), (100, 50), (-10, 30)$

In Problems 29–34, determine the viewing window used.

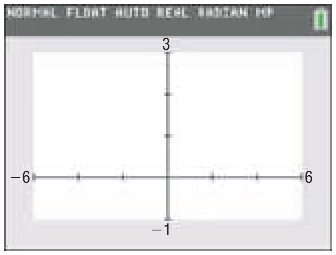
29.



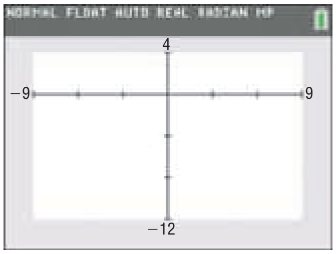
30.



31.



32.

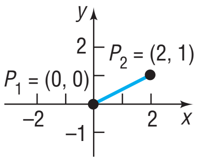


33.

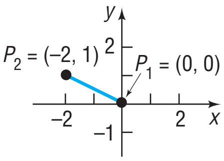
34.

In Problems 35–46, find the distance $d(P_1, P_2)$ between the points P_1 and P_2.

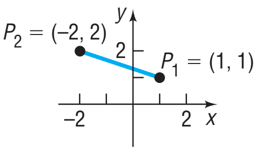
35.



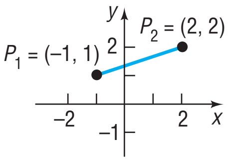
36.



37.



38.



39. $P_1 = (3, -4); \quad P_2 = (5, 4)$

40. $P_1 = (-1, 0); \quad P_2 = (2, 4)$

41. $P_1 = (-5, -3); \quad P_2 = (11, 9)$

42. $P_1 = (2, -3); \quad P_2 = (10, 3)$

43. $P_1 = (4, -3); \quad P_2 = (6, 4)$

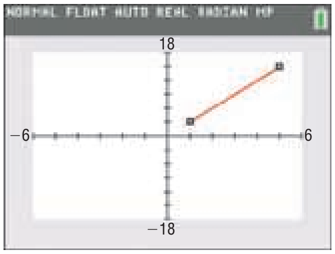
44. $P_1 = (-4, -3); \quad P_2 = (6, 2)$

45. $P_1 = (a, b); \quad P_2 = (0, 0)$

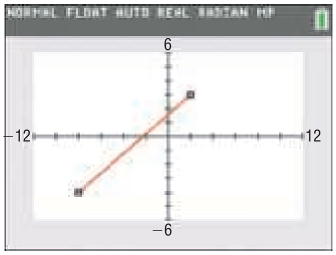
46. $P_1 = (a, a); \quad P_2 = (0, 0)$

In Problems 47–50, find the length of the line segment. Assume that the endpoints of each line segment have integer coordinates.

47.



48.



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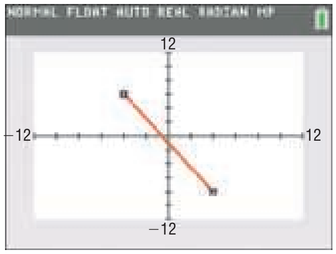
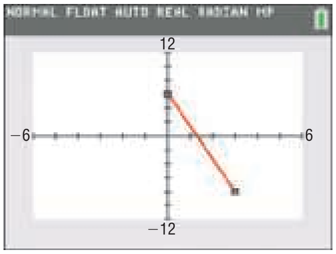


50.



In Problems 51–56, plot each point and form the triangle ABC. Verify that the triangle is a right triangle. Find its area.

51. $A = (-2, 5)$; $B = (1, 3)$; $C = (-1, 0)$

52. $A = (-2, 5)$; $B = (12, 3)$; $C = (10, -11)$

53. $A = (-5, 3)$; $B = (6, 0)$; $C = (5, 5)$

54. $A = (-6, 3)$; $B = (3, -5)$; $C = (-1, 5)$

55. $A = (4, -3)$; $B = (0, -3)$; $C = (4, 2)$

56. $A = (4, -3)$; $B = (4, 1)$; $C = (2, 1)$

In Problems 57–64, find the midpoint of the line segment joining the points P_1 and P_2.

57. $P_1 = (3, -4)$; $P_2 = (5, 4)$

58. $P_1 = (-2, 0)$; $P_2 = (2, 4)$

59. $P_1 = (-5, -3)$; $P_2 = (11, 9)$

60. $P_1 = (2, -3)$; $P_2 = (10, 3)$

61. $P_1 = (4, -3)$; $P_2 = (6, 1)$

62. $P_1 = (-4, -3)$; $P_2 = (2, 2)$

63. $P_1 = (a, b)$; $P_2 = (0, 0)$

64. $P_1 = (a, a)$; $P_2 = (0, 0)$

In Problems 65–70, tell whether the given points are on the graph of the equation.

65. Equation: $y = x^4 - \sqrt{x}$
Points: $(0, 0)$; $(1, 1)$; $(-1, 0)$

66. Equation: $y = x^3 - 2\sqrt{x}$
Points: $(0, 0)$; $(1, 1)$; $(1, -1)$

67. Equation: $y^2 = x^2 + 9$
Points: $(0, 3)$; $(3, 0)$; $(-3, 0)$

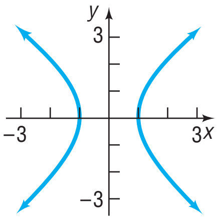
68. Equation: $y^3 = x + 1$
Points: $(1, 2)$; $(0, 1)$; $(-1, 0)$

69. Equation: $x^2 + y^2 = 4$
Points: $(0, 2)$; $(-2, 2)$; $(\sqrt{2}, \sqrt{2})$

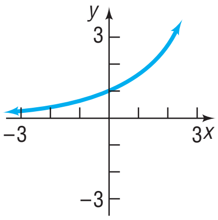
70. Equation: $x^2 + 4y^2 = 4$
Points: $(0, 1)$; $(2, 0)$; $\left(2, \frac{1}{2}\right)$

In Problems 71–78, the graph of an equation is given. List the intercepts of the graph.

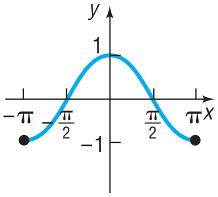
71.



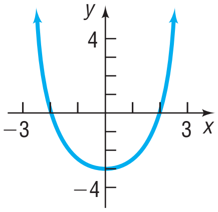
72.



73.



74.



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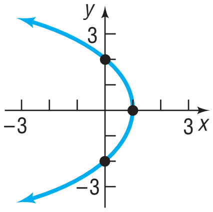

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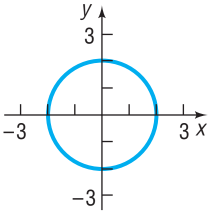

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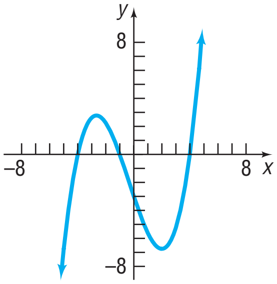

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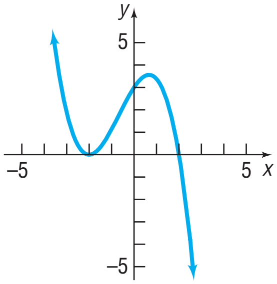


In Problems 79–90, graph each equation by plotting points. Verify your results using a graphing utility.

79. $y = x + 2$

80. $y = x - 6$

81. $y = 2x + 8$

82. $y = 3x - 9$

83. $y = x^2 - 1$

84. $y = x^2 - 9$

85. $y = -x^2 + 4$

86. $y = -x^2 + 1$

87. $2x + 3y = 6$

88. $5x + 2y = 10$

89. $9x^2 + 4y = 36$

90. $4x^2 + y = 4$

In Problems 91–98, graph each equation using a graphing utility. Use a graphing utility to approximate the intercepts rounded to two decimal places. Use the TABLE feature to help to establish the viewing window.

91. $y = 2x - 13$

92. $y = -3x + 14$

93. $y = 2x^2 - 15$

94. $y = -3x^2 + 19$

95. $3x - 2y = 43$

96. $4x + 5y = 82$

97. $5x^2 + 3y = 37$

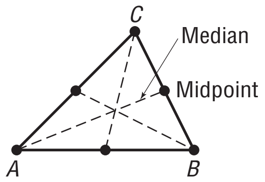
98. $2x^2 - 3y = 35$

99. If the point $(2, 5)$ is shifted 3 units right and 2 units down, what are its new coordinates?

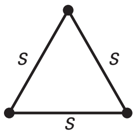
100. If the point $(-1, 6)$ is shifted 2 units left and 4 units up, what are its new coordinates?

Applications and Extensions

101. The **medians** of a triangle are the line segments from each vertex to the midpoint of the opposite side (see the figure). Find the lengths of the medians of the triangle with vertices at $A = (0, 0)$, $B = (6, 0)$, and $C = (4, 4)$.



102. An **equilateral triangle** is one in which all three sides are of equal length. If two vertices of an equilateral triangle are $(0, 4)$ and $(0, 0)$, find the third vertex. How many of these triangles are possible?



*In Problems 103–106, find the length of each side of the triangle determined by the three points P_1, P_2, and P_3. State whether the triangle is an isosceles triangle, a right triangle, neither of these, or both. (An **isosceles triangle** is one in which at least two of the sides are of equal length.)*

103. $P_1 = (2, 1);\quad P_2 = (-4, 1);\quad P_3 = (-4, -3)$

104. $P_1 = (-1, 4);\quad P_2 = (6, 2);\quad P_3 = (4, -5)$

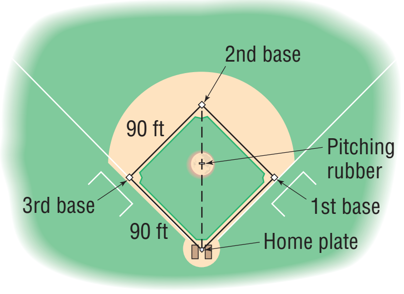
105. $P_1 = (-2, -1);\quad P_2 = (0, 7);\quad P_3 = (3, 2)$

106. $P_1 = (7, 2);\quad P_2 = (-4, 0);\quad P_3 = (4, 6)$

107. Completing a Line Segment Plot the points $A = (-1, 8)$ and $M = (2, 3)$ in the xy-plane. If M is the midpoint of a line segment AB, find the coordinates of B.

108. Completing a Line Segment Plot the points $M = (5, -4)$ and $B = (7, -2)$ in the xy-plane. If M is the midpoint of a line segment AB, find the coordinates of A.

109. Baseball A major league baseball "diamond" is actually a square, 90 feet on a side (see the figure). What is the distance directly from home plate to second base (the diagonal of the square)?



110. Little League Baseball The layout of a Little League playing field is a square, 60 feet on a side. How far is it directly from home plate to second base (the diagonal of the square)?

Source: *Little League Baseball, Official Regulations and Playing Rules, 2014.*

111. Baseball Refer to Problem 109. Overlay a rectangular coordinate system on a major league baseball diamond so that the origin is at home plate, the positive x-axis lies in the direction from home plate to first base, and the positive y-axis lies in the direction from home plate to third base.

(a) What are the coordinates of first base, second base, and third base? Use feet as the unit of measurement.

(b) If the right fielder is located at $(310, 15)$, how far is it from there to second base?

(c) If the center fielder is located at $(300, 300)$, how far is it from there to third base?

112. Little League Baseball Refer to Problem 110. Overlay a rectangular coordinate system on a Little League baseball diamond so that the origin is at home plate, the positive x-axis lies in the direction from home plate to first base, and the positive y-axis lies in the direction from home plate to third base.

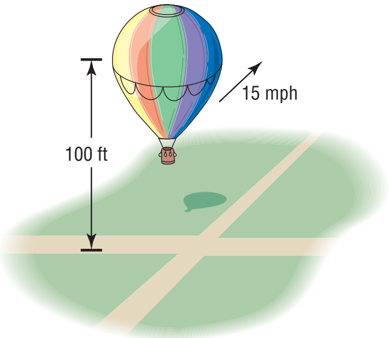
(a) What are the coordinates of first base, second base, and third base? Use feet as the unit of measurement.

(b) If the right fielder is located at $(180, 20)$, how far is it from there to second base?

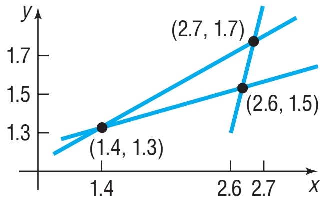
(c) If the center fielder is located at $(220, 220)$, how far is it from there to third base?

113. Distance between Moving Objects A Ford Focus and a Freightliner truck leave an intersection at the same time. The Focus heads east at an average speed of 30 miles per hour, while the truck heads south at an average speed of 40 miles per hour. Find an expression for their distance apart d (in miles) at the end of t hours.

114. Distance of a Moving Object from a Fixed Point A hot-air balloon, headed due east at an average speed of 15 miles per hour at a constant altitude of 100 feet, passes over an intersection (see the figure). Find an expression for its distance d (measured in feet) from the intersection t seconds later.



115. Drafting Error When a draftsman draws three lines that are to intersect at one point, the lines may not intersect as intended and subsequently will form an **error triangle**. If this error triangle is long and thin, one estimate for the location of the desired point is the midpoint of the shortest side. The figure shows one such error triangle.

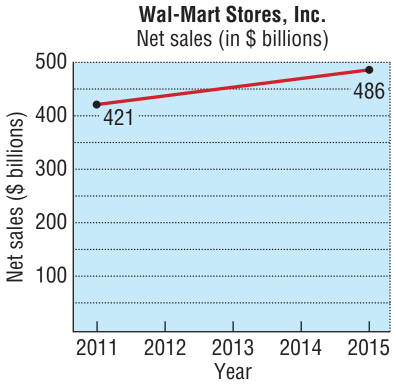


(a) Find an estimate for the desired intersection point.

(b) Find the length of the median for the midpoint found in part (a). See Problem 101.

116. Net Sales The figure illustrates how net sales of Wal-Mart Stores, Inc., have grown from 2011 through 2015. Use the midpoint formula to estimate the net sales of Wal-Mart Stores, Inc., in 2013. How does your result compare to the reported value of $469 billion?

Source: *Wal-Mart Stores, Inc., 2015 Annual Report*



117. Poverty Threshold Poverty thresholds are determined by the U.S. Census Bureau. A poverty threshold represents the minimum annual household income for a family not to be considered poor. In 2004, the poverty threshold for a family of four with two children under the age of 18 years was $19,157. In 2014, the poverty threshold for a family of four with two children under the age of 18 years was $24,008. Assuming poverty thresholds increase in a straight-line fashion, use the midpoint formula to estimate the poverty threshold of a family of four with two children under the age of 18 in 2009. How does your result compare to the actual poverty threshold in 2009 of $21,756?

Source: *U.S. Census Bureau*

Explaining Concepts: Discussion and Writing

In Problem 118, you may use a graphing utility, but it is not required.

118. (a) Graph $y = \sqrt{x^2}$, $y = x$, $y = |x|$, and $y = (\sqrt{x})^2$, noting which graphs are the same.

(b) Explain why the graphs of $y = \sqrt{x^2}$ and $y = |x|$ are the same.

(c) Explain why the graphs of $y = x$ and $y = (\sqrt{x})^2$ are not the same.

(d) Explain why the graphs of $y = \sqrt{x^2}$ and $y = x$ are not the same.

119. Make up an equation satisfied by the ordered pairs $(2, 0)$, $(4, 0)$, and $(0, 1)$. Compare your equation with a friend's equation. Comment on any similarities.

120. Draw a graph that contains the points $(-2, -1)$, $(0, 1)$, $(1, 3)$, and $(3, 5)$. Compare your graph with those of other students. Are most of the graphs almost straight lines? How many are "curved"? Discuss the various ways that these points might be connected.

121. Explain what is meant by a complete graph.

122. Write a paragraph that describes a Cartesian plane. Then write a second paragraph that describes how to plot points in the Cartesian plane. Your paragraphs should include the terms "coordinate axes," "ordered pair," "coordinates," "plot," "x-coordinate," and "y-coordinate."

'Are You Prepared?' Answers

1. 0 **2.** 8 **3.** 5 **4.** $11^2 + 60^2 = 61^2$ **5.** $\frac{1}{2}bh$ **6.** True

1.2 Solving Equations Using a Graphing Utility; Linear and Rational Equations

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Properties of Real Numbers (Section R.1, pp. 10–15)
- Domain of a Variable (Section R.2, p. 22)
- Evaluate Algebraic Expressions (Section R.2, pp. 21–22)
- Rational Expressions (Section R.7, pp. 63–71)

Now Work the **'Are You Prepared?'** problems on page 107.

OBJECTIVES
1 Solve Equations Using a Graphing Utility (p. 100)
2 Solve Linear Equations (p. 102)
3 Solve Rational Equations (p. 103)
4 Solve Problems That Can Be Modeled by Linear Equations (p. 105)

An **equation in one variable** is a statement in which two expressions, at least one containing the variable, are set equal. The expressions are called the **sides** of the equation. Since an equation is a statement, it may be true or false, depending on the value of the variable. Unless otherwise restricted, the admissible values of the variable are those in the domain of the variable. Those admissible values of the variable, if any, that result in a true statement are called **solutions**, or **roots**, of the equation. To **solve an equation** means to find all the solutions of the equation.

For example, the following are all equations in one variable, x:

$$x + 5 = 9 \qquad x^2 + 5x = 2x - 2 \qquad \frac{x^2 - 4}{x + 1} = 0 \qquad x^2 + 9 = 5$$

The first of these statements, $x + 5 = 9$, is true when $x = 4$ and false for any other choice of x. That is, 4 is a solution of the equation $x + 5 = 9$. We also say that 4 **satisfies** the equation $x + 5 = 9$, because, when 4 is substituted for x, a true statement results.

Sometimes an equation will have more than one solution. For example, the equation

$$\frac{x^2 - 4}{x + 1} = 0$$

has $x = -2$ and $x = 2$ as solutions.

Usually, we will write the solutions of an equation in set notation. This set is called the **solution set** of the equation. For example, the solution set of the equation $x^2 - 9 = 0$ is $\{-3, 3\}$.

Unless indicated otherwise, we will limit ourselves to real solutions—that is, solutions that are real numbers. Some equations have no real solution. For example, $x^2 + 9 = 5$ has no real solution, because there is no real number whose square, when added to 9, equals 5.

An equation that is satisfied for every choice of the variable for which both sides are defined is called an **identity**. For example, the equation

$$3x + 5 = x + 3 + 2x + 2$$

is an identity, because this statement is true for any real number x.

In this text, we present two methods for solving equations: algebraic and graphical. We shall see that some equations can be solved using algebraic techniques to obtain *exact* solutions. For other equations, however, there are no algebraic techniques that lead to an exact solution. For such equations, a graphing utility can often be used to investigate possible solutions.

One goal of this text is to determine when equations can be solved algebraically. If an algebraic method for solving an equation exists, we shall use it to obtain an exact solution. A graphing utility can then be used to support the algebraic result. However, if no algebraic techniques are available to solve an equation, a graphing utility will be used to obtain approximate solutions.

1 Solve Equations Using a Graphing Utility

When a graphing utility is used to solve an equation, usually *approximate* solutions are obtained. Unless otherwise stated, we shall follow the practice of giving approximate solutions as decimals *rounded to two decimal places*.

The ZERO (or ROOT) feature of a graphing utility can be used to find the solutions of an equation when one side of the equation is 0. In using this feature to solve equations, make use of the fact that when the graph of an equation in two variables, x and y, crosses or touches the x-axis then $y = 0$. For this reason, any value of x for which $y = 0$ will be a solution to the equation. That is, solving an equation for x when one side of the equation is 0 is equivalent to finding where the graph of the corresponding equation in two variables crosses or touches the x-axis.

EXAMPLE 1 Using ZERO (or ROOT) to Approximate Solutions of an Equation

Find the solution(s) of the equation $x^3 - x + 1 = 0$. Round answers to two decimal places.

Solution The solutions of the equation $x^3 - x + 1 = 0$ are the same as the x-intercepts of the graph of $Y_1 = x^3 - x + 1$. Begin by graphing Y_1. Figure 28 shows the graph.

From the graph there appears to be one x-intercept (solution to the equation) between -2 and -1. Using the ZERO (or ROOT) feature of a graphing utility, determine that the x-intercept, and thus the solution to the equation, is $x = -1.32$ rounded to two decimal places. See Figure 29.

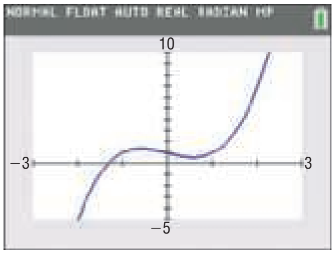


Figure 28 $Y_1 = x^3 - x + 1$

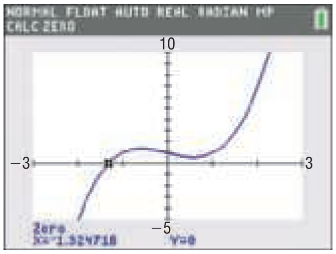


Figure 29 ■

Now Work PROBLEM 19

Note: Graphing utilities use a process in which they search for a solution until the answer is found within a certain tolerance level (such as within 0.0001). Therefore, the y-coordinate may sometimes be a nonzero value such as 1.1527 E-8, which is 1.1527×10^{-8}, very close to zero. ■

A second method for solving equations using a graphing utility involves the INTERSECT feature of the graphing utility. This feature is used most effectively when neither side of the equation is 0.

EXAMPLE 2 Using INTERSECT to Approximate Solutions of an Equation

Find the solution(s) to the equation $4x^4 - 3 = 2x + 1$. Round answers to two decimal places.

Solution Begin by graphing each side of the equation as follows: graph $Y_1 = 4x^4 - 3$ and $Y_2 = 2x + 1$. See Figure 30.

At a point of intersection of the graphs, the value of the y-coordinate is the same for Y_1 and Y_2. Thus, the x-coordinate of the point of intersection represents a solution to the equation. Do you see why? The INTERSECT feature on a graphing utility determines a point of intersection of the graphs. Using this feature, find that the graphs intersect at $(-0.87, -0.73)$ and $(1.12, 3.23)$ rounded to two decimal places. See Figure 31(a) and (b). The solutions of the equation are $x = -0.87$ and $x = 1.12$ rounded to two decimal places.

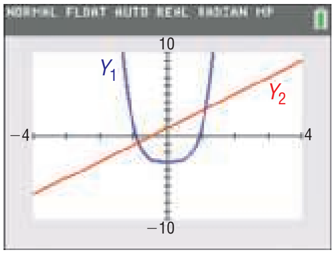
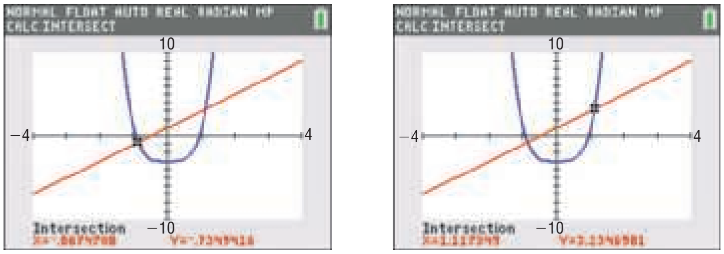


Figure 30 $Y_1 = 4x^4 - 3;\ Y_2 = 2x + 1$



(a)

(b)

Figure 31

Now Work PROBLEM 21

SUMMARY

Steps for Approximating Solutions of Equations Using ZERO (or ROOT)

STEP 1: Write the equation in the form {expression in x} = 0.

STEP 2: Graph Y_1 = {expression in x}. Be sure that the graph is complete. That is, be sure that all the x-intercepts are shown on the screen.

STEP 3: Use ZERO (or ROOT) to determine each x-intercept of the graph.

Steps for Approximating Solutions of Equations Using INTERSECT

STEP 1: Graph Y_1 = {expression in x on the left side of the equation}
Y_2 = {expression in x on the right side of the equation}

Be sure that the graphs are complete. That is, be sure that all the points of intersection are shown on the screen.

STEP 2: Use INTERSECT to determine the x-coordinate of each point of intersection.

Solving Equations Algebraically

One method for solving equations algebraically requires that a series of *equivalent equations* be developed from the original equation until an obvious solution results.

For example, consider the following succession of equivalent equations:

$$\begin{aligned} 2x + 3 &= 13 \\ 2x &= 10 \\ x &= 5 \end{aligned}$$

We conclude that the solution set of the original equation is {5}.

How are equivalent equations obtained? In general, there are five ways to do so.

Procedures That Result in Equivalent Equations

1. Interchange the two sides of the equation:

$$3 = x \quad \text{is equivalent to} \quad x = 3$$

2. Simplify the sides of the equation by combining like terms, eliminating parentheses, and so on:

$$x + 2 + 6 = 2x + 3(x + 1)$$

$$\text{is equivalent to} \quad x + 8 = 5x + 3$$

3. Add or subtract the same expression on both sides of the equation:

$$3x - 5 = 4$$

$$\text{is equivalent to} \quad (3x - 5) + 5 = 4 + 5$$

4. Multiply or divide both sides of the equation by the same nonzero expression:

$$\frac{3x}{x-1} = \frac{6}{x-1} \qquad x \neq 1$$

is equivalent to $\dfrac{3x}{x-1} \cdot (x-1) = \dfrac{6}{x-1} \cdot (x-1)$

5. If one side of the equation is 0 and the other side can be factored, then we may use the Zero-Product Property* and set each factor equal to 0:

$$x(x-3) = 0$$

is equivalent to $x = 0$ or $x - 3 = 0$

WARNING Squaring both sides of an equation does not necessarily lead to an equivalent equation. For example, $x = 3$ has one solution, but $x^2 = 9$ has two solutions, $x = -3$ and $x = 3$. ■

Whenever it is possible to solve an equation in your head, do so. For example:

The solution of $2x = 8$ is $x = 4$.

The solution of $3x - 15 = 0$ is $x = 5$.

Now Work PROBLEM 11

2 Solve Linear Equations

Linear equations are equations such as

$$3x + 12 = 0 \qquad \frac{3}{4}x - \frac{1}{5} = 0 \qquad 0.62x - 0.3 = 0$$

A **linear equation in one variable** is equivalent to an equation of the form

$$\boxed{ax + b = 0}$$

where a and b are real numbers and $a \neq 0$.

Sometimes a linear equation is called a **first-degree equation**, because the left side is a polynomial in x of degree 1.

EXAMPLE 3 Solving a Linear Equation

Solve the equation: $3(x-2) = 5(x-1)$

Algebraic Solution

$$\begin{aligned}
3(x-2) &= 5(x-1) \\
3x - 6 &= 5x - 5 && \textbf{Use the Distributive Property.} \\
3x - 6 - 5x &= 5x - 5 - 5x && \textbf{Subtract } \boldsymbol{5x} \textbf{ from each side.} \\
-2x - 6 &= -5 && \textbf{Simplify.} \\
-2x - 6 + 6 &= -5 + 6 && \textbf{Add 6 to each side.} \\
-2x &= 1 && \textbf{Simplify.} \\
\frac{\cancel{-2}x}{\cancel{-2}} &= \frac{1}{-2} && \textbf{Divide each side by } \boldsymbol{-2}\textbf{.} \\
x &= -\frac{1}{2} && \textbf{Simplify.}
\end{aligned}$$

Graphing Solution

Graph $Y_1 = 3(x-2)$ and $Y_2 = 5(x-1)$. See Figure 32. Using INTERSECT, the point of intersection is found to be $(-0.5, -7.5)$. The solution of the equation is $x = -0.5$.

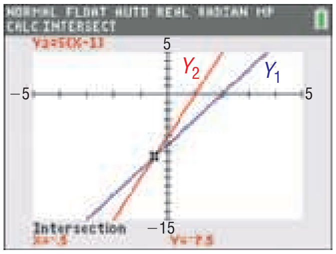


Figure 32 $Y_1 = 3(x-2)$; $Y_2 = 5(x-1)$

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*The Zero-Product Property says that if $ab = 0$, then $a = 0$, $b = 0$, or both equal 0.

✓**Check:** Let $x = -\frac{1}{2}$ in the expression in x on the left side of the equation and simplify. Let $x = -\frac{1}{2}$ in the expression in x on the right side of the equation and simplify. If the two expressions are equal, the solution checks.

$$3(x-2) = 3\left(-\frac{1}{2} - 2\right) = 3\left(-\frac{5}{2}\right) = -\frac{15}{2}$$
$$5(x-1) = 5\left(-\frac{1}{2} - 1\right) = 5\left(-\frac{3}{2}\right) = -\frac{15}{2}$$

Since the two expressions are equal, the solution $x = -\frac{1}{2}$ checks.

The solution set is $\left\{-\frac{1}{2}\right\}$. ■

Now Work PROBLEM 37

EXAMPLE 4 Solving an Equation That Leads to a Linear Equation

Solve the equation: $(2x-1)(x-1) = (x-5)(2x-5)$

Algebraic Solution

$$\begin{aligned}
(2x-1)(x-1) &= (x-5)(2x-5) & \\
2x^2 - 3x + 1 &= 2x^2 - 15x + 25 & &\text{Multiply and combine like terms.}\\
2x^2 - 3x + 1 - 2x^2 &= 2x^2 - 15x + 25 - 2x^2 & &\text{Subtract } 2x^2 \text{ from each side.}\\
-3x + 1 &= -15x + 25 & &\text{Simplify.}\\
-3x + 1 - 1 &= -15x + 25 - 1 & &\text{Subtract 1 from each side.}\\
-3x &= -15x + 24 & &\text{Simplify.}\\
-3x + 15x &= -15x + 24 + 15x & &\text{Add } 15x \text{ to each side.}\\
12x &= 24 & &\text{Simplify.}\\
\frac{12x}{12} &= \frac{24}{12} & &\text{Divide each side by 12.}\\
x &= 2 & &\text{Simplify.}
\end{aligned}$$

Graphing Solution

Graph $Y_1 = (2x-1)(x-1)$ and $Y_2 = (x-5)(2x-5)$. See Figure 33. Using INTERSECT, the point of intersection is found to be $(2, 3)$. The solution of the equation is $x = 2$.

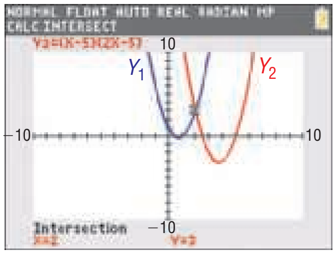


Figure 33 $Y_1 = (2x-1)(x-1)$; $Y_2 = (x-5)(2x-5)$

✓**Check:** $(2x-1)(x-1) = (2 \cdot 2 - 1)(2-1) = (3)(1) = 3$

$(x-5)(2x-5) = (2-5)(2 \cdot 2 - 5) = (-3)(-1) = 3$

Since the two expressions are equal, the solution checks. The solution set is $\{2\}$. ■

Now Work PROBLEM 55

3 Solve Rational Equations

A **rational equation** is an equation that contains a rational expression. Examples of rational equations are

$$\frac{3}{x+1} = \frac{2}{x-1} + 7 \quad \text{and} \quad \frac{x-5}{x-4} = \frac{3}{x+2}$$

To solve a rational equation, multiply both sides of the equation by the least common multiple of the denominators of the rational expressions that make up the rational equation. We shall only show the algebraic solution to rational equations. We leave it to you to verify the results using a graphing utility.

EXAMPLE 5 Solving a Rational Equation

Solve the equation: $\dfrac{3}{x-2} = \dfrac{1}{x-1} + \dfrac{7}{(x-1)(x-2)}$

Solution The domain of the variable is $\{x \mid x \neq 1, x \neq 2\}$. Clear the equation of rational expressions by multiplying both sides by the least common multiple of the denominators of the three rational expressions, $(x-1)(x-2)$.

$$\frac{3}{x-2} = \frac{1}{x-1} + \frac{7}{(x-1)(x-2)}$$

$$(x-1)\cancel{(x-2)}\frac{3}{\cancel{x-2}} = (x-1)(x-2)\left[\frac{1}{x-1} + \frac{7}{(x-1)(x-2)}\right]$$

Multiply both sides by $(x-1)(x-2)$. Divide out common factors on the left.

$$3x-3 = \cancel{(x-1)}(x-2)\frac{1}{\cancel{x-1}} + \cancel{(x-1)}\cancel{(x-2)}\frac{7}{\cancel{(x-1)}\cancel{(x-2)}}$$

Use the Distributive Property on each side; divide out common factors on the right.

$$3x-3 = (x-2)+7$$

$$3x-3 = x+5$$

Combine like terms.

$$2x = 8$$

Add 3 to each side; subtract x from each side.

$$x = 4$$

Divide each side by 2.

✓**Check:** $\dfrac{3}{x-2} = \dfrac{3}{4-2} = \dfrac{3}{2}$

$$\frac{1}{x-1} + \frac{7}{(x-1)(x-2)} = \frac{1}{4-1} + \frac{7}{(4-1)(4-2)} = \frac{1}{3} + \frac{7}{3 \cdot 2} = \frac{2}{6} + \frac{7}{6} = \frac{9}{6} = \frac{3}{2}$$

Since the two expressions are equal, the solution checks. The solution set is $\{4\}$. ■

Now Work PROBLEM 71

Sometimes the process of creating equivalent equations leads to apparent solutions that are not solutions of the original equation. These are called **extraneous solutions**.

EXAMPLE 6 A Rational Equation with No Solution

Solve the equation: $\dfrac{3x}{x-1} + 2 = \dfrac{3}{x-1}$

Solution The domain of the variable is $\{x \mid x \neq 1\}$. Since the two rational expressions in the equation have the same denominator, $x-1$, simplify by multiplying both sides by $x-1$. The resulting equation is equivalent to the original equation, since we are multiplying by $x-1$, which is not 0. (Remember, $x \neq 1$.)

$$\frac{3x}{x-1} + 2 = \frac{3}{x-1}$$

$$\left(\frac{3x}{x-1} + 2\right) \cdot (x-1) = \frac{3}{\cancel{x-1}} \cdot \cancel{(x-1)}$$

Multiply both sides by $x-1$; divide out common factors on the right.

$$\frac{3x}{\cancel{x-1}} \cdot \cancel{(x-1)} + 2 \cdot (x-1) = 3$$

Use the Distributive Property and divide out common factors on the left side.

$$3x + 2x - 2 = 3$$

Simplify.

$$5x - 2 = 3$$

Combine like terms.

$$5x = 5$$

Add 2 to each side.

$$x = 1$$

Divide both sides by 5.

The solution appears to be 1. But recall that $x = 1$ is not in the domain of the variable, so $x = 1$ is an extraneous solution. The equation has no solution. The solution set is $\varnothing$. ■

Now Work PROBLEM 61

4 Solve Problems That Can Be Modeled by Linear Equations

Although each situation has unique features, we can provide an outline of the steps to follow when solving applied problems.

Note: The icon 🌐 is a *Model It!* icon. It indicates that the discussion or problem involves modeling. ■

Note: It is a good practice to choose a variable that reminds you of the unknown. For example, use t for time. ■

Steps for Solving Applied Problems

STEP 1: Read the problem carefully, perhaps two or three times. Pay particular attention to the question being asked in order to identify what you are looking for. Identify any relevant formulas you may need ($d = rt$, $A = \pi r^2$, etc.). If you can, determine realistic possibilities for the answer.

STEP 2: Assign a letter (variable) to represent what you are looking for, and, if necessary, express any remaining unknown quantities in terms of this variable.

STEP 3: Make a list of all the known facts, and translate them into mathematical expressions. These may take the form of an equation (or, later, an inequality) involving the variable. The equation (or inequality) is called the **model**. If possible, draw an appropriately labeled diagram to assist you. Sometimes a table or chart helps.

STEP 4: Solve the equation for the variable, and then answer the question using a complete sentence.

STEP 5: Check the answer with the facts in the problem. If it agrees, congratulations! If it does not agree, try again.

EXAMPLE 7 Solving an Applied Problem: Investments

A total of \$18,000 is invested, some in stocks and some in bonds. If the amount invested in bonds is half that invested in stocks, how much is invested in each category?

Step-by-Step Solution

Step 1: Determine what you are looking for.

We are being asked to find the amount of two investments. These amounts must total \$18,000, because a total of \$18,000 is invested.

Step 2: Assign a variable to represent what you are looking for. If necessary, express any remaining unknown quantities in terms of this variable.

Let s represent the amount invested in stocks. Now, for example, if \$10,000 were invested in stocks, $\$18{,}000 - \$10{,}000 = \$8000$ would be left to invest in bonds. In general, if s represents the amount in stocks, then $18{,}000 - s$ is the amount invested in bonds.

Step 3: Translate the English into mathematical statements. It may be helpful to draw a figure that represents the situation. Also, a table can be used to organize the information. Use the information to build your model.

Set up a table.

Amount in Stocks	Amount in Bonds	Reason
s	$18{,}000 - s$	Total invested is \$18,000

We also know that:

Total amount invested in bonds is one-half that in stocks

$$18{,}000 - s = \frac{1}{2}(s) \qquad \text{The Model}$$

Step 4: Solve the equation and answer the original question.

$$18{,}000 - s = \frac{1}{2}s$$

$$18{,}000 = s + \frac{1}{2}s \qquad \text{Add } s \text{ to both sides.}$$

$$18{,}000 = \frac{3}{2}s \qquad \text{Simplify.}$$

$$\left(\frac{2}{3}\right)18{,}000 = \left(\frac{2}{3}\right)\left(\frac{3}{2}s\right) \qquad \textbf{Multiply both sides by } \frac{2}{3}\textbf{.}$$

$$12{,}000 = s \qquad \textbf{Simplify.}$$

So \$12,000 is invested in stocks and \$18,000 − \$12,000 = \$6000 is invested in bonds.

Step 5: Check your answer with the facts presented in the problem.

The total invested is \$12,000 + \$6000 = \$18,000, and the amount invested in bonds, \$6000, is half the amount invested in stocks, \$12,000. ■

Now Work PROBLEM 95

EXAMPLE 8

Determining an Hourly Wage

Shannon grossed \$725 one week by working 52 hours. Her employer pays time-and-a-half for all hours worked in excess of 40 hours. With this information, can you determine Shannon's regular hourly wage?

Solution

STEP 1: We are looking for an hourly wage. Our answer will be in dollars per hour.

STEP 2: Let w represent the regular hourly wage, measured in dollars per hour. Then $1.5w$ is the overtime hourly wage.

STEP 3: Set up a table:

	Hours Worked	Hourly Wage	Salary
Regular	40	w	$40w$
Overtime	12	$1.5w$	$12(1.5w) = 18w$

The sum of regular salary plus overtime salary will equal \$725. From the table, $40w + 18w = 725$. This is our model.

STEP 4:

$$\begin{aligned} 40w + 18w &= 725 \\ 58w &= 725 \\ w &= 12.50 \end{aligned}$$

Shannon's regular hourly wage is \$12.50 per hour.

STEP 5: Forty hours yields a salary of $40(12.50) = \$500$, and 12 hours of overtime yields a salary of $12(1.5)(12.50) = \$225$, for a total of \$725. ■

Now Work PROBLEM 99

Historical Feature

Solving equations is among the oldest of mathematical activities, and efforts to systematize this activity determined much of the shape of modern mathematics.

Consider the following problem and its solution using only words: Solve the problem of how many apples Jim has, given that

> "Bob's five apples and Jim's apples together make twelve apples" by thinking,
> "Jim's apples are all twelve apples less Bob's five apples" and then concluding,
> "Jim has seven apples."

The mental steps translated into algebra are

$$\begin{aligned} 5 + x &= 12 \\ x &= 12 - 5 \\ &= 7 \end{aligned}$$

The solution of this problem using only words is the earliest form of algebra. Such problems were solved exactly this way in Babylonia in 1800 BC. We know almost nothing of mathematical work before this date, although most authorities believe the sophistication of the earliest known texts indicates that a long period of previous development must have occurred. The method of writing out equations in words persisted for thousands of years, and although it now seems extremely cumbersome, it was used very effectively by many generations of mathematicians. The Arabs developed a good deal of the theory of cubic equations while writing out all the equations in words. About AD 1500, the tendency to abbreviate words into written equations began to lead in the direction of modern notation; for example, the Latin word *et* (meaning *and*) developed into the plus sign, +. Although the occasional use of letters to represent variables dates back to AD 1200, the practice did not become common until about AD 1600. Development thereafter was rapid, and by 1635 algebraic notation did not differ essentially from what we use now.

1.2 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Simplify: $-3(x-5)$ (p. 11)

2. Evaluate: $2x - 3 - 5(x+1)$ for $x = 3$. (pp. 21–22)

3. Is $x = 4$ in the domain of $\frac{3}{x-4}$? (p. 22)

4. Find the least common multiple of the denominators of $\frac{1}{x^2-1}$ and $\frac{3}{x^2+4x+3}$. (pp. 67–68)

Concepts and Vocabulary

5. Multiplying both sides of an equation by any number except ________ results in an equivalent equation.
(a) -1 (b) 0 (c) 1 (d) π

6. An equation that is satisfied for every value of the variable for which both sides are defined is called a(n) ____________.

7. An equation of the form $ax + b = 0$ is called a(n) ___________ equation or a(n) _________-degree equation.

8. An admissible value for the variable that makes the equation a true statement is called a(n) ________ of the equation.
(a) degree (b) identity (c) model (d) solution

9. ***True or False*** Some equations have no solution.

10. ***True or False*** When solving equations using a graphing utility, the solutions are always exact.

Skill Building

In Problems 11–18, mentally solve each equation.

11. $7x = 21$

12. $6x = -24$

13. $3x + 15 = 0$

14. $6x + 18 = 0$

15. $2x - 3 = 0$

16. $3x + 4 = 0$

17. $\frac{1}{3}x = \frac{5}{12}$

18. $\frac{2}{3}x = \frac{9}{2}$

In Problems 19–30, use a graphing utility to approximate the real solutions, if any, of each equation rounded to two decimal places. All solutions lie between −10 and 10.

19. $x^3 - 4x + 2 = 0$

20. $x^3 - 8x + 1 = 0$

21. $-2x^4 + 5 = 3x - 2$

22. $-x^4 + 1 = 2x^2 - 3$

23. $x^4 - 2x^3 + 3x - 1 = 0$

24. $3x^4 - x^3 + 4x^2 - 5 = 0$

25. $-x^3 - \frac{5}{3}x^2 + \frac{7}{2}x + 2 = 0$

26. $-x^4 + 3x^3 + \frac{7}{3}x^2 - \frac{15}{2}x + 2 = 0$

27. $-\frac{2}{3}x^4 - 2x^3 + \frac{5}{2}x = -\frac{2}{3}x^2 + \frac{1}{2}$

28. $\frac{1}{4}x^3 - 5x = \frac{1}{5}x^2 - 4$

29. $x^4 - 5x^2 + 2x + 11 = 0$

30. $-3x^4 + 8x^2 - 2x - 9 = 0$

In Problems 31–76, solve each equation algebraically. Verify your results using a graphing utility.

31. $3x + 4 = x$

32. $2x + 9 = 5x$

33. $2t - 6 = 3 - t$

34. $5y + 6 = -18 - y$

35. $6 - x = 2x + 9$

36. $3 - 2x = 2 - x$

37. $2(3 + 2x) = 3(x - 4)$

38. $3(2 - x) = 2x - 1$

39. $8x - (3x + 2) = 3x - 10$

40. $7 - (2x - 1) = 10$

41. $2(3x - 5) + 6(x - 3) = -3(4 - 5x) + 5x - 6$

42. $5(x - 2) - 2(3x + 1) = 4(1 - 2x) + x$

43. $\frac{3}{2}x + 2 = \frac{1}{2} - \frac{1}{2}x$

44. $\frac{1}{3}x = 2 - \frac{2}{3}x$

45. $\frac{2}{3}p = \frac{1}{2}p - \frac{1}{3}$

46. $\frac{1}{2} - \frac{1}{3}p = \frac{4}{3}$

47. $0.9t = 0.4 + 0.1t$

48. $0.9t = 1 + t$

49. $\frac{x+1}{3} + \frac{x+2}{7} = 2$

50. $\frac{2x+1}{3} + 16 = 3x$

51. $\frac{2}{y} + \frac{4}{y} = 3$

52. $\frac{4}{y} - 5 = \frac{5}{2y}$

53. $\frac{1}{2} + \frac{2}{x} = \frac{3}{4}$

54. $\frac{3}{x} - \frac{1}{3} = \frac{1}{6}$

55. $(x + 7)(x - 1) = (x + 1)^2$

56. $(x + 2)(x - 3) = (x + 3)^2$

57. $x(2x - 3) = (2x + 1)(x - 4)$

58. $x(1 + 2x) = (2x - 1)(x - 2)$

59. $z(z^2 + 1) = 3 + z^3$

60. $w(4 - w^2) = 8 - w^3$

61. $\dfrac{x}{x - 2} + 3 = \dfrac{2}{x - 2}$

62. $\dfrac{2x}{x + 3} = \dfrac{-6}{x + 3} - 2$

63. $\dfrac{2x}{x^2 - 4} = \dfrac{4}{x^2 - 4} - \dfrac{3}{x + 2}$

64. $\dfrac{x}{x^2 - 9} + \dfrac{4}{x + 3} = \dfrac{3}{x^2 - 9}$

65. $\dfrac{x}{x + 2} = \dfrac{3}{2}$

66. $\dfrac{3x}{x - 1} = 2$

67. $\dfrac{5}{2x - 3} = \dfrac{3}{x + 5}$

68. $\dfrac{-4}{x + 4} = \dfrac{-3}{x + 6}$

69. $\dfrac{6t + 7}{4t - 1} = \dfrac{3t + 8}{2t - 4}$

70. $\dfrac{8w + 5}{10w - 7} = \dfrac{4w - 3}{5w + 7}$

71. $\dfrac{4}{x - 2} = \dfrac{-3}{x + 5} + \dfrac{7}{(x + 5)(x - 2)}$

72. $\dfrac{-4}{2x + 3} + \dfrac{1}{x - 1} = \dfrac{1}{(2x + 3)(x - 1)}$

73. $\dfrac{2}{y + 3} + \dfrac{3}{y - 4} = \dfrac{5}{y + 6}$

74. $\dfrac{5}{5z - 11} + \dfrac{4}{2z - 3} = \dfrac{-3}{5 - z}$

75. $\dfrac{x}{x^2 - 1} - \dfrac{x + 3}{x^2 - x} = \dfrac{-3}{x^2 + x}$

76. $\dfrac{x + 1}{x^2 + 2x} - \dfrac{x + 4}{x^2 + x} = \dfrac{-3}{x^2 + 3x + 2}$

Applications and Extensions

77. If $(a, 2)$ is a point on the graph of $y = 5x + 4$, what is a?

78. If $(2, b)$ is a point on the graph of $y = x^2 + 3x$, what is b?

79. If (a, b) is a point on the graph of $2x + 3y = 6$, write an equation that relates a to b.

80. If $(2, 0)$ and $(0, 5)$ are points on the graph of $y = mx + b$, what are m and b?

In Problems 81–86, solve each equation. The letters a, b, and c are constants.

81. $ax - b = c, \quad a \neq 0$

82. $1 - ax = b, \quad a \neq 0$

83. $\dfrac{x}{a} + \dfrac{x}{b} = c, \quad a \neq 0, b \neq 0, a \neq -b$

84. $\dfrac{a}{x} + \dfrac{b}{x} = c, \quad c \neq 0, \quad a + b \neq 0$

85. $\dfrac{1}{x - a} + \dfrac{1}{x + a} = \dfrac{2}{x - 1}$

86. $\dfrac{b + c}{x + a} = \dfrac{b - c}{x - a}, \quad c \neq 0, a \neq 0$

87. Find the number a for which $x = 4$ is a solution of the equation

$$x + 2a = 16 + ax - 6a$$

88. Find the number b for which $x = 2$ is a solution of the equation

$$x + 2b = x - 4 + 2bx$$

Problems 89–94 list some formulas that occur in applications. Solve each formula for the indicated variable.

89. Electricity $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$ for R

90. Finance $A = P(1 + rt)$ for r

91. Mechanics $F = \dfrac{mv^2}{R}$ for R

92. Chemistry $PV = nRT$ for T

93. Mathematics $S = \dfrac{a}{1 - r}$ for r

94. Mechanics $v = -gt + v_0$ for t

---

95. Finance A total of \$20,000 is to be invested, some in bonds and some in certificates of deposit (CDs). If the amount invested in bonds is to exceed that in CDs by \$3000, how much will be invested in each type of investment?

96. Finance A total of \$10,000 is to be divided between Sean and George, with George to receive \$3000 less than Sean. How much will each receive?

97. Finance An inheritance of \$900,000 is to be divided among Scott, Alice, and Tricia in the following manner: Alice is to receive $\frac{3}{4}$ of what Scott gets, while Tricia gets $\frac{1}{2}$ of what Scott gets. How much does each receive?

98. Sharing the Cost of a Pizza Judy and Tom agree to share the cost of an \$18 pizza based on how much each ate. If Tom ate $\frac{2}{3}$ the amount that Judy ate, how much should each pay? [**Hint:** Some pizza may be left.]

99. Computing Hourly Wages Sandra, who is paid time-and-a-half for hours worked in excess of 40 hours, had gross weekly wages of \$598 for 48 hours worked. What is her regular hourly rate?

100. Computing Hourly Wages Leigh is paid time-and-a-half for hours worked in excess of 40 hours and double-time for hours worked on Sunday. If Leigh had gross weekly wages of \$798 for working 50 hours, 4 of which were on Sunday, what is her regular hourly rate?

101. Computing Grades Going into the final exam, which will count as two tests, Brooke has test scores of 80, 83, 71, 61, and 95. What score does Brooke need on the final in order to have an average score of 80?

102. Computing Grades Going into the final exam, which will count as two-thirds of the final grade, Mike has test scores of 86, 80, 84, and 90. What minimum score does Mike need on the final in order to earn a B, which requires an average score of 80? What does he need to earn an A, which requires an average of 90?

103. Business: Discount Pricing A builder of homes reduced the price of a model by 15%. If the new price is \$170,000, what was its original price? How much can be saved by purchasing the model?

104. Business: Discount Pricing At a year-end clearance, a car dealer reduces the list price of last year's models by 12%. If a certain four-door model has a discounted price of \$28,160, what was its list price? How much can be saved by purchasing last year's model?

105. Personal Finance: Concession Markup A movie theater marks up the candy it sells by 275%. If a box of candy sells for \$3.00 at the theater, how much did the theater pay for the box?

106. Personal Finance: Cost of a Car The suggested list price of a new car is \$34,000. The dealer's cost is 85% of list. How much will you pay if the dealer is willing to accept \$100 over cost for the car?

107. Business: Theater Attendance The manager of the Coral Theater wants to know whether the majority of its patrons is adults or children. During a week in July, 5200 tickets were sold and the receipts totaled \$32,200. The adult admission is \$8.50, and the children's admission is \$6.00. How many adult patrons were there?

108. Business: Discount Pricing A wool suit, discounted by 30% for a clearance sale, has a price tag of \$399. What was the suit's original price?

109. Geometry The perimeter of a rectangle is 60 feet. Find its length and width if the length is 8 feet longer than the width.

110. Geometry The perimeter of a rectangle is 42 meters. Find its length and width if the length is twice the width.

111. Smartphones In January, 2015, there were 97.89 million people in the United States who owned a smartphone that ran the Google Android operating system (OS). The Google Android OS was used on 53.2% of all smartphones. How many people in the United States owned a smartphone in January, 2015?

Source: *comScore Networks*

112. Social Networking A September, 2014, survey indicated that 53% of U.S. adults aged 18–29 used the social media platform Instagram. If this was 21 percentage points less than twice the percent that used Twitter, what percent of U.S. adults aged 18–29 used Twitter?

Discussion and Writing

113. One step in the following list contains an error. Identify it and explain what is wrong.

$$x = 2 \tag{1}$$
$$3x - 2x = 2 \tag{2}$$
$$3x = 2x + 2 \tag{3}$$
$$x^2 + 3x = x^2 + 2x + 2 \tag{4}$$
$$x^2 + 3x - 10 = x^2 + 2x - 8 \tag{5}$$
$$(x - 2)(x + 5) = (x - 2)(x + 4) \tag{6}$$
$$x + 5 = x + 4 \tag{7}$$
$$5 = 4 \tag{8}$$

114. The equation

$$\frac{5}{x+3} + 3 = \frac{8+x}{x+3}$$

has no solution, yet when we go through the process of solving it we obtain $x = -3$. Write a paragraph to explain what causes this to happen.

115. Make up an equation that has no solution and give it to a fellow student to solve. Ask the fellow student to write a critique of your equation.

116. Explain the difference between the directions "solve," "evaluate," and "simplify." Write an example using each direction with the expression $3(x + 2) - x$.

'Are You Prepared?' Answers

1. $-3x + 15$

2. -17

3. No

4. $(x + 1)(x - 1)(x + 3)$

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Zero-Product Property (Section R.1, p. 14)
- Factoring (Section R.5, pp. 50–56)
- Square Roots (Section R.2, pp. 24–25)
- Completing the Square (Section R.5, p. 57)

Now Work the **'Are You Prepared?'** problems on page 117.

OBJECTIVES
1 Solve Quadratic Equations by Factoring (p. 110)
2 Solve Quadratic Equations Using the Square Root Method (p. 112)
3 Solve Quadratic Equations by Completing the Square (p. 113)
4 Solve Quadratic Equations Using the Quadratic Formula (p. 113)
5 Solve Problems That Can Be Modeled by Quadratic Equations (p. 116)

Quadratic equations are equations such as

$$2x^2 + x + 8 = 0 \qquad 3x^2 - 5x = 0 \qquad x^2 - 9 = 0$$

A **quadratic equation** is an equation equivalent to one of the form

$$ax^2 + bx + c = 0 \tag{1}$$

where a, b, and c are real numbers and $a \neq 0$.

A quadratic equation written in the form $ax^2 + bx + c = 0$ is in **standard form**.

Sometimes, a quadratic equation is called a **second-degree equation** because, when it is in standard form, the left side is a polynomial of degree 2. We shall discuss four algebraic ways of solving quadratic equations: by factoring, by the square root method, by completing the square, and by using the quadratic formula.

1 Solve Quadratic Equations by Factoring

When a quadratic equation is written in standard form, it may be possible to factor the expression on the left side into the product of two first-degree polynomials. The Zero-Product Property can then be used by setting each factor equal to 0 and solving the resulting linear equations. With this approach we obtain the *exact* solutions of the quadratic equation. This approach leads us to a basic premise in mathematics. Whenever a problem is encountered, use techniques that reduce the problem to one you already know how to solve. In this instance, we are reducing quadratic equations to linear equations using the technique of factoring.

Let's look at an example.

EXAMPLE 1 Solving a Quadratic Equation by Factoring and by Graphing

Solve the equation: $2x^2 - x - 3 = 0$

Algebraic Solution

The equation is in standard form. The left side may be factored as

$$2x^2 - x - 3 = 0$$
$$(2x - 3)(x + 1) = 0 \quad \textbf{Factor.}$$

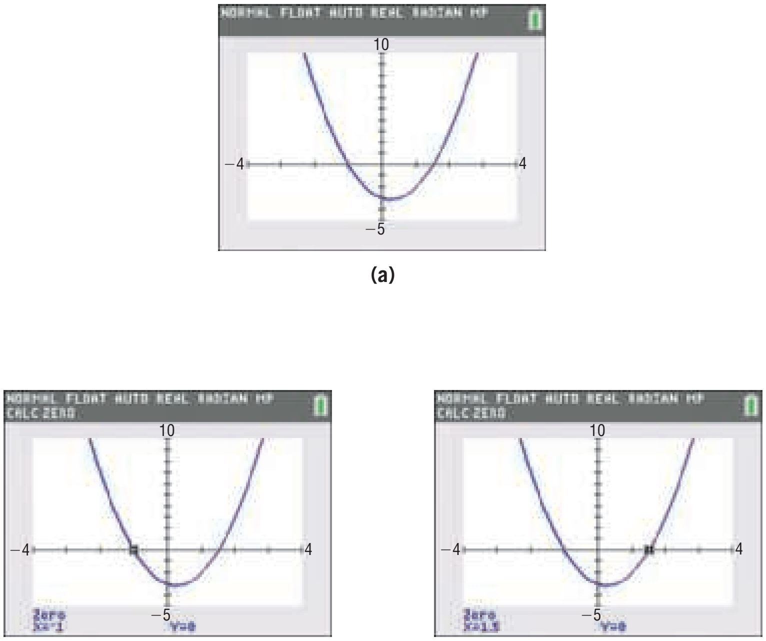
Graphing Solution

Graph $Y_1 = 2x^2 - x - 3$. See Figure 34(a). From the graph it appears there are two solutions to the equation (since the graph crosses the x-axis in two places). Using ZERO, the x-intercepts, and therefore solutions to the equation, are -1 and 1.5. See Figures 34(b) and (c). The solution set is $\{-1, 1.5\}$.

Use the Zero-Product Property and set each factor equal to zero.

$$\begin{aligned} 2x - 3 &= 0 \quad \text{or} \quad & x + 1 &= 0 \\ 2x &= 3 \quad \text{or} \quad & x &= -1 \\ x &= \frac{3}{2} & & \end{aligned}$$

The solution set is $\left\{-1, \frac{3}{2}\right\}$. ■



(a)

(b)

(c)

Figure 34 ■

Now Work PROBLEM 13

When the left side factors into two linear equations with the same solution, the quadratic equation is said to have a **repeated solution**. This solution is also called a **root of multiplicity 2**, or a **double root**.

EXAMPLE 2 Solving a Quadratic Equation by Factoring and by Graphing

Solve the equation: $9x^2 + 1 = 6x$

Algebraic Solution

Put the equation in standard form by subtracting $6x$ from each side.

$$\begin{aligned} 9x^2 + 1 &= 6x \\ 9x^2 - 6x + 1 &= 0 \end{aligned}$$

Factor the left side of the equation.

$$\begin{aligned} (3x - 1)(3x - 1) &= 0 \\ 3x - 1 = 0 \quad \text{or} \quad 3x - 1 &= 0 \qquad \textbf{Zero-Product Property} \\ x = \frac{1}{3} \quad \text{or} \quad x &= \frac{1}{3} \end{aligned}$$

The equation has only the repeated solution $\frac{1}{3}$.

The solution set is $\left\{\frac{1}{3}\right\}$. ■

Graphing Solution

Graph $Y_1 = 9x^2 + 1$ and $Y_2 = 6x$. See Figure 35. Using INTERSECT, the only point of intersection is (0.33, 2), so the solution of the equation is $x = 0.33$, rounded to two decimal places. The solution set is $\{0.33\}$. This solution is approximate.

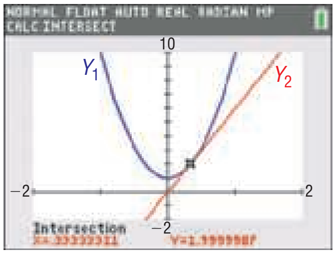


Figure 35 $Y_1 = 9x^2 + 1$; $Y_2 = 6x$ ■

Now Work PROBLEM 23

2 Solve Quadratic Equations Using the Square Root Method

Suppose that we wish to solve the quadratic equation

$$x^2 = p \qquad \textbf{(2)}$$

where p is a nonnegative number. Proceeding as in the earlier examples,

$$x^2 - p = 0 \qquad \text{Put in standard form.}$$

$$(x - \sqrt{p})(x + \sqrt{p}) = 0 \qquad \text{Factor (over the real numbers).}$$

$$x = \sqrt{p} \quad \text{or} \quad x = -\sqrt{p} \qquad \text{Solve.}$$

we have the following result:

$$\text{If } x^2 = p \text{ and } p \geq 0, \text{ then } x = \sqrt{p} \text{ or } x = -\sqrt{p}. \qquad \textbf{(3)}$$

When statement (3) is used, it is called the **Square Root Method**. Note that in statement (3), if $p > 0$ the equation $x^2 = p$ has two solutions, $x = \sqrt{p}$ and $x = -\sqrt{p}$. We usually abbreviate these solutions as $x = \pm\sqrt{p}$, which is read as "x equals plus or minus the square root of p."

For example, the two solutions of the equation

$$x^2 = 4$$

are

$$x = \pm\sqrt{4} \qquad \text{Use the Square Root Method.}$$

and, since $\sqrt{4} = 2$, we have

$$x = \pm 2$$

The solution set is $\{-2, 2\}$.

EXAMPLE 3 Solving Quadratic Equations Using the Square Root Method

Solve each equation: (a) $x^2 = 5$ (b) $(x - 2)^2 = 16$

Solution (a) $x^2 = 5$

$$x = \pm\sqrt{5} \qquad \text{Use the Square Root Method.}$$

$$x = \sqrt{5} \quad \text{or} \quad x = -\sqrt{5}$$

The solution set is $\{-\sqrt{5}, \sqrt{5}\}$.

(b) $(x - 2)^2 = 16$

$$x - 2 = \pm\sqrt{16} \qquad \text{Use the Square Root Method.}$$

$$x - 2 = \pm 4 \qquad \sqrt{16} = 4$$

$$x - 2 = 4 \quad \text{or} \quad x - 2 = -4$$

$$x = 6 \quad \text{or} \quad x = -2$$

The solution set is $\{-2, 6\}$.

✓ **Check:** Verify the solutions using a graphing utility. Are the solutions provided by the utility exact? ■

Now Work PROBLEM 33

3 Solve Quadratic Equations by Completing the Square

EXAMPLE 4

Solving a Quadratic Equation by Completing the Square

Solve by completing the square: $x^2 + 5x + 4 = 0$

Solution Always begin this procedure by rearranging the equation so that the constant is on the right side.

Note: If the coefficient of the square term is not 1, divide through by the coefficient of the square term before attempting to complete the square. For example, to solve $2x^2 - 8x = 5$ by completing the square, divide both sides of the equation by 2 and obtain $x^2 - 4x = \frac{5}{2}$. ■

$$x^2 + 5x + 4 = 0$$
$$x^2 + 5x = -4$$

Since the coefficient of x^2 is 1, we can complete the square on the left side by adding $\left(\frac{1}{2} \cdot 5\right)^2 = \frac{25}{4}$. Remember, in an equation, whatever is added to the left side must also be added to the right side. So add $\frac{25}{4}$ to *both* sides.

$$x^2 + 5x + \frac{25}{4} = -4 + \frac{25}{4} \qquad \text{Add } \frac{25}{4} \text{ to both sides.}$$
$$\left(x + \frac{5}{2}\right)^2 = \frac{9}{4} \qquad \text{Factor; simplify.}$$
$$x + \frac{5}{2} = \pm\sqrt{\frac{9}{4}} \qquad \text{Use the Square Root Method.}$$
$$x + \frac{5}{2} = \pm\frac{3}{2} \qquad \sqrt{\frac{9}{4}} = \frac{\sqrt{9}}{\sqrt{4}} = \frac{3}{2}$$
$$x = -\frac{5}{2} \pm \frac{3}{2}$$
$$x = -\frac{5}{2} + \frac{3}{2} = -1 \quad \text{or} \quad x = -\frac{5}{2} - \frac{3}{2} = -4$$

The solution set is $\{-4, -1\}$.

✓ **Check:** Verify the solutions using a graphing utility. ■

The solution of the equation in Example 4 can also be obtained by factoring. Rework Example 4 using factoring.

Now Work PROBLEM 37

4 Solve Quadratic Equations Using the Quadratic Formula

The method of completing the square can be used to obtain a general formula for solving the quadratic equation

Note: There is no loss in generality to assume that $a > 0$, since if $a < 0$ we can multiply both sides by -1 to obtain an equivalent equation with a positive leading coefficient. ■

$$ax^2 + bx + c = 0, \qquad a > 0$$

As in Example 4, rearrange the equation as

$$ax^2 + bx = -c$$

Since $a > 0$, divide both sides by a to get

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Now the coefficient of x^2 is 1. To complete the square on the left side, add the square of $\frac{1}{2}$ of the coefficient of x; that is, add

$$\left(\frac{1}{2} \cdot \frac{b}{a}\right)^2 = \frac{b^2}{4a^2}$$

to each side. Then

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2}{4a^2} - \frac{4ac}{4a^2} = \frac{b^2 - 4ac}{4a^2} \qquad \textbf{(4)}$$

Provided that $b^2 - 4ac \geq 0$, we now can use the Square Root Method to get

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} \qquad \text{The square root of a quotient equals the quotient of the square roots. Also, } \sqrt{4a^2} = 2a \text{ since } a > 0.$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{Add } -\frac{b}{2a} \text{ to both sides.}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Combine the quotients on the right.}$$

What if $b^2 - 4ac$ is negative? Then equation (4) states that the left expression (a real number squared) equals the right expression (a negative number). Since this is impossible for real numbers, we conclude that if $b^2 - 4ac < 0$, the quadratic equation has no *real* solution.*

THEOREM

Quadratic Formula

Consider the quadratic equation

$$\boxed{ax^2 + bx + c = 0 \qquad a \neq 0}$$

If $b^2 - 4ac < 0$, this equation has no real solution.
If $b^2 - 4ac \geq 0$, the real solution(s) of this equation is (are) given by the **quadratic formula**

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \textbf{(5)}}$$

■

The quantity $\boldsymbol{b^2 - 4ac}$ is called the **discriminant** of the quadratic equation, because its value tells us whether the equation has real solutions. In fact, it also tells us how many solutions to expect.

Discriminant of a Quadratic Equation

For a quadratic equation $ax^2 + bx + c = 0$:

1. If $b^2 - 4ac > 0$, there are two unequal real solutions.
2. If $b^2 - 4ac = 0$, there is a repeated real solution, a root of multiplicity 2.
3. If $b^2 - 4ac < 0$, there is no real solution.

When asked to find the real solutions, if any, of a quadratic equation, always evaluate the discriminant first to see if there are any real solutions.

---

*We consider quadratic equations where $b^2 - 4ac$ is negative in the next section.

EXAMPLE 5 Solving a Quadratic Equation Using the Quadratic Formula

Find the real solutions, if any, of the equation $3x^2 - 5x + 1 = 0$.

Algebraic Solution

The equation is in standard form, so compare it to $ax^2 + bx + c = 0$ to find a, b, and c.

$$3x^2 - 5x + 1 = 0$$

$$ax^2 + bx + c = 0 \qquad \mathbf{a = 3, b = -5, c = 1}$$

With $a = 3$, $b = -5$, and $c = 1$, evaluate the discriminant $b^2 - 4ac$.

$$b^2 - 4ac = (-5)^2 - 4(3)(1) = 25 - 12 = 13$$

Since $b^2 - 4ac > 0$, there are two real solutions.

Use the quadratic formula with $a = 3, b = -5, c = 1$, and $b^2 - 4ac = 13$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{13}}{2(3)} = \frac{5 \pm \sqrt{13}}{6}$$

The solution set is $\left\{\frac{5 - \sqrt{13}}{6}, \frac{5 + \sqrt{13}}{6}\right\}$. These solutions are exact.

Graphing Solution

Figure 36 shows the graph of the equation

$$Y_1 = 3x^2 - 5x + 1$$

There are two x-intercepts: one between 0 and 1, the other between 1 and 2. Using ZERO (or ROOT), we find the solutions to the equation are 0.23 and 1.43, rounded to two decimal places. These solutions are approximate.

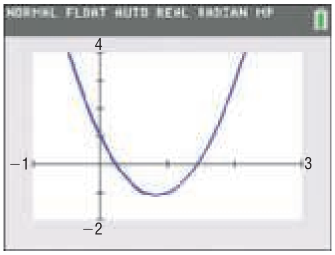


Figure 36 $Y_1 = 3x^2 - 5x + 1$

Now Work PROBLEM 43

EXAMPLE 6 Solving a Quadratic Equation Using the Quadratic Formula

Find the real solutions, if any, of the equation $3x^2 + 2 = 4x$.

Algebraic Solution

The equation, as given, is not in standard form.

$$3x^2 + 2 = 4x$$

$$3x^2 - 4x + 2 = 0 \qquad \textbf{Subtract } 4x \textbf{ from both sides to put the equation in standard form.}$$

$$ax^2 + bx + c = 0 \qquad \textbf{Compare to standard form.}$$

With $a = 3$, $b = -4$, and $c = 2$, the discriminant is

$$\begin{aligned} b^2 - 4ac = (-4)^2 - 4(3)(2) &= 16 - 24 \\ &= -8 \end{aligned}$$

Since $b^2 - 4ac < 0$, the equation has no real solution.

Graphing Solution

Use the standard form of the equation and graph

$$Y_1 = 3x^2 - 4x + 2$$

See Figure 37. Notice that there are no x-intercepts, so the equation has no real solution, as expected based on the value of the discriminant.

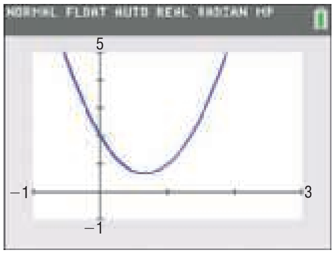


Figure 37 $Y_1 = 3x^2 - 4x + 2$

Now Work PROBLEM 49

SUMMARY

Procedure for Solving a Quadratic Equation Algebraically

To solve a quadratic equation algebraically, first put it in standard form:

$$ax^2 + bx + c = 0$$

Then

STEP 1: Identify a, b, and c.

STEP 2: Evaluate the discriminant, $b^2 - 4ac$.

STEP 3: (a) If the discriminant is negative, the equation has no real solution.

(b) If the discriminant is zero, the equation has one real solution, a repeated root.

(c) If the discriminant is positive, the equation has two distinct real solutions.

If you can easily spot factors, use the factoring method to solve the equation. Otherwise, use the quadratic formula or the method of completing the square.

5 Solve Problems That Can Be Modeled by Quadratic Equations

Many applied problems require the solution of a quadratic equation. Let's look at one that you will probably see again in a slightly different form if you study calculus.

EXAMPLE 7 **Constructing a Box**

From each corner of a square piece of sheet metal, remove a square of side 9 centimeters. Turn up the edges to form an open box. If the box is to hold 144 cubic centimeters (cm^3), what should be the dimensions of the piece of sheet metal?

Solution Use Figure 38 as a guide. We have labeled the length of a side of the square piece of sheet metal, x. The box will be of height 9 centimeters, and its square base will have $x - 18$ as the length of each side. The volume V (Length × Width × Height) of the box is therefore

$$V = (x - 18)(x - 18) \cdot 9 = 9(x - 18)^2 \qquad \textbf{The Model}$$

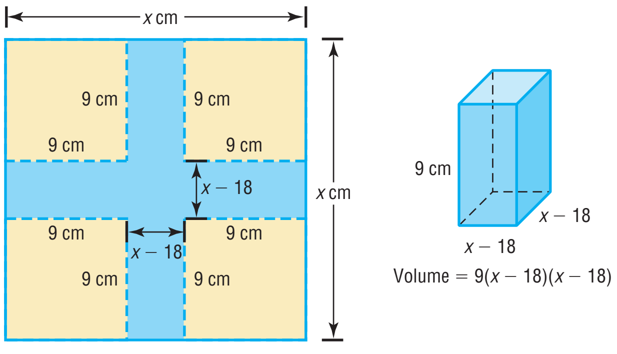


Figure 38

Since the volume of the box is to be 144 cm^3, we have

$$\begin{aligned} 9(x - 18)^2 &= 144 && V = 144 \\ (x - 18)^2 &= 16 && \text{Divide each side by 9.} \\ x - 18 &= \pm 4 && \text{Use the Square Root Method.} \\ x &= 18 \pm 4 \\ x = 22 \quad &\text{or} \quad x = 14 \end{aligned}$$

Discard the solution $x = 14$ (do you see why?) and conclude that the sheet metal should be 22 centimeters by 22 centimeters.

✓ **Check:** If we take a piece of sheet metal that measures 22 centimeters by 22 centimeters, cut out a 9-centimeter square from each corner, and fold up the edges, we get a box whose dimensions are 9 cm by 4 cm by 4 cm, with volume 9 cm × 4 cm × 4 cm = 144 cm^3, as required. ■

Now Work PROBLEM 95

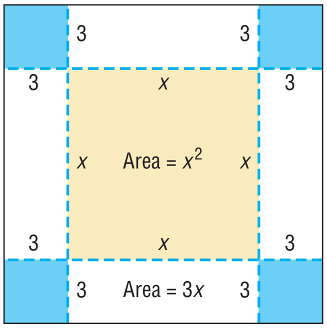
Historical Feature

Problems using quadratic equations are found in the oldest known mathematical literature. Babylonians and Egyptians were solving such problems before 1800 BC. Euclid solved quadratic equations geometrically in his *Data* (300 BC), and the Hindus and Arabs gave rules for solving any quadratic equation with real roots. Because negative numbers were not freely used before AD 1500, there were several different types of quadratic equations, each with its own rule. Thomas Harriot (1560–1621) introduced the method of factoring to obtain solutions, and François Viète (1540–1603) introduced a method that is essentially completing the square.

Until modern times it was usual to neglect the negative roots (if there were any), and equations involving square roots of negative quantities were regarded as unsolvable until the 1500s.

Historical Problems

1. *One solution of al-Khwārizmī* Solve $x^2 + 12x = 85$ by drawing the square shown. The area of the four white rectangles and the yellow square is $x^2 + 12x$. We then set this expression equal to 85 to get the equation $x^2 + 12x = 85$. If we add the four blue squares, we will have a larger square of known area. Complete the solution.



2. *Viète's method* Solve $x^2 + 12x - 85 = 0$ by letting $x = u + z$.

   Then

$$(u + z)^2 + 12(u + z) - 85 = 0$$
$$u^2 + (2z + 12)u + (z^2 + 12z - 85) = 0$$

   Now select z so that $2z + 12 = 0$ and finish the solution.

3. *Another method to get the quadratic formula* Look at equation (4) on page 114. Rewrite the right side as $\left(\dfrac{\sqrt{b^2 - 4ac}}{2a}\right)^2$ and then subtract it from each side. The right side is now 0 and the left side is a difference of two squares. If you factor this difference of two squares, you will easily be able to get the quadratic formula and, moreover, the quadratic expression is factored, which is sometimes useful.

1.3 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Factor: $x^2 - 5x - 6$ (pp. 53–54)
2. Factor: $2x^2 - x - 3$ (pp. 55–56)
3. The solution set of the equation $(x - 3)(3x + 5) = 0$ is ________. (p. 14)
4. Simplify: $\sqrt{8^2 - 4 \cdot 2 \cdot 3}$ (pp. 75–76)
5. Complete the square of the expression $x^2 + 5x$. Factor the new expression. (p. 57)

Concepts and Vocabulary

6. When a quadratic equation has a repeated solution, it is called a(n) ____________ root or a root of ________________.
7. The quantity $b^2 - 4ac$ is called the ______________ of a quadratic equation. If it is ___________, the equation has no real solution.
8. ***True or False*** Quadratic equations always have two real solutions.
9. A quadratic equation is sometimes called a ________ equation.
   (a) first-degree (b) second-degree
   (c) third-degree (d) fourth-degree
10. Which of the following quadratic equations is in standard form?
    (a) $x^2 - 7x = 5$ (b) $9 = x^2$
    (c) $(x + 5)(x - 4) = 0$ (d) $0 = 5x^2 - 6x - 1$

Skill Building

In Problems 11–30, solve each equation by factoring. Verify your solution using a graphing utility.

11. $x^2 - 9x = 0$

12. $x^2 + 4x = 0$

13. $x^2 - 25 = 0$

14. $x^2 - 9 = 0$

15. $z^2 + z - 6 = 0$

16. $v^2 + 7v + 6 = 0$

17. $2x^2 - 5x - 3 = 0$

18. $3x^2 + 5x + 2 = 0$

19. $3t^2 - 48 = 0$

20. $2y^2 - 50 = 0$

21. $x(x - 8) + 12 = 0$

22. $x(x + 4) = 12$

23. $4x^2 + 9 = 12x$

24. $25x^2 + 16 = 40x$

25. $6(p^2 - 1) = 5p$

26. $2(2u^2 - 4u) + 3 = 0$

27. $6x - 5 = \frac{6}{x}$

28. $x + \frac{12}{x} = 7$

29. $\frac{4(x-2)}{x-3} + \frac{3}{x} = \frac{-3}{x(x-3)}$

30. $\frac{5}{x+4} = 4 + \frac{3}{x-2}$

In Problems 31–36, solve each equation by the Square Root Method. Verify your solution using a graphing utility.

31. $x^2 = 25$

32. $x^2 = 36$

33. $(x - 1)^2 = 4$

34. $(x + 2)^2 = 1$

35. $(2y + 3)^2 = 9$

36. $(3z - 2)^2 = 4$

In Problems 37–42, solve each equation by completing the square. Verify your solution using a graphing utility.

37. $x^2 + 4x = 21$

38. $x^2 - 6x = 13$

39. $x^2 - \frac{1}{2}x - \frac{3}{16} = 0$

40. $x^2 + \frac{2}{3}x - \frac{1}{3} = 0$

41. $3x^2 + x - \frac{1}{2} = 0$

42. $2x^2 - 3x - 1 = 0$

In Problems 43–66, find the real solutions, if any, of each equation. Use the quadratic formula. Verify your solution using a graphing utility.

43. $x^2 - 4x + 2 = 0$

44. $x^2 + 4x + 2 = 0$

45. $x^2 - 4x - 1 = 0$

46. $x^2 + 6x + 1 = 0$

47. $2x^2 - 5x + 3 = 0$

48. $2x^2 + 5x + 3 = 0$

49. $4y^2 - y + 2 = 0$

50. $4t^2 + t + 1 = 0$

51. $4x^2 = 1 - 2x$

52. $2x^2 = 1 - 2x$

53. $4x^2 = 9x$

54. $5x = 4x^2$

55. $9t^2 - 6t + 1 = 0$

56. $4u^2 - 6u + 9 = 0$

57. $\frac{3}{4}x^2 - \frac{1}{4}x - \frac{1}{2} = 0$

58. $\frac{2}{3}x^2 - x - 3 = 0$

59. $\frac{5}{3}x^2 - x = \frac{1}{3}$

60. $\frac{3}{5}x^2 - x = \frac{1}{5}$

61. $2x(x + 2) = 3$

62. $3x(x + 2) = 1$

63. $4 - \frac{1}{x} - \frac{2}{x^2} = 0$

64. $4 + \frac{1}{x} - \frac{1}{x^2} = 0$

65. $\frac{3x}{x-2} + \frac{1}{x} = 4$

66. $\frac{2x}{x-3} + \frac{1}{x} = 4$

In Problems 67–72, use the discriminant to determine whether each quadratic equation has two unequal real solutions, a repeated real solution (a double root), or no real solution, without solving the equation.

67. $2x^2 - 6x + 7 = 0$

68. $x^2 + 4x + 7 = 0$

69. $9x^2 - 30x + 25 = 0$

70. $25x^2 - 20x + 4 = 0$

71. $3x^2 + 5x - 8 = 0$

72. $2x^2 - 3x - 7 = 0$

Mixed Practice

In Problems 73–88, find the real solutions, if any, of each equation. Use any method. Verify your solution using a graphing utility.

73. $x^2 - 5 = 0$

74. $x^2 - 6 = 0$

75. $16x^2 - 8x + 1 = 0$

76. $9x^2 - 12x + 4 = 0$

77. $10x^2 - 19x - 15 = 0$

78. $6x^2 + 7x - 20 = 0$

79. $2 + z = 6z^2$

80. $2 = y + 6y^2$

81. $x^2 + \sqrt{2}x = \frac{1}{2}$

82. $\frac{1}{2}x^2 = \sqrt{2}x + 1$

83. $x^2 + x = 4$

84. $x^2 + x = 1$

85. $5x(x - 1) = -7x^2 + 2$

86. $10x(x + 2) = -3x + 5$

87. $\frac{x}{x - 2} + \frac{2}{x + 1} = \frac{7x + 1}{x^2 - x - 2}$

88. $\frac{3x}{x + 2} + \frac{1}{x - 1} = \frac{4 - 7x}{x^2 + x - 2}$

Applications and Extensions

89. Pythagorean Theorem How many right triangles have a hypotenuse that measures $2x + 3$ meters and legs that measure $2x - 5$ meters and $x + 7$ meters? What are the dimensions of the triangle(s)?

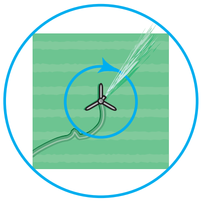
90. Pythagorean Theorem How many right triangles have a hypotenuse that measures $4x + 5$ inches and legs that measure $3x + 13$ inches and x inches? What are the dimensions of the triangle(s)?

91. Dimensions of a Window The area of the opening of a rectangular window is to be 143 square feet. If the length is to be 2 feet more than the width, what are the dimensions?

92. Dimensions of a Window The area of a rectangular window is to be 306 square centimeters. If the length exceeds the width by 1 centimeter, what are the dimensions?

93. Geometry Find the dimensions of a rectangle whose perimeter is 26 meters and whose area is 40 square meters.

94. Watering a Field An adjustable water sprinkler that sprays water in a circular pattern is placed at the center of a square field whose area is 1250 square feet (see the figure). What is the shortest radius setting that can be used if the field is to be completely enclosed within the circle?



95. Constructing a Box An open box is to be constructed from a square piece of sheet metal by removing a square of side 1 foot from each corner and turning up the edges. If the box is to hold 4 cubic feet, what should be the dimensions of the sheet metal?

96. Constructing a Box Rework Problem 95 if the piece of sheet metal is a rectangle whose length is twice its width.

97. Physics A ball is thrown vertically upward from the top of a building 96 feet tall with an initial velocity of 80 feet per second. The distance s (in feet) of the ball from the ground after t seconds is $s = 96 + 80t - 16t^2$.

(a) After how many seconds does the ball strike the ground?
(b) After how many seconds will the ball pass the top of the building on its way down?

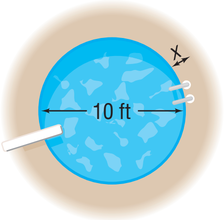
98. Physics An object is propelled vertically upward with an initial velocity of 20 meters per second. The distance s (in meters) of the object from the ground after t seconds is $s = -4.9t^2 + 20t$.

(a) When will the object be 15 meters above the ground?
(b) When will it strike the ground?
(c) Will the object reach a height of 100 meters?

99. Reducing the Size of a Candy Bar A jumbo chocolate bar with a rectangular shape measures 12 centimeters in length, 7 centimeters in width, and 3 centimeters in thickness. Due to escalating costs of cocoa, management decides to reduce the volume of the bar by 10%. To accomplish this reduction, management decides that the new bar should have the same 3 centimeter thickness, but the length and width of each should be reduced an equal number of centimeters. What should be the dimensions of the new candy bar?

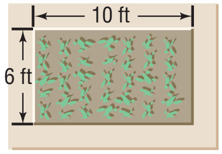
100. Reducing the Size of a Candy Bar Rework Problem 99 if the reduction is to be 20%.

101. Constructing a Border around a Pool A circular pool measures 10 feet across. One cubic yard of concrete is to be used to create a circular border of uniform width around the pool. If the border is to have a depth of 3 inches, how wide will the border be? (1 cubic yard = 27 cubic feet) See the illustration.



102. Constructing a Border around a Pool Rework Problem 101 if the depth of the border is 4 inches.

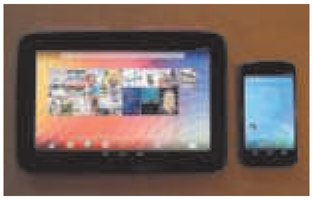
103. Constructing a Border around a Garden A landscaper, who just completed a rectangular flower garden measuring 6 feet by 10 feet, orders 1 cubic yard of premixed cement, all of which is to be used to create a border of uniform width around the garden. If the border is to have a depth of 3 inches, how wide will the border be? (1 cubic yard = 27 cubic feet)



104. Dimensions of a Patio A contractor orders 8 cubic yards of premixed cement, all of which is to be used to pour a patio that will be 4 inches thick. If the length of the patio is specified to be twice the width, what will be the patio dimensions? (1 cubic yard = 27 cubic feet)

105. Comparing Tablets The screen size of a tablet is determined by the length of the diagonal of the rectangular screen. The 9.7-inch iPad Air^{TM} comes in a 4:3 format, which means that the ratio of the length to the width of the rectangular screen is 4:3. What is the area of the iPad's screen? What is the area of a 10-inch Google Nexus^{TM} if its screen is in a 16:10 format? Which screen is larger? (**Hint:** If x is the length of a 4:3 format screen, then $\frac{3}{4}x$ is the width.)

iPad mini 4:3



Google Nexus 16:10

106. Comparing Tablets Refer to Problem 105. Find the screen area of a 7.9-inch iPad mini with Retina^{TM} in a 4:3 format, and compare it with an 8-inch Dell Venue Pro^{TM} if its screen is in a 16:9 format. Which screen is larger?

107. Field Design A football field is sloped from the center toward the sides for drainage. The height h, in feet, of the field, x feet from the side, is given by $h = -0.00025x^2 + 0.04x$. Find the height of the field a distance of 35 feet from the side. Round to the nearest tenth of a foot.

108. College Value The difference, d, in median earnings, in \$1000s, between high school graduates and college graduates can be approximated by $d = -0.002x^2 + 0.319x + 7.512$, where x is the number of years after 1965. Based on this model, estimate to the nearest year when the difference in median earnings was \$15,000. (***Source:*** *Current Population Survey*)

109. Student Working A study found that a student's GPA, g, is related to the number of hours worked each week, h, by the equation $g = -0.0006h^2 + 0.015h + 3.04$. Estimate the number of hours worked each week for a student with a GPA of 2.97. Round to the nearest whole hour.

110. Fraternity Purchase A fraternity wants to buy a new LED Smart TV that costs \$1470. If 7 members of the fraternity are not able to contribute, the share for the remaining members increases by \$5. How many members are in the fraternity?

111. The sum of the consecutive integers 1, 2, 3, . . . n is given by the formula $\frac{1}{2}n(n + 1)$. How many consecutive integers, starting with 1, must be added to get a sum of 666?

112. Geometry If a polygon of n sides has $\frac{1}{2}n(n - 3)$ diagonals, how many sides will a polygon with 65 diagonals have? Is there a polygon with 80 diagonals?

113. Show that the sum of the roots of a quadratic equation is $-\frac{b}{a}$.

114. Show that the product of the roots of a quadratic equation is $\frac{c}{a}$.

115. Find k such that the equation $kx^2 + x + k = 0$ has a repeated real solution.

116. Find k such that the equation $x^2 - kx + 4 = 0$ has a repeated real solution.

117. Show that the real solutions of the equation $ax^2 + bx + c = 0$ are the negatives of the real solutions of the equation $ax^2 - bx + c = 0$. Assume that $b^2 - 4ac \geq 0$.

118. Show that the real solutions of the equation $ax^2 + bx + c = 0$ are the reciprocals of the real solutions of the equation $cx^2 + bx + a = 0$. Assume that $b^2 - 4ac \geq 0$.

Explaining Concepts: Discussion and Writing

119. Which of the following pairs of equations are equivalent? Explain.

(a) $x^2 = 9$; $x = 3$ (b) $x = \sqrt{9}$; $x = 3$

(c) $(x - 1)(x - 2) = (x - 1)^2$; $x - 2 = x - 1$

120. Describe three ways that you might solve a quadratic equation. State your preferred method; explain why you chose it.

121. Explain the benefits of evaluating the discriminant of a quadratic equation before attempting to solve it.

122. Create three quadratic equations: one having two distinct solutions, one having no real solution, and one having exactly one real solution.

123. The word *quadratic* seems to imply four (*quad*), yet a quadratic equation is an equation that involves a polynomial of degree 2. Investigate the origin of the term *quadratic* as it is used in the expression *quadratic equation*. Write a brief essay on your findings.

'Are You Prepared?' Answers

1. $(x - 6)(x + 1)$ **2.** $(2x - 3)(x + 1)$ **3.** $\left\{-\frac{5}{3}, 3\right\}$ **4.** $2\sqrt{10}$ **5.** $x^2 + 5x + \frac{25}{4} = \left(x + \frac{5}{2}\right)^2$

1.4 Complex Numbers; Quadratic Equations in the Complex Number System*

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Classification of Numbers (Section R.1, pp. 4–5)
- Rationalizing Denominators (Section R.8, p. 76)

Now Work the **'Are You Prepared?'** problems on page 128.

OBJECTIVES **1** Add, Subtract, Multiply, and Divide Complex Numbers (p. 122)
2 Solve Quadratic Equations in the Complex Number System (p. 125)

Complex Numbers

One property of a real number is that its square is nonnegative (greater than or equal to 0). For example, there is no real number x for which

$$x^2 = -1$$

To remedy this situation, we introduce a new number called the *imaginary unit.*

DEFINITION

The **imaginary unit**, which we denote by i, is the number whose square is -1. That is,

$$\boxed{i^2 = -1}$$

This should not surprise you. If our universe were to consist only of integers, there would be no number x for which $2x = 1$. This was remedied by introducing numbers such as $\frac{1}{2}$ and $\frac{2}{3}$, the *rational numbers*. If our universe were to consist only of rational numbers, there would be no x whose square equals 2. That is, there would be no number x for which $x^2 = 2$. To remedy this, we introduced numbers such as $\sqrt{2}$ and $\sqrt[3]{5}$, the *irrational numbers*. Recall that the *real numbers* consist of the rational numbers and the irrational numbers. Now, if our universe were to consist only of real numbers, then there would be no number x whose square is -1. To remedy this, we introduced a number i, whose square is -1.

In the progression outlined, each time we encountered a situation that was unsuitable, a new number system was introduced to remedy this situation. And each new number system contained the earlier number system as a subset. The number system that results from introducing the number i is called the **complex number system**.

DEFINITION

Complex numbers are numbers of the form $\boldsymbol{a + bi}$, where a and b are real numbers. The real number a is called the **real part** of the number $a + bi$; the real number b is called the **imaginary part** of $a + bi$; and i is the imaginary unit, so $i^2 = -1$.

For example, the complex number $-5 + 6i$ has the real part -5 and the imaginary part 6.

When a complex number is written in the form $a + bi$, where a and b are real numbers, it is in **standard form**. However, if the imaginary part of a complex number is negative, such as in the complex number $3 + (-2)i$, we agree to write it instead in the form $3 - 2i$.

Also, the complex number $a + 0i$ is usually written simply as a. This serves to remind us that the real numbers are a subset of the complex numbers. The complex number $0 + bi$ is usually written as bi. Sometimes the complex number bi is called a **pure imaginary number**.

---

*This section may be omitted without any loss of continuity.

1 Add, Subtract, Multiply, and Divide Complex Numbers

Equality, addition, subtraction, and multiplication of complex numbers are defined so as to preserve the familiar rules of algebra for real numbers. Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.

Equality of Complex Numbers

$$a + bi = c + di \quad \text{if and only if} \quad a = c \text{ and } b = d \qquad \textbf{(1)}$$

Two complex numbers are added by forming the complex number whose real part is the sum of the real parts and whose imaginary part is the sum of the imaginary parts.

Sum of Complex Numbers

$$(a + bi) + (c + di) = (a + c) + (b + d)i \qquad \textbf{(2)}$$

To subtract two complex numbers, use this rule:

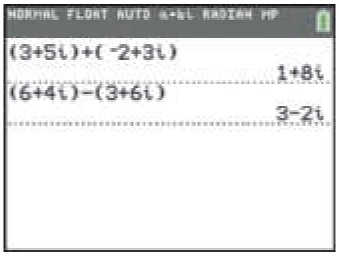
Difference of Complex Numbers

$$(a + bi) - (c + di) = (a - c) + (b - d)i \qquad \textbf{(3)}$$

EXAMPLE 1 Adding and Subtracting Complex Numbers

(a) $(3 + 5i) + (-2 + 3i) = [3 + (-2)] + (5 + 3)i = 1 + 8i$

(b) $(6 + 4i) - (3 + 6i) = (6 - 3) + (4 - 6)i = 3 + (-2)i = 3 - 2i$ ■



```
(3+5i)+(-2+3i)
                   1+8i
(6+4i)-(3+6i)
                   3-2i
```

Figure 39

Some graphing calculators have the capability of handling complex numbers. For example, Figure 39 shows the results of Example 1 using a TI-84 Plus C graphing calculator.

Now Work PROBLEM 15

Products of complex numbers are calculated as illustrated in Example 2.

EXAMPLE 2 Multiplying Complex Numbers

$$\begin{aligned}(5 + 3i) \cdot (2 + 7i) &= 5 \cdot (2 + 7i) + 3i(2 + 7i) && \textbf{Distributive Property}\\ &= 10 + 35i + 6i + 21i^2 && \textbf{Distributive Property}\\ &= 10 + 41i + 21(-1) && \boldsymbol{i^2 = -1}\\ &= -11 + 41i\end{aligned}$$ ■

Based on the procedure of Example 2, the **product** of two complex numbers is defined as follows:

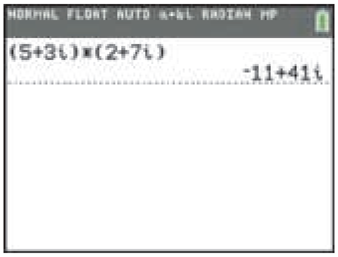
Product of Complex Numbers

$$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i \qquad \textbf{(4)}$$

Do not bother to memorize formula (4). Instead, whenever it is necessary to multiply two complex numbers, follow the usual rules for multiplying two binomials, as in Example 2, remembering that $i^2 = -1$. For example,

$$(2i)(2i) = 4i^2 = 4(-1) = -4$$
$$(2 + i)(1 - i) = 2 - 2i + i - i^2 = 3 - i$$

Graphing calculators may also be used to multiply complex numbers. Figure 40 shows the result obtained in Example 2 using a TI-84 Plus C graphing calculator.



```
NORMAL FLOAT AUTO a+bi RADIAN MP
(5+3i)*(2+7i)
                      -11+41i
```

Figure 40

Now Work PROBLEM 21

Algebraic properties for addition and multiplication, such as the Commutative, Associative, and Distributive Properties, hold for complex numbers. However, the property that every nonzero complex number has a multiplicative inverse, or reciprocal, requires a closer look.

Conjugates

DEFINITION

If $z = a + bi$ is a complex number, then its **conjugate**, denoted by $\bar{z}$, is defined as

$$\boxed{\bar{z} = \overline{a + bi} = a - bi}$$

Note: The conjugate of a complex number can be found by changing the sign of the imaginary part.

For example, $\overline{2 + 3i} = 2 - 3i$ and $\overline{-6 - 2i} = -6 + 2i$.

EXAMPLE 3 Multiplying a Complex Number by Its Conjugate

Find the product of the complex number $z = 3 + 4i$ and its conjugate $\bar{z}$.

Solution Since $\bar{z} = 3 - 4i$, we have

$$z\bar{z} = (3 + 4i)(3 - 4i) = 9 - 12i + 12i - 16i^2 = 9 + 16 = 25$$

The result obtained in Example 3 has an important generalization.

THEOREM

The product of a complex number and its conjugate is a nonnegative real number. That is, if $z = a + bi$, then

$$\boxed{z\bar{z} = a^2 + b^2 \qquad \textbf{(5)}}$$

Proof If $z = a + bi$, then

$$z\bar{z} = (a + bi)(a - bi) = a^2 - abi + abi - (bi)^2 = a^2 - b^2i^2 = a^2 + b^2 \qquad ■$$

To express the reciprocal of a nonzero complex number z in standard form, multiply the numerator and denominator of $\frac{1}{z}$ by its conjugate $\bar{z}$. That is, if $z = a + bi$ is a nonzero complex number, then

$$\frac{1}{a + bi} = \frac{1}{z} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{z\bar{z}} \underset{\substack{\uparrow \\ \text{Use (5).}}}{=} \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$$

EXAMPLE 4 Writing the Reciprocal of a Complex Number in Standard Form

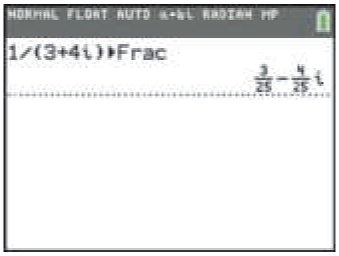
Write $\frac{1}{3 + 4i}$ in standard form $a + bi$; that is, find the reciprocal of $3 + 4i$.

Solution Multiply the numerator and denominator of $\frac{1}{3+4i}$ by the conjugate of $3+4i$, the complex number $3-4i$. The result is

$$\frac{1}{3+4i} = \frac{1}{3+4i} \cdot \frac{3-4i}{3-4i} = \frac{3-4i}{9+16} = \frac{3}{25} - \frac{4}{25}i$$ ■

A graphing calculator can be used to verify the result of Example 4. See Figure 41.

To express the quotient of two complex numbers in standard form, multiply the numerator and denominator of the quotient by the conjugate of the denominator.



1/(3+4i)▸Frac

$\frac{3}{25}-\frac{4}{25}i$

Figure 41

EXAMPLE 5 Writing Quotients of Complex Numbers in Standard Form

Write each of the following in standard form.

(a) $\frac{1+4i}{5-12i}$ (b) $\frac{2-3i}{4-3i}$

Solution (a)
$$\frac{1+4i}{5-12i} = \frac{1+4i}{5-12i} \cdot \frac{5+12i}{5+12i} = \frac{5+12i+20i+48i^2}{25+144}$$
$$= \frac{-43+32i}{169} = -\frac{43}{169} + \frac{32}{169}i$$

(b)
$$\frac{2-3i}{4-3i} = \frac{2-3i}{4-3i} \cdot \frac{4+3i}{4+3i} = \frac{8+6i-12i-9i^2}{16+9}$$
$$= \frac{17-6i}{25} = \frac{17}{25} - \frac{6}{25}i$$ ■

Now Work PROBLEM 29

EXAMPLE 6 Writing Other Expressions in Standard Form

If $z = 2-3i$ and $w = 5+2i$, write each of the following expressions in standard form.

(a) $\frac{z}{w}$ (b) $\overline{z+w}$ (c) $z+\overline{z}$

Solution (a)
$$\frac{z}{w} = \frac{z \cdot \overline{w}}{w \cdot \overline{w}} = \frac{(2-3i)(5-2i)}{(5+2i)(5-2i)} = \frac{10-4i-15i+6i^2}{25+4}$$
$$= \frac{4-19i}{29} = \frac{4}{29} - \frac{19}{29}i$$

(b) $\overline{z+w} = \overline{(2-3i)+(5+2i)} = \overline{7-i} = 7+i$

(c) $z+\overline{z} = (2-3i)+(2+3i) = 4$ ■

The conjugate of a complex number has certain general properties that will be useful later.

For a real number $a = a+0i$, the conjugate is $\overline{a} = \overline{a+0i} = a-0i = a$.

THEOREM

The conjugate of a real number is the real number itself. ■

Other properties that are direct consequences of the definition of the conjugate are given next. In each statement, z and w represent complex numbers.

THEOREM

The conjugate of the conjugate of a complex number is the complex number itself.

$$(\overline{\overline{z}}) = z \qquad \textbf{(6)}$$

The conjugate of the sum of two complex numbers equals the sum of their conjugates.

$$\overline{z + w} = \overline{z} + \overline{w} \qquad \textbf{(7)}$$

The conjugate of the product of two complex numbers equals the product of their conjugates.

$$\overline{z \cdot w} = \overline{z} \cdot \overline{w} \qquad \textbf{(8)}$$

The proofs of equations (6), (7), and (8) are left as exercises. See Problems 94–96.

Powers of *i*

The powers of i follow a pattern that is useful to know.

$$\begin{array}{ll} i^1 = i & i^5 = i^4 \cdot i = 1 \cdot i = i \\ i^2 = -1 & i^6 = i^4 \cdot i^2 = -1 \\ i^3 = i^2 \cdot i = -i & i^7 = i^4 \cdot i^3 = -i \\ i^4 = i^2 \cdot i^2 = (-1)(-1) = 1 & i^8 = i^4 \cdot i^4 = 1 \end{array}$$

And so on. The powers of i repeat with every fourth power.

EXAMPLE 7 Evaluating Powers of *i*

(a) $i^{27} = i^{24} \cdot i^3 = (i^4)^6 \cdot i^3 = 1^6 \cdot i^3 = -i$

(b) $i^{101} = i^{100} \cdot i^1 = (i^4)^{25} \cdot i = 1^{25} \cdot i = i$

EXAMPLE 8 Writing the Power of a Complex Number in Standard Form

Write $(2 + i)^3$ in standard form.

Solution Use the special product formula for $(x + a)^3$.

$$(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$$

Using this special product formula,

$$\begin{aligned} (2 + i)^3 &= 2^3 + 3 \cdot i \cdot 2^2 + 3 \cdot i^2 \cdot 2 + i^3 \\ &= 8 + 12i + 6(-1) + (-i) \\ &= 2 + 11i \end{aligned}$$

Note: Another way to find $(2 + i)^3$ is to multiply out $(2 + i)^2\,(2 + i)$.

Now Work PROBLEMS 35 AND 43

2 Solve Quadratic Equations in the Complex Number System

Quadratic equations with a negative discriminant have no real number solution. However, if we extend our number system to allow complex numbers, quadratic equations will always have a solution. Since the solution to a quadratic equation involves the square root of the discriminant, we begin with a discussion of square roots of negative numbers.

DEFINITION If N is a positive real number, we define the **principal square root of $-N$**, denoted by $\sqrt{-N}$, as

$$\boxed{\sqrt{-N} = \sqrt{N}i}$$

where i is the imaginary unit and $i^2 = -1$. ■

EXAMPLE 9 Evaluating the Square Root of a Negative Number

(a) $\sqrt{-1} = \sqrt{1}i = i$ (b) $\sqrt{-16} = \sqrt{16}i = 4i$

(c) $\sqrt{-8} = \sqrt{8}i = 2\sqrt{2}i$ ■

Now Work PROBLEM 51

EXAMPLE 10 Using the Square Root Method in the Complex Number System

Solve each equation in the complex number system.

(a) $x^2 = 4$ (b) $x^2 = -9$

Solution (a) $x^2 = 4$

$$x = \pm\sqrt{4} = \pm 2$$

The equation has the solution set $\{-2, 2\}$.

(b) $x^2 = -9$

$$x = \pm\sqrt{-9} = \pm\sqrt{9}i = \pm 3i$$

The equation has the solution set $\{-3i, 3i\}$. ■

WARNING When working with square roots of negative numbers, do not set the square root of a product equal to the product of the square roots (which can be done with positive numbers). To see why, look at this calculation: We know that $\sqrt{100} = 10$. However, it is also true that $100 = (-25)(-4)$, so

$$\begin{aligned} 10 &= \sqrt{100} \\ &= \sqrt{(-25)(-4)} \\ &\neq \sqrt{-25}\sqrt{-4} \end{aligned}$$

because

$$\begin{aligned} &\sqrt{-25} \cdot \sqrt{-4} \\ &= (\sqrt{25}i)(\sqrt{4}i) \\ &= (5i)(2i) \\ &= 10i^2 = -10 \end{aligned}$$ ■

Now Work PROBLEM 59

Because we have defined the square root of a negative number, we can now restate the quadratic formula without restriction.

THEOREM Quadratic Formula

In the complex number system, the solutions of the quadratic equation $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$, are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \textbf{(9)}$$ ■

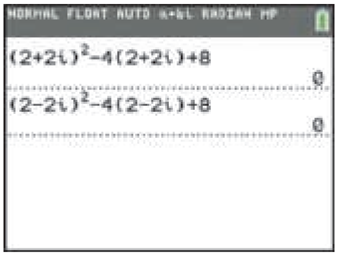
EXAMPLE 11 Solving a Quadratic Equation in the Complex Number System

Solve the equation $x^2 - 4x + 8 = 0$ in the complex number system.

Solution Here $a = 1$, $b = -4$, $c = 8$, and $b^2 - 4ac = 16 - 4(1)(8) = -16$. Using equation (9),

$$x = \frac{-(-4) \pm \sqrt{-16}}{2(1)} = \frac{4 \pm \sqrt{16}i}{2} = \frac{4 \pm 4i}{2} = 2 \pm 2i$$

The equation has the solution set $\{2 - 2i, 2 + 2i\}$.



```
NORMAL FLOAT AUTO a+bi RADIAN MP
(2+2i)²-4(2+2i)+8
                                0
(2-2i)²-4(2-2i)+8
                                0
```

Figure 42

✓Check:

$$2 + 2i: \quad (2 + 2i)^2 - 4(2 + 2i) + 8 = 4 + \cancel{8i} + 4i^2 - \cancel{8} - \cancel{8i} + \cancel{8}$$
$$= 4 - 4 = 0$$
$$2 - 2i: \quad (2 - 2i)^2 - 4(2 - 2i) + 8 = 4 - \cancel{8i} + 4i^2 - \cancel{8} + \cancel{8i} + \cancel{8}$$
$$= 4 - 4 = 0$$

■

Figure 42 shows the check of the solution using a TI-84 Plus C graphing calculator. Graph $Y_1 = x^2 - 4x + 8$. How many x-intercepts are there?

Now Work PROBLEM 65

The discriminant $b^2 - 4ac$ of a quadratic equation still serves as a way to determine the character of the solutions.

Character of the Solutions of a Quadratic Equation

In the complex number system, consider a quadratic equation $ax^2 + bx + c = 0$ with real coefficients.

1. If $b^2 - 4ac > 0$, the equation has two unequal real solutions.
2. If $b^2 - 4ac = 0$, the equation has a repeated real solution, a double root.
3. If $b^2 - 4ac < 0$, the equation has two complex solutions that are not real. The solutions are conjugates of each other.

The third conclusion above is a consequence of the fact that if $b^2 - 4ac = -N < 0$, then by the quadratic formula, the solutions are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-b + \sqrt{-N}}{2a} = \frac{-b + \sqrt{N}i}{2a} = \frac{-b}{2a} + \frac{\sqrt{N}}{2a}i$$

and

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-b - \sqrt{-N}}{2a} = \frac{-b - \sqrt{N}i}{2a} = \frac{-b}{2a} - \frac{\sqrt{N}}{2a}i$$

which are conjugates of each other.

EXAMPLE 12

Determining the Character of the Solutions of a Quadratic Equation

Without solving, determine the character of the solutions of each equation.

(a) $3x^2 + 4x + 5 = 0$

(b) $2x^2 + 4x + 1 = 0$

(c) $9x^2 - 6x + 1 = 0$

Solution

(a) Here $a = 3, b = 4,$ and $c = 5,$ so $b^2 - 4ac = 4^2 - 4(3)(5) = -44$. The solutions are two complex numbers that are not real and are conjugates of each other.

(b) Here $a = 2, b = 4,$ and $c = 1$, so $b^2 - 4ac = 4^2 - 4(2)(1) = 8$. The solutions are two unequal real numbers.

(c) Here $a = 9, b = -6,$ and $c = 1,$ so $b^2 - 4ac = (-6)^2 - 4(9)(1) = 0$. The solution is a repeated real number—that is, a double root. ■

Now Work PROBLEM 79

1.4 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Name the integers and the rational numbers in the set $\left\{-3, 0, \sqrt{2}, \frac{6}{5}, \pi\right\}$. (pp. 4–5)

2. ***True or False*** Rational numbers and irrational numbers are in the set of real numbers. (pp. 4–5)

3. Rationalize the denominator of $\frac{3}{2 + \sqrt{3}}$. (p. 76)

Concepts and Vocabulary

4. In the complex number $5 + 2i$, the number 5 is called the _______ part; the number 2 is called the ____________ part; the number i is called the ____________ ______.

5. ***True or False*** The conjugate of $2 + 5i$ is $-2 - 5i$.

6. ***True or False*** All real numbers are complex numbers.

7. ***True or False*** If $2 - 3i$ is a solution of a quadratic equation with real coefficients, then $-2 + 3i$ is also a solution.

8. Which of the following is the principal square root of -4?
(a) $-2i$ (b) $2i$ (c) -2 (d) 2

9. Which operation involving complex numbers requires the use of a conjugate?
(a) division (b) multiplication
(c) subtraction (d) addition

10. Powers of i repeat every ______ power.
(a) second (b) third (c) fourth (d) fifth

Skill Building

In Problems 11–48, write each expression in the standard form a + bi. Verify your results using a graphing utility.

11. $(2 - 3i) + (6 + 8i)$
12. $(4 + 5i) + (-8 + 2i)$
13. $(-3 + 2i) - (4 - 4i)$
14. $(3 - 4i) - (-3 - 4i)$
15. $(2 - 5i) - (8 + 6i)$
16. $(-8 + 4i) - (2 - 2i)$
17. $3(2 - 6i)$
18. $-4(2 + 8i)$
19. $2i(2 - 3i)$
20. $3i(-3 + 4i)$
21. $(3 - 4i)(2 + i)$
22. $(5 + 3i)(2 - i)$
23. $(-6 + i)(-6 - i)$
24. $(-3 + i)(3 + i)$
25. $\frac{10}{3 - 4i}$
26. $\frac{13}{5 - 12i}$
27. $\frac{2 + i}{i}$
28. $\frac{2 - i}{-2i}$
29. $\frac{6 - i}{1 + i}$
30. $\frac{2 + 3i}{1 - i}$
31. $\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2$
32. $\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)^2$
33. $(1 + i)^2$
34. $(1 - i)^2$
35. i^{23}
36. i^{14}
37. i^{-15}
38. i^{-23}
39. $i^6 - 5$
40. $4 + i^3$
41. $6i^3 - 4i^5$
42. $4i^3 - 2i^2 + 1$
43. $(1 + i)^3$
44. $(3i)^4 + 1$
45. $i^7(1 + i^2)$
46. $2i^4(1 + i^2)$
47. $i^6 + i^4 + i^2 + 1$
48. $i^7 + i^5 + i^3 + i$

In Problems 49–58, perform the indicated operations and express your answer in the form a + bi.

49. $\sqrt{-4}$
50. $\sqrt{-9}$
51. $\sqrt{-25}$
52. $\sqrt{-64}$
53. $\sqrt{-12}$
54. $\sqrt{-18}$
55. $\sqrt{-200}$
56. $\sqrt{-45}$
57. $\sqrt{(3 + 4i)(4i - 3)}$
58. $\sqrt{(4 + 3i)(3i - 4)}$

In Problems 59–78, solve each equation in the complex number system. Check your results using a graphing utility.

59. $x^2 + 4 = 0$
60. $x^2 - 4 = 0$
61. $x^2 - 16 = 0$
62. $x^2 + 25 = 0$
63. $x^2 - 6x + 13 = 0$
64. $x^2 + 4x + 8 = 0$
65. $x^2 - 6x + 10 = 0$
66. $x^2 - 2x + 5 = 0$
67. $8x^2 - 4x + 1 = 0$
68. $10x^2 + 6x + 1 = 0$
69. $5x^2 + 1 = 2x$
70. $13x^2 + 1 = 6x$
71. $x^2 + x + 1 = 0$
72. $x^2 - x + 1 = 0$
73. $x^3 - 8 = 0$
74. $x^3 + 27 = 0$
75. $x^4 = 16$
76. $x^4 = 1$
77. $x^4 + 13x^2 + 36 = 0$
78. $x^4 + 3x^2 - 4 = 0$

In Problems 79–84, without solving, determine the character of the solutions of each equation in the complex number system. Verify your answer using a graphing utility.

79. $3x^2 - 3x + 4 = 0$
80. $2x^2 - 4x + 1 = 0$
81. $2x^2 + 3x = 4$

82. $x^2 + 6 = 2x$

83. $9x^2 - 12x + 4 = 0$

84. $4x^2 + 12x + 9 = 0$

85. $2 + 3i$ is a solution of a quadratic equation with real coefficients. Find the other solution.

86. $4 - i$ is a solution of a quadratic equation with real coefficients. Find the other solution.

In Problems 87–90, $z = 3 - 4i$ and $w = 8 + 3i$. Write each expression in the standard form $a + bi$.

87. $z + \overline{z}$

88. $w - \overline{w}$

89. $z\overline{z}$

90. $\overline{z - w}$

Applications and Extensions

91. Electrical Circuits The impedance Z, in ohms, of a circuit element is defined as the ratio of the phasor voltage V, in volts, across the element to the phasor current I, in amperes, through the element. That is, $Z = \dfrac{V}{I}$. If the voltage across a circuit element is $18 + i$ volts and the current through the element is $3 - 4i$ amperes, determine the impedance.

92. Parallel Circuits In an ac circuit with two parallel pathways, the total impedance Z, in ohms, satisfies the formula $\dfrac{1}{Z} = \dfrac{1}{Z_1} + \dfrac{1}{Z_2}$, where Z_1 is the impedance of the first pathway and Z_2 is the impedance of the second pathway. Determine the total impedance if the impedances of the two pathways are $Z_1 = 2 + i$ ohms and $Z_2 = 4 - 3i$ ohms.

93. Use $z = a + bi$ to show that $z + \overline{z} = 2a$ and $z - \overline{z} = 2bi$.

94. Use $z = a + bi$ to show that $\overline{\overline{z}} = z$.

95. Use $z = a + bi$ and $w = c + di$ to show that $\overline{z + w} = \overline{z} + \overline{w}$.

96. Use $z = a + bi$ and $w = c + di$ to show that $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$.

Explaining Concepts: Discussion and Writing

97. Explain to a friend how you would add two complex numbers and how you would multiply two complex numbers. Explain any differences between the two explanations.

98. Write a brief paragraph that compares the method used to rationalize denominators and the method used to write the quotient of two complex numbers in standard form.

99. Use an Internet search engine to investigate the origins of complex numbers. Write a paragraph describing what you find, and present it to the class.

100. Explain how the method of multiplying two complex numbers is related to multiplying two binomials.

101. What Went Wrong? A student multiplied $\sqrt{-9}$ and $\sqrt{-9}$ as follows:

$$\begin{aligned}\sqrt{-9} \cdot \sqrt{-9} &= \sqrt{(-9)(-9)} \\ &= \sqrt{81} \\ &= 9\end{aligned}$$

The instructor marked the problem incorrect. Why?

'Are You Prepared?' Answers

1. Integers: $\{-3, 0\}$; rational numbers: $\left\{-3, 0, \dfrac{6}{5}\right\}$

2. True

3. $3(2 - \sqrt{3})$

1.5 Radical Equations; Equations Quadratic in Form; Absolute Value Equations; Factorable Equations

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Absolute Value (Section R.2, p. 20)
- Square Roots (Section R.2, pp. 24–25)
- Factoring Polynomials (Section R.5, pp. 50–56)
- *n*th Roots; Rational Exponents (Section R.8, pp. 74–78)

Now Work the **'Are You Prepared?'** problems on page 135.

OBJECTIVES

1 Solve Radical Equations (p. 129)

2 Solve Equations Quadratic in Form (p. 131)

3 Solve Absolute Value Equations (p. 133)

4 Solve Equations by Factoring (p. 134)

1 Solve Radical Equations

When the variable in an equation occurs in a square root, cube root, and so on—that is, when it occurs in a radical—the equation is called a **radical equation**. Sometimes a suitable operation will change a radical equation to one that is linear or quadratic. A commonly used procedure is to isolate the most complicated radical on one side of the equation and then eliminate it by raising each side to a power equal to the index of the radical. Care must be taken, however, because apparent solutions that are not, in fact, solutions of the original equation may result. Recall that these are called **extraneous solutions**. In radical equations, extraneous solutions may occur when the index of the radical is even. Therefore, we need to check all answers when working with radical equations, and we check them in the *original* equation.

EXAMPLE 1 Solving a Radical Equation

Find the real solutions of the equation: $\sqrt[3]{2x-4}-2=0$

Algebraic Solution

The equation contains a radical whose index is 3. Isolate it on the left side:

$$\sqrt[3]{2x-4}-2=0$$

$\sqrt[3]{2x-4}=2$ **Add 2 to both sides.**

Because the index of the radical is 3, raise each side to the third power and solve.

$\left(\sqrt[3]{2x-4}\right)^3=2^3$ **Raise each side to the power 3.**

$2x-4=8$ **Simplify.**

$2x=12$ **Add 4 to both sides.**

$x=6$ **Divide both sides by 2.**

Graphing Solution

Figure 43 shows the graph of the equation $Y_1=\sqrt[3]{2x-4}-2$. From the graph, there is one x-intercept near 6. Using ZERO (or ROOT), find that the x-intercept is 6. The only solution is $x=6$.

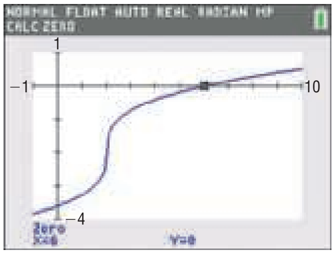


Figure 43 $Y_1=\sqrt[3]{2x-4}-2$

✓**Check:** $\sqrt[3]{2(6)-4}-2=\sqrt[3]{12-4}-2=\sqrt[3]{8}-2=2-2=0$

The solution set is $\{6\}$. ■

Now Work PROBLEM 13

EXAMPLE 2 Solving a Radical Equation

Find the real solutions of the equation: $\sqrt{x-1}=x-7$

Algebraic Solution

Square both sides since the index of a square root is 2.

$$\sqrt{x-1}=x-7$$

$\left(\sqrt{x-1}\right)^2=(x-7)^2$ **Square both sides.**

$x-1=x^2-14x+49$ **Remove parentheses.**

$x^2-15x+50=0$ **Put in standard form.**

$(x-10)(x-5)=0$ **Factor.**

$x=10$ or $x=5$ **Apply the Zero-Product Property and solve.**

Graphing Solution

Graph $Y_1=\sqrt{x-1}$ and $Y_2=x-7$. See Figure 44. From the graph, there is one point of intersection. Using INTERSECT, the point of intersection is $(10,3)$, so the solution is $x=10$.

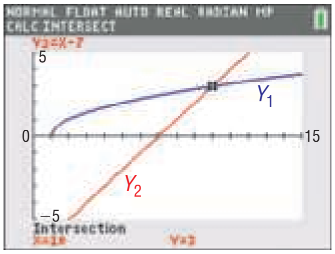


Figure 44 $Y_1=\sqrt{x-1};\ Y_2=x-7$

There is a discrepancy between the algebraic solution and graphing solution. Let's check the results of our algebraic solution.

Check: $x = 10$: $\sqrt{x-1} = \sqrt{10-1} = \sqrt{9} = 3$ and $x - 7 = 10 - 7 = 3$

$x = 5$: $\sqrt{x-1} = \sqrt{5-1} = \sqrt{4} = 2$ and $x - 7 = 5 - 7 = -2$

The apparent algebraic solution $x = 5$ is extraneous; the only solution of the equation is $x = 10$. The solution set is $\{10\}$. ■

Now Work PROBLEM 25

Sometimes, it is necessary to raise each side to a power more than once in order to solve a radical equation algebraically.

EXAMPLE 3 Solving a Radical Equation

Find the real solutions of the equation: $\sqrt{2x+3} - \sqrt{x+2} = 2$

Algebraic Solution

First, isolate the more complicated radical expression (in this case, $\sqrt{2x+3}$) on the left side:

$$\sqrt{2x+3} = \sqrt{x+2} + 2$$

Now square both sides (the index of the radical is 2).

$$\left(\sqrt{2x+3}\right)^2 = \left(\sqrt{x+2} + 2\right)^2 \quad \text{Square both sides.}$$

$$2x + 3 = \left(\sqrt{x+2}\right)^2 + 4\sqrt{x+2} + 4 \quad \text{Remove parentheses.}$$

$$2x + 3 = x + 2 + 4\sqrt{x+2} + 4 \quad \text{Simplify.}$$

$$2x + 3 = x + 6 + 4\sqrt{x+2} \quad \text{Combine like terms.}$$

Because the equation still contains a radical, isolate the remaining radical on the right side and again square both sides.

$$x - 3 = 4\sqrt{x+2} \quad \text{Isolate the radical on the right side.}$$

$$(x-3)^2 = 16(x+2) \quad \text{Square both sides.}$$

$$x^2 - 6x + 9 = 16x + 32 \quad \text{Remove parentheses.}$$

$$x^2 - 22x - 23 = 0 \quad \text{Put in standard form.}$$

$$(x-23)(x+1) = 0 \quad \text{Factor.}$$

$$x = 23 \quad \text{or} \quad x = -1 \quad \text{Apply the Zero-Product Property and solve.}$$

Graphing Solution

Graph $Y_1 = \sqrt{2x+3} - \sqrt{x+2}$ and $Y_2 = 2$. See Figure 45. From the graph there is one point of intersection. Using INTERSECT, the point of intersection is $(23, 2)$, so the solution is $x = 23$.

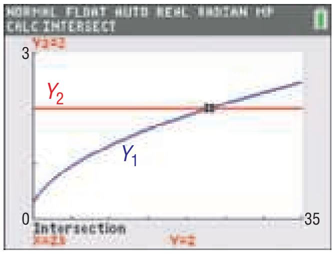


Figure 45 $Y_1 = \sqrt{2x+3} - \sqrt{x+2}$; $Y_2 = 2$

Check: $x = 23$: $\sqrt{2(23)+3} - \sqrt{23+2} = \sqrt{49} - \sqrt{25} = 7 - 5 = 2$

$x = -1$: $\sqrt{2(-1)+3} - \sqrt{-1+2} = \sqrt{1} - \sqrt{1} = 1 - 1 = 0$

The apparent solution $x = -1$ is extraneous; the only solution is $x = 23$. The solution set of the equation is $\{23\}$. ■

Now Work PROBLEM 35

2 Solve Equations Quadratic in Form

The equation $x^4 + x^2 - 12 = 0$ is not quadratic in x, but it is quadratic in x^2. That is, if we let $u = x^2$, we get $u^2 + u - 12 = 0$, a quadratic equation. This equation can be solved for u and, in turn, by using $u = x^2$, we can find the solutions x of the original equation.

In general, if an appropriate substitution u transforms an equation into one of the form

$$au^2 + bu + c = 0, \qquad a \neq 0$$

then the original equation is called an **equation of the quadratic type** or an **equation quadratic in form**.

The difficulty of solving such an equation lies in the determination that the equation is, in fact, quadratic in form. After you are told an equation is quadratic in form, it is easy enough to see it, but some practice is needed to enable you to recognize them on your own.

EXAMPLE 4 Solving an Equation That Is Quadratic in Form

Find the real solutions of the equation: $(x + 2)^2 + 11(x + 2) - 12 = 0$

Solution For this equation, let $u = x + 2$. Then $u^2 = (x + 2)^2$, and the original equation,

$$(x + 2)^2 + 11(x + 2) - 12 = 0$$

becomes

$$\begin{aligned} u^2 + 11u - 12 &= 0 && \text{Let } u = (x + 2). \\ (u + 12)(u - 1) &= 0 && \text{Factor.} \\ u = -12 \quad \text{or} \quad u &= 1 && \text{Solve.} \end{aligned}$$

WARNING Do not stop after finding values for u. Remember to finish solving for the original variable. ■

But we want to solve for x. Because $u = x + 2$, we have

$$\begin{aligned} x + 2 = -12 \quad &\text{or} \quad x + 2 = 1 \\ x = -14 \quad & \quad\quad\; x = -1 \end{aligned}$$

✓**Check:** $x = -14$: $(-14 + 2)^2 + 11(-14 + 2) - 12$
$= (-12)^2 + 11(-12) - 12 = 144 - 132 - 12 = 0$
$x = -1$: $(-1 + 2)^2 + 11(-1 + 2) - 12 = 1 + 11 - 12 = 0$

The original equation has the solution set $\{-14, -1\}$. ■

✓**Check:** Verify the solution of Example 4 using a graphing utility.

EXAMPLE 5 Solving an Equation That Is Quadratic in Form

Find the real solutions of the equation: $x + 2\sqrt{x} - 3 = 0$

Solution For the equation $x + 2\sqrt{x} - 3 = 0$, let $u = \sqrt{x}$. Then $u^2 = x$, and the original equation,

$$x + 2\sqrt{x} - 3 = 0$$

becomes

$$\begin{aligned} u^2 + 2u - 3 &= 0 && \text{Let } u = \sqrt{x}. \text{ Then } u^2 = x. \\ (u + 3)(u - 1) &= 0 && \text{Factor.} \\ u = -3 \quad \text{or} \quad u &= 1 && \text{Solve.} \end{aligned}$$

Since $u = \sqrt{x}$, we have $\sqrt{x} = -3$ or $\sqrt{x} = 1$. The first of these, $\sqrt{x} = -3$, has no real solution, since the square root of a real number is never negative. The second, $\sqrt{x} = 1$, has the solution $x = 1$.

✓**Check:** $1 + 2\sqrt{1} - 3 = 1 + 2 - 3 = 0$

The solution set of the original equation is $\{1\}$. ■

✓**Check:** Verify the solution to Example 5 using a graphing utility.

Another method for solving Example 5 would be to treat it as a radical equation. Solve it this way for practice.

The idea should now be clear. If an equation contains an expression and that same expression squared, make a substitution for the expression. You may get a quadratic equation.

Now Work PROBLEM 55

3 Solve Absolute Value Equations

Recall that on the real number line, the absolute value of a equals the distance from the origin to the point whose coordinate is a. For example, there are two points whose distance from the origin is 5 units, -5 and 5. Thus the equation $|x| = 5$ will have the solution set $\{-5, 5\}$.

Another way to obtain this result is to use the algebraic definition of absolute value, $|a| = a$ if $a \geq 0$, $|a| = -a$ if $a < 0$. The equation $|u| = a$ leads to two equations, depending on whether u is nonnegative (greater than or equal to zero) or negative.

$$|u| = a$$

If $u < 0$

$$|u| = a$$

$$-u = a \quad |u| = -u \text{ when } u < 0$$

$$u = -a \quad \text{Multiply both sides by } -1.$$

If $u \geq 0$

$$|u| = a$$

$$u = a \quad |u| = u \text{ when } u \geq 0$$

So we have the following result.

THEOREM

If a is a positive real number and if u is any algebraic expression, then

$$\boxed{|u| = a \quad \text{is equivalent to} \quad u = a \quad \text{or} \quad u = -a \qquad \textbf{(1)}}$$

EXAMPLE 6 **Solving an Equation Involving Absolute Value**

Solve the equation: $|2x - 3| + 2 = 7$

Algebraic Solution

$$|2x - 3| + 2 = 7$$

$$|2x - 3| = 5 \quad \text{Subtract 2 from each side.}$$

$$2x - 3 = 5 \quad \text{or} \quad 2x - 3 = -5 \quad \text{Apply (1).}$$

$$2x = 8 \quad \text{or} \quad 2x = -2 \quad \text{Add 3 to both sides.}$$

$$x = 4 \quad \text{or} \quad x = -1 \quad \text{Divide both sides by 2.}$$

The solution set is $\{-1, 4\}$. ■

Graphing Solution

For this equation, graph $Y_1 = |2x - 3| + 2$ and $Y_2 = 7$ on the same screen and find their point(s) of intersection, if any. See Figure 46. Using the INTERSECT command (twice), find the points of intersection to be $(-1, 7)$ and $(4, 7)$. The solution set is $\{-1, 4\}$.

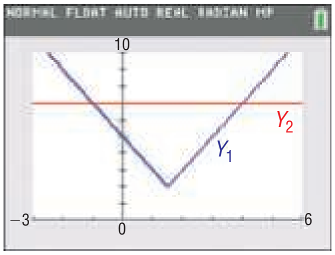


Figure 46 $Y_1 = |2x - 3| + 2$; $Y_2 = 7$ ■

Now Work PROBLEM 71

4 Solve Equations by Factoring

We have already solved certain quadratic equations using factoring. Let's look at examples of other kinds of equations that can be solved by factoring.

EXAMPLE 7 Solving Equations by Factoring

Solve the equation: $x^4 = 4x^2$

Algebraic Solution

Begin by collecting all terms on one side. This results in 0 on one side and an expression to be factored on the other.

$$x^4 = 4x^2$$
$$x^4 - 4x^2 = 0$$
$$x^2(x^2 - 4) = 0 \quad \textbf{Factor.}$$
$$x^2 = 0 \quad \text{or} \quad x^2 - 4 = 0 \quad \textbf{Apply the Zero-Product Property.}$$
$$x^2 = 4$$
$$x = 0 \quad \text{or} \quad x = -2 \quad \text{or} \quad x = 2 \quad \textbf{Use the Square Root Method.}$$

Graphing Solution

Graph $Y_1 = x^4$ and $Y_2 = 4x^2$ on the same screen and find their point(s) of intersection, if any. See Figure 47. Using the INTERSECT command (three times), find the points of intersection to be $(-2, 16)$, $(0, 0)$, and $(2, 16)$. The solutions are $x = -2$, $x = 0$, and $x = 2$.

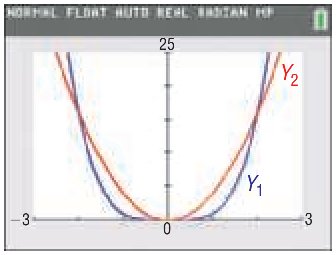


Figure 47 $Y_1 = x^4$; $Y_2 = 4x^2$

Check:
$x = -2$: $(-2)^4 = 16$ and $4(-2)^2 = 16$ **−2 is a solution.**
$x = 0$: $0^4 = 0$ and $4 \cdot 0^2 = 0$ **0 is a solution.**
$x = 2$: $2^4 = 16$ and $4 \cdot 2^2 = 16$ **2 is a solution.**

The solution set is $\{-2, 0, 2\}$. ■

EXAMPLE 8 Solving Equations by Factoring

Solve the equation: $x^3 - x^2 - 4x + 4 = 0$

Algebraic Solution

Do you recall the method of factoring by grouping? (If not, review pp. 54–55.) Group the terms of $x^3 - x^2 - 4x + 4 = 0$ as follows:

$$(x^3 - x^2) - (4x - 4) = 0$$

Factor out x^2 from the first grouping and 4 from the second.

$$x^2(x - 1) - 4(x - 1) = 0$$

This reveals the common factor $(x - 1)$, so we have

$$(x^2 - 4)(x - 1) = 0$$
$$(x - 2)(x + 2)(x - 1) = 0 \quad \textbf{Factor again.}$$
$$x - 2 = 0 \quad \text{or} \quad x + 2 = 0 \quad \text{or} \quad x - 1 = 0 \quad \textbf{Set each factor equal to 0.}$$
$$x = 2 \qquad x = -2 \qquad x = 1 \quad \textbf{Solve.}$$

Graphing Solution

Graph $Y_1 = x^3 - x^2 - 4x + 4$. See Figure 48. Using ZERO (three times), the values of x for which $y = 0$ are -2, 1, and 2.

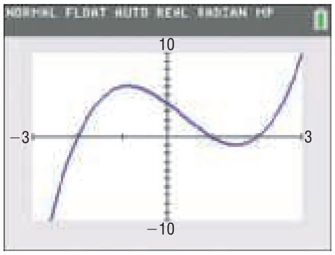


Figure 48 $Y_1 = x^3 - x^2 - 4x + 4$

Check:
$x = -2$: $(-2)^3 - (-2)^2 - 4(-2) + 4 = -8 - 4 + 8 + 4 = 0$ **−2 is a solution.**
$x = 1$: $1^3 - 1^2 - 4(1) + 4 = 1 - 1 - 4 + 4 = 0$ **1 is a solution.**
$x = 2$: $2^3 - 2^2 - 4(2) + 4 = 8 - 4 - 8 + 4 = 0$ **2 is a solution.**

The solution set is $\{-2, 1, 2\}$. ■

Now Work PROBLEM 93

1.5 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. ***True or False*** The principal square root of any nonnegative real number is always nonnegative. (p. 24)

2. $(\sqrt[3]{x})^3 =$ _____. (p. 75)

3. Factor $2x^2 - 7x - 4$. (pp. 55–56)

4. Factor $x^3 + 4x^2 - 9x - 36$. (pp. 54–55)

5. Use a real number line to describe why $|-4| = 4$. (p. 20)

Concepts and Vocabulary

6. ***True or False*** Factoring can be used to solve only quadratic equations or equations that are quadratic in form.

7. If u is an expression involving x, the equation $au^2 + bu + c = 0$, $a \neq 0$, is called an equation ________________.

8. ***True or False*** Radical equations sometimes have no real solution.

9. An apparent solution that does not satisfy the original equation is called a(n) ________ solution.
(a) extraneous (b) imaginary
(c) radical (d) conditional

10. Solving which equation is likely to require squaring each side more than once?
(a) $\sqrt{x+2} = \sqrt{3x-5}$ (b) $x^4 - 3x^2 = 10$
(c) $\sqrt{x+1} + \sqrt{x-4} = 8$ (d) $\sqrt{3x+1} = 5$

Skill Building

In Problems 11–44, find the real solutions of each equation. Verify your results using a graphing utility.

11. $\sqrt{y+3} = 5$
12. $\sqrt{t-3} = 7$
13. $\sqrt{2t-1} = 1$
14. $\sqrt{3t+4} = 2$
15. $\sqrt{3t+4} = -6$
16. $\sqrt{5t+3} = -2$
17. $\sqrt[3]{1-2x} - 3 = 0$
18. $\sqrt[3]{1-2x} - 1 = 0$
19. $\sqrt[4]{5x-4} = 2$
20. $\sqrt[5]{2x-3} = -1$
21. $\sqrt[5]{x^2+2x} = -1$
22. $\sqrt[4]{x^2+16} = \sqrt{5}$
23. $x = 8\sqrt{x}$
24. $x = 3\sqrt{x}$
25. $\sqrt{15-2x} = x$
26. $\sqrt{12-x} = x$
27. $\sqrt{3(x+10)} - 4 = x$
28. $\sqrt{1-x} - 3 = x + 2$
29. $\sqrt{x^2-x-4} = x + 2$
30. $\sqrt{x^2-x-8} = x + 5$
31. $3 + \sqrt{3x+1} = x$
32. $2 + \sqrt{12-2x} = x$
33. $\sqrt{2x+3} - \sqrt{x+1} = 1$
34. $\sqrt{3x+7} + \sqrt{x+2} = 1$
35. $\sqrt{3x+1} - \sqrt{x-1} = 2$
36. $\sqrt{3x-5} - \sqrt{x+7} = 2$
37. $\sqrt{3-2\sqrt{x}} = \sqrt{x}$
38. $\sqrt{10+3\sqrt{x}} = \sqrt{x}$
39. $(3x+1)^{1/2} = 4$
40. $(3x-5)^{1/2} = 2$
41. $(5x-2)^{1/3} = 2$
42. $(2x+1)^{1/3} = -1$
43. $(x^2+9)^{1/2} = 5$
44. $(x^2-16)^{1/2} = 9$

In Problems 45–70, find the real solutions of each equation. Verify your results using a graphing utility.

45. $t^4 - 16 = 0$
46. $y^4 - 4 = 0$
47. $x^4 - 5x^2 + 4 = 0$
48. $x^4 - 10x^2 + 24 = 0$
49. $3x^4 - 2x^2 - 1 = 0$
50. $2x^4 - 5x^2 - 12 = 0$
51. $x^6 + 7x^3 - 8 = 0$
52. $x^6 - 7x^3 - 8 = 0$
53. $(x+2)^2 + 7(x+2) + 12 = 0$
54. $(2x+5)^2 - (2x+5) - 6 = 0$
55. $2(s+1)^2 - 5(s+1) = 3$
56. $3(1-y)^2 + 5(1-y) + 2 = 0$
57. $x - 4\sqrt{x} = 0$
58. $x - 8\sqrt{x} = 0$
59. $x + \sqrt{x} = 20$
60. $x + \sqrt{x} = 6$
61. $t^{1/2} - 2t^{1/4} + 1 = 0$
62. $z^{1/2} - 4z^{1/4} + 4 = 0$
63. $4x^{1/2} - 9x^{1/4} + 4 = 0$
64. $x^{1/2} - 3x^{1/4} + 2 = 0$
65. $\dfrac{1}{(x+1)^2} = \dfrac{1}{x+1} + 2$
66. $\dfrac{1}{(x-1)^2} + \dfrac{1}{x-1} = 12$
67. $3x^{-2} - 7x^{-1} - 6 = 0$
68. $2x^{-2} - 3x^{-1} - 4 = 0$
69. $2x^{2/3} - 5x^{1/3} - 3 = 0$
70. $3x^{4/3} + 5x^{2/3} - 2 = 0$

In Problems 71–88, solve each equation. Verify your results using a graphing utility.

71. $|2x+3| = 5$
72. $|3x-1| = 2$
73. $|1-4t| + 8 = 13$

74. $|1 - 2z| + 6 = 9$

75. $|-2x| = 8$

76. $|-x| = 1$

77. $4 - |2x| = 3$

78. $5 - \left|\frac{1}{2}x\right| = 3$

79. $\frac{2}{3}|x| = 9$

80. $\frac{3}{4}|x| = 9$

81. $\left|\frac{x}{3} + \frac{2}{5}\right| = 2$

82. $\left|\frac{x}{2} - \frac{1}{3}\right| = 1$

83. $|u - 2| = -\frac{1}{2}$

84. $|2 - v| = -1$

85. $|x^2 - 9| = 0$

86. $|x^2 - 16| = 0$

87. $|x^2 - 2x| = 3$

88. $|x^2 + x| = 12$

In Problems 89–98, find the real solutions of each equation by factoring. Verify your results using a graphing utility.

89. $x^3 - 9x = 0$

90. $x^4 - 81x^2 = 0$

91. $x^3 + x^2 - 20x = 0$

92. $x^3 + 6x^2 - 7x = 0$

93. $x^3 + x^2 - x - 1 = 0$

94. $x^3 + 4x^2 - x - 4 = 0$

95. $x^3 - 3x^2 - 4x + 12 = 0$

96. $x^3 - 3x^2 - x + 3 = 0$

97. $2x^3 + 4 = x^2 + 8x$

98. $3x^3 + 4x^2 = 27x + 36$

In Problems 99–102, find the real solutions of each equation. Use a calculator to express any solutions rounded to two decimal places.

99. $x - 4x^{1/2} + 2 = 0$

100. $x^{2/3} + 4x^{1/3} + 2 = 0$

101. $x^4 + \sqrt{3}x^2 - 3 = 0$

102. $x^4 + \sqrt{2}x^2 - 2 = 0$

Mixed Practice

In Problems 103–122, find the real solutions of each equation. Verify your results using a graphing utility.

103. $3x^2 + 7x - 20 = 0$

104. $2x^2 - 13x + 21 = 0$

105. $5a^3 - 45a = -2a^2 + 18$

106. $3z^3 - 12z = -5z^2 + 20$

107. $-3|5x - 2| + 9 = 0$

108. $\frac{1}{4}|2x - 3| = \frac{3}{2}$

109. $4(w - 3) = w + 3$

110. $6(k + 3) - 2k = 12$

111. $\left(\frac{v}{v + 1}\right)^2 + \frac{2v}{v + 1} = 8$

112. $\left(\frac{y}{y - 1}\right)^2 = \frac{6y}{y - 1} + 7$

113. $|-3x + 2| = x + 10$

114. $|4x - 3| = x + 2$

115. $\sqrt{2x + 5} - x = 1$

116. $\sqrt{3x + 1} - 2x = -6$

117. $3m^2 + 6m = -1$

118. $4y^2 - 8y = 3$

119. $|x^2 + x - 1| = 1$

120. $|x^2 + 3x - 2| = 2$

121. $\sqrt[4]{5x^2 - 6} = x$

122. $\sqrt[4]{4 - 3x^2} = x$

In Problems 123–126, find all complex solutions of each equation.

123. $t^4 - 16 = 0$

124. $y^4 - 81 = 0$

125. $x^6 - 9x^3 + 8 = 0$

126. $z^6 + 28z^3 + 27 = 0$

Applications and Extensions

127. If $k = \frac{x + 3}{x - 3}$ and $k^2 - k = 12$, find x.

128. If $k = \frac{x + 3}{x - 4}$ and $k^2 - 3k = 28$, find x.

129. Find all points having an x-coordinate of 2 whose distance from the point $(-2, -1)$ is 5.

130. Find all points having a y-coordinate of -3 whose distance from the point $(1, 2)$ is 13.

131. Find all points on the x-axis that are 5 units from the point $(4, -3)$.

132. Find all points on the y-axis that are 5 units from the point $(4, 4)$.

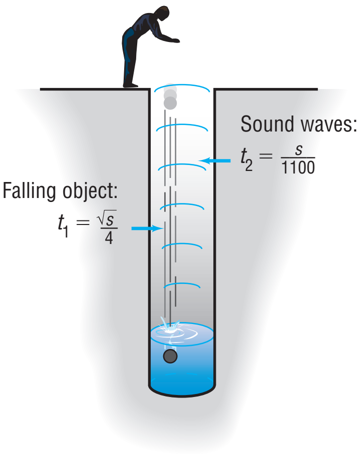
133. Solve: $|8 - 3x| = |2x - 7|$

134. Solve: $|5x + 3| = |12 - 4x|$

135. Physics: Using Sound to Measure Distance The distance to the surface of the water in a well can sometimes be found by dropping an object into the well and measuring the time elapsed until a sound is heard. If t_1 is the time (measured in seconds) that it takes for the object to strike the water, then t_1 will obey the equation $s = 16t_1^2$, where s is the distance (measured in feet). It follows that $t_1 = \frac{\sqrt{s}}{4}$. Suppose that t_2 is the time that it takes for the sound of the impact to reach your ears. Because sound travels at a speed of approximately 1100 feet per second, the time t_2 for the sound to travel the distance s will be $t_2 = \frac{s}{1100}$. See the illustration. Now $t_1 + t_2$ is the total time that elapses from the moment that the object is dropped to the moment that a sound is heard. We have the equation

$$\text{Total time elapsed} = \frac{\sqrt{s}}{4} + \frac{s}{1100}$$

Find the distance to the water's surface if the total time elapsed from dropping a rock to hearing it hit water is 4 seconds.



136. Crushing Load A civil engineer relates the thickness T, in inches, and height H, in feet, of a square wooden pillar to its crushing load L, in tons, using the model $T = \sqrt[4]{\frac{LH^2}{25}}$. If a square wooden pillar is 4 inches thick and 10 feet high, what is its crushing load?

137. Foucault's Pendulum The period of a pendulum is the time it takes the pendulum to make one full swing back and forth. The period T, in seconds, is given by the formula $T = 2\pi\sqrt{\frac{l}{32}}$, where l is the length, in feet, of the pendulum. In 1851, Jean-Bernard-Léon Foucault demonstrated the axial rotation of Earth using a large pendulum that he hung in the Panthéon in Paris. The period of Foucault's pendulum was approximately 16.5 seconds. What was its length?

Explaining Concepts: Discussion and Writing

138. Make up a radical equation that has no solution.

139. Make up a radical equation that has an extraneous solution.

140. Discuss the step in the solving process for radical equations that leads to the possibility of extraneous solutions. Why is there no such possibility for linear and quadratic equations?

141. The equation $|x| = -2$ has no real solution. Why?

142. What Went Wrong? On an exam, Jane solved the equation $\sqrt{2x+3} - x = 0$ and wrote that the solution set was $\{-1, 3\}$. Jane received 3 out of 5 points for the problem. Jane asks you why she received 3 out of 5 points. Provide an explanation.

'Are You Prepared?' Answers

1. True

2. x

3. $(2x + 1)(x - 4)$

4. $(x - 3)(x + 3)(x + 4)$

5. The distance from the origin to -4 on a real number line is 4 units.

1.6 Problem Solving: Interest, Mixture, Uniform Motion, Constant Rate Jobs

OBJECTIVES

1 Translate Verbal Descriptions into Mathematical Expressions (p. 138)
2 Solve Interest Problems (p. 139)
3 Solve Mixture Problems (p. 140)
4 Solve Uniform Motion Problems (p. 141)
5 Solve Constant Rate Job Problems (p. 143)

Applied (word) problems do not come in the form "Solve the equation. . . ." Instead, they supply information using words, a verbal description of the real problem. So, to solve applied problems, we must be able to translate the verbal description into the language of mathematics. This can be done by using variables to represent unknown quantities and then finding relationships (such as equations) that involve these variables. The process of doing all this is called **mathematical modeling**.

Any solution to the mathematical problem must be checked against the mathematical problem, the verbal description, and the real problem. See Figure 49 on page 138 for an illustration of the **modeling process**.

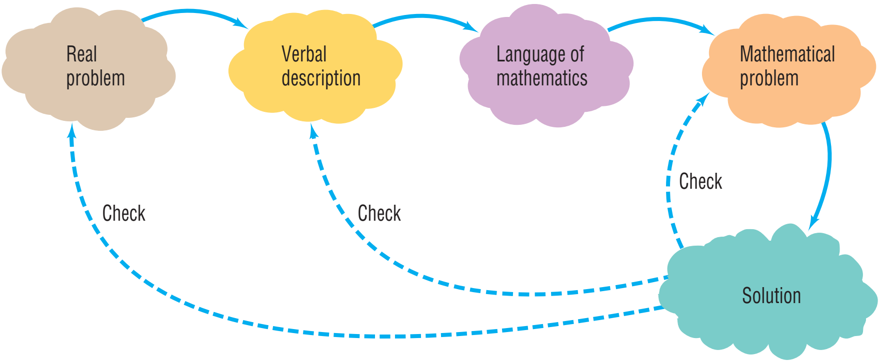


Figure 49 Modeling Process

1 Translate Verbal Descriptions into Mathematical Expressions

EXAMPLE 1

Translating Verbal Descriptions into Mathematical Expressions

(a) For uniform motion, the average speed of an object equals the distance traveled divided by the time required.

Translation: If r is the speed, d the distance, and t the time, then $r = \frac{d}{t}$.

(b) Let x denote a number.

The number 5 times as large as x is $5x$.

The number 3 less than x is $x - 3$.

The number that exceeds x by 4 is $x + 4$.

The number that, when added to x, gives 5 is $5 - x$. ■

Now Work PROBLEM 9

Always check the units used to measure the variables of an applied problem. In Example 1(a), if r is measured in miles per hour, then the distance d must be expressed in miles, and the time t must be expressed in hours. It is a good practice to check units to be sure that they are consistent and make sense.

The steps for solving applied problems, given earlier, are repeated next.

Steps for Solving Applied Problems

STEP 1: Read the problem carefully, perhaps two or three times. Pay particular attention to the question being asked in order to identify what you are looking for. Identify any relevant formulas you may need ($d = rt$, $A = \pi r^2$, etc.) If you can, determine realistic possibilities for the answer.

STEP 2: Assign a letter (variable) to represent what you are looking for, and, if necessary, express any remaining unknown quantities in terms of this variable.

STEP 3: Make a list of all the known facts, and translate them into mathematical expressions. These may take the form of an equation or an inequality involving the variable (the model). If possible, draw an appropriately labeled diagram to assist you. Sometimes a table or chart helps.

STEP 4: Solve the equation for the variable, and then answer the question using a complete sentence.

STEP 5: Check the answer with the facts in the problem. If it agrees, congratulations! If it does not agree, try again.

2 Solve Interest Problems

Interest is money paid for the use of money. The total amount borrowed (whether by an individual from a bank in the form of a loan or by a bank from an individual in the form of a savings account) is called the **principal**. The **rate of interest**, expressed as a percent, is the amount charged for the use of the principal for a given period of time, usually on a yearly (that is, per annum) basis.

Simple Interest Formula

If a principal of P dollars is borrowed for a period of t years at a per annum interest rate r, expressed as a decimal, the interest I charged is

$$\boxed{I = Prt \qquad \textbf{(1)}}$$

Interest charged according to formula (1) is called **simple interest**. When using formula (1), be sure to express r as a decimal. For example, if the rate of interest is 4%, then $r = 0.04$.

EXAMPLE 2 Finance: Computing Interest on a Loan

Suppose that Juanita borrows \$500 for 6 months at the simple interest rate of 9% per annum. What is the interest that Juanita will be charged on the loan? How much does Juanita owe after 6 months?

Solution The rate of interest is given per annum, so the actual time that the money is borrowed must be expressed in years. The interest charged would be the principal, \$500, times the rate of interest $(9\% = 0.09)$, times the time in years, $\frac{1}{2}$ (6 months $= \frac{1}{2}$ year):

$$\text{Interest charged} = I = Prt = (500)(0.09)\left(\frac{1}{2}\right) = \$22.50$$

After 6 months, Juanita will owe what she borrowed plus the interest:

$$\$500 + \$22.50 = \$522.50$$

■

EXAMPLE 3 Financial Planning

Candy has \$70,000 to invest and wants an annual return of \$2800, which requires an overall rate of return of 4%. She can invest in a safe, government-insured certificate of deposit, but it pays only 2%. To obtain 4%, she agrees to invest some of her money in noninsured corporate bonds paying 7%. How much should be placed in each investment to achieve her goal?

Solution **STEP 1:** The question is asking for two dollar amounts: the principal to invest in the corporate bonds and the principal to invest in the certificate of deposit.

STEP 2: Let b represent the amount (in dollars) to be invested in the bonds. Then $70{,}000 - b$ is the amount that will be invested in the certificate. (Do you see why?)

Note: We could have also let c represent the amount invested in the certificate and $70{,}000 - c$ the amount invested in bonds. ■

STEP 3: Set up a table:

	Principal (\$)	Rate	Time (yr)	Interest (\$)
Bonds	b	7% = 0.07	1	$0.07b$
Certificate	$70{,}000 - b$	2% = 0.02	1	$0.02(70{,}000 - b)$
Total	70,000	4% = 0.04	1	$0.04(70{,}000) = 2800$

Since the combined interest from the investments is equal to the total interest, we have

$$\begin{aligned}\text{Bond interest} + \text{Certificate interest} &= \text{Total interest}\\ 0.07b + 0.02(70{,}000 - b) &= 2800\end{aligned}$$

(Note that the units are consistent: the unit is dollars on each side.)

Step 4:
$$\begin{aligned}0.07b + 1400 - 0.02b &= 2800\\ 0.05b &= 1400\\ b &= 28{,}000\end{aligned}$$

Candy should place \$28,000 in the bonds and \$70,000 − \$28,000 = \$42,000 in the certificate.

Step 5: The interest on the bonds after 1 year is 0.07(\$28,000) = \$1960; the interest on the certificate after 1 year is 0.02(\$42,000) = \$840. The total annual interest is \$2800, the required amount. ■

Now Work PROBLEM 19

3 Solve Mixture Problems

Oil refineries sometimes produce gasoline that is a blend of two or more types of fuel; bakeries occasionally blend two or more types of flour for their bread. These problems are referred to as **mixture problems** because they combine two or more quantities to form a mixture.

EXAMPLE 4 Blending Coffees

The manager of a Starbucks store decides to experiment with a new blend of coffee. She will mix some B grade Colombian coffee that sells for \$5 per pound with some A grade Arabica coffee that sells for \$10 per pound to get 100 pounds of the new blend. The selling price of the new blend is to be \$7 per pound, and there is to be no difference in revenue from selling the new blend versus selling the other types. How many pounds of the B grade Colombian and A grade Arabica coffees are required?

Solution Let c represent the number of pounds of the B grade Colombian coffee. Then $100 - c$ equals the number of pounds of the A grade Arabica coffee. See Figure 50.

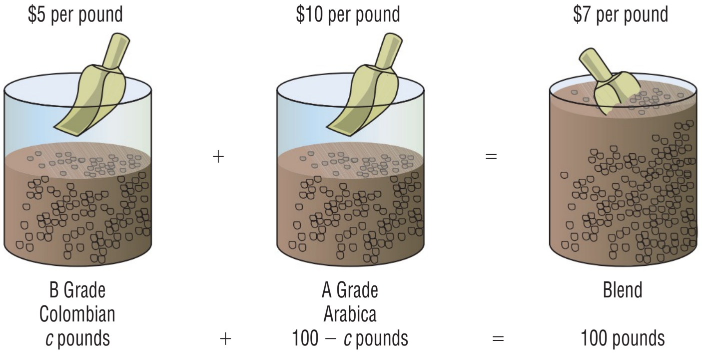


Figure 50 Mixing coffees

Since there is to be no difference in revenue between selling the A and B grades separately versus the blend, we have

$$\begin{array}{ccccc}\text{Revenue from B grade} & + & \text{Revenue from A grade} & = & \text{Revenue from blend}\\ \left\{\begin{matrix}\text{Price per pound}\\ \text{of B grade}\end{matrix}\right\}\left\{\begin{matrix}\text{Pounds of}\\ \text{B grade}\end{matrix}\right\} & + & \left\{\begin{matrix}\text{Price per pound}\\ \text{of A grade}\end{matrix}\right\}\left\{\begin{matrix}\text{Pounds of}\\ \text{A grade}\end{matrix}\right\} & = & \left\{\begin{matrix}\text{Price per pound}\\ \text{of blend}\end{matrix}\right\}\left\{\begin{matrix}\text{Pounds of}\\ \text{blend}\end{matrix}\right\}\\ \$5 \cdot c & + & \$10 \cdot (100 - c) & = & \$7 \cdot 100\end{array}$$

Now solve the equation

$$\begin{aligned} 5c + 10(100 - c) &= 700 \\ 5c + 1000 - 10c &= 700 \\ -5c &= -300 \\ c &= 60 \end{aligned}$$

The manager should blend 60 pounds of B grade Colombian coffee with $100 - 60 = 40$ pounds of A grade Arabica coffee to get the desired blend.

✓ **Check:** The 60 pounds of B grade coffee would sell for $(\$5)(60) = \$300$, and the 40 pounds of A grade coffee would sell for $(\$10)(40) = \$400$; the total revenue, $700, equals the revenue obtained from selling the blend, as desired. ■

Now Work PROBLEM 23

4 Solve Uniform Motion Problems

Objects that move at a constant speed are said to be in **uniform motion**. When the average speed of an object is known, it can be interpreted as that object's constant speed. For example, a bicyclist traveling at an average speed of 25 miles per hour can be considered to be in uniform motion with a constant speed of 25 miles per hour.

Uniform Motion Formula

If an object moves at an average speed (rate) r, the distance d covered in time t is given by the formula

$$\boxed{d = rt \qquad \textbf{(2)}}$$

That is, Distance = Rate · Time.

EXAMPLE 5

Physics: Uniform Motion

Tanya, who is a long-distance runner, runs at an average speed of 8 miles per hour (mi/h). Two hours after Tanya leaves your house, you leave in your Honda and follow the same route. If your average speed is 40 mi/h how long will it be before you catch up to Tanya? How far will each of you be from your home?

Solution Refer to Figure 51. We use t to represent the time (in hours) that it takes the Honda to catch up to Tanya. When this occurs, the total time elapsed for Tanya is $t + 2$ hours because she left 2 hours earlier.

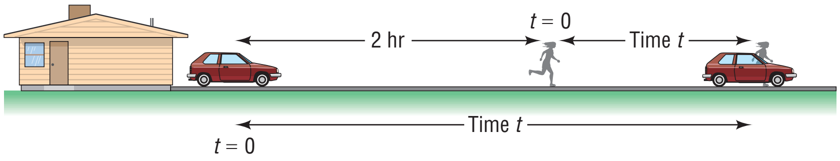


Figure 51

Set up the following table:

	Rate (mi/h)	Time (h)	Distance (mi)
Tanya	8	$t + 2$	$8(t + 2)$
Honda	40	t	$40t$

The distance traveled is the same for both, which leads to the equation

$$\begin{aligned} 8(t + 2) &= 40t \\ 8t + 16 &= 40t \\ 32t &= 16 \\ t &= \frac{1}{2} \text{ hour} \end{aligned}$$

It will take the Honda $\frac{1}{2}$ hour to catch up to Tanya. Each will have gone 20 miles.

✓**Check:** In 2.5 hours, Tanya travels a distance of $(2.5)(8) = 20$ miles. In $\frac{1}{2}$ hour, the Honda travels a distance of $\left(\frac{1}{2}\right)(40) = 20$ miles. ■

EXAMPLE 6 Physics: Uniform Motion

A motorboat heads upstream a distance of 24 miles on a river whose current is running at 3 miles per hour (mi/h). The trip up and back takes 6 hours. Assuming that the motorboat maintained a constant speed relative to the water, what was its speed?

Solution See Figure 52. Use r to represent the constant speed of the motorboat relative to the water. Then the true speed going upstream is $r - 3$ mi/h, and the true speed going downstream is $r + 3$ mi/h. Since Distance = Rate · Time, then Time = $\frac{\text{Distance}}{\text{Rate}}$. Set up a table.

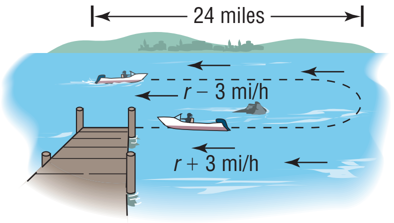


Figure 52

	Distance (mi)	Rate (mi/h)	Time (h) $= \frac{\text{Distance}}{\text{Rate}}$
Upstream	24	$r - 3$	$\frac{24}{r-3}$
Downstream	24	$r + 3$	$\frac{24}{r+3}$

The total time up and back is 6 hours, which gives the equation

$$\frac{24}{r-3} + \frac{24}{r+3} = 6$$

$$\frac{24(r+3) + 24(r-3)}{(r-3)(r+3)} = 6 \qquad \text{Add the quotients on the left.}$$

$$\frac{48r}{r^2 - 9} = 6 \qquad \text{Simplify.}$$

$$48r = 6(r^2 - 9) \qquad \text{Multiply both sides by } r^2 - 9.$$

$$6r^2 - 48r - 54 = 0 \qquad \text{Place in standard form.}$$

$$r^2 - 8r - 9 = 0 \qquad \text{Divide by 6.}$$

$$(r - 9)(r + 1) = 0 \qquad \text{Factor.}$$

$$r = 9 \quad \text{or} \quad r = -1 \qquad \text{Apply the Zero-Product Property and solve.}$$

Discard the solution $r = -1$ mi/h, and conclude that the speed of the motorboat relative to the water is 9 mi/h. ■

Now Work PROBLEM 29

5 Solve Constant Rate Job Problems

Here we look at jobs that are performed at a **constant rate**. The assumption is that if a job can be done in t units of time, then $\frac{1}{t}$ of the job is done in 1 unit of time. In other words, if a job takes 4 hours, then $\frac{1}{4}$ of the job is done in 1 hour.

EXAMPLE 7 Working Together to Do a Job

At 10 AM, Danny is asked by his father to weed the garden. From past experience, Danny knows that this will take him 4 hours, working alone. His older brother Mike, when it is his turn to do this job, requires 6 hours. Since Mike wants to go golfing with Danny and has a reservation for 1 PM, he agrees to help Danny. Assuming no gain or loss of efficiency, when will they finish if they work together? Can they make the golf date?

Solution Set up Table 4. In 1 hour, Danny does $\frac{1}{4}$ of the job, and in 1 hour, Mike does $\frac{1}{6}$ of the job. Let t be the time (in hours) that it takes them to do the job together. In 1 hour, then, $\frac{1}{t}$ of the job is completed. Reason as follows:

$$\begin{pmatrix}\text{Part done by Danny}\\ \text{in 1 hour}\end{pmatrix} + \begin{pmatrix}\text{Part done by Mike}\\ \text{in 1 hour}\end{pmatrix} = \begin{pmatrix}\text{Part done together}\\ \text{in 1 hour}\end{pmatrix}$$

Table 4

	Hours to Do Job	Part of Job Done in 1 Hour
Danny	4	$\frac{1}{4}$
Mike	6	$\frac{1}{6}$
Together	t	$\frac{1}{t}$

From Table 2,

$$\frac{1}{4} + \frac{1}{6} = \frac{1}{t} \qquad \text{The model.}$$

$$\frac{3}{12} + \frac{2}{12} = \frac{1}{t} \qquad \text{LCD = 12 on the left.}$$

$$\frac{5}{12} = \frac{1}{t} \qquad \text{Simplify.}$$

$$5t = 12 \qquad \text{Multiply both sides by } 12t.$$

$$t = \frac{12}{5} \qquad \text{Divide each side by 5.}$$

Working together, Mike and Danny can do the job in $\frac{12}{5}$ hours, or 2 hours, 24 minutes. They should make the golf date, since they will finish at 12:24 PM. ■

Now Work PROBLEM 35

1.6 Assess Your Understanding

Concepts and Vocabulary

1. The process of using variables to represent unknown quantities and then finding relationships that involve these variables is referred to as ______________ __________.

2. The money paid for the use of money is _________.

3. Objects that move at a constant speed are said to be in _________ _________.

4. ***True or False*** The amount charged for the use of principal for a given period of time is called the rate of interest.

5. ***True or False*** If an object moves at an average speed r, the distance d covered in time t is given by the formula $d = rt$.

6. Suppose that you want to mix two coffees in order to obtain 100 pounds of a blend. If x represents the number of pounds of coffee A, write an algebraic expression that represents the number of pounds of coffee B.
(a) $100 - x$ (b) $x - 100$ (c) $100x$ (d) $100 + x$

7. Which of the following is the simple interest formula?
(a) $I = \frac{rt}{P}$ (b) $I = Prt$ (c) $I = \frac{P}{rt}$ (d) $I = P + rt$

8. If it takes 5 hours to complete a job, what fraction of the job is done in 1 hour?
(a) $\frac{4}{5}$ (b) $\frac{5}{4}$ (c) $\frac{1}{5}$ (d) $\frac{1}{4}$

Applications and Extensions

In Problems 9–18, translate each sentence into a mathematical equation. Be sure to identify the meaning of all symbols.

9. Geometry The area of a circle is the product of the number π and the square of the radius.

10. Geometry The circumference of a circle is the product of the number π and twice the radius.

11. Geometry The area of a square is the square of the length of a side.

12. Geometry The perimeter of a square is four times the length of a side.

13. Physics Force equals the product of mass and acceleration.

14. Physics Pressure is force per unit area.

15. Physics Work equals force times distance.

16. Physics Kinetic energy is one-half the product of the mass and the square of the velocity.

17. Business The total variable cost of manufacturing x dishwashers is \$150 per dishwasher times the number of dishwashers manufactured.

18. Business The total revenue derived from selling x dishwashers is \$250 per dishwasher times the number of dishwashers sold.

19. Financial Planning Betsy, a recent retiree, requires \$6000 per year in extra income. She has \$50,000 to invest and can invest in B-rated bonds paying 15% per year or in a certificate of deposit (CD) paying 7% per year. How much money should Betsy invest in each to realize exactly \$6000 in interest per year?

20. Financial Planning After 2 years, Betsy (see Problem 19) finds that she will now require \$7000 per year. Assuming that the remaining information is the same, how should the money be reinvested?

21. Banking A bank loaned out \$12,000, part of it at the rate of 8% per year and the rest at the rate of 18% per year. If the interest received totaled \$1000, how much was loaned at 8%?

22. Banking Wendy, a loan officer at a bank, has \$1,000,000 to lend and is required to obtain an average return of 18% per year. If she can lend at the rate of 19% or at the rate of 16%, how much can she lend at the 16% rate and still meet her requirement?

23. Blending Teas The manager of a store that specializes in selling tea decides to experiment with a new blend. She will mix some Earl Grey tea that sells for \$5 per pound with some Orange Pekoe tea that sells for \$3 per pound to get 100 pounds of the new blend. The selling price of the new blend is to be \$4.50 per pound, and there is to be no difference in revenue between selling the new blend and selling the other types. How many pounds of the Earl Grey tea and of the Orange Pekoe tea are required?

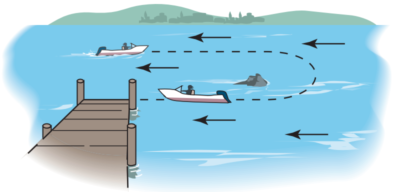
24. Business: Blending Coffee A coffee manufacturer wants to market a new blend of coffee that sells for \$3.90 per pound by mixing two coffees that sell for \$2.75 and \$5 per pound, respectively. What amounts of each coffee should be blended to obtain the desired mixture?

[**Hint:** Assume that the total weight of the desired blend is 100 pounds.]

25. Business: Mixing Nuts A nut store normally sells cashews for \$9.00 per pound and almonds for \$3.50 per pound. But at the end of the month the almonds had not sold well, so, in order to sell 60 pounds of almonds, the manager decided to mix the 60 pounds of almonds with some cashews and sell the mixture for \$7.50 per pound. How many pounds of cashews should be mixed with the almonds to ensure no change in the revenue?

26. Business: Mixing Candy A candy store sells boxes of candy containing caramels and cremes. Each box sells for \$12.50 and holds 30 pieces of candy (all pieces are the same size). If the caramels cost \$0.25 to produce and the cremes cost \$0.45 to produce, how many of each should be in a box to yield a profit of \$3?

27. Physics: Uniform Motion A motorboat can maintain a constant speed of 16 miles per hour relative to the water. The boat travels upstream to a certain point in 20 minutes; the return trip takes 15 minutes. What is the speed of the current? See the figure.



28. Physics: Uniform Motion A motorboat heads upstream on a river that has a current of 3 miles per hour. The trip upstream takes 5 hours, and the return trip takes 2.5 hours. What is the speed of the motorboat? (Assume that the boat maintains a constant speed relative to the water.)

29. Physics: Uniform Motion A motorboat maintained a constant speed of 15 miles per hour relative to the water in going 10 miles upstream and then returning. The total time for the trip was 1.5 hours. Use this information to find the speed of the current.

30. Physics: Uniform Motion Two cars enter the Florida Turnpike at Commercial Boulevard at 8:00 AM, each heading for Wildwood. One car's average speed is 10 miles per hour more than the other's. The faster car arrives at Wildwood at 11:00 AM, $\frac{1}{2}$ hour before the other car. What was the average speed of each car? How far did each travel?

31. Moving Walkways The speed of a moving walkway is typically about 2.5 feet per second. Walking on such a moving walkway, it takes Karen a total of 40 seconds to travel 50 feet with the movement of the walkway and then back again against the movement of the walkway. What is Karen's normal walking speed?

Source: *Answers.com*

32. High-Speed Walkways Toronto's Pearson International Airport has a high-speed version of a moving walkway. If Liam walks while riding this moving walkway, he can travel 280 meters in 60 seconds less time than if he stands still on the moving walkway. If Liam walks at a normal rate of 1.5 meters per second, what is the speed of the walkway?

Source: *Answers.com*

33. Tennis A regulation doubles tennis court has an area of 2808 square feet. If it is 6 feet longer than twice its width, determine the dimensions of the court.

Source: *United States Tennis Association*

34. **Laser Printers** It takes an HP LaserJet M451dw laser printer 16 minutes longer to complete an 840-page print job by itself than it takes an HP LaserJet CP4025dn to complete the same job by itself. Together the two printers can complete the job in 15 minutes. How long does it take each printer to complete the print job alone? What is the speed of each printer?

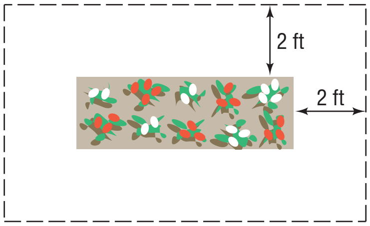
Source: *Hewlett-Packard*

35. **Working Together on a Job** Trent can deliver his newspapers in 30 minutes. It takes Lois 20 minutes to do the same route. How long would it take them to deliver the newspapers if they worked together?

36. **Working Together on a Job** Patrice, by himself, can paint four rooms in 10 hours. If he hires April to help, they can do the same job together in 6 hours. If he lets April work alone, how long will it take her to paint four rooms?

37. **Enclosing a Garden** A gardener has 46 feet of fencing to be used to enclose a rectangular garden that has a border 2 feet wide surrounding it. See the figure.

(a) If the length of the garden is to be twice its width, what will be the dimensions of the garden?
(b) What is the area of the garden?
(c) If the length and width of the garden are to be the same, what will be the dimensions of the garden?
(d) What will be the area of the square garden?

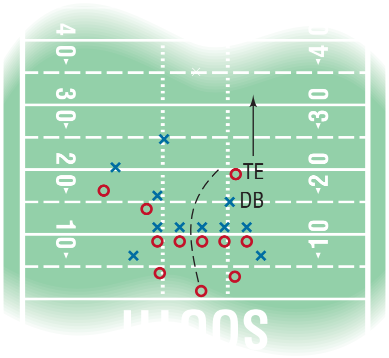


38. **Construction** A pond is enclosed by a wooden deck that is 3 feet wide. The fence surrounding the deck is 100 feet long.

(a) If the pond is square, what are its dimensions?
(b) If the pond is rectangular and the length of the pond is to be three times its width, what are its dimensions?
(c) If the pond is circular, what is its diameter?
(d) Which pond has the larger area?

39. **Football** A tight end can run the 100-yard dash in 12 seconds. A defensive back can do it in 10 seconds. The tight end catches a pass at his own 20-yard line with the defensive back at the 15-yard line. (See the figure.) If no other players are nearby, at what yard line will the defensive back catch up to the tight end?

[**Hint:** At time $t = 0$, the defensive back is 5 yards behind the tight end.]



40. **Computing Business Expense** Therese, an outside salesperson, uses her car for both business and pleasure. Last year, she traveled 30,000 miles, using 900 gallons of gasoline. Her car gets 40 miles per gallon on the highway and 25 in the city. She can deduct all highway travel, but no city travel, on her taxes. How many miles should Therese deduct as a business expense?

41. **Mixing Water and Antifreeze** How much water should be added to 1 gallon of pure antifreeze to obtain a solution that is 60% antifreeze?

42. **Mixing Water and Antifreeze** The cooling system of a certain foreign-made car has a capacity of 15 liters. If the system is filled with a mixture that is 40% antifreeze, how much of this mixture should be drained and replaced by pure antifreeze so that the system is filled with a solution that is 60% antifreeze?

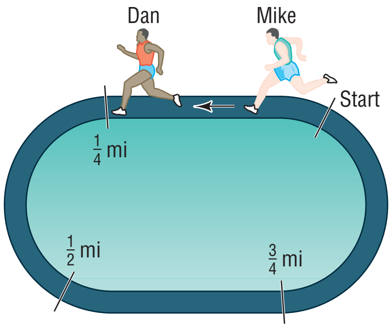
43. **Chemistry: Salt Solutions** How much water must be evaporated from 32 ounces of a 4% salt solution to make a 6% salt solution?

44. **Chemistry: Salt Solutions** How much water must be evaporated from 240 gallons of a 3% salt solution to produce a 5% salt solution?

45. **Purity of Gold** The purity of gold is measured in karats, with pure gold being 24 karats. Other purities of gold are expressed as proportional parts of pure gold. Thus, 18-karat gold is $\frac{18}{24}$, or 75% pure gold; 12-karat gold is $\frac{12}{24}$, or 50% pure gold; and so on. How much 12-karat gold should be mixed with pure gold to obtain 60 grams of 16-karat gold?

46. **Chemistry: Sugar Molecules** A sugar molecule has twice as many atoms of hydrogen as it does oxygen and one more atom of carbon than of oxygen. If a sugar molecule has a total of 45 atoms, how many are oxygen? How many are hydrogen?

47. **Running a Race** Mike can run the mile in 6 minutes, and Dan can run the mile in 9 minutes. If Mike gives Dan a head start of 1 minute, how far from the start will Mike pass Dan? How long does it take? See the figure.



48. **Range of an Airplane** An air rescue plane averages 300 miles per hour in still air. It carries enough fuel for 5 hours of flying time. If, upon takeoff, it encounters a head wind of 30 mi/h, how far can it fly and return safely? (Assume that the wind remains constant.)

49. **Emptying Oil Tankers** An oil tanker can be emptied by the main pump in 4 hours. An auxiliary pump can empty the tanker in 9 hours. If the main pump is started at 9 AM, when should the auxiliary pump be started so that the tanker is emptied by noon?

50. **Cement Mix** A 20-pound bag of Economy brand cement mix contains 25% cement and 75% sand. How much pure cement must be added to produce a cement mix that is 40% cement?

51. **Filling a Tub** A bathroom tub will fill in 15 minutes with both faucets open and the stopper in place. With both faucets closed and the stopper removed, the tub will empty in 20 minutes. How long will it take for the tub to fill if both faucets are open and the stopper is removed?

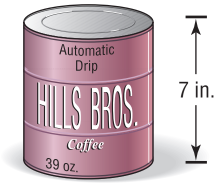
52. **Using Two Pumps** A 5-horsepower (hp) pump can empty a pool in 5 hours. A smaller, 2-hp pump empties the same pool in 8 hours. The pumps are used together to begin emptying this pool. After two hours, the 2-hp pump breaks down. How long will it take the larger pump to finish emptying the pool?

53. **A Biathlon** Suppose that you have entered an 87-mile biathlon that consists of a run and a bicycle race. During your run, your average speed is 6 miles per hour, and during your bicycle race, your average speed is 25 miles per hour. You finish the race in 5 hours. What is the distance of the run? What is the distance of the bicycle race?

54. **Cyclists** Two cyclists leave a city at the same time, one going east and the other going west. The westbound cyclist bikes 5 mph faster than the eastbound cyclist. After 6 hours they are 246 miles apart. How fast is each cyclist riding?

55. **Comparing Olympic Heroes** In the 2012 Olympics, Usain Bolt of Jamaica won the gold medal in the 100-meter race with a time of 9.69 seconds. In the 1896 Olympics, Thomas Burke of the United States won the gold medal in the 100-meter race in 12.0 seconds. If they ran in the same race, repeating their respective times, by how many meters would Bolt beat Burke?

56. **Constructing a Coffee Can** A 39-ounce can of Hills Bros.® coffee requires 188.5 square inches of aluminum. If its height is 7 inches, what is its radius? [**Hint:** The surface area S of a right cylinder is $S = 2\pi r^2 + 2\pi rh$, where r is the radius and h is the height.]



Explaining Concepts: Discussion and Writing

57. **Critical Thinking** You are the manager of a clothing store and have just purchased 100 dress shirts for $20.00 each. After 1 month of selling the shirts at the regular price, you plan to have a sale giving 40% off the original selling price. However, you still want to make a profit of $4 on each shirt at the sale price. What should you price the shirts at initially to ensure this? If, instead of 40% off at the sale, you give 50% off, by how much is your profit reduced?

58. **Critical Thinking** Make up a word problem that requires solving a linear equation as part of its solution. Exchange problems with a friend. Write a critique of your friend's problem.

59. **Critical Thinking** Without solving, explain what is wrong with the following mixture problem: How many liters of 25% ethanol should be added to 20 liters of 48% ethanol to obtain a solution of 58% ethanol? Now go through an algebraic solution. What happens?

60. **Computing Average Speed** In going from Chicago to Atlanta, a car averages 45 miles per hour, and in going from Atlanta to Miami, it averages 55 miles per hour. If Atlanta is halfway between Chicago and Miami, what is the average speed from Chicago to Miami? Discuss an intuitive solution. Write a paragraph defending your intuitive solution. Then solve the problem algebraically. Is your intuitive solution the same as the algebraic one? If not, find the flaw.

61. **Speed of a Plane** On a recent flight from Phoenix to Kansas City, a distance of 919 nautical miles, the plane arrived 20 minutes early. On leaving the aircraft, I asked the captain, "What was our tail wind?" He replied, "I don't know, but our ground speed was 550 knots." Has enough information been provided for you to find the tail wind? If possible, find the tail wind. (1 knot = 1 nautical mile per hour)

1.7 Solving Inequalities

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Inequalities (Section R.2, pp. 19–20)
- Absolute Value (Section R.2, p. 20)
- Union of Sets (Section R.1, pp. 2–3)

Now Work the **'Are You Prepared?'** problems on page 154.

OBJECTIVES
1 Use Interval Notation (p. 147)
2 Use Properties of Inequalities (p. 148)
3 Solve Linear Inequalities Algebraically and Graphically (p. 150)
4 Solve Combined Inequalities Algebraically and Graphically (p. 151)
5 Solve Absolute Value Inequalities Algebraically and Graphically (p. 152)

Suppose that a and b are two real numbers and $a < b$. The notation $a < x < b$ means that x is a number *between* a and b. So the expression $a < x < b$ is equivalent to the two inequalities $a < x$ and $x < b$. Similarly, the expression $a \le x \le b$ is equivalent to the two inequalities $a \leq x$ and $x \leq b$. The remaining two possibilities, $a \leq x < b$ and $a < x \leq b$, are defined similarly.

Although it is acceptable to write $3 \geq x \geq 2$, it is preferable to reverse the inequality symbols and write instead $2 \leq x \leq 3$ so that the values go from smaller to larger, reading from left to right.

A statement such as $2 \leq x \leq 1$ is false because there is no number x for which $2 \leq x$ and $x \leq 1$. Finally, never mix inequality symbols, as in $2 \leq x \geq 3$.

1 Use Interval Notation

Let a and b represent two real numbers with $a < b$:

In Words

The notation $[a, b]$ represents all real numbers between a and b, inclusive. The notation (a, b) represents all real numbers between a and b, not including either a or b.

A **closed interval**, denoted by $[a, b]$, consists of all real numbers x for which $a \leq x \leq b$.

An **open interval**, denoted by (a, b), consists of all real numbers x for which $a < x < b$.

The **half-open**, or **half-closed**, **intervals** are $(a, b]$, consisting of all real numbers x for which $a < x \leq b$, and $[a, b)$, consisting of all real numbers x for which $a \leq x < b$.

In each of these definitions, a is called the **left endpoint** and b the **right endpoint** of the interval.

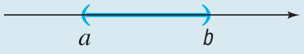
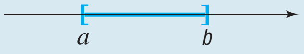
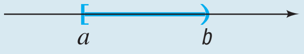
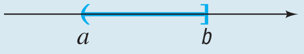
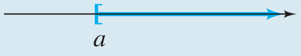
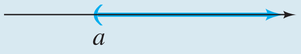
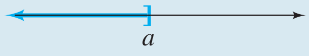
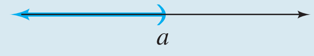
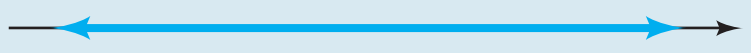
The symbol ∞ (read as "infinity") is not a real number, but notation used to indicate unboundedness in the positive direction. The symbol $-\infty$ (read as "minus infinity" or "negative infinity") also is not a real number, but notation used to indicate unboundedness in the negative direction. The symbols ∞ and $-\infty$ are used to define five other kinds of intervals:

$[a, \infty)$	consists of all real numbers x for which $x \geq a$
(a, ∞)	consists of all real numbers x for which $x > a$
$(-\infty, a]$	consists of all real numbers x for which $x \leq a$
$(-\infty, a)$	consists of all real numbers x for which $x < a$
$(-\infty, \infty)$	consists of all real numbers x

Note that ∞ and $-\infty$ are never included as endpoints since they are not real numbers.

Table 5 summarizes interval notation, corresponding inequality notation, and their graphs.

Table 5

Interval	Inequality	Graph
The open interval (a, b)	$a < x < b$	
The closed interval $[a, b]$	$a \leq x \leq b$	
The half-open interval $[a, b)$	$a \leq x < b$	
The half-open interval $(a, b]$	$a < x \leq b$	
The interval $[a, \infty)$	$x \geq a$	
The interval (a, ∞)	$x > a$	
The interval $(-\infty, a]$	$x \leq a$	
The interval $(-\infty, a)$	$x < a$	
The interval $(-\infty, \infty)$	All real numbers	

EXAMPLE 1 Writing Inequalities Using Interval Notation

Write each inequality using interval notation.

(a) $1 \le x \le 3$
(b) $-4 < x < 0$
(c) $x > 5$
(d) $x \le 1$

Solution

(a) $1 \le x \le 3$ represents all numbers x between 1 and 3, inclusive. In interval notation, we write $[1, 3]$.

(b) In interval notation, $-4 < x < 0$ is written $(-4, 0)$.

(c) $x > 5$ consists of all numbers x greater than 5. In interval notation, we write $(5, \infty)$.

(d) In interval notation, $x \le 1$ is written $(-\infty, 1]$. ■

EXAMPLE 2 Writing Intervals Using Inequality Notation

Write each interval as an inequality involving x.

(a) $[1, 4)$
(b) $(2, \infty)$
(c) $[2, 3]$
(d) $(-\infty, -3]$

Solution

(a) $[1, 4)$ consists of all numbers x for which $1 \le x < 4$.

(b) $(2, \infty)$ consists of all numbers x for which $x > 2$.

(c) $[2, 3]$ consists of all numbers x for which $2 \le x \le 3$.

(d) $(-\infty, -3]$ consists of all numbers x for which $x \le -3$. ■

Now Work PROBLEMS 13, 23, AND 31

2 Use Properties of Inequalities

The product of two positive real numbers is positive, the product of two negative real numbers is positive, and the product of 0 and 0 is 0. For any real number a, the value of a^2 is 0 or positive; that is, a^2 is nonnegative. This is called the **nonnegative property**.

In Words
The square of a real number is never negative.

Nonnegative Property

For any real number a,

$$a^2 \ge 0 \qquad \textbf{(1)}$$

When the same number is added to both sides of an inequality, an equivalent inequality is obtained. For example, since $3 < 5$, then $3 + 4 < 5 + 4$ or $7 < 9$. This is called the **addition property** of inequalities.

In Words
The addition property states that the sense, or direction, of an inequality remains unchanged if the same number is added to each side.

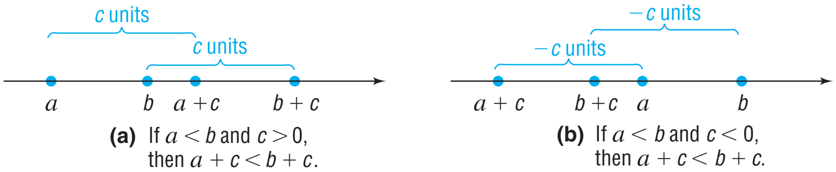
Addition Property of Inequalities

For real numbers a, b, and c:

$$\text{If } a < b, \text{ then } a + c < b + c. \qquad \textbf{(2a)}$$

$$\text{If } a > b, \text{ then } a + c > b + c. \qquad \textbf{(2b)}$$

Figure 53 illustrates the addition property (2a). In Figure 53(a), we see that a lies to the left of b. If c is positive, then $a + c$ and $b + c$ lie c units to the right of a and c units to the right of b, respectively. Consequently, $a + c$ must lie to the left of $b + c$; that is, $a + c < b + c$. Figure 53(b) illustrates the situation if c is negative.



(a) If $a < b$ and $c > 0$, then $a + c < b + c$.

(b) If $a < b$ and $c < 0$, then $a + c < b + c$.

Figure 53 Addition property of inequalities

Draw an illustration similar to Figure 53 that illustrates the addition property (2b).

EXAMPLE 3 Addition Property of Inequalities

(a) If $x < -5$, then $x + 5 < -5 + 5$ or $x + 5 < 0$.

(b) If $x > 2$, then $x + (-2) > 2 + (-2)$ or $x - 2 > 0$. ■

Now Work PROBLEM 39

EXAMPLE 4 Multiplying an Inequality by a Positive Number

Express as an inequality the result of multiplying each side of the inequality $3 < 7$ by 2.

Solution Begin with

$$3 < 7$$

Multiplying each side by 2 yields the numbers 6 and 14, so we have

$$6 < 14$$ ■

EXAMPLE 5 Multiplying an Inequality by a Negative Number

Express as an inequality the result of multiplying each side of the inequality $9 > 2$ by -4.

Solution Begin with

$$9 > 2$$

Multiplying each side by -4 yields the numbers -36 and -8, so we have

$$-36 < -8$$ ■

Note that the effect of multiplying both sides of $9 > 2$ by the negative number -4 is that the direction of the inequality symbol is reversed.

Examples 4 and 5 illustrate the following general **multiplication properties** for inequalities:

In Words
The multiplication properties state that the sense, or direction, of an inequality *remains the same* if each side is multiplied by a *positive* real number, whereas the direction is *reversed* if each side is multiplied by a *negative* real number.

Multiplication Properties for Inequalities

For real numbers a, b, and c:

If $a < b$ and if $c > 0$, then $ac < bc$.
If $a < b$ and if $c < 0$, then $ac > bc$. **(3a)**

If $a > b$ and if $c > 0$, then $ac > bc$.
If $a > b$ and if $c < 0$, then $ac < bc$. **(3b)**

EXAMPLE 6 Multiplication Property of Inequalities

(a) If $2x < 6$, then $\frac{1}{2}(2x) < \frac{1}{2}(6)$ or $x < 3$.

(b) If $\frac{x}{-3} > 12$, then $-3\left(\frac{x}{-3}\right) < -3(12)$ or $x < -36$.

(c) If $-4x > -8$, then $\frac{-4x}{-4} < \frac{-8}{-4}$ or $x < 2$. ■

Now Work PROBLEM 45

3 Solve Linear Inequalities Algebraically and Graphically

An **inequality in one variable** is a statement involving two expressions, at least one containing the variable, separated by one of the inequality symbols, $<$, $\le$, $>$, or $\ge$. To **solve an inequality** means to find all values of the variable for which the statement is true. These values are called **solutions** of the inequality.

For example, the following are all inequalities involving one variable, x:

$$x + 5 < 8 \qquad 2x - 3 \ge 4 \qquad x^2 - 1 \le 3 \qquad \frac{x+1}{x-2} > 0$$

As with equations, one method for solving an inequality is to replace it by a series of equivalent inequalities until an inequality with an obvious solution, such as $x < 3$, is obtained. Equivalent inequalities are obtained by applying some of the same operations as those used to find equivalent equations. The addition property and the multiplication properties form the basis for the following procedures.

Procedures That Leave the Inequality Symbol Unchanged

1. Simplify both sides of the inequality by combining like terms and eliminating parentheses:

$$x + 2 + 6 > 2x + 5(x + 1)$$

$$\text{is equivalent to} \qquad x + 8 > 7x + 5$$

2. Add or subtract the same expression on both sides of the inequality:

$$3x - 5 < 4$$

$$\text{is equivalent to} \qquad (3x - 5) + 5 < 4 + 5$$

3. Multiply or divide both sides of the inequality by the same *positive* expression:

$$4x > 16 \quad \text{is equivalent to} \quad \frac{4x}{4} > \frac{16}{4}$$

Procedures That Reverse the Sense or Direction of the Inequality Symbol

1. Interchange the two sides of the inequality:

$$3 < x \quad \text{is equivalent to} \quad x > 3$$

2. Multiply or divide both sides of the inequality by the same *negative* expression:

$$-2x > 6 \quad \text{is equivalent to} \quad \frac{-2x}{-2} < \frac{6}{-2}$$

To solve an inequality using a graphing utility, follow these steps:

Steps for Solving Inequalities Graphically

STEP 1: Write the inequality in one of the following forms:

$$Y_1 < Y_2, \qquad Y_1 > Y_2, \qquad Y_1 \le Y_2, \qquad Y_1 \ge Y_2$$

STEP 2: Graph Y_1 and Y_2 on the same screen.

STEP 3: If the inequality is of the form $Y_1 < Y_2$, determine on what interval Y_1 is below Y_2.

If the inequality is of the form $Y_1 > Y_2$, determine on what interval Y_1 is above Y_2.

If the inequality is not strict ($\le$ or $\ge$), include the x-coordinates of the points of intersection in the solution.

As the examples that follow illustrate, we solve linear inequalities using many of the same steps that we would use to solve linear equations. The goal is to get the variable on one side of the inequality and a constant on the other. In writing the solution of an inequality, either set notation or interval notation may be used, whichever is more convenient.

EXAMPLE 7 **Solving an Inequality Algebraically and Graphically**

Solve the inequality $4x + 7 \ge 2x - 3$, and graph the solution set.

Algebraic Solution

$$\begin{aligned}
4x + 7 &\ge 2x - 3 \\
4x + 7 - 7 &\ge 2x - 3 - 7 && \text{Subtract 7 from both sides.} \\
4x &\ge 2x - 10 && \text{Simplify.} \\
4x - 2x &\ge 2x - 10 - 2x && \text{Subtract } 2x \text{ from both sides.} \\
2x &\ge -10 && \text{Simplify.} \\
\frac{2x}{2} &\ge \frac{-10}{2} && \text{Divide both sides by 2. (The direction of the inequality symbol is unchanged.)} \\
x &\ge -5 && \text{Simplify.}
\end{aligned}$$

The solution set is $\{x \mid x \ge -5\}$ or, using interval notation, all numbers in the interval $[-5, \infty)$.

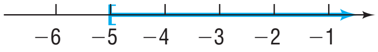


Figure 55 $\{x \mid x \ge -5\}$

Graphing Solution

Graph $Y_1 = 4x + 7$ and $Y_2 = 2x - 3$ on the same screen. See Figure 54. Using the INTERSECT command, we find that Y_1 and Y_2 intersect at $x = -5$. The graph of Y_1 is above that of Y_2, $Y_1 > Y_2$, to the right of the point of intersection. Since the inequality is not strict, the solution set is $\{x \mid x \ge -5\}$ or, using interval notation, $[-5, \infty)$.

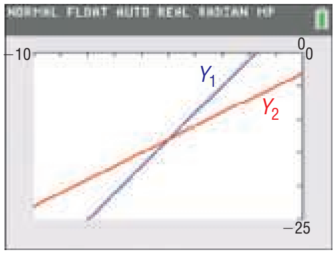


Figure 54 $Y_1 = 4x + 7$; $Y_2 = 2x - 3$

See Figure 55 for the graph of the solution set. ■

Now Work PROBLEM 51

4 Solve Combined Inequalities Algebraically and Graphically

EXAMPLE 8 **Solving a Combined Inequality Algebraically and Graphically**

Solve the inequality: $-1 \le \dfrac{3 - 5x}{2} \le 9$

Algebraic Solution

$$-1 \leq \frac{3-5x}{2} \leq 9$$

$$2(-1) \leq 2\left(\frac{3-5x}{2}\right) \leq 2(9)$$ **Multiply each part by 2 to remove the denominator.**

$$-2 \leq 3-5x \leq 18$$ **Simplify.**

$$-2-3 \leq 3-5x-3 \leq 18-3$$ **Subtract 3 from each part to isolate the term containing *x*.**

$$-5 \leq -5x \leq 15$$ **Simplify.**

$$\frac{-5}{-5} \geq \frac{-5x}{-5} \geq \frac{15}{-5}$$ **Divide each part by −5 (reverse the sense of each inequality symbol).**

$$1 \geq x \geq -3$$ **Simplify.**

$$-3 \leq x \leq 1$$ **Reverse the order so the numbers get larger as you read from left to right.**

The solution set is $\{x \mid -3 \leq x \leq 1\}$. In interval notation, the solution is $[-3, 1]$.

Graphing Solution

To solve a combined inequality graphically, graph each part: $Y_1 = -1$, $Y_2 = \frac{3-5x}{2}$, $Y_3 = 9$. We seek the values of x for which the graph of Y_2 is between the graphs of Y_1 and Y_3. See Figure 56. The point of intersection of Y_1 and Y_2 is $(1, -1)$, and the point of intersection of Y_2 and Y_3 is $(-3, 9)$. The inequality is true for all values of x between these two intersection points. Since the inequality is nonstrict, the solution set is $\{x \mid -3 \leq x \leq 1\}$ or, using interval notation, $[-3, 1]$.

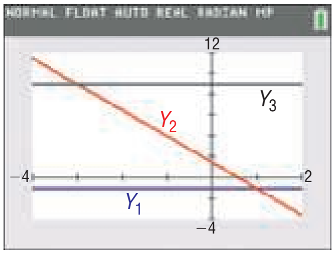


Figure 56 $Y_1 = -1;\ Y_2 = \frac{3-5x}{2};\ Y_3 = 9$

See Figure 57 for the graph of the solution set.

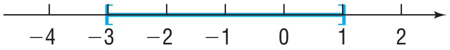


Figure 57 $\{x \mid -3 \leq x \leq 1\}$ ■

Now Work PROBLEM 67

5 Solve Absolute Value Inequalities Algebraically and Graphically

EXAMPLE 9 Solving an Inequality Involving Absolute Value

Solve the inequality: $|x| < 4$

Algebraic Solution

We are looking for all points whose coordinate x is a distance less than 4 units from the origin. See Figure 58 for an illustration.

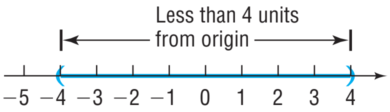


Figure 58 $|x| < 4$

Because any x between -4 and 4 satisfies the condition $|x| < 4$, the solution set consists of all numbers x for which $-4 < x < 4$, that is, all x in the interval $(-4, 4)$. ■

Graphing Solution

Graph $Y_1 = |x|$ and $Y_2 = 4$ on the same screen. See Figure 59. Using the INTERSECT command (twice), find that Y_1 and Y_2 intersect at $x = -4$ and at $x = 4$. The graph of Y_1 is below that of Y_2, $Y_1 < Y_2$, between the points of intersection. Because the inequality is strict, the solution set is $\{x \mid -4 < x < 4\}$ or, using interval notation, $(-4, 4)$.

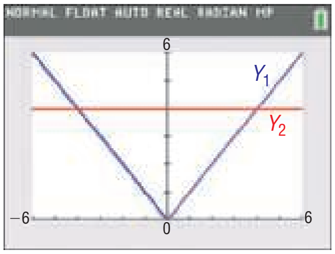


Figure 59 $Y_1 = |x|;\ Y_2 = 4$ ■

We are led to the following results:

THEOREM

If a is any positive number and if u is any algebraic expression, then

$\lvert u \rvert < a$	is equivalent to	$-a < u < a$	**(4)**
$\lvert u \rvert \le a$	is equivalent to	$-a \le u \le a$	**(5)**

In other words, $|u| < a$ is equivalent to $-a < u$ and $u < a$.

EXAMPLE 10 Solving an Inequality Involving Absolute Value

Solve the inequality $|2x + 4| \le 3$, and graph the solution set.

Algebraic Solution

$$|2x + 4| \le 3$$

This follows the form of statement (5); the expression $u = 2x + 4$ is inside the absolute value bars.

$$-3 \le 2x + 4 \le 3$$

Apply statement (5).

$$-3 - 4 \le 2x + 4 - 4 \le 3 - 4$$

Subtract 4 from each part.

$$-7 \le 2x \le -1$$

Simplify.

$$\frac{-7}{2} \le \frac{2x}{2} \le \frac{-1}{2}$$

Divide each part by 2.

$$-\frac{7}{2} \le x \le -\frac{1}{2}$$

Simplify.

The solution set is $\left\{x \middle| -\frac{7}{2} \le x \le -\frac{1}{2}\right\}$, that is, all x in the interval $\left[-\frac{7}{2}, -\frac{1}{2}\right]$.

Graphing Solution

Graph $Y_1 = |2x + 4|$ and $Y_2 = 3$ on the same screen. See Figure 60. Using the INTERSECT command (twice), find that Y_1 and Y_2 intersect at $x = -3.5$ and at $x = -0.5$. The graph of Y_1 is below that of Y_2, $Y_1 < Y_2$, between the points of intersection. Because the inequality is not strict, the solution set is $\{x | -3.5 \le x \le -0.5\}$ or, using interval notation, $[-3.5, -0.5]$.

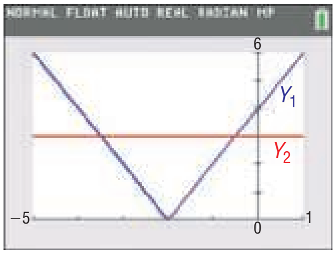


Figure 60 $Y_1 = |2x + 4|$; $Y_2 = 3$

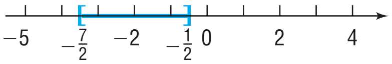


Figure 61 $\left\{x \middle| -\frac{7}{2} \le x \le -\frac{1}{2}\right\}$

See Figure 61 for a graph of the solution set. ■

Now Work PROBLEM 79

EXAMPLE 11 Solving an Inequality Involving Absolute Value

Solve the inequality $|x| > 3$.

Algebraic Solution

We are looking for all points whose coordinate x is a distance greater than 3 units from the origin. Figure 62 illustrates the situation.

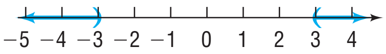


Figure 62 $|x| > 3$

Graphing Solution

Graph $Y_1 = |x|$ and $Y_2 = 3$ on the same screen. See Figure 63 on page 154. Using the INTERSECT command (twice), find that Y_1 and Y_2 intersect at $x = -3$ and at $x = 3$. The graph of Y_1 is above that of Y_2, $Y_1 > Y_2$, to the left of $x = -3$ and to the right of $x = 3$. Because the inequality is strict, the solution set is $\{x | x < -3 \text{ or } x > 3\}$. Using interval notation, the solution is $(-\infty, -3) \cup (3, \infty)$.*

---

*Recall that the symbol $\cup$ represents the union of two sets and means "or."

We conclude that any x less than -3 or greater than 3 satisfies the condition $|x| > 3$. Consequently, the solution set is $\{x \mid x < -3 \text{ or } x > 3\}$. Using interval notation, the solution is $(-\infty, -3) \cup (3, \infty)$.

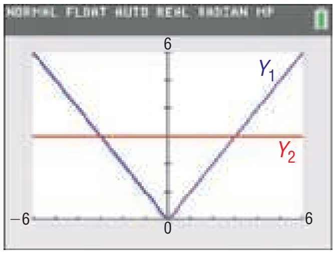


Figure 63 $Y_1 = |x|;\ Y_2 = 3$

THEOREM

If a is any positive number and u is any algebraic expression, then

$\lvert u \rvert > a$	is equivalent to	$u < -a$ or $u > a$	**(6)**
$\lvert u \rvert \geq a$	is equivalent to	$u \leq -a$ or $u \geq a$	**(7)**

WARNING A common error to be avoided is to attempt to write the solution $x < 1$ or $x > 4$ as $1 > x > 4$, which is incorrect, since there are no numbers x for which $1 > x$ and $x > 4$. Another common error is to "mix" the symbols and write $1 < x > 4$, which makes no sense.

EXAMPLE 12 Solving an Inequality Involving Absolute Value

Solve the inequality $|2x - 5| > 3$, and graph the solution set.

Algebraic Solution

$|2x - 5| > 3$ **This follows the form of statement (6); the expression $u = 2x - 5$ is inside the absolute value bars.**

$$2x - 5 < -3 \quad \text{or} \quad 2x - 5 > 3 \qquad \textbf{Apply statement (6).}$$

$$2x - 5 + 5 < -3 + 5 \quad \text{or} \quad 2x - 5 + 5 > 3 + 5 \qquad \textbf{Add 5 to each part.}$$

$$2x < 2 \quad \text{or} \quad 2x > 8 \qquad \textbf{Simplify.}$$

$$\frac{2x}{2} < \frac{2}{2} \quad \text{or} \quad \frac{2x}{2} > \frac{8}{2} \qquad \textbf{Divide each part by 2.}$$

$$x < 1 \quad \text{or} \quad x > 4 \qquad \textbf{Simplify.}$$

The solution set is $\{x \mid x < 1 \text{ or } x > 4\}$. Using interval notation, the solution is $(-\infty, 1) \cup (4, \infty)$.

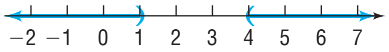


Figure 65 $\{x \mid x < 1 \text{ or } x > 4\}$

Graphing Solution

Graph $Y_1 = |2x - 5|$ and $Y_2 = 3$ on the same screen. See Figure 64. Using the INTERSECT command (twice), find that Y_1 and Y_2 intersect at $x = 1$ and at $x = 4$. The graph of Y_1 is above that of Y_2, $Y_1 > Y_2$, to the left of $x = 1$ and to the right of $x = 4$. Because the inequality is strict, the solution set is $\{x \mid x < 1 \text{ or } x > 4\}$. Using interval notation, the solution is $(-\infty, 1) \cup (4, \infty)$.

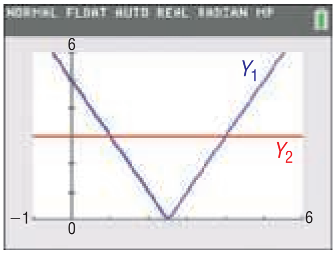


Figure 64 $Y_1 = |2x - 5|;\ Y_2 = 3$

See Figure 65 for a graph of the solution set.

Now Work PROBLEM 81

1.7 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Graph the inequality: $x \geq -2$. (p. 20)

2. ***True or False*** The absolute value of a negative number is positive. (p. 20)

3. If $A = \{a, e, i, o, u\}$ and $B = \{a, b, c, d, e\}$, what is $A \cup B$? (pp. 2–3)

Concepts and Vocabulary

4. If each side of an inequality is multiplied by a(n) __________ number, then the sense of the inequality symbol is reversed.

5. A(n) ________ _________, denoted $[a, b]$, consists of all real numbers x for which $a \le x \le b$.

6. $|u| \le a$ is equivalent to _______________.

7. The interval __________ consists of all real numbers for which $x \le a$.

In Problems 8–10, assume that $a < b$ and $c < 0$.

8. ***True or False*** $a + c < b + c$

9. ***True or False*** $ac > bc$

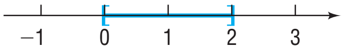
10. ***True or False*** $\frac{a}{c} < \frac{b}{c}$

11. Which of the following has no solution?
(a) $|x| < -5$ (b) $|x| \le 0$ (c) $|x| > 0$ (d) $|x| \ge 0$

12. Which pair of inequalities is equivalent to $0 < x \le 3$?
(a) $x > 0$ and $x \ge 3$ (b) $x < 0$ and $x \ge 3$
(c) $x > 0$ and $x \le 3$ (d) $x < 0$ and $x \le 3$

Skill Building

In Problems 13–18, express the graph shown in color using interval notation. Also express each as an inequality involving x.

13.


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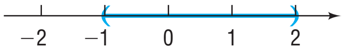

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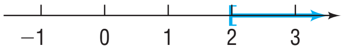

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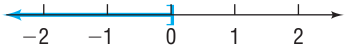
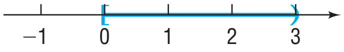
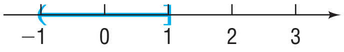


17.


18.


In Problems 19–22, an inequality is given. Write the inequality obtained by:

(a) Adding 3 to each side of the given inequality.
(b) Subtracting 5 from each side of the given inequality.
(c) Multiplying each side of the given inequality by 3.
(d) Multiplying each side of the given inequality by −2.

19. $3 < 5$
20. $2 > 1$
21. $2x + 1 < 2$
22. $1 - 2x > 5$

In Problems 23–30, write each inequality using interval notation, and illustrate each inequality using the real number line.

23. $0 \le x \le 4$
24. $-1 < x < 5$
25. $4 \le x < 6$
26. $-2 < x \le 0$
27. $x \ge 4$
28. $x \le 5$
29. $x < -4$
30. $x > 1$

In Problems 31–38, write each interval as an inequality involving x, and illustrate each inequality using the real number line.

31. $[2, 5]$
32. $(1, 2)$
33. $(-3, -2)$
34. $[0, 1)$
35. $[4, \infty)$
36. $(-\infty, 2]$
37. $(-\infty, -3)$
38. $(-8, \infty)$

In Problems 39–50, fill in the blank with the correct inequality symbol.

39. If $x < 5$, then $x - 5$ _________ 0.
40. If $x < -4$, then $x + 4$ _________ 0.
41. If $x > -4$, then $x + 4$ _________ 0.
42. If $x > 6$, then $x - 6$ _________ 0.
43. If $x \ge -4$, then $3x$ _________ -12.
44. If $x \le 3$, then $2x$ _________ 6.
45. If $x > 6$, then $-2x$ _________ -12.
46. If $x > -2$, then $-4x$ _________ 8.
47. If $2x < 6$, then x _________ 3.
48. If $3x \le 12$, then x _________ 4.
49. If $-\frac{1}{2}x \le 3$, then x _________ -6.
50. If $-\frac{1}{4}x > 1$, then x _________ -4.

In Problems 51–88, solve each inequality algebraically. Express your answer using set notation or interval notation. Graph the solution set. Verify your results using a graphing utility.

51. $3x - 7 > 2$
52. $2x + 5 > 1$
53. $1 - 2x \le 3$
54. $2 - 3x \le 5$
55. $3x - 1 \ge 3 + x$
56. $2x - 2 \ge 3 + x$
57. $-2(x + 3) < 8$
58. $-3(1 - x) < 12$
59. $4 - 3(1 - x) \le 3$
60. $8 - 4(2 - x) \le -2x$
61. $\frac{1}{2}(x - 4) > x + 8$
62. $3x + 4 > \frac{1}{3}(x - 2)$
63. $0 \le 2x - 6 \le 4$
64. $4 \le 2x + 2 \le 10$
65. $-5 \le 4 - 3x \le 2$
66. $-3 \le 3 - 2x \le 9$
67. $-3 < \frac{2x - 1}{4} < 0$
68. $0 < \frac{3x + 2}{2} < 4$
69. $1 < 1 - \frac{1}{2}x < 4$
70. $0 < 1 - \frac{1}{3}x < 1$
71. $(x + 2)(x - 3) > (x - 1)(x + 1)$
72. $(x - 1)(x + 1) > (x - 3)(x + 4)$
73. $|2x| < 8$
74. $|3x| < 15$

75. $|3x| > 12$
76. $|2x| > 6$
77. $|3t - 2| \leq 4$
78. $|2u + 5| \leq 7$
79. $|x - 2| + 2 < 3$
80. $|x + 4| + 3 < 5$
81. $|x - 3| \geq 2$
82. $|x + 4| \geq 2$
83. $|1 - 2x| > |-3|$
84. $|2 - 3x| > |-1|$
85. $|1 - 4x| - 7 < -2$
86. $|1 - 2x| - 4 < -1$
87. $|2x + 1| < -1$
88. $|3x - 4| \geq 0$

Mixed Practice

In Problems 89–108, solve each inequality algebraically. Express your answer using set notation or interval notation. Graph the solution set. Verify your results using a graphing utility.

89. $3 - 4x < 11$
90. $1 - 3x \leq 7$
91. $|2x + 1| - 5 \geq -1$
92. $|5x + 2| - 3 > 9$
93. $\frac{x}{2} \geq 1 - \frac{x}{4}$
94. $\frac{x}{3} \geq 2 + \frac{x}{6}$
95. $-\frac{1}{3} \leq \frac{x + 1}{6} < \frac{4}{3}$
96. $-\frac{3}{2} < \frac{x - 3}{4} \leq \frac{5}{4}$
97. $x(4x + 3) \leq (2x + 1)^2$
98. $x(9x - 5) \leq (3x - 1)^2$
99. $|(3x - 2) - 7| < \frac{1}{2}$
100. $|(4x - 1) - 11)| < \frac{1}{4}$
101. $-3 < 5 - 2x \leq 11$
102. $2 \leq 3 - 2(x + 1) < 8$
103. $7 - |x - 1| > 4$
104. $9 - |x + 3| \geq 5$
105. $-3 < x + 5 < 2x$
106. $2 < x - 3 < 2x$
107. $x + 2 < 2x - 1 < 5x$
108. $2x - 1 < 3x + 5 < 5x - 7$

Applications and Extensions

109. Express the fact that x differs from 2 by less than $\frac{1}{2}$ as an inequality involving an absolute value. Solve for x.

110. Express the fact that x differs from -1 by less than 1 as an inequality involving an absolute value. Solve for x.

111. Express the fact that x differs from -3 by more than 2 as an inequality involving an absolute value. Solve for x.

112. Express the fact that x differs from 2 by more than 3 as an inequality involving an absolute value. Solve for x.

113. A young adult may be defined as someone older than 21 but less than 30 years of age. Express this statement using inequalities.

114. Middle-aged may be defined as being 40 or more and less than 60. Express this statement using inequalities.

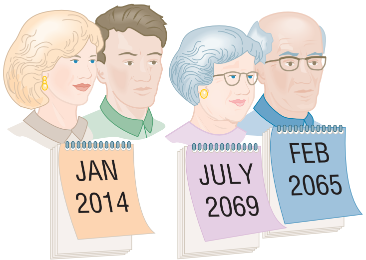
115. Body Temperature Normal human body temperature is 98.6°F. If a temperature x that differs from normal by at least 1.5°F is considered unhealthy, write the condition for an unhealthy temperature x as an inequality involving an absolute value, and solve for x.

116. Household Voltage In the United States, normal household voltage is 115 volts. However, it is not uncommon for actual voltage to differ from normal voltage by at most 5 volts. Express this situation as an inequality involving an absolute value. Use x as the actual voltage and solve for x.

117. Life Expectancy The Social Security Administration determined that an average 30-year-old male in 2014 could expect to live at least 51.9 more years and an average 30-year-old female in 2014 could expect to live at least 55.6 more years.

(a) To what age can an average 30-year-old male expect to live? Express your answer as an inequality.
(b) To what age can an average 30-year-old female expect to live? Express your answer as an inequality.
(c) Who can expect to live longer, a male or a female? By how many years?

Source: *National Vital Statistics Report, 2014.*



118. General Chemistry For a certain ideal gas, the volume V (in cubic centimeters) equals 20 times the temperature T in kelvins (K). If the temperature varies from 353 K to 393 K, inclusive, what is the corresponding range of the volume of the gas?

119. Real Estate A real estate agent agrees to sell a large apartment complex according to the following commission schedule: \$45,000 plus 25% of the selling price in excess of \$900,000. Assuming that the complex will sell at some price between \$900,000 and \$1,100,000, inclusive, over what range does the agent's commission vary? How does the commission vary as a percent of selling price?

120. Sales Commission A used car salesperson is paid a commission of \$25 plus 40% of the selling price in excess of owner's cost. The owner claims that used cars typically sell for at least owner's cost plus \$70 and at most owner's cost plus \$300. For each sale made, over what range can the salesperson expect the commission to vary?

121. Federal Tax Withholding The percentage method of withholding for federal income tax (2014) states that a single person whose weekly wages, after subtracting withholding allowances, are over \$753, but not over \$1762, shall have \$97.75 plus 25% of the excess over \$753 withheld. Over what range does the amount withheld vary if the weekly wages vary from \$900 to \$1100, inclusive?

Source: *Employer's Tax Guide. Internal Revenue Service, 2014.*

122. Exercising Sue wants to lose weight. For healthy weight loss, the American College of Sports Medicine (ACSM) recommends 200 to 300 minutes of exercise per week. For the first six days of the week, Sue exercised 40, 45, 0, 50, 25, and 35 minutes. How long should Sue exercise on the seventh day in order to stay within the ACSM guidelines?

123. Electricity Rates Suppose Commonwealth Edison Company's charge for electricity in 2014 was 8.21¢ per kilowatt-hour. In addition, each monthly bill contained a customer charge of \$15.37. If that year's bills ranged from a low of \$72.84 to a high of \$237.04, over what range did usage vary (in kilowatt-hours)?

Source: *Commonwealth Edison Co., 2014.*

124. Water Bills The Village of Oak Lawn charges homeowners \$57.07 per quarter-year plus \$5.81 per 1000 gallons for water usage in excess of 10,000 gallons. In 2014, one homeowner's quarterly bill ranged from a high of \$150.03 to a low of \$97.74. Over what range did water usage vary?

Source: *Village of Oak Lawn, Illinois, 2014.*

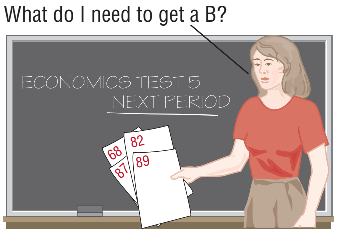
125. Markup of a Used Car The markup over dealer's cost of a used car ranges from 12% to 18%. If the sticker price is \$8800, over what range will the dealer's cost vary?

126. IQ Tests A standard intelligence test has an average score of 100. According to statistical theory, of the people who take the test, the 2.5% with the highest scores will have scores of more than 1.96σ above the average, where σ (sigma, a number called the *standard deviation*) depends on the nature of the test. If $\sigma = 12$ for this test and there is (in principle) no upper limit to the score possible on the test, write the interval of possible test scores of the people in the top 2.5%.

127. Computing Grades In your Economics 101 class, you have scores of 68, 82, 87, and 89 on the first four of five tests. To get a grade of B, the average of the first five test scores must be greater than or equal to 80 and less than 90.

(a) Find the range of the score that you need on the last test to get a B.

(b) What score do you need if the fifth test counts double?



128. "Light" Foods For food products to be labeled "light," the U.S. Food and Drug Administration requires that the altered product must either contain at least one-third fewer calories than the regular product or contain at least one-half less fat than the regular product. If a serving of Miracle Whip® Light contains 20 calories and 1.5 grams of fat, then what must be true about either the number of calories or the grams of fat in a serving of regular Miracle Whip®?

129. Reading Books A HuffPost/YouGov poll found that in 2013 Americans read an average of 13.6 books per year. Suppose HuffPost/YouGov is 99% confident that the result from this poll is off by fewer than 1.8 books from the actual average x. Express this situation as an inequality involving absolute value, and solve the inequality for x to determine the interval in which the actual average is likely to fall. [**Note:** In statistics, this interval is called a 99% **confidence interval**.]

130. Speed of Sound According to data from the Hill Aerospace Museum (Hill Air Force Base, Utah), the speed of sound varies depending on altitude, barometric pressure, and temperature. For example, at 20,000 feet, 13.75 inches of mercury, and -12.3°F, the speed of sound is about 707 miles per hour, but the speed can vary from this result by as much as 55 miles per hour as conditions change.

(a) Express this situation as an inequality involving an absolute value.

(b) Using x for the speed of sound, solve for x to find an interval for the speed of sound.

131. Arithmetic Mean If $a < b$, show that $a < \frac{a+b}{2} < b$. The number $\frac{a+b}{2}$ is called the **arithmetic mean** of a and b.

132. Refer to Problem 131. Show that the arithmetic mean of a and b is equidistant from a and b.

133. Geometric Mean If $0 < a < b$, show that $a < \sqrt{ab} < b$. The number $\sqrt{ab}$ is called the **geometric mean** of a and b.

134. Refer to Problems 131 and 133. Show that the geometric mean of a and b is less than the arithmetic mean of a and b.

135. Harmonic Mean For $0 < a < b$, let h be defined by

$$\frac{1}{h} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)$$

Show that $a < h < b$. The number h is called the **harmonic mean** of a and b.

136. Refer to Problems 131, 133, and 135. Show that the harmonic mean of a and b equals the geometric mean squared divided by the arithmetic mean.

Explaining Concepts: Discussion and Writing

137. Make up an inequality that has no solution. Make up one that has exactly one solution.

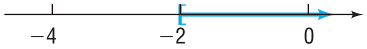
138. How would you explain to a fellow student the underlying reason for the multiplication properties for inequalities (page 149)? That is, the sense or direction of an inequality remains the same if each side is multiplied by a positive real number, while the direction is reversed if each side is multiplied by a negative real number.

139. The inequality $x^2 + 1 < -5$ has no solution. Explain why.

140. Do you prefer to use inequality notation or interval notation to express the solution to an inequality? Give your reasons. Are there particular circumstances when you prefer one to the other? Cite examples.

'Are You Prepared?' Answers

1.



2. True

3. $A \cup B = \{a, b, c, d, e, i, o, u\}$

Chapter Review

Things to Know

Formulas

Distance formula (p. 85)	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Midpoint formula (p. 88)	$M = (x, y) = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

Quadratic equation and quadratic formula

The real solutions of the equation $ax^2 + bx + c = 0$, $a \neq 0$, are given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, provided $b^2 - 4ac \geq 0$. If $b^2 - 4ac < 0$, there are no real solutions. (p. 114)

In the complex number system, the solutions of the equation $ax^2 + bx + c = 0$, $a \neq 0$, are given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. (p. 126)

Discriminant (pp. 114 and 127)

If $b^2 - 4ac > 0$, there are two unequal real solutions.

If $b^2 - 4ac = 0$, there is one repeated real solution—a double root, or a root of multiplicity 2.

If $b^2 - 4ac < 0$, there are no real solutions, but there are two distinct complex solutions that are not real; the complex solutions are conjugates of each other.

Interval notation (p. 147)

$[a, b]$	$\{x \mid a \leq x \leq b\}$	(a, b)	$\{x \mid a < x < b\}$	$(-\infty, a]$	$\{x \mid x \leq a\}$
$[a, b)$	$\{x \mid a \leq x < b\}$	$[a, \infty)$	$\{x \mid x \geq a\}$	$(-\infty, a)$	$\{x \mid x < a\}$
$(a, b]$	$\{x \mid a < x \leq b\}$	(a, ∞)	$\{x \mid x > a\}$	$(-\infty, \infty)$	All real numbers

Properties of inequalities

Addition property (p. 148)	If $a < b$, then $a + c < b + c$. If $a > b$, then $a + c > b + c$.
Multiplication properties (p. 149)	(a) If $a < b$ and if $c > 0$, then $ac < bc$. If $a < b$ and if $c < 0$, then $ac > bc$. (b) If $a > b$ and if $c > 0$, then $ac > bc$. If $a > b$ and if $c < 0$, then $ac < bc$.

Absolute value

If $|u| = a$, $a > 0$, then $u = -a$ or $u = a$. (p. 133)

If $|u| \leq a$, $a > 0$, then $-a \leq u \leq a$. (p. 153)

If $|u| \geq a$, $a > 0$, then $u \leq -a$ or $u \geq a$. (p. 154)

Objectives

Section		You should be able to	Example(s)	Review Exercises
1.1	1	Use the distance formula (p. 85)	1, 2, 3	45(a)–47(a), 53, 54
	2	Use the midpoint formula (p. 87)	4	45(b)–47(b)
	3	Graph equations by plotting points (p. 88)	5, 6, 7	50–52
	4	Graph equations using a graphing utility (p. 90)	8, 9	48, 50–52
	5	Use a graphing utility to create tables (p. 92)	10	48
	6	Find intercepts from a graph (p. 93)	11	49
	7	Use a graphing utility to approximate intercepts (p. 93)	12	50–52
1.2	1	Solve equations using a graphing utility (p. 100)	1, 2	27, 28
	2	Solve linear equations (p. 102)	3, 4	1–3
	3	Solve rational equations (p. 103)	5, 6	4, 25
	4	Solve problems that can be modeled by linear equations (p. 105)	7, 8	69
1.3	1	Solve quadratic equations by factoring (p. 110)	1, 2	5, 7, 18, 19
	2	Solve quadratic equations using the Square Root Method (p. 112)	3	26
	3	Solve quadratic equations by completing the square (p. 113)	4	5, 7, 8, 10, 11, 18, 19
	4	Solve quadratic equations using the quadratic formula (p. 113)	5, 6	5, 7, 8, 10, 11, 18, 19
	5	Solve problems that can be modeled by quadratic equations (p. 116)	7	63, 67, 71, 72
1.4	1	Add, subtract, multiply, and divide complex numbers (p. 122)	1–8	36–40
	2	Solve quadratic equations in the complex number system (p. 125)	10, 11	41–44
1.5	1	Solve radical equations (p. 129)	1–3	9, 13–15, 20
	2	Solve equations quadratic in form (p. 131)	4, 5	12, 17
	3	Solve absolute value equations (p. 133)	6	21, 22
	4	Solve equations by factoring (p. 134)	7, 8	23, 24
1.6	1	Translate verbal descriptions into mathematical expressions (p. 138)	1	55
	2	Solve interest problems (p. 139)	2, 3	56, 57
	3	Solve mixture problems (p. 140)	4	65, 66
	4	Solve uniform motion problems (p. 141)	5, 6	58, 60–62, 68, 73
	5	Solve constant rate job problems (p. 143)	7	64, 70
1.7	1	Use interval notation (p. 147)	1, 2	29–35
	2	Use properties of inequalities (p. 148)	3–6	29–35
	3	Solve linear inequalities algebraically and graphically (p. 150)	7	29
	4	Solve combined inequalities algebraically and graphically (p. 151)	8	30, 31, 59
	5	Solve absolute value inequalities algebraically and graphically (p. 152)	9–12	32–35

Review Exercises

In Problems 1–26, find all the real solutions, if any, of each equation. (Where they appear, a, b, m, and n are positive constants.) When possible, verify your results using a graphing utility.

1. $2 - \frac{x}{3} = 8$

2. $-2(5 - 3x) + 8 = 4 + 5x$

3. $\frac{3x}{4} - \frac{x}{3} = \frac{1}{12}$

4. $\frac{x}{x-1} = \frac{6}{5}, \quad x \neq 1$

5. $x(1 - x) = 6$

6. $\frac{1-3x}{4} = \frac{x+6}{3} + \frac{1}{2}$

7. $(x - 1)(2x + 3) = 3$

8. $2x + 3 = 4x^2$

9. $\sqrt[3]{x^2 - 1} = 2$

10. $x(x + 1) + 2 = 0$

11. $3x^2 - x + 1 = 0$

12. $x^4 - 5x^2 + 4 = 0$

13. $\sqrt{2x - 3} + x = 3$

14. $\sqrt[4]{2x + 3} = 2$

15. $\sqrt{x + 1} + \sqrt{x - 1} = \sqrt{2x + 1}$

16. $2x^{1/2} - 3 = 0$

17. $x^{-6} - 7x^{-3} - 8 = 0$

18. $x^2 + m^2 = 2mx + (nx)^2, \quad n \neq 1, n \neq -1$

19. $10a^2x^2 - 2abx - 36b^2 = 0$

20. $\sqrt{x^2 + 3x + 7} - \sqrt{x^2 - 3x + 9} + 2 = 0$

21. $|2x + 3| = 7$

22. $|2 - 3x| + 2 = 9$

23. $2x^3 = 3x^2$

24. $2x^3 + 5x^2 - 8x - 20 = 0$

25. $\dfrac{1}{x-1} + \dfrac{3}{x+2} = \dfrac{11}{x^2 + x - 2}$

26. $(x - 2)^2 = 9$

In Problems 27–28, use a graphing utility to approximate the solutions of each equation rounded to two decimal places. All solutions lie between −10 and 10.

27. $x^3 - 5x + 3 = 0$

28. $x^4 - 3 = 2x + 1$

In Problems 29–35, solve each inequality. Express your answer using set notation or interval notation. Graph the solution set. Verify your results using a graphing utility.

29. $\dfrac{2x - 3}{5} + 2 \leq \dfrac{x}{2}$

30. $-9 \leq \dfrac{2x + 3}{-4} \leq 7$

31. $2 < \dfrac{3 - 3x}{12} < 6$

32. $|3x + 4| < \dfrac{1}{2}$

33. $|2x - 5| \geq 9$

34. $2 + |2 - 3x| \leq 4$

35. $1 - |2 - 3x| < -4$

In Problems 36–40, use the complex number system and write each expression in the standard form $a + bi$.

36. $(6 + 3i) - (2 - 4i)$

37. $4(3 - i) + 3(-5 + 2i)$

38. $\dfrac{3}{3 + i}$

39. i^{50}

40. $(2 + 3i)^3$

In Problems 41–44, solve each equation in the complex number system.

41. $x^2 + x + 1 = 0$

42. $2x^2 + x - 2 = 0$

43. $x^2 + 3 = x$

44. $x(1 + x) = 2$

In Problems 45–47, find the following for each pair of points:

(a) *The distance between the points*

(b) *The midpoint of the line segment connecting the points*

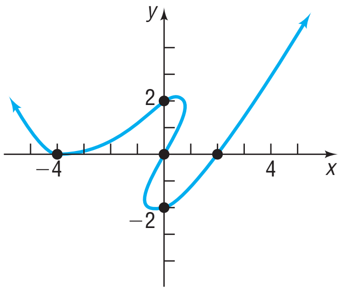
45. $(0, 0)$; $(4, 2)$

46. $(1, -1)$; $(-2, 3)$

47. $(4, -4)$; $(4, 8)$

48. Graph $y = -x^2 + 15$ using a graphing utility. Create a table of values to determine a good initial viewing window.

49. List the intercepts of the following graph.



In Problems 50–52, graph each equation by plotting points. Verify your results using a graphing utility. Approximate the intercepts using a graphing utility, and label them on the graph.

50. $2x - 3y = 6$

51. $y = x^2 - 9$

52. $x^2 + 2y = 16$

---

53. Show that the points $A = (3, 4)$, $B = (1, 1)$, and $C = (-2, 3)$ are the vertices of an isosceles triangle.

54. Find two numbers y such that the distance from $(-3, 2)$ to $(5, y)$ is 10.

55. Translate the following statement into a mathematical expression: The perimeter p of a rectangle is the sum of two times the length l and two times the width w.

56. **Banking** A bank lends \$9000 at 7% simple interest. At the end of 1 year, how much interest is owed on the loan?

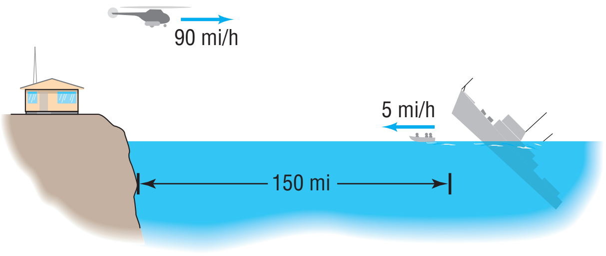
57. **Financial Planning** Steve, a recent retiree, requires \$5000 per year in extra income. He has \$70,000 to invest and can invest in A-rated bonds paying 8% per year or in a certificate of deposit (CD) paying 5% per year. How much money should be invested in each to realize exactly \$5000 in interest per year?

58. **Lightning and Thunder** A flash of lightning is seen, and the resulting thunderclap is heard 3 seconds later. If the speed of sound averages 1100 feet per second, how far away is the storm?

59. Physics: Intensity of Light The intensity I (in candlepower) of a certain light source obeys the equation $I = \frac{900}{x^2}$, where x is the distance (in meters) from the light. Over what range of distances can an object be placed from this light source so that the range of intensity of light is from 1600 to 3600 candlepower, inclusive?

60. Extent of Search and Rescue A search plane has a cruising speed of 250 miles per hour and carries enough fuel for at most 5 hours of flying. If there is a wind that averages 30 miles per hour and the direction of the search is with the wind one way and against it the other, how far can the search plane travel before it has to turn back?

61. Rescue at Sea A life raft, set adrift from a sinking ship 150 miles offshore, travels directly toward a Coast Guard station at the rate of 5 miles per hour. At the time that the raft is set adrift, a rescue helicopter is dispatched from the Coast Guard station. If the helicopter's average speed is 90 miles per hour, how long will it take the helicopter to reach the life raft?



62. Physics: Uniform Motion Two bees leave two locations 150 meters apart and fly, without stopping, back and forth between these two locations at average speeds of 3 meters per second and 5 meters per second, respectively. How long is it until the bees meet for the first time? How long is it until they meet for the second time?

63. Physics An object is thrown down from the top of a building 1280 feet tall with an initial velocity of 32 feet per second. The distance s (in feet) of the object from the ground after t seconds is $s = 1280 - 32t - 16t^2$.

(a) When will the object strike ground?
(b) What is the height of the object after 4 seconds?

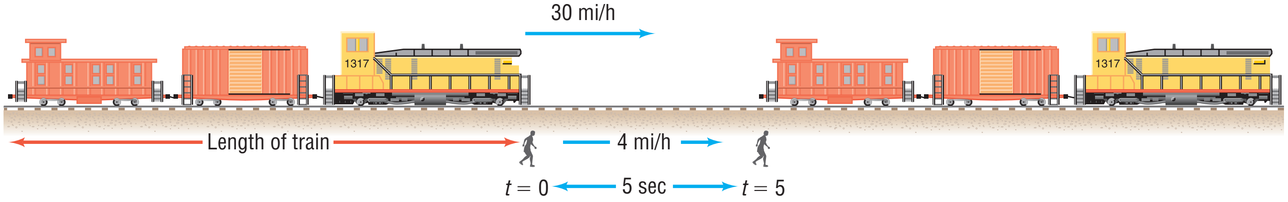
64. Working Together to Get a Job Done Clarissa and Shawna, working together, can paint the exterior of a house in 6 days. Clarissa by herself can complete this job in 5 days less than Shawna. How long will it take Clarissa to complete the job by herself?

65. Business: Blending Coffee A coffee house has 20 pounds of a coffee that sells for \$4 per pound. How many pounds of a coffee that sells for \$8 per pound should be mixed with the 20 pounds of \$4-per-pound coffee to obtain a blend that will sell for \$5 per pound? How much of the \$5-per-pound coffee is there to sell?

66. Chemistry: Salt Solutions How much water must be evaporated from 64 ounces of a 2% salt solution to make a 10% salt solution?

67. Geometry The hypotenuse of a right triangle measures 13 centimeters. Find the lengths of the legs if their sum is 17 centimeters.

68. Physics: Uniform Motion A man is walking at an average speed of 4 miles per hour alongside a railroad track. A freight train, going in the same direction at an average speed of 30 miles per hour, requires 5 seconds to pass the man. How long is the freight train? Give your answer in feet.



69. Framing a Painting An artist has 50 inches of oak trim to frame a painting. The frame is to have a border 3 inches wide surrounding the painting.

(a) If the painting is square, what are its dimensions? What are the dimensions of the frame?
(b) If the painting is rectangular with a length twice its width, what are the dimensions of the painting? What are the dimensions of the frame?

70. Using Two Pumps An 8-horsepower (hp) pump can fill a tank in 8 hours. A smaller, 3-hp pump fills the same tank in 12 hours. The pumps are used together to begin filling this tank. After 4 hours, the 8-hp pump breaks down. How long will it take the smaller pump to fill the tank?

71. Pleasing Proportion One formula stating the relationship between the length l and width w of a rectangle of "pleasing proportion" is $l^2 = w(l + w)$. How should a 4 foot by 8 foot sheet of plasterboard be cut so that the result is a rectangle of "pleasing proportion" with a width of 4 feet?

72. Business: Determining the Cost of a Charter A group of 20 senior citizens can charter a bus for a one-day excursion trip for \$15 per person. The charter company agrees to reduce the price of each ticket by 10¢ for each additional passenger in excess of 20 who goes on the trip, up to a maximum of 44 passengers (the capacity of the bus). If the final bill from the charter company was \$482.40, how many seniors went on the trip, and how much did each pay?

73. Evening Up a Race In a 100-meter race, Todd crosses the finish line 5 meters ahead of Scott. To even things up, Todd suggests to Scott that they race again, this time with Todd lining up 5 meters behind the start.

(a) Assuming that Todd and Scott run at the same pace as before, does the second race end in a tie?
(b) If not, who wins?
(c) By how many meters does he win?
(d) How far back should Todd start so that the race ends in a tie?

After running the race a second time, Scott, to even things up, suggests to Todd that he (Scott) line up 5 meters in front of the start.

(e) Assuming again that they run at the same pace as in the first race, does the third race result in a tie?
(f) If not, who wins?
(g) By how many meters?
(h) How far up should Scott start so that the race ends in a tie?

74. Explain the differences among the following three problems. Are there any similarities in their solution?

(a) Write the expression as a single quotient:

$$\frac{x}{x-2}+\frac{x}{x^2-4}$$

(b) Solve: $\dfrac{x}{x-2}+\dfrac{x}{x^2-4}=0$

(c) Solve: $\dfrac{x}{x-2}+\dfrac{x}{x^2-4}<0$

Chapter Test

CHAPTER Test Prep VIDEOS

The Chapter Test Prep Videos are step-by-step solutions available in MyMathLab®, or on this text's YouTube Channel. Flip back to the Resources for Success page for a link to this text's YouTube channel.

1. Suppose the points $(-2, -3)$ and $(4, 5)$ are the endpoints of a line segment.

(a) Find the distance between the two points.
(b) Find the midpoint of the line segment connecting the two points.

In Problems 2–9, solve each equation algebraically in the real number system. Answers should be exact.

2. $2x^2+6x=x-3$

3. $x+1=\sqrt{x+7}$

4. $2-\dfrac{3}{m}=\dfrac{2}{m+2}$

5. $5x-8=-4(x-1)+6$

6. $5|3-2b|-7=8$

7. $x^4+x^2=3x^2+8$

8. $x^2-4x+2=0$

9. $2x^2+x-1=x(x+7)+2$

In Problems 10 and 11, graph each equation by plotting points. Use a graphing utility to approximate the intercepts and label them on the graph.

10. $2x-7y=21$

11. $y=x^2-5$

In Problems 12–14, use a graphing utility to approximate the real solutions of each equation rounded to two decimal places. All solutions lie between −10 and 10.

12. $2x^3-x^2-2x+1=0$

13. $x^4-5x^2-8=0$

14. $-x^3+7x-2=x^2+3x-3$

In Problems 15–18, solve each inequality. Express your answer in set-builder notation or interval notation. Graph the solution set.

15. $\dfrac{2x+3}{4}<-2$

16. $|2x+3|-4\ge 3$

17. $-7<3-5x\le 8$

18. $|3x+4|<8$

In Problems 19–21, write each expression in the standard form $a+bi$.

19. $2(3-7i)-(4+11i)$

20. $(3+10i)(8+i)$

21. $\dfrac{2+i}{5-3i}$

22. Solve the equation $4x^2-4x+5=0$ in the complex number system.

23. Jamie is a cashier at a local supermarket. On average, she can check out a customer in 5 minutes. Scott, a cashier trainee, takes an average of 8.5 minutes to check out customers. How long would it take Jamie and Scott to check out 65 customers if they work together at different registers? Round to two decimal places if necessary.

24. A health food retailer sells a mixture of dried cherries, cranberries, and pecans for \$15.00 per pound and dried banana chips for \$2.25 per pound. The retailer decides to create a new mix by adding banana chips to the original mix. How many pounds of banana chips must be mixed with 40 pounds of the original mix to obtain a new mixture that sells for \$10.25 per pound with no loss in revenue? Round to two decimal places if necessary.

25. A Mountainsmith Auspex 4000 backpack originally sold for \$275.00 but is advertised at 42% off. What is the sale price of the backpack?

26. Glenn invests \$10,000 in a certificate of deposit (CD) that pays simple interest of 4% per annum. How much interest will Glenn earn after 3 months?

Chapter Projects

I. **Financing a Purchase** At some point in your life, you are likely to need to borrow money to finance a purchase. For example, most of us will finance the purchase of a car or a home. What is the mathematics behind financing a purchase? When you borrow money from a bank, the bank uses a rather complex equation (or formula) to determine how much you need to pay each month to repay the loan. There are a number of variables that determine the monthly payment. These variables include the amount borrowed, the interest rate, and the length of the loan. The interest rate is based on current economic conditions, the length of the loan, the type of item being purchased, and your credit history.

The following formula gives the monthly payment P required to pay off a loan amount L at an annual interest rate r, expressed as a decimal, but usually given as a percent. The time t, measured in months, is the length of the loan. For example, a 30-year loan requires $12 \times 30 = 360$ monthly payments.

$$P = L\left[\frac{\frac{r}{12}}{1-\left(1+\frac{r}{12}\right)^{-t}}\right]$$

P = monthly payment
L = loan amount
r = annual rate of interest expressed as a decimal
t = length of loan, in months

1. Interest rates change daily. Many websites post current interest rates on loans. Go to *www.bankrate.com* (or some other website that posts lenders' interest rates) and find the current best interest rate on a 60-month new-car purchase loan. Use this rate to determine the monthly payment on a $30,000 automobile loan.
2. Determine the total amount paid for the loan by multiplying the loan payment by the term of the loan.
3. Determine the total amount of interest paid by subtracting the loan amount from the total amount paid from question 2.
4. More often than not, we decide how much of a payment we can afford and use that information to determine the loan amount. Suppose you can afford a monthly payment of $500. Use the interest rate from question 1 to determine the maximum amount you can borrow. If you have $5000 to put down on the car, what is the maximum value of a car you can purchase?
5. Repeat questions 1 through 4 using a 72-month new-car purchase loan, a 60-month used-car purchase loan, and a 72-month used-car purchase loan.

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6. We can use the power of a spreadsheet, such as Excel, to create a loan amortization schedule. A loan amortization schedule is a list of the monthly payments, a breakdown of interest and principal, along with a current loan balance. Create a loan amortization schedule for each of the four loan scenarios discussed above, using the following as a guide. You may want to use an Internet search engine to research specific keystrokes for creating an amortization schedule in a spreadsheet. We supply a sample spreadsheet with formulas included as a guide. Use the spreadsheet to verify your results from questions 1 through 5.

Loan Information		**Payment Number**	**Payment Amount**	**Interest**	**Principal**	**Balance**	**Total Interest Paid**
Loan Amount	$30,000.00	1	=PMT($B$3/12,$B$5,–$B$2,0)	=B2*$B$3/12	=E2–F2	=B2–G2	=B2*$B$3/12
Annual Interest Rate	0.045	2	=PMT($B$3/12,$B$5,–$B$2,0)	=H2*$B$3/12	=E3–F3	=H2–G3	=I2+F3
Length of Loan (years)	5	3	=PMT($B$3/12,$B$5,–$B$2,0)	=H3*$B$3/12	=E4–F4	=H3–G4	=I3+F4
Number of Payments	=B4*12	⋮	⋮	⋮	⋮	⋮	⋮

7. Go to an online automobile website such as *www.cars.com* , *www.edmunds.com* , or *www.autobytel.com* . Research the types of vehicles you can afford for a monthly payment of $500. Decide on a vehicle you would purchase based on your analysis in questions 1–6. Be sure to justify your decision, and include the impact the term of the loan has on your decision. You might consider other factors in your decision, such as expected maintenance costs and insurance costs.

Citation: Excel ©2013 Microsoft Corporation. Used with permission from Microsoft.

The following project is also available on the Instructor's Resource Center (IRC):

II. **Project at Motorola: How Many Cellular Phones Can I Make?** An industrial engineer uses a model involving equations to be sure production levels meet customer demand.

2 Graphs

How to Value a House

Two things to consider in valuing a home are, first, how does it compare to similar homes that have sold recently? Is the asking price fair? And second, what value do you place on the advertised features and amenities? Yes, other people might value them highly, but do you?

Zestimate home valuation, RealestateABC.com, and Reply.com are among the many algorithmic (generated by a computer model) starting points in figuring out the value of a home. They show you how the home is priced relative to other homes in the area, but you need to add in all the things that only someone who has seen the house knows. You can do that using My Estimator, and then you create your own estimate and see how it stacks up against the asking price.

Looking at "Comps"

Knowing whether an asking price is fair will be important when you're ready to make an offer on a house. It will be even more important when your mortgage lender hires an appraiser to determine whether the house is worth the loan you want.

Check with your agent, Zillow.com, propertyshark.com, or other websites to see recent sales of homes in the area that are similar, or comparable, to what you're looking for. Print them out and keep these "comps" in a three-ring binder; you'll be referring to them quite a bit.

Note that "recent sales" usually means within the last six months. A sales price from a year ago may bear little or no relation to what is going on in your area right now. In fact, some lenders will not accept comps older than three months.

Market activity also determines how easy or difficult it is to find accurate comps. In a "hot" or busy market, with sales happening all the time, you're likely to have lots of comps to choose from. In a less active market, finding reasonable comps becomes harder. And if the home you're looking at has special design features, finding a comparable property is harder still. It's also necessary to know what's going on in a given sub-segment. Maybe large, high-end homes are selling like hotcakes, but owners of smaller houses are staying put, or vice versa.

Source: *http://allmyhome.blogspot.com/2008/07/how-to-value-house.html*

—See the Internet-based Chapter Project—

Outline

••• A Look Back

Chapter R reviewed algebra essentials and geometry essentials. Chapter 1 introduced the Cartesian plane to graph equations in two variables and examined equations in one variable.

A Look Ahead •••

Here we continue our discussion of the Cartesian plane by introducing additional techniques that can be used to obtain the graph of an equation with two variables. We also introduce two specific graphs: lines and circles. We conclude the chapter by looking at variation.

2.1 Intercepts; Symmetry; Graphing Key Equations

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Introduction to Graphing Equations (Section 1.1, p. 88–92)
- Solving Quadratic Equations (Section 1.3, pp. 110–116)
- Finding Intercepts Using a Graphing Utility (Section 1.1, pp. 93–94)
- Finding Intercepts from a Graph (Section 1.1, p. 93)

Now Work the **'Are You Prepared?'** problems on page 170.

OBJECTIVES
1 Find Intercepts Algebraically from an Equation (p. 165)
2 Test an Equation for Symmetry (p. 166)
3 Know How to Graph Key Equations (p. 168)

1 Find Intercepts Algebraically from an Equation

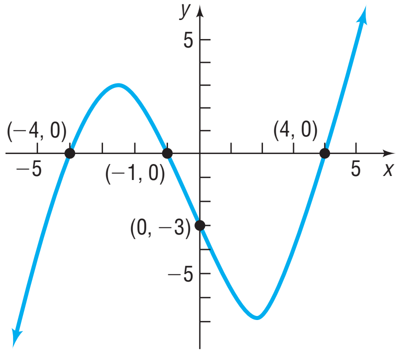


Figure 1

In Section 1.1, we discussed how to find intercepts from a graph and how to approximate intercepts from an equation using a graphing utility. Now we discuss how to find intercepts from an equation algebraically. To better understand the procedure, look at Figure 1. From the graph, note that the intercepts are $(-4, 0), (-1, 0), (4, 0)$, and $(0, -3)$. The x-intercepts are $-4, -1$, and 4. The y-intercept is -3. Notice that x-intercepts have y-coordinates that equal 0; y-intercepts have x-coordinates that equal 0. This leads to the following procedure for finding intercepts.

Procedure for Finding Intercepts

1. To find the x-intercept(s), if any, of the graph of an equation, let $y = 0$ in the equation and solve for x, where x is a real number.
2. To find the y-intercept(s), if any, of the graph of an equation, let $x = 0$ in the equation and solve for y, where y is a real number.

EXAMPLE 1 **Finding Intercepts from an Equation**

Find the x-intercept(s) and the y-intercept(s) of the graph of $y = x^2 - 4$. Then graph $y = x^2 - 4$ by plotting points.

Solution To find the x-intercept(s), let $y = 0$ and obtain the equation

$$\begin{array}{rl} x^2 - 4 = 0 & \quad y = x^2 - 4 \text{ with } y = 0 \\ (x+2)(x-2) = 0 & \quad \text{Factor.} \\ x + 2 = 0 \quad \text{or} \quad x - 2 = 0 & \quad \text{Zero-Product Property} \\ x = -2 \quad \text{or} \quad\quad x = 2 & \quad \text{Solve.} \end{array}$$

The equation has two solutions, -2 and 2. The x-intercepts are -2 and 2.

To find the y-intercept(s), let $x = 0$ in the equation.

$$\begin{aligned} y &= x^2 - 4 \\ &= 0^2 - 4 = -4 \end{aligned}$$

The y-intercept is -4.

Since $x^2 \geq 0$ for all x, we deduce from the equation $y = x^2 - 4$ that $y \geq -4$ for all x. This information, the intercepts, and the points from Table 1 on the next page enable us to graph $y = x^2 - 4$ by hand. See Figure 2.

Table 1

x	$y = x^2 - 4$	(x, y)
-3	$(-3)^2 - 4 = 5$	$(-3, 5)$
-1	-3	$(-1, -3)$
1	-3	$(1, -3)$
3	5	$(3, 5)$

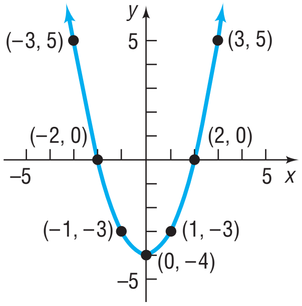
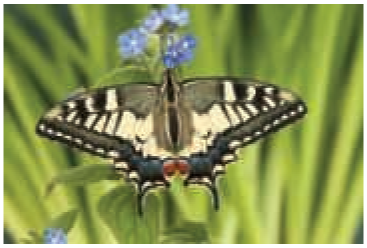


Figure 2 $y = x^2 - 4$ ■



Now Work PROBLEM 15

2 Test an Equation for Symmetry

Another helpful tool for graphing equations by hand involves *symmetry*, particularly symmetry with respect to the x-axis, the y-axis, and the origin.

Symmetry often occurs in nature. Consider the picture of the butterfly. Do you see the symmetry?

DEFINITION

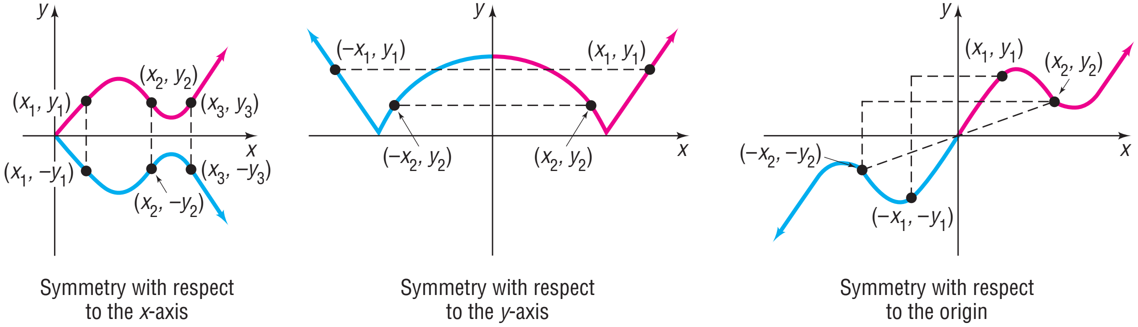
A graph is said to be **symmetric with respect to the x-axis** if, for every point (x, y) on the graph, the point $(x, -y)$ is also on the graph.

A graph is said to be **symmetric with respect to the y-axis** if, for every point (x, y) on the graph, the point $(-x, y)$ is also on the graph.

A graph is said to be **symmetric with respect to the origin** if, for every point (x, y) on the graph, the point $(-x, -y)$ is also on the graph. ■

Figure 3 illustrates the definition. Notice that, when a graph is symmetric with respect to the x-axis, the part of the graph above the x-axis is a reflection or mirror image of the part below it, and vice versa. When a graph is symmetric with respect to the y-axis, the part of the graph to the right of the y-axis is a reflection of the part to the left of it, and vice versa. Symmetry with respect to the origin may be viewed in two ways:

1. As a reflection about the y-axis, followed by a reflection about the x-axis
2. As a projection along a line through the origin so that the distances from the origin are equal



Symmetry with respect to the x-axis

Symmetry with respect to the y-axis

Symmetry with respect to the origin

Figure 3

EXAMPLE 2 **Symmetric Points**

(a) If a graph is symmetric with respect to the x-axis and the point $(4, 2)$ is on the graph, then the point $(4, -2)$ is also on the graph.

(b) If a graph is symmetric with respect to the y-axis and the point $(4, 2)$ is on the graph, then the point $(-4, 2)$ is also on the graph.

(c) If a graph is symmetric with respect to the origin and the point $(4, 2)$ is on the graph, then the point $(-4, -2)$ is also on the graph. ■

Now Work PROBLEM 23

When the graph of an equation is symmetric with respect to the x-axis, the y-axis, or the origin, the number of points that you need to plot in order to see the pattern is reduced. For example, if the graph of an equation is symmetric with respect to the y-axis, then once points to the right of the y-axis are plotted, an equal number of points on the graph can be obtained by reflecting them about the y-axis. Because of this, before we graph an equation, we first want to determine whether it has any symmetry. The following tests are used for this purpose.

Tests for Symmetry

To test the graph of an equation for symmetry with respect to the

x-Axis Replace y by $-y$ in the equation. If an equivalent equation results, the graph of the equation is symmetric with respect to the x-axis.

y-Axis Replace x by $-x$ in the equation. If an equivalent equation results, the graph of the equation is symmetric with respect to the y-axis.

Origin Replace x by $-x$ and y by $-y$ in the equation. If an equivalent equation results, the graph of the equation is symmetric with respect to the origin.

EXAMPLE 3 Finding Intercepts and Testing an Equation for Symmetry

For the equation $y = \frac{x^2 - 4}{x^2 + 1}$: (a) find the intercepts and (b) test for symmetry.

Solution (a) To obtain the x-intercept(s), let $y = 0$ in the equation and solve for x.

$$\frac{x^2 - 4}{x^2 + 1} = 0 \qquad \textbf{Let } y = 0.$$

$$x^2 - 4 = 0 \qquad \textbf{Multiply both sides by } x^2 + 1.$$

$$x = -2 \quad \text{or} \quad x = 2 \qquad \textbf{Factor and use the Zero-Product Property.}$$

To obtain the y-intercept(s), let $x = 0$ in the equation and solve for y.

$$y = \frac{x^2 - 4}{x^2 + 1} = \frac{0^2 - 4}{0^2 + 1} = \frac{-4}{1} = -4$$

The x-intercepts are -2 and 2; the y-intercept is -4.

(b) Now test the equation for symmetry with respect to the x-axis, the y-axis, and the origin.

x-Axis: To test for symmetry with respect to the x-axis, replace y by $-y$. Since

$$-y = \frac{x^2 - 4}{x^2 + 1} \text{ is not equivalent to } y = \frac{x^2 - 4}{x^2 + 1}$$

the graph of the equation is not symmetric with respect to the x-axis.

y-Axis: To test for symmetry with respect to the y-axis, replace x by $-x$. Since

$$y = \frac{(-x)^2 - 4}{(-x)^2 + 1} = \frac{x^2 - 4}{x^2 + 1} \text{ is equivalent to } y = \frac{x^2 - 4}{x^2 + 1}$$

the graph of the equation is symmetric with respect to the y-axis.

Origin: To test for symmetry with respect to the origin, replace x by $-x$ and y by $-y$.

$$-y = \frac{(-x)^2 - 4}{(-x)^2 + 1} \qquad \textbf{Replace } x \textbf{ by } -x \textbf{ and } y \textbf{ by } -y.$$

$$-y = \frac{x^2 - 4}{x^2 + 1} \qquad \textbf{Simplify.}$$

Since the result is not equivalent to the original equation, the graph of the equation $y = \frac{x^2 - 4}{x^2 + 1}$ is not symmetric with respect to the origin. ■

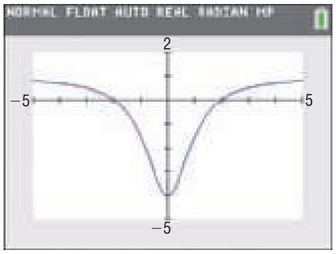


Figure 4 $y = \frac{x^2 - 4}{x^2 + 1}$

Seeing the Concept

Figure 4 shows the graph of $y = \frac{x^2 - 4}{x^2 + 1}$ using a TI-84 Plus C graphing calculator. Do you see the symmetry with respect to the *y*-axis? Also, did you notice that the point (2, 0) is on the graph along with $(-2, 0)$? How could we have found the second *x*-intercept using symmetry? ■

Now Work PROBLEM 53

3 Know How to Graph Key Equations

There are certain equations whose graphs we should be able to easily visualize in our mind's eye. For example, you should know the graph of $y = x^2$ discussed in Example 7 in Section 1.1. The next three examples use intercepts, symmetry, and point plotting to obtain the graphs of additional key equations. It is important to know the graphs of these key equations because we use them later.

EXAMPLE 4 Graphing the Equation $y = x^3$ by Finding Intercepts and Checking for Symmetry

Graph the equation $y = x^3$ by hand by plotting points. Find any intercepts and check for symmetry first.

Solution First, find the intercepts of $y = x^3$. When $x = 0$, then $y = 0$; and when $y = 0$, then $x = 0$. The origin $(0, 0)$ is the only intercept. Now test $y = x^3$ for symmetry.

x-Axis: Replace y by $-y$. Since $-y = x^3$ is not equivalent to $y = x^3$, the graph is not symmetric with respect to the *x*-axis.

y-Axis: Replace x by $-x$. Since $y = (-x)^3 = -x^3$ is not equivalent to $y = x^3$, the graph is not symmetric with respect to the *y*-axis.

Origin: Replace x by $-x$ and y by $-y$. Since $-y = (-x)^3 = -x^3$ is equivalent to $y = x^3$ (multiply both sides by -1), the graph is symmetric with respect to the origin.

To graph by hand, use the equation to obtain several points on the graph. Because of the symmetry with respect to the origin, we only need to locate points on the graph for which $x \geq 0$. See Table 2. Points on the graph could also be obtained using the TABLE feature on a graphing utility. See Table 3. Do you see the symmetry with respect to the origin from the table? Figure 5 shows the graph.

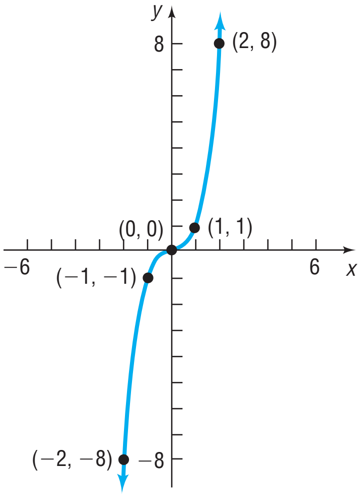
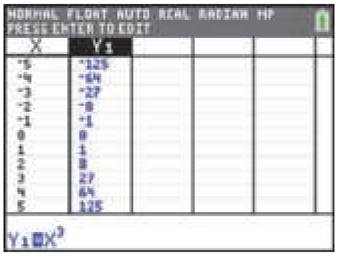


Figure 5 $y = x^3$

Table 2

x	$y = x^3$	(x, y)
0	0	(0, 0)
1	1	(1, 1)
2	8	(2, 8)
3	27	(3, 27)

Table 3



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PRESS ENTER TO EDIT

X	Y1
-5	-125
-4	-64
-3	-27
-2	-8
-1	-1
0	0
1	1
2	8
3	27
4	64
5	125

Y1■X³

■

EXAMPLE 5 Graphing the Equation $x = y^2$

(a) Graph the equation $x = y^2$. Find any intercepts and check for symmetry first.
(b) Graph $x = y^2$ where $y \geq 0$.

Solution (a) The lone intercept is $(0, 0)$. The graph is symmetric with respect to the x-axis since $x = (-y)^2$ is equivalent to $x = y^2$. The graph is not symmetric with respect to the y-axis or the origin.

To graph $x = y^2$ by hand, use the equation to obtain several points on the graph. Because the equation is solved for x, it is easier to assign values to y and use the equation to determine the corresponding values of x. Because of the symmetry, start by finding points whose y-coordinates are nonnegative. Then use the symmetry to find additional points on the graph. See Table 4. For example, since $(1, 1)$ is on the graph, so is $(1, -1)$. Since $(4, 2)$ is on the graph, so is $(4, -2)$, and so on. Plot these points and connect them with a smooth curve to obtain Figure 6.

Table 4

y	$x = y^2$	(x, y)
0	0	(0, 0)
1	1	(1, 1)
2	4	(4, 2)
3	9	(9, 3)

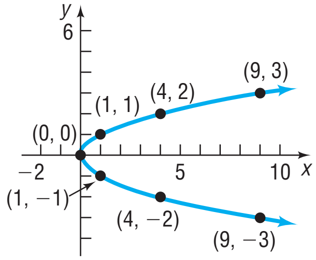


Figure 6 $x = y^2$

To graph the equation $x = y^2$ using a graphing utility, write the equation in the form $y = \{\text{expression in } x\}$. We proceed to solve for y.

$$x = y^2$$
$$y^2 = x$$
$$y = \pm\sqrt{x} \quad \text{Square Root Method}$$

To graph $x = y^2$, graph both $Y_1 = \sqrt{x}$ and $Y_2 = -\sqrt{x}$ on the same screen. Figure 7 shows the result. Table 5 shows various values of y for a given value of x when $Y_1 = \sqrt{x}$ and $Y_2 = -\sqrt{x}$. Notice that when $x < 0$ we get an error. Can you explain why?

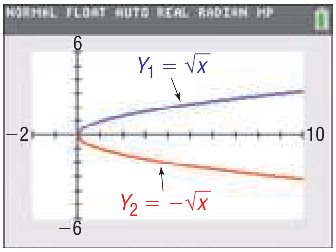
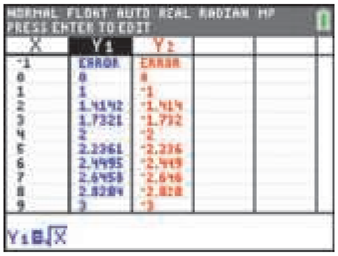


Figure 7

Table 5



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PRESS ENTER TO EDIT

X	Y_1	Y_2
−1	ERROR	ERROR
0	0	0
1	1	−1
2	1.4142	−1.414
3	1.7321	−1.732
4	2	−2
5	2.2361	−2.236
6	2.4495	−2.449
7	2.6458	−2.646
8	2.8284	−2.828
9	3	−3

$Y_1 = \sqrt{X}$

(b) If we restrict y so that $y \geq 0$, the equation $x = y^2$, $y \geq 0$, may be written as $y = \sqrt{x}$. The portion of the graph of $x = y^2$ in quadrant I plus the origin is the graph of $y = \sqrt{x}$. See Figure 8.

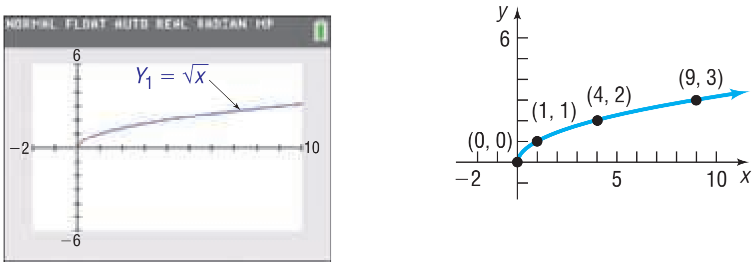


Figure 8 $y = \sqrt{x}$

EXAMPLE 6 Graphing the Equation $y = \dfrac{1}{x}$

Graph the equation $y = \dfrac{1}{x}$. Find any intercepts and check for symmetry first.

Solution Check for intercepts first. If we let $x = 0$, we obtain a 0 in the denominator, which is not defined. We conclude that there is no y-intercept. If we let $y = 0$, we get the

Table 6

x	$y = \frac{1}{x}$	(x, y)
$\frac{1}{10}$	10	$\left(\frac{1}{10}, 10\right)$
$\frac{1}{3}$	3	$\left(\frac{1}{3}, 3\right)$
$\frac{1}{2}$	2	$\left(\frac{1}{2}, 2\right)$
1	1	$(1, 1)$
2	$\frac{1}{2}$	$\left(2, \frac{1}{2}\right)$
3	$\frac{1}{3}$	$\left(3, \frac{1}{3}\right)$
10	$\frac{1}{10}$	$\left(10, \frac{1}{10}\right)$

equation $\frac{1}{x} = 0$, which has no solution. We conclude that there is no x-intercept. The graph of $y = \frac{1}{x}$ does not cross or touch the coordinate axes.

Next check for symmetry.

x-Axis: Replacing y by $-y$ yields $-y = \frac{1}{x}$, which is not equivalent to $y = \frac{1}{x}$.

y-Axis: Replacing x by $-x$ yields $y = \frac{1}{-x} = -\frac{1}{x}$, which is not equivalent to $y = \frac{1}{x}$.

Origin: Replacing x by $-x$ and y by $-y$ yields $-y = -\frac{1}{x}$, which is equivalent to $y = \frac{1}{x}$. The graph is symmetric only with respect to the origin.

Use the equation to form Table 6 and obtain some points on the graph. Because of symmetry, we only find points (x, y) for which x is positive. From Table 6 we infer that, if x is a large and positive number, then $y = \frac{1}{x}$ is a positive number close to 0. We also infer that if x is a positive number close to 0 then $y = \frac{1}{x}$ is a large and positive number. Armed with this information, we can graph the equation. Figure 9 illustrates some of these points and the graph of $y = \frac{1}{x}$. Observe how the absence of intercepts and the existence of symmetry with respect to the origin were utilized. Figure 10 confirms our algebraic analysis using a TI-84 Plus C.

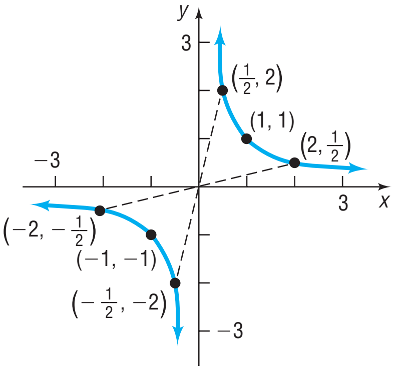


Figure 9 $y = \frac{1}{x}$

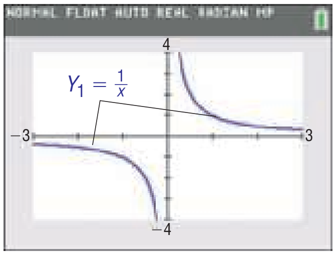


Figure 10 $y = \frac{1}{x}$

2.1 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Graph $y = 2x - 4$ by plotting points. Based on the graph, determine the intercepts. (pp. 88–90, 93)
2. Solve: $x^2 - 4x - 12 = 0$ (pp. 110–116)

Concepts and Vocabulary

3. The points, if any, at which a graph crosses or touches the coordinate axes are called ________.
4. If for every point (x, y) on the graph of an equation the point $(-x, y)$ is also on the graph, then the graph is symmetric with respect to the ________.
5. If the graph of an equation is symmetric with respect to the y-axis and -4 is an x-intercept of this graph, then ________ is also an x-intercept.
6. If the graph of an equation is symmetric with respect to the origin and $(3, -4)$ is a point on the graph, then ________ is also a point on the graph.
7. ***True or False*** To find the y-intercepts of the graph of an equation, let $x = 0$ and solve for y.
8. ***True or False*** If a graph is symmetric with respect to the x-axis, then it cannot be symmetric with respect to the y-axis.

9. To find the x-intercept(s), if any, of the graph of an equation, let __________ in the equation and solve for x.
(a) $y = 0$ (b) $x = 0$
(c) $y = x$ (d) $x = -y$

10. To test whether the graph of an equation is symmetric with respect to the origin, replace __________ in the equation and simplify. If an equivalent equation results, then the graph is symmetric with respect to the origin.
(a) x by $-x$ (b) y by $-y$
(c) x by $-x$ and y by $-y$ (d) x by $-y$ and y by $-x$

Skill Building

In Problems 11–22, find the intercepts and graph each equation by plotting points. Be sure to label the intercepts.

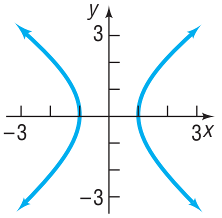
11. $y = x + 2$
12. $y = x - 6$
13. $y = 2x + 8$
14. $y = 3x - 9$
15. $y = x^2 - 1$
16. $y = x^2 - 9$
17. $y = -x^2 + 4$
18. $y = -x^2 + 1$
19. $2x + 3y = 6$
20. $5x + 2y = 10$
21. $9x^2 + 4y = 36$
22. $4x^2 + y = 4$

In Problems 23–32, plot each point. Then plot the point that is symmetric to it with respect to (a) the x-axis; (b) the y-axis; (c) the origin.

23. $(3, 4)$
24. $(5, 3)$
25. $(-2, 1)$
26. $(4, -2)$
27. $(5, -2)$
28. $(-1, -1)$
29. $(-3, -4)$
30. $(4, 0)$
31. $(0, -3)$
32. $(-3, 0)$

In Problems 33–44, the graph of an equation is given. (a) Find the intercepts. (b) Indicate whether the graph is symmetric with respect to the x-axis, the y-axis, or the origin.

33.



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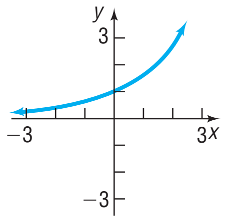

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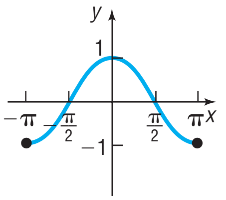

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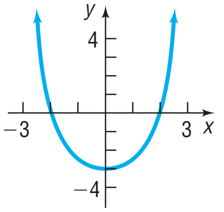

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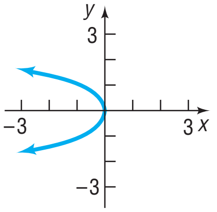

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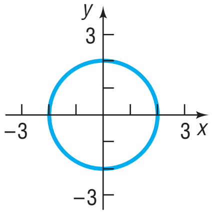

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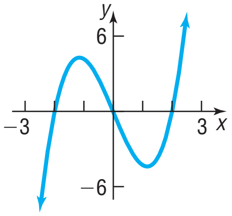

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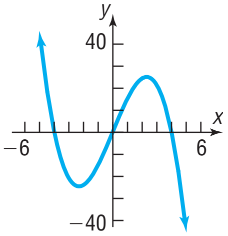

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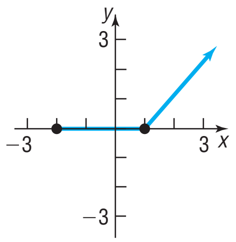

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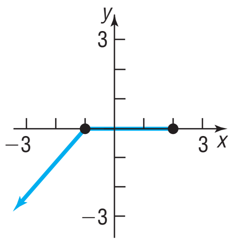

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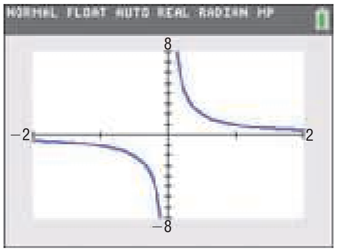

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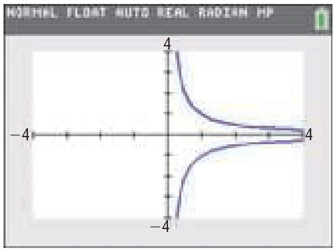
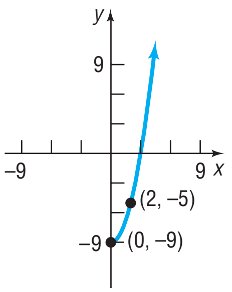
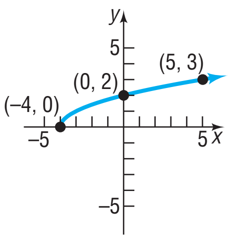


In Problems 45–48, draw a complete graph so that it has the type of symmetry indicated.

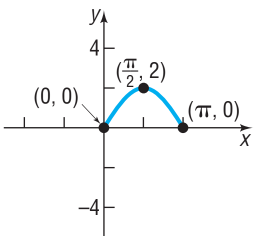
45. y-axis



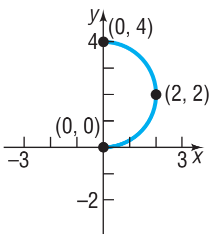
46. x-axis



47. Origin



48. y-axis



In Problems 49–64, list the intercepts and test for symmetry.

49. $y^2 = x + 4$

50. $y^2 = x + 9$

51. $y = \sqrt[3]{x}$

52. $y = \sqrt[5]{x}$

53. $y = x^4 - 8x^2 - 9$

54. $y = x^4 - 2x^2 - 8$

55. $9x^2 + 4y^2 = 36$

56. $4x^2 + y^2 = 4$

57. $y = x^3 - 27$

58. $y = x^4 - 1$

59. $y = x^2 - 3x - 4$

60. $y = x^2 + 4$

61. $y = \dfrac{3x}{x^2 + 9}$

62. $y = \dfrac{x^2 - 4}{2x}$

63. $y = \dfrac{-x^3}{x^2 - 9}$

64. $y = \dfrac{x^4 + 1}{2x^5}$

In Problems 65–68, draw a quick sketch of each equation.

65. $y = x^3$

66. $x = y^2$

67. $y = \sqrt{x}$

68. $y = \dfrac{1}{x}$

69. If $(3, b)$ is a point on the graph of $y = 4x + 1$, what is b?

70. If $(-2, b)$ is a point on the graph of $2x + 3y = 2$, what is b?

71. If $(a, 4)$ is a point on the graph of $y = x^2 + 3x$, what is a?

72. If $(a, -5)$ is a point on the graph of $y = x^2 + 6x$, what is a?

Mixed Practice

In Problems 73–80, (a) find the intercepts of each equation, (b) test each equation for symmetry with respect to the x-axis, the y-axis, and the origin, and (c) graph each equation by hand by plotting points. Be sure to label the intercepts on the graph and use any symmetry to assist in drawing the graph. Verify your results using a graphing utility.

73. $y = x^2 - 5$

74. $y = x^2 - 8$

75. $x - y^2 = -9$

76. $x + y^2 = 4$

77. $x^2 + y^2 = 9$

78. $x^2 + y^2 = 16$

79. $y = x^3 - 4x$

80. $y = x^3 - x$

Applications and Extensions

81. Given that the point (1, 2) is on the graph of an equation that is symmetric with respect to the origin, what other point is on the graph?

82. If the graph of an equation is symmetric with respect to the y-axis and 6 is an x-intercept of this graph, name another x-intercept.

83. If the graph of an equation is symmetric with respect to the origin and -4 is an x-intercept of this graph, name another x-intercept.

84. If the graph of an equation is symmetric with respect to the x-axis and 2 is a y-intercept, name another y-intercept.

85. Microphones In studios and on stages, cardioid microphones are often preferred for the richness they add to voices and for their ability to reduce the level of sound from the sides and rear of the microphone. Suppose one such cardioid pattern is given by the equation $(x^2 + y^2 - x)^2 = x^2 + y^2$.

(a) Find the intercepts of the graph of the equation.

(b) Test for symmetry with respect to the x-axis, y-axis, and origin.

Source: *www.notaviva.com*

86. Solar Energy The solar electric generating systems at Kramer Junction, California, use parabolic troughs to heat a heat-transfer fluid to a high temperature. This fluid is used to generate steam that drives a power conversion system to produce electricity. For troughs 7.5 feet wide, an equation for the cross-section is $16y^2 = 120x - 225$.

(a) Find the intercepts of the graph of the equation.

(b) Test for symmetry with respect to the x-axis, y-axis, and origin.

Source: *U.S. Department of Energy*

Explaining Concepts: Discussion and Writing

87. (a) Graph $y = \sqrt{x^2}$, $y = x$, $y = |x|$, and $y = (\sqrt{x})^2$, noting which graphs are the same.

(b) Explain why the graphs of $y = \sqrt{x^2}$ and $y = |x|$ are the same.

(c) Explain why the graphs of $y = x$ and $y = (\sqrt{x})^2$ are not the same.

(d) Explain why the graphs of $y = \sqrt{x^2}$ and $y = x$ are not the same.

88. Explain what is meant by a complete graph.

89. Draw a graph of an equation that contains two x-intercepts; at one the graph crosses the x-axis, and at the other the graph touches the x-axis.

90. Make up an equation with the intercepts $(2, 0)$, $(4, 0)$, and $(0, 1)$. Compare your equation with a friend's equation. Comment on any similarities.

91. Draw a graph that contains the points $(-2, -1)$, $(0, 1)$, $(1, 3)$, and $(3, 5)$. Compare your graph with those of other students. Are most of the graphs almost straight lines? How many are "curved"? Discuss the various ways that these points might be connected.

92. An equation is being tested for symmetry with respect to the x-axis, the y-axis, and the origin. Explain why, if two of these symmetries are present, the remaining one must also be present.

93. Draw a graph that contains the points $(-2, 5)$, $(-1, 3)$, and $(0, 2)$ that is symmetric with respect to the y-axis. Compare your graph with those of other students; comment on any similarities. Can a graph contain these points and be symmetric with respect to the x-axis? the origin? Why or why not?

Retain Your Knowledge

Problems 94–97 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

94. Find the value of $\dfrac{x + y}{x - y}$ if $x = 6$ and $y = -2$.

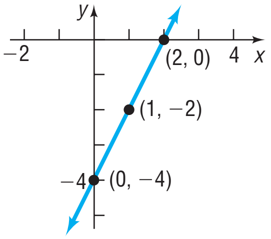
95. Factor $3x^2 - 30x + 75$ completely.

96. Simplify: $\sqrt{-196}$

97. Solve $x^2 - 8x + 4 = 0$ by completing the square.

'Are You Prepared?' Answers

1.



x-intercept: 2; y-intercept: -4

2. $\{-2, 6\}$

2.2 Lines

OBJECTIVES
1 Calculate and Interpret the Slope of a Line (p. 173)
2 Graph Lines Given a Point and the Slope (p. 176)
3 Find the Equation of a Vertical Line (p. 176)
4 Use the Point–Slope Form of a Line; Identify Horizontal Lines (p. 177)
5 Write the Equation of a Line in Slope–Intercept Form (p. 178)
6 Find the Equation of a Line Given Two Points (p. 179)
7 Graph Lines Written in General Form Using Intercepts (p. 180)
8 Find Equations of Parallel Lines (p. 181)
9 Find Equations of Perpendicular Lines (p. 182)

In this section we study a certain type of equation that contains two variables, called a *linear equation,* and its graph, a *line*.

1 Calculate and Interpret the Slope of a Line

Consider the staircase illustrated in Figure 11. Each step contains exactly the same horizontal **run** and the same vertical **rise**. The ratio of the rise to the run, called the *slope*, is a numerical measure of the steepness of the staircase. For example, if the run is increased and the rise remains the same, the staircase becomes less steep. If the run is kept the same but the rise is increased, the staircase becomes more steep. This important characteristic of a line is best defined using rectangular coordinates.

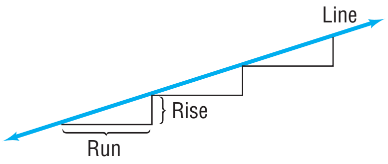


Figure 11

DEFINITION

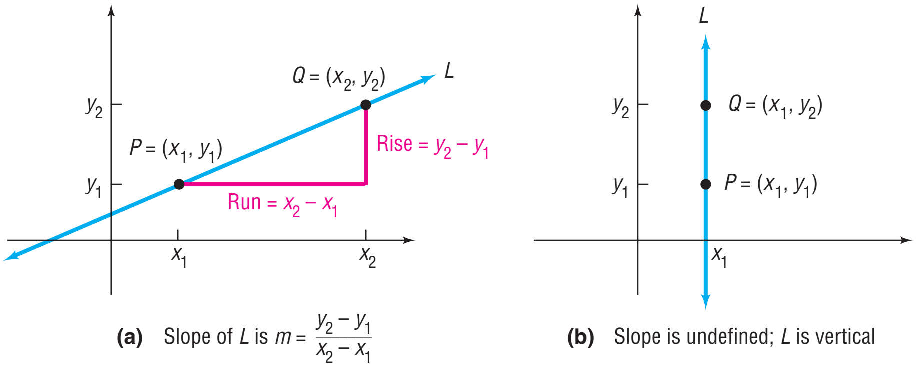
Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be two distinct points. If $x_1 \neq x_2$, the **slope *m*** of the nonvertical line L containing P and Q is defined by the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad x_1 \neq x_2 \qquad \textbf{(1)}$$

If $x_1 = x_2$, L is a **vertical line** and the slope m of L is **undefined** (since this results in division by 0).

Figure 12(a) provides an illustration of the slope of a nonvertical line; Figure 12(b) illustrates a vertical line.

Figure 12



(a) Slope of L is $m = \frac{y_2 - y_1}{x_2 - x_1}$ **(b)** Slope is undefined; L is vertical

As Figure 12(a) illustrates, the slope m of a nonvertical line may be viewed as

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Rise}}{\text{Run}} \quad \text{or} \quad m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\Delta y}{\Delta x}$$

In Words

The symbol Δ is the Greek letter delta. In mathematics, Δ is read "change in," so $\frac{\Delta y}{\Delta x}$ is read "change in y divided by change in x."

That is, the slope m of a nonvertical line measures the amount y changes when x changes from x_1 to x_2. The expression $\frac{\Delta y}{\Delta x}$ is called the **average rate of change** of y with respect to x.

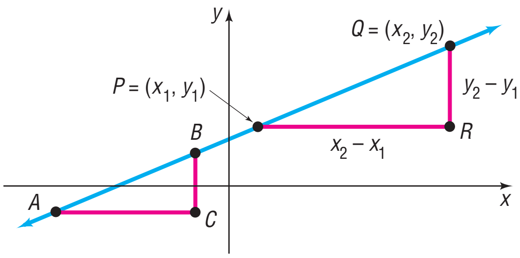
Two comments about computing the slope of a nonvertical line may prove helpful:

1. Any two distinct points on the line can be used to compute the slope of the line. (See Figure 13 for justification.) Since any two distinct points can be used to compute the slope of a line, the average rate of change of a line is always the same number.

Figure 13

Triangles *ABC* and *PQR* are similar (equal angles), so ratios of corresponding sides are equal. Then

Slope using P and $Q = \frac{y_2 - y_1}{x_2 - x_1} = \frac{d(B, C)}{d(A, C)}$ = Slope using A and B



2. The slope of a line may be computed from $P = (x_1, y_1)$ to $Q = (x_2, y_2)$ or from Q to P because

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

EXAMPLE 1 Finding and Interpreting the Slope of a Line Given Two Points

The slope m of the line containing the points $(1, 2)$ and $(5, -3)$ may be computed as

$$m = \frac{-3 - 2}{5 - 1} = \frac{-5}{4} = -\frac{5}{4} \quad \text{or as} \quad m = \frac{2 - (-3)}{1 - 5} = \frac{5}{-4} = -\frac{5}{4}$$

For every 4-unit change in x, y will change by -5 units. That is, if x increases by 4 units, then y will decrease by 5 units. The average rate of change of y with respect to x is $-\frac{5}{4}$. ■

Now Work PROBLEMS **13** AND **19**

Square Screens

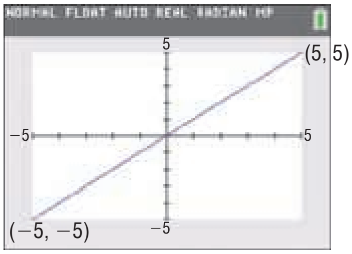


Figure 14 $y = x$

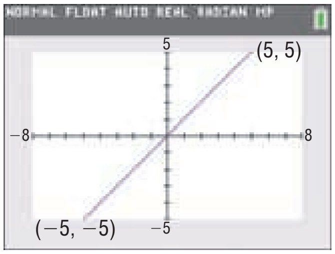


Figure 15 $y = x$

To get an undistorted view of slope, the same scale must be used on each axis. However, most graphing utilities have a rectangular screen. Because of this, using the same interval for both x and y will result in a distorted view. For example, Figure 14 shows the graph of the line $y = x$ connecting the points $(-5, -5)$ and $(5, 5)$. We expect the line to bisect the first and third quadrants, but it doesn't. We need to adjust the selections for Xmin, Xmax, Ymin, and Ymax so that a **square screen** results. On a TI-84 Plus C, this is accomplished by setting the ratio of x to y at 8 : 5.*

Figure 15 shows the graph of the line $y = x$ on a square screen using a TI-84 Plus C. Notice that the line now bisects the first and third quadrants. Compare this illustration to Figure 14.

To get a better idea of the meaning of the slope m of a line, consider the following.

Exploration On the same square screen, graph the following equations:

$Y_1 = 0$	**Slope of the line is 0.**
$Y_2 = \frac{1}{4}x$	**Slope of the line is $\frac{1}{4}$.**
$Y_3 = \frac{1}{2}x$	**Slope of the line is $\frac{1}{2}$.**
$Y_4 = x$	**Slope of the line is 1.**
$Y_5 = 2x$	**Slope of the line is 2.**
$Y_6 = 6x$	**Slope of the line is 6.**

See Figure 16. ■

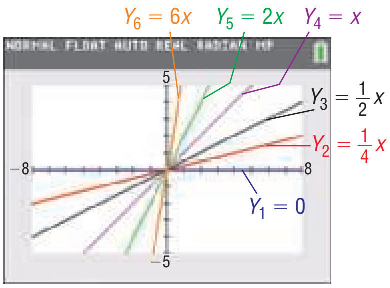


Figure 16 $y = mx$

Exploration On the same square screen, graph the following equations:

$Y_1 = 0$	**Slope of the line is 0.**
$Y_2 = -\frac{1}{4}x$	**Slope of the line is $-\frac{1}{4}$.**
$Y_3 = -\frac{1}{2}x$	**Slope of the line is $-\frac{1}{2}$.**
$Y_4 = -x$	**Slope of the line is -1.**
$Y_5 = -2x$	**Slope of the line is -2.**
$Y_6 = -6x$	**Slope of the line is -6.**

See Figure 17. ■

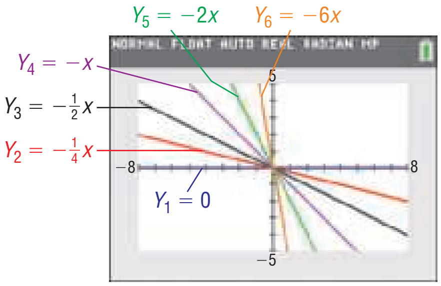


Figure 17 $y = mx$

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*Most graphing utilities have a feature that automatically squares the viewing window. Consult your owner's manual for the appropriate keystrokes.

Figures 16 and 17 on the previous page illustrate the following facts:

1. When the slope of a line is positive, the line slants upward from left to right.
2. When the slope of a line is negative, the line slants downward from left to right.
3. When the slope is 0, the line is horizontal.

Figures 16 and 17 also illustrate that the closer the line is to the vertical position, the greater the magnitude of the slope. Thus, a line with slope 6 is steeper than a line whose slope is 3.

2 Graph Lines Given a Point and the Slope

EXAMPLE 2 Graphing a Line Given a Point and a Slope

Draw a graph of the line that contains the point $(3, 2)$ and has a slope of:

(a) $\dfrac{3}{4}$ (b) $-\dfrac{4}{5}$

Solution

(a) Slope $= \dfrac{\text{Rise}}{\text{Run}}$. The slope $\dfrac{3}{4}$ means that for every horizontal movement (run) of 4 units to the right, there will be a vertical movement (rise) of 3 units. Start at the given point $(3, 2)$ and move 4 units to the right and 3 units up, arriving at the point $(7, 5)$. Drawing the line through this point and the point $(3, 2)$ gives the graph. See Figure 18.

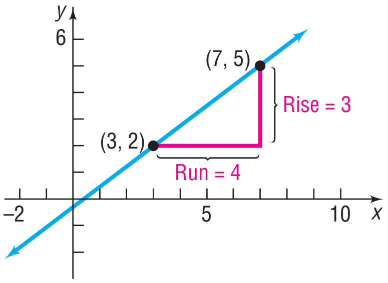

Figure 18

(b) The fact that the slope is

$$-\frac{4}{5} = \frac{-4}{5} = \frac{\text{Rise}}{\text{Run}}$$

means that for every horizontal movement of 5 units to the right there will be a corresponding vertical movement of -4 units (a downward movement of 4 units). Start at the given point $(3, 2)$ and move 5 units to the right and then 4 units down, arriving at the point $(8, -2)$. Drawing the line through these points gives the graph. See Figure 19.

Alternatively, consider that

$$-\frac{4}{5} = \frac{4}{-5} = \frac{\text{Rise}}{\text{Run}}$$

so for every horizontal movement of -5 units (a movement to the left of 5 units), there will be a corresponding vertical movement of 4 units (upward). This approach leads to the point $(-2, 6)$, which is also on the graph of the line in Figure 19. ■

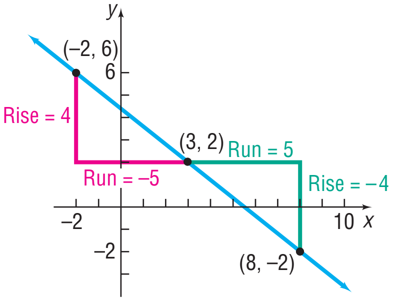

Figure 19

Now Work PROBLEM 25

3 Find the Equation of a Vertical Line

EXAMPLE 3 Graphing a Line

Graph the equation: $x = 3$

Solution

To graph $x = 3$ by hand, find all points (x, y) in the plane for which $x = 3$. No matter what y-coordinate is used, the corresponding x-coordinate always equals 3. Consequently, the graph of the equation $x = 3$ is a vertical line with x-intercept 3 and undefined slope. See Figure 20(a).

In order for an equation to be graphed using a graphing utility, the equation must be expressed in the form $y = \{\text{expression in } x\}$. But $x = 3$ cannot be put into this form. To overcome this, most graphing utilities have special commands for drawing vertical lines. DRAW, LINE, PLOT, and VERT are among the more common ones. Consult your manual to determine the correct methodology for your graphing utility. Figure 20(b) shows the graph on a TI-84 Plus C.

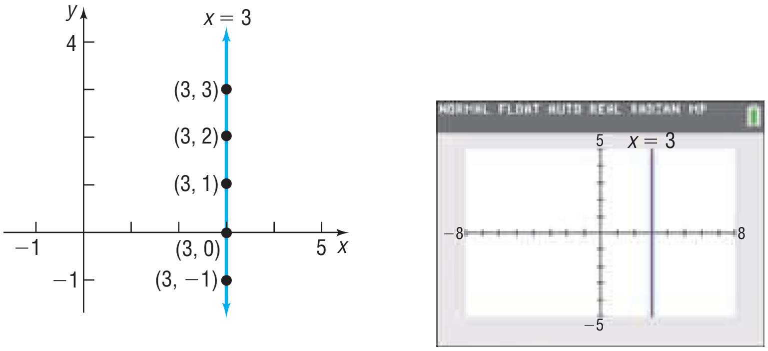

Figure 20 (a) (b)

Example 3 suggests the following result:

THEOREM

Equation of a Vertical Line

A vertical line is given by an equation of the form

$$\boxed{x = a}$$

where a is the x-intercept.

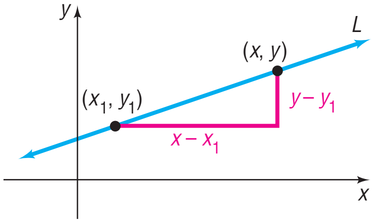

Figure 21

4 Use the Point–Slope Form of a Line; Identify Horizontal Lines

Let L be a nonvertical line with slope m that contains the point (x_1, y_1). See Figure 21. For any other point (x, y) on L, we have

$$m = \frac{y - y_1}{x - x_1} \quad \text{or} \quad y - y_1 = m(x - x_1)$$

THEOREM

Point–Slope Form of an Equation of a Line

An equation of a nonvertical line with slope m that contains the point (x_1, y_1) is

$$\boxed{y - y_1 = m(x - x_1) \qquad \textbf{(2)}}$$

EXAMPLE 4

Using the Point–Slope Form of a Line

An equation of the line with slope 4 and containing the point $(1, 2)$ can be found by using the point–slope form with $m = 4$, $x_1 = 1$, and $y_1 = 2$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2 &= 4(x - 1) && \mathbf{m = 4, x_1 = 1, y_1 = 2} \\ y &= 4x - 2 && \textbf{Solve for } \mathbf{y}. \end{aligned}$$

See Figure 22 for the graph.

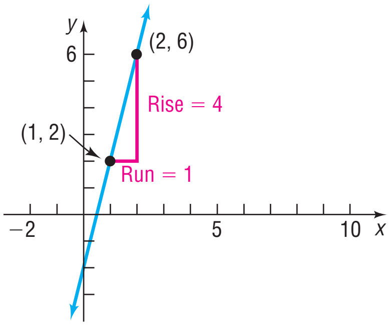

Figure 22 $y = 4x - 2$

Now Work PROBLEMS 39 AND 47

EXAMPLE 5 Finding the Equation of a Horizontal Line

Find an equation of the horizontal line containing the point $(3, 2)$.

Solution Because all the y-values are equal on a horizontal line, the slope of a horizontal line is 0. To get an equation, use the point–slope form with $m = 0$, $x_1 = 3$, and $y_1 = 2$.

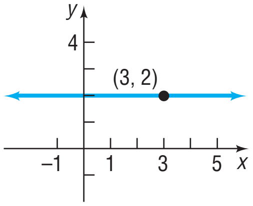

Figure 23 $y = 2$

$$y - y_1 = m(x - x_1)$$
$$y - 2 = 0 \cdot (x - 3) \qquad m = 0, x_1 = 3, \text{ and } y_1 = 2$$
$$y - 2 = 0$$
$$y = 2$$

See Figure 23 for the graph. ■

Example 5 suggests the following result:

THEOREM **Equation of a Horizontal Line**

A horizontal line is given by an equation of the form

$$\boxed{y = b}$$

where b is the y-intercept.

Now Work PROBLEM 59

5 Write the Equation of a Line in Slope–Intercept Form

Another useful equation of a line is obtained when the slope m and y-intercept b are known. In this event, both the slope m of the line and the point $(0, b)$ on the line are known; then use the point–slope form, equation (2), to obtain the following equation:

$$y - b = m(x - 0) \quad \text{or} \quad y = mx + b$$

THEOREM **Slope–Intercept Form of an Equation of a Line**

An equation of a line with slope m and y-intercept b is

$$\boxed{y = mx + b \qquad (3)}$$

Now Work PROBLEM 51 (EXPRESS ANSWER IN SLOPE–INTERCEPT FORM)

Seeing the Concept

To see the role that the slope m plays, graph the following lines on the same screen.

$Y_1 = 2$ $\quad m = 0$

$Y_2 = x + 2$ $\quad m = 1$

$Y_3 = -x + 2$ $\quad m = -1$

$Y_4 = 3x + 2$ $\quad m = 3$

$Y_5 = -3x + 2$ $\quad m = -3$

See Figure 24. What do you conclude about the lines $y = mx + 2$? ■

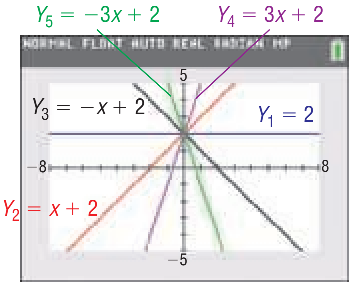

Figure 24 $y = mx + 2$

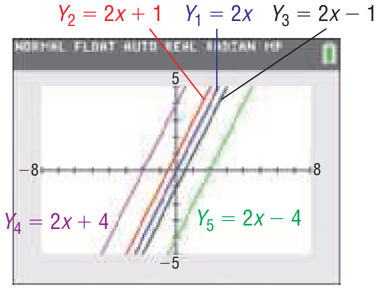

Figure 25 $y = 2x + b$

Seeing the Concept

To see the role of the y-intercept b, graph the following lines on the same screen.

$Y_1 = 2x$ $\quad b = 0$

$Y_2 = 2x + 1$ $\quad b = 1$

$Y_3 = 2x - 1$ $\quad b = -1$

$Y_4 = 2x + 4$ $\quad b = 4$

$Y_5 = 2x - 4$ $\quad b = -4$

See Figure 25. What do you conclude about the lines $y = 2x + b$? ■

When the equation of a line is written in slope–intercept form, it is easy to find the slope m and y-intercept b of the line. For example,

$$\begin{aligned} y &= -2x + 7 \\ &\quad\ \uparrow \quad\ \ \uparrow \\ y &= \ \ mx + b \end{aligned}$$

The slope of this line is -2 and its y-intercept is 7.

Now Work PROBLEM 73

EXAMPLE 6 **Finding the Slope and y-Intercept**

Find the slope m and y-intercept b of the equation $2x + 4y = 8$. Graph the equation.

Solution To obtain the slope and y-intercept, write the equation in slope–intercept form by solving for y.

$$2x + 4y = 8$$

$$4y = -2x + 8$$

$$y = -\frac{1}{2}x + 2 \qquad y = mx + b$$

The coefficient of x, $-\frac{1}{2}$, is the slope, and the constant, 2, is the y-intercept. Graph the line with the y-intercept 2 and the slope $-\frac{1}{2}$. Starting at the point $(0, 2)$, go to the right 2 units and then down 1 unit to the point $(2, 1)$. Draw the line through these points. See Figure 26. ■

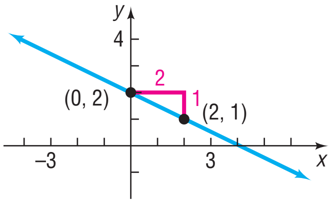

Figure 26 $y = -\frac{1}{2}x + 2$

Now Work PROBLEM 79

6 Find the Equation of a Line Given Two Points

EXAMPLE 7 **Finding an Equation of a Line Given Two Points**

Find an equation of the line containing the points $(2, 3)$ and $(-4, 6)$. Graph the line.

Solution First compute the slope of the line with $(x_1, y_1) = (2, 3)$ and $(x_2, y_2) = (-4, 6)$.

$$m = \frac{6 - 3}{-4 - 2} = \frac{3}{-6} = -\frac{1}{2} \qquad m = \frac{y_2 - y_1}{x_2 - x_1}$$

Use the point $(x_1, y_1) = (2, 3)$ and the slope $m = -\frac{1}{2}$ to get the point–slope form of the equation of the line.

$$y - 3 = -\frac{1}{2}(x - 2) \qquad y - y_1 = m(x - x_1)$$

$$y = -\frac{1}{2}x + 4 \qquad y = mx + b$$

See Figure 27 for the graph. ■

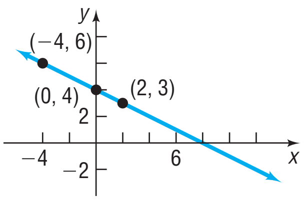

Figure 27 $y = -\frac{1}{2}x + 4$

In the solution to Example 7, we could have used the other point, $(-4, 6)$, instead of the point $(2, 3)$. The equation that results, when written in slope–intercept form, is the equation obtained in Example 7. (Try it for yourself.)

Now Work PROBLEM 53

7 Graph Lines Written in General Form Using Intercepts

Refer to Example 6. The form of the equation of the line $2x + 4y = 8$ is called the *general form*.

DEFINITION

The equation of a line is in **general form*** when it is written as

$$Ax + By = C \qquad \textbf{(4)}$$

where A, B, and C are real numbers and A and B are not both 0.

If $B = 0$ in equation (4), then $A \neq 0$ and the graph of the equation is a vertical line: $x = \dfrac{C}{A}$. If $B \neq 0$ in equation (4), then we can solve the equation for y and write the equation in slope–intercept form as we did in Example 6.

Another approach to graphing equation (4) would be to find its intercepts. Remember, the intercepts of the graph of an equation are the points where the graph crosses or touches a coordinate axis.

EXAMPLE 8 Graphing an Equation in General Form Using Its Intercepts

Graph the equation $2x + 4y = 8$ by finding its intercepts.

Solution To obtain the x-intercept, let $y = 0$ in the equation and solve for x.

$$\begin{aligned} 2x + 4y &= 8 \\ 2x + 4(0) &= 8 && \textbf{Let } y = 0. \\ 2x &= 8 \\ x &= 4 && \textbf{Divide both sides by 2.} \end{aligned}$$

The x-intercept is 4 and the point $(4, 0)$ is on the graph of the equation.

To obtain the y-intercept, let $x = 0$ in the equation and solve for y.

$$\begin{aligned} 2x + 4y &= 8 \\ 2(0) + 4y &= 8 && \textbf{Let } x = 0. \\ 4y &= 8 \\ y &= 2 && \textbf{Divide both sides by 4.} \end{aligned}$$

The y-intercept is 2 and the point $(0, 2)$ is on the graph of the equation.

Plot the points $(4, 0)$ and $(0, 2)$ and draw the line through the points. See Figure 28. ■

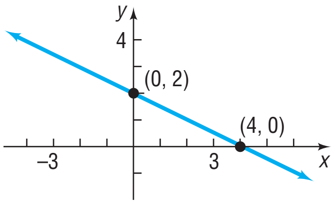

Figure 28 $2x + 4y = 8$

Now Work PROBLEM 93

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*Some texts use the term **standard form**.

Every line has an equation that is equivalent to an equation written in general form. For example, a vertical line whose equation is

$$x = a$$

can be written in the general form

$$1 \cdot x + 0 \cdot y = a \quad \mathbf{A = 1, B = 0, C = a}$$

A horizontal line whose equation is

$$y = b$$

can be written in the general form

$$0 \cdot x + 1 \cdot y = b \quad \mathbf{A = 0, B = 1, C = b}$$

Lines that are neither vertical nor horizontal have general equations of the form

$$Ax + By = C \quad \mathbf{A \neq 0 \text{ and } B \neq 0}$$

Because the equation of every line can be written in general form, any equation equivalent to equation (4) is called a **linear equation**.

8 Find Equations of Parallel Lines

When two lines (in the plane) do not intersect (that is, they have no points in common), they are **parallel**. Look at Figure 29. There we have drawn two parallel lines and have constructed two right triangles by drawing sides parallel to the coordinate axes. The right triangles are similar. (Do you see why? Two angles are equal.) Because the triangles are similar, the ratios of corresponding sides are equal.

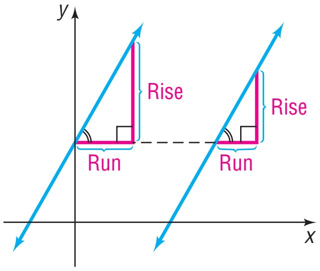


Figure 29 Parallel lines

THEOREM

Criteria for Parallel Lines

Two nonvertical lines are parallel if and only if their slopes are equal and they have different y-intercepts.

The use of the words "if and only if" in the preceding theorem means that actually two statements are being made, one the converse of the other.

If two nonvertical lines are parallel, then their slopes are equal and they have different y-intercepts.

If two nonvertical lines have equal slopes and they have different y-intercepts, then they are parallel.

EXAMPLE 9 Showing That Two Lines Are Parallel

Show that the lines given by the following equations are parallel.

$$L_1: \; 2x + 3y = 6 \qquad L_2: \; 4x + 6y = 0$$

Solution To determine whether these lines have equal slopes and different y-intercepts, write each equation in slope–intercept form.

$$\begin{aligned} L_1: \; 2x + 3y &= 6 & \qquad L_2: \; 4x + 6y &= 0 \\ 3y &= -2x + 6 & 6y &= -4x \\ y &= -\frac{2}{3}x + 2 & y &= -\frac{2}{3}x \end{aligned}$$

$$\text{Slope} = -\frac{2}{3}; \; y\text{-intercept} = 2 \qquad \text{Slope} = -\frac{2}{3}; \; y\text{-intercept} = 0$$

Because these lines have the same slope, $-\frac{2}{3}$, but different y-intercepts, the lines are parallel. See Figure 30.

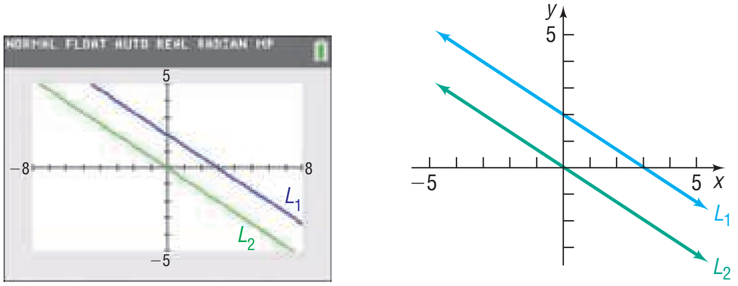


Figure 30 Parallel lines ■

EXAMPLE 10 Finding a Line That Is Parallel to a Given Line

Find an equation for the line that contains the point $(2, -3)$ and is parallel to the line $2x + y = 6$.

Solution Since the two lines are to be parallel, the slope of the line being sought equals the slope of the line $2x + y = 6$. Begin by writing the equation of the line $2x + y = 6$ in slope–intercept form.

$$2x + y = 6$$

$$y = -2x + 6 \quad \textbf{Place in the form } y = mx + b.$$

The slope is -2. Since the line being sought also has slope -2 and contains the point $(2, -3)$, use the point–slope form to obtain its equation.

$$\begin{aligned} y - y_1 &= m(x - x_1) && \textbf{Point–slope form} \\ y - (-3) &= -2(x - 2) && m = -2, x_1 = 2, y_1 = -3 \\ y + 3 &= -2x + 4 && \textbf{Simplify.} \\ y &= -2x + 1 && \textbf{Slope–intercept form} \\ 2x + y &= 1 && \textbf{General form} \end{aligned}$$

This line is parallel to the line $2x + y = 6$ and contains the point $(2, -3)$. See Figure 31. ■

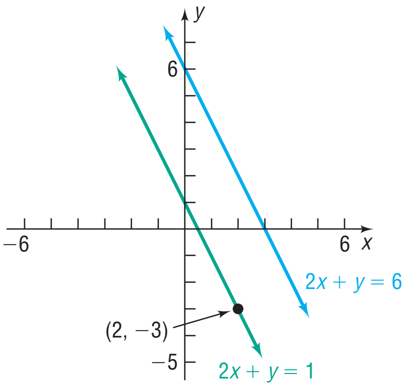


Figure 31

Now Work PROBLEM 61

9 Find Equations of Perpendicular Lines

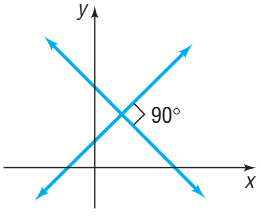


Figure 32 Perpendicular lines

When two lines intersect at a right angle (90°), they are **perpendicular**. See Figure 32.

The following result gives a condition, in terms of their slopes, for two lines to be perpendicular.

THEOREM

Criterion for Perpendicular Lines

Two nonvertical lines are perpendicular if and only if the product of their slopes is -1.

Here we shall prove the "only if" part of the statement:

If two nonvertical lines are perpendicular, then the product of their slopes is -1.

In Problem 130 you are asked to prove the "if" part of the theorem:

If two nonvertical lines have slopes whose product is -1, then the lines are perpendicular.

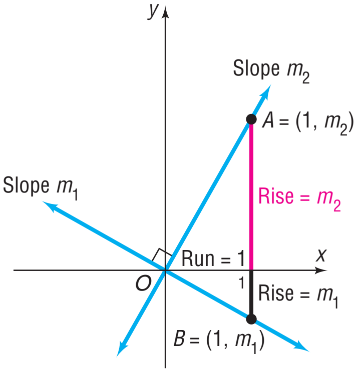


Figure 33

Proof Let m_1 and m_2 denote the slopes of the two lines. There is no loss in generality (that is, neither the angle nor the slopes are affected) if we situate the lines so that they meet at the origin. See Figure 33. The point $A = (1, m_2)$ is on the line having slope m_2, and the point $B = (1, m_1)$ is on the line having slope m_1. (Do you see why this must be true?)

Suppose that the lines are perpendicular. Then triangle OAB is a right triangle. As a result of the Pythagorean Theorem, it follows that

$$[d(O, A)]^2 + [d(O, B)]^2 = [d(A, B)]^2 \tag{5}$$

Using the distance formula, the squares of these distances are

$$[d(O, A)]^2 = (1 - 0)^2 + (m_2 - 0)^2 = 1 + m_2^2$$
$$[d(O, B)]^2 = (1 - 0)^2 + (m_1 - 0)^2 = 1 + m_1^2$$
$$[d(A, B)]^2 = (1 - 1)^2 + (m_2 - m_1)^2 = m_2^2 - 2m_1m_2 + m_1^2$$

Using these facts in equation (5), we get

$$(1 + m_2^2) + (1 + m_1^2) = m_2^2 - 2m_1m_2 + m_1^2$$

which, upon simplification, can be written as

$$m_1m_2 = -1$$

If the lines are perpendicular, the product of their slopes is -1. ■

You may find it easier to remember the condition for two nonvertical lines to be perpendicular by observing that the equality $m_1m_2 = -1$ means that m_1 and m_2 are negative reciprocals of each other; that is, either $m_1 = -\frac{1}{m_2}$ or $m_2 = -\frac{1}{m_1}$.

EXAMPLE 11 Finding the Slope of a Line Perpendicular to Another Line

If a line has slope $\frac{3}{2}$, any line having slope $-\frac{2}{3}$ is perpendicular to it. ■

EXAMPLE 12 Finding the Equation of a Line Perpendicular to a Given Line

Find an equation of the line that contains the point $(1, -2)$ and is perpendicular to the line $x + 3y = 6$. Graph the two lines.

Solution First write the equation of the given line in slope–intercept form to find its slope.

$$x + 3y = 6$$
$$3y = -x + 6 \qquad \text{Proceed to solve for } y.$$
$$y = -\frac{1}{3}x + 2 \qquad \text{Place in the form } y = mx + b.$$

The given line has slope $-\frac{1}{3}$. Any line perpendicular to this line will have slope 3. Because the point $(1, -2)$ is on this line with slope 3, use the point–slope form of the equation of a line.

$$y - y_1 = m(x - x_1) \qquad \text{Point–slope form}$$
$$y - (-2) = 3(x - 1) \qquad m = 3, x_1 = 1, y_1 = -2$$

To obtain other forms of the equation, proceed as follows:

$$y + 2 = 3(x - 1)$$

$$y + 2 = 3x - 3 \qquad \text{Simplify.}$$

$$y = 3x - 5 \qquad \text{Slope–intercept form}$$

$$3x - y = 5 \qquad \text{General form}$$

Figure 34 shows the graphs.

WARNING Be sure to use a square screen when you graph perpendicular lines. Otherwise, the angle between the two lines will appear distorted. ■

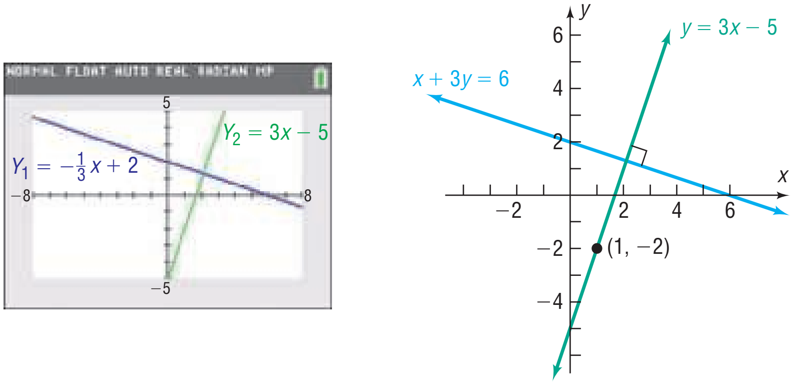


Figure 34

Now Work PROBLEM 67

2.2 Assess Your Understanding

Concepts and Vocabulary

1. The slope of a vertical line is ____________; the slope of a horizontal line is _________.

2. For the line $2x + 3y = 6$, the x-intercept is ________ and the y-intercept is ________.

3. ***True or False*** The equation $3x + 4y = 6$ is written in general form.

4. ***True or False*** The slope of the line $2y = 3x + 5$ is 3.

5. ***True or False*** The point $(1, 2)$ is on the line $2x + y = 4$.

6. Two nonvertical lines have slopes m_1 and m_2, respectively. The lines are parallel if ____________ and the ____________ are unequal; the lines are perpendicular if ____________.

7. The lines $y = 2x + 3$ and $y = ax + 5$ are parallel if $a =$ ________.

8. The lines $y = 2x - 1$ and $y = ax + 2$ are perpendicular if $a =$ ________.

9. ***True or False*** Perpendicular lines have slopes that are reciprocals of one another.

10. Choose the formula for finding the slope m of a nonvertical line that contains the two distinct points (x_1, y_1) and (x_2, y_2).

(a) $m = \dfrac{y_2 - x_2}{y_1 - x_1} \quad x_1 \neq y_1$

(b) $m = \dfrac{y_2 - x_1}{x_2 - y_1} \quad y_1 \neq x_2$

(c) $m = \dfrac{x_2 - x_1}{y_2 - y_1} \quad y_1 \neq y_2$

(d) $m = \dfrac{y_2 - y_1}{x_2 - x_1} \quad x_1 \neq x_2$

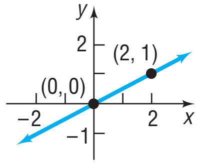
11. If a line slants downward from left to right, then which of the following describes its slope?
(a) positive (b) zero (c) negative (d) undefined

12. Choose the correct statement about the graph of the line $y = -3$.
(a) The graph is vertical with x-intercept -3.
(b) The graph is horizontal with y-intercept -3.
(c) The graph is vertical with y-intercept -3.
(d) The graph is horizontal with x-intercept -3.

Skill Building

In Problems 13–16, (a) find the slope of the line and (b) interpret the slope.

13.



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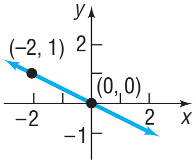

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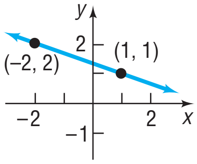

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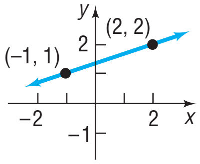


In Problems 17–24, plot each pair of points and determine the slope of the line containing them. Graph the line by hand.

17. $(2, 3);\ (4, 0)$ **18.** $(4, 2);\ (3, 4)$ **19.** $(-2, 3);\ (2, 1)$ **20.** $(-1, 1);\ (2, 3)$

21. $(-3, -1);\ (2, -1)$ **22.** $(4, 2);\ (-5, 2)$ **23.** $(-1, 2);\ (-1, -2)$ **24.** $(2, 0);\ (2, 2)$

In Problems 25–32, graph the line containing the point P and having slope m.

25. $P = (1, 2);\ m = 3$ **26.** $P = (2, 1);\ m = 4$ **27.** $P = (2, 4);\ m = -\frac{3}{4}$ **28.** $P = (1, 3);\ m = -\frac{2}{5}$

29. $P = (-1, 3);\ m = 0$ **30.** $P = (2, -4);\ m = 0$ **31.** $P = (0, 3)$; slope undefined **32.** $P = (-2, 0)$; slope undefined

In Problems 33–38, the slope and a point on a line are given. Use this information to locate three additional points on the line. Answers may vary.

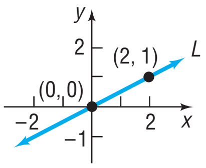
[**Hint:** It is not necessary to find the equation of the line. See Example 2.]

33. Slope 4; point $(1, 2)$ **34.** Slope 2; point $(-2, 3)$ **35.** Slope $-\frac{3}{2}$; point $(2, -4)$

36. Slope $\frac{4}{3}$; point $(-3, 2)$ **37.** Slope -2; point $(-2, -3)$ **38.** Slope -1; point $(4, 1)$

In Problems 39–46, find an equation of the line L.

39.



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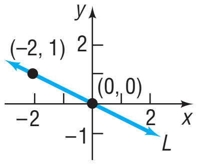

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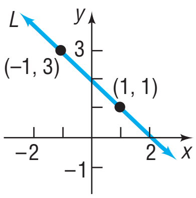

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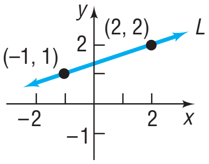

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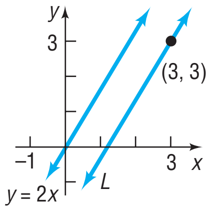
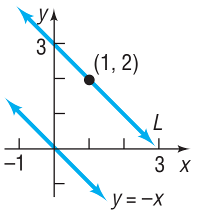


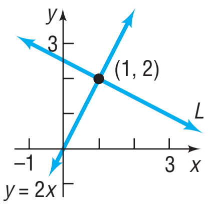
L is parallel to y = 2x

44.



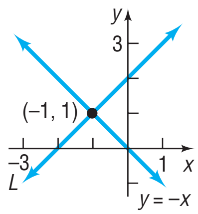
L is parallel to y = −x

45.



L is perpendicular to y = 2x

46.



L is perpendicular to y = −x

In Problems 47–72, find an equation for the line with the given properties. Express your answer using either the general form or the slope–intercept form of the equation of a line, whichever you prefer.

47. Slope = 3; containing the point $(-2, 3)$ **48.** Slope = 2; containing the point $(4, -3)$

49. Slope = $-\frac{2}{3}$; containing the point $(1, -1)$ **50.** Slope = $\frac{1}{2}$; containing the point $(3, 1)$

51. Slope = −3; y-intercept = 3 **52.** Slope = −5; y-intercept = −7

53. Containing the points $(1, 3)$ and $(-1, 2)$ **54.** Containing the points $(-3, 4)$ and $(2, 5)$

55. x-intercept = 2; y-intercept = −1 **56.** x-intercept = −4; y-intercept = 4

57. Slope undefined; containing the point $(2, 4)$ **58.** Slope undefined; containing the point $(3, 8)$

59. Horizontal; containing the point $(-3, 2)$ **60.** Vertical; containing the point $(4, -5)$

61. Parallel to the line $y = 4x$; containing the point $(-1, 2)$ **62.** Parallel to the line $y = -3x$; containing the point $(-1, 2)$

63. Parallel to the line $5x - y = -2$; containing the point $(0, 0)$ **64.** Parallel to the line $x - 2y = -5$; containing the point $(0, 0)$

65. Parallel to the line $x = 5$; containing the point $(4, 2)$

66. Parallel to the line $y = 5$; containing the point $(4, 2)$

67. Perpendicular to the line $y = \frac{1}{6}x + 4$; containing the point $(1, -6)$

68. Perpendicular to the line $y = 8x - 3$; containing the point $(10, -2)$

69. Perpendicular to the line $2x + 5y = 2$; containing the point $(-3, -6)$

70. Perpendicular to the line $x - 3y = -12$; containing the point $(0, 4)$

71. Perpendicular to the line $x = 8$; containing the point $(3, 4)$

72. Perpendicular to the line $y = 8$; containing the point $(3, 4)$

In Problems 73–92, find the slope and y-intercept of each line. Graph the line by hand. Verify your graph using a graphing utility.

73. $y = 2x + 3$

74. $y = -3x + 4$

75. $\frac{1}{4}y = x - 1$

76. $\frac{1}{3}x + y = 2$

77. $y = \frac{1}{2}x + 2$

78. $y = 2x + \frac{1}{2}$

79. $x + 4y = 4$

80. $-x + 3y = 6$

81. $2x - 3y = 6$

82. $3x + 2y = 6$

83. $x + y = 1$

84. $x - y = 2$

85. $x = -4$

86. $y = -1$

87. $y = 5$

88. $x = 2$

89. $y - x = 0$

90. $x + y = 0$

91. $2y - 3x = 0$

92. $3x + 2y = 0$

In Problems 93–102, (a) find the intercepts of the graph of each equation and (b) graph the equation.

93. $2x + 3y = 6$

94. $3x - 2y = 6$

95. $-4x + 5y = 40$

96. $6x - 4y = 24$

97. $7x + 2y = 21$

98. $5x + 3y = 18$

99. $\frac{1}{2}x + \frac{1}{3}y = 1$

100. $x - \frac{2}{3}y = 4$

101. $0.2x - 0.5y = 1$

102. $-0.3x + 0.4y = 1.2$

103. Find an equation of the x-axis.

104. Find an equation of the y-axis.

In Problems 105–108, the equations of two lines are given. Determine whether the lines are parallel, perpendicular, or neither.

105. $y = 2x - 3$
$y = 2x + 4$

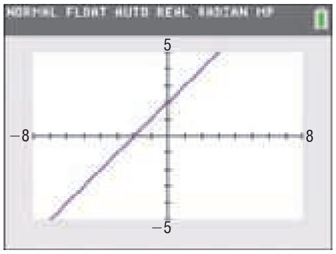
106. $y = \frac{1}{2}x - 3$
$y = -2x + 4$

107. $y = 4x + 5$
$y = -4x + 2$

108. $y = -2x + 3$
$y = -\frac{1}{2}x + 2$

In Problems 109–112, write an equation of each line. Express your answer using either the general form or the slope–intercept form of the equation of a line, whichever you prefer.

109.



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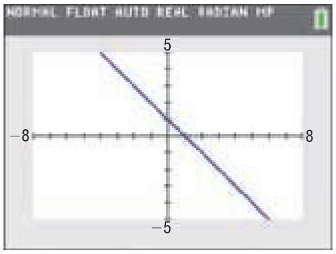

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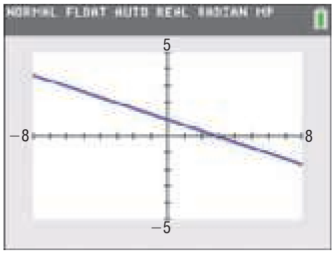

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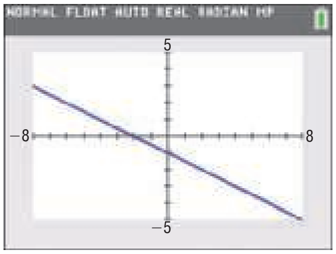


Applications and Extensions

113. Geometry Use slopes to show that the triangle whose vertices are $(-2, 5)$, $(1, 3)$, and $(-1, 0)$ is a right triangle.

114. Geometry Use slopes to show that the quadrilateral whose vertices are $(1, -1)$, $(4, 1)$, $(2, 2)$, and $(5, 4)$ is a parallelogram.

115. Geometry Use slopes to show that the quadrilateral whose vertices are $(-1, 0)$, $(2, 3)$, $(1, -2)$, and $(4, 1)$ is a rectangle.

116. Geometry Use slopes and the distance formula to show that the quadrilateral whose vertices are $(0, 0)$, $(1, 3)$, $(4, 2)$, and $(3, -1)$ is a square.

117. Truck Rentals A truck rental company rents a moving truck for one day by charging \$39 plus \$0.60 per mile. Write a linear equation that relates the cost C, in dollars, of renting the truck to the number x of miles driven. What is the cost of renting the truck if the truck is driven 110 miles? 230 miles?

118. Cost Equation The **fixed costs** of operating a business are the costs incurred regardless of the level of production. Fixed costs include rent, fixed salaries, and costs of leasing machinery. The **variable costs** of operating a business are the costs that change with the level of output. Variable costs include raw materials, hourly wages, and electricity. Suppose that a manufacturer of jeans has fixed daily costs of $500 and variable costs of $8 for each pair of jeans manufactured. Write a linear equation that relates the daily cost C, in dollars, of manufacturing the jeans to the number x of jeans manufactured. What is the cost of manufacturing 400 pairs of jeans? 740 pairs?

119. Cost of Driving a Car The annual fixed costs for owning a small sedan are $1461, assuming the car is completely paid for. The cost to drive the car is approximately $0.16 per mile. Write a linear equation that relates the cost C and the number x of miles driven annually.

Source: *AAA, 2014*

120. Wages of a Car Salesperson Dan receives $375 per week for selling new and used cars at a car dealership in Oak Lawn, Illinois. In addition, he receives 5% of the profit on any sales that he generates. Write a linear equation that represents Dan's weekly salary S when he has sales that generate a profit of x dollars.

121. Electricity Rates in Illinois Commonwealth Edison Company supplies electricity to residential customers for a monthly customer charge of $15.14 plus 7.57 cents per kilowatt-hour for up to 800 kilowatt-hours (kW-h).

(a) Write a linear equation that relates the monthly charge C, in dollars, to the number x of kilowatt-hours used in a month, $0 \le x \le 800$.
(b) Graph this equation.
(c) What is the monthly charge for using 200 kilowatt-hours?
(d) What is the monthly charge for using 500 kilowatt-hours?
(e) Interpret the slope of the line.

Source: *Commonwealth Edison Company,* January 2015.

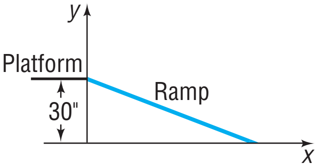
122. Electricity Rates in Florida Florida Power & Light Company supplies electricity to residential customers for a monthly customer charge of $7.57 plus 9.01 cents per kilowatt-hour for up to 1000 kilowatt-hours.
(a) Write a linear equation that relates the monthly charge C, in dollars, to the number x of kilowatt-hours used in a month, $0 \le x \le 1000$.
(b) Graph this equation.
(c) What is the monthly charge for using 200 kilowatt-hours?
(d) What is the monthly charge for using 500 kilowatt-hours?
(e) Interpret the slope of the line.

Source: *Florida Power & Light Company,* March 2015.

123. Measuring Temperature The relationship between Celsius (°C) and Fahrenheit (°F) degrees of measuring temperature is linear. Find a linear equation relating °C and °F if 0°C corresponds to 32°F and 100°C corresponds to 212°F. Use the equation to find the Celsius measure of 70°F.

124. Measuring Temperature The Kelvin (K) scale for measuring temperature is obtained by adding 273 to the Celsius temperature.
(a) Write a linear equation relating K and °C.
(b) Write a linear equation relating K and °F (see Problem 123).

125. Access Ramp A wooden access ramp is being built to reach a platform that sits 30 inches above the floor. The ramp drops 2 inches for every 25-inch run.



(a) Write a linear equation that relates the height y of the ramp above the floor to the horizontal distance x from the platform.
(b) Find and interpret the x-intercept of the graph of your equation.
(c) Design requirements stipulate that the maximum run be 30 feet and that the maximum slope be a drop of 1 inch for each 12 inches of run. Will this ramp meet the requirements? Explain.
(d) What slopes could be used to obtain the 30-inch rise and still meet design requirements?

Source: *www.adaptiveaccess.com/wood_ramps.php*

126. Cigarette Use A report in the Child Trends DataBase indicated that, in 2000, 20.6% of twelfth grade students reported daily use of cigarettes. In 2013, 8.5% of twelfth grade students reported daily use of cigarettes.
(a) Write a linear equation that relates the percent y of twelfth grade students who smoke cigarettes daily to the number x of years after 2000.
(b) Find the intercepts of the graph of your equation.
(c) Do the intercepts have any meaningful interpretation?
(d) Use your equation to predict the percent for the year 2025. Is this result reasonable?

Source: *www.childtrends.org*

127. Product Promotion A cereal company finds that the number of people who will buy one of its products in the first month that the product is introduced is linearly related to the amount of money it spends on advertising. If it spends $40,000 on advertising, then 100,000 boxes of cereal will be sold, and if it spends $60,000, then 200,000 boxes will be sold.
(a) Write a linear equation that relates the amount A spent on advertising to the number x of boxes the company aims to sell.
(b) How much expenditure on advertising is needed to sell 300,000 boxes of cereal?
(c) Interpret the slope.

128. Show that the line containing the points (a, b) and (b, a), $a \neq b$, is perpendicular to the line $y = x$. Also show that the midpoint of (a, b) and (b, a) lies on the line $y = x$.

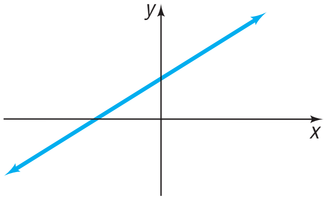
129. The equation $2x - y = C$ defines a **family of lines**, one line for each value of C. On one set of coordinate axes, graph the members of the family when $C = -4$, $C = 0$, and $C = 2$. Can you draw a conclusion from the graph about each member of the family?

130. Prove that if two nonvertical lines have slopes whose product is -1, then the lines are perpendicular. **[Hint:** Refer to Figure 33 and use the converse of the Pythagorean Theorem.]

Explaining Concepts: Discussion and Writing

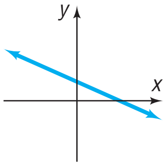
131. Which of the following equations might have the graph shown? (More than one answer is possible.)

(a) $2x + 3y = 6$
(b) $-2x + 3y = 6$
(c) $3x - 4y = -12$
(d) $x - y = 1$
(e) $x - y = -1$
(f) $y = 3x - 5$
(g) $y = 2x + 3$
(h) $y = -3x + 3$



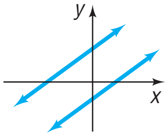
132. Which of the following equations might have the graph shown? (More than one answer is possible.)

(a) $2x + 3y = 6$
(b) $2x - 3y = 6$
(c) $3x + 4y = 12$
(d) $x - y = 1$
(e) $x - y = -1$
(f) $y = -2x - 1$
(g) $y = -\frac{1}{2}x + 10$
(h) $y = x + 4$



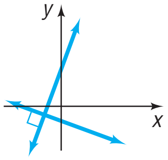
133. The figure shows the graph of two parallel lines. Which of the following pairs of equations might have such a graph?

(a) $x - 2y = 3$
  $x + 2y = 7$
(b) $x + y = 2$
  $x + y = -1$
(c) $x - y = -2$
  $x - y = 1$
(d) $x - y = -2$
  $2x - 2y = -4$
(e) $x + 2y = 2$
  $x + 2y = -1$



134. The figure shows the graph of two perpendicular lines. Which of the following pairs of equations might have such a graph?

(a) $y - 2x = 2$
  $y + 2x = -1$
(b) $y - 2x = 0$
  $2y + x = 0$
(c) $2y - x = 2$
  $2y + x = -2$
(d) $y - 2x = 2$
  $x + 2y = -1$
(e) $2x + y = -2$
  $2y + x = -2$



135. ***m* is for Slope** The accepted symbol used to denote the slope of a line is the letter m. Investigate the origin of this symbolism. Begin by consulting a French dictionary and looking up the French word *monter*. Write a brief essay on your findings.

136. **Grade of a Road** The term *grade* is used to describe the inclination of a road. How is this term related to the notion of slope of a line? Is a 4% grade very steep? Investigate the grades of some mountainous roads and determine their slopes. Write a brief essay on your findings.

137. **Carpentry** Carpenters use the term *pitch* to describe the steepness of staircases and roofs. How is pitch related to slope? Investigate typical pitches used for stairs and for roofs. Write a brief essay on your findings.

138. Can the equation of every line be written in slope–intercept form? Why?

139. Does every line have exactly one x-intercept and one y-intercept? Are there any lines that have no intercepts?

140. What can you say about two lines that have equal slopes and equal y-intercepts?

141. What can you say about two lines with the same x-intercept and the same y-intercept? Assume that the x-intercept is not 0.

142. If two distinct lines have the same slope but different x-intercepts, can they have the same y-intercept?

143. If two distinct lines have the same y-intercept but different slopes, can they have the same x-intercept?

144. Which form of the equation of a line do you prefer to use? Justify your position with an example that shows that your choice is better than another. Have reasons.

145. **What Went Wrong?** A student is asked to find the slope of the line joining $(-3, 2)$ and $(1, -4)$. He states that the slope is $\frac{3}{2}$. Is he correct? If not, what went wrong?

Retain Your Knowledge

Problems 146–149 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

146. Simplify $\left(\frac{x^2y^{-3}}{x^4y^5}\right)^{-2}$. Assume $x \neq 0$ and $y \neq 0$. Express the answer so that all exponents are positive.

147. The lengths of the legs of a right triangle are $a = 8$ and $b = 15$. Find the hypotenuse.

148. Solve the equation: $(x - 3)^2 + 25 = 49$

149. Solve $|2x - 5| + 7 < 10$. Express the answer using set notation or interval notation. Graph the solution set.

2.3 Circles

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Completing the Square (Chapter R, Section R.5, p. 57)
- Square Root Method (Section 1.3, p. 112)

Now Work the **'Are You Prepared?'** problems on page 193.

OBJECTIVES
1 Write the Standard Form of the Equation of a Circle (p. 189)
2 Graph a Circle by Hand and by Using a Graphing Utility (p. 190)
3 Work with the General Form of the Equation of a Circle (p. 192)

1 Write the Standard Form of the Equation of a Circle

One advantage of a coordinate system is that it enables us to translate a geometric statement into an algebraic statement, and vice versa. Consider, for example, the following geometric statement that defines a circle.

DEFINITION

A **circle** is a set of points in the xy-plane that are a fixed distance r from a fixed point (h, k). The fixed distance r is called the **radius**, and the fixed point (h, k) is called the **center** of the circle.

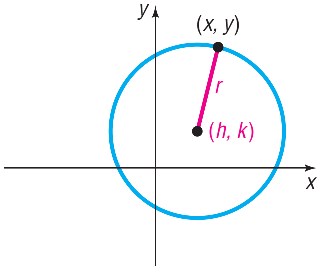


Figure 35
$(x - h)^2 + (y - k)^2 = r^2$

Figure 35 shows the graph of a circle. To find the equation, let (x, y) represent the coordinates of any point on a circle with radius r and center (h, k). Then the distance between the points (x, y) and (h, k) must always equal r. That is, by the distance formula,

$$\sqrt{(x - h)^2 + (y - k)^2} = r$$

or, equivalently,

$$(x - h)^2 + (y - k)^2 = r^2$$

DEFINITION

The **standard form of an equation of a circle** with radius r and center (h, k) is

$$\boxed{(x - h)^2 + (y - k)^2 = r^2 \qquad \textbf{(1)}}$$

THEOREM

The standard form of an equation of a circle of radius r with center at the origin $(0, 0)$ is

$$\boxed{x^2 + y^2 = r^2}$$

DEFINITION

If the radius $r = 1$, the circle whose center is at the origin is called the **unit circle** and has the equation

$$\boxed{x^2 + y^2 = 1}$$

See Figure 36. Notice that the graph of the unit circle is symmetric with respect to the x-axis, the y-axis, and the origin.

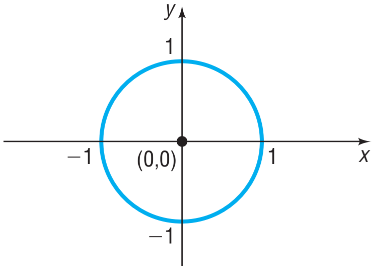


Figure 36
Unit circle $x^2 + y^2 = 1$

EXAMPLE 1 Writing the Standard Form of the Equation of a Circle

Write the standard form of the equation of the circle with radius 5 and center $(-3, 6)$.

Solution Substitute the values $r = 5$, $h = -3$, and $k = 6$ into equation (1).

$$(x - h)^2 + (y - k)^2 = r^2$$
$$(x + 3)^2 + (y - 6)^2 = 25$$

Now Work PROBLEM 9

2 Graph a Circle by Hand and by Using a Graphing Utility

EXAMPLE 2 Graphing a Circle by Hand and by Using a Graphing Utility

Graph the equation: $(x + 3)^2 + (y - 2)^2 = 16$

Solution Since the equation is in the form of equation (1), its graph is a circle. To graph the equation by hand, compare the given equation to the standard form of the equation of a circle. The comparison yields information about the circle.

$$(x + 3)^2 + (y - 2)^2 = 16$$
$$(x - (-3))^2 + (y - 2)^2 = 4^2$$
$$\uparrow \qquad\qquad \uparrow \qquad \uparrow$$
$$(x - h)^2 + (y - k)^2 = r^2$$

We see that $h = -3$, $k = 2$, and $r = 4$. The circle has center $(-3, 2)$ and a radius of 4 units. To graph this circle, first plot the center $(-3, 2)$. Since the radius is 4, locate four points on the circle by plotting points 4 units to the left, to the right, up, and down from the center. These four points can then be used as guides to obtain the graph. See Figure 37.

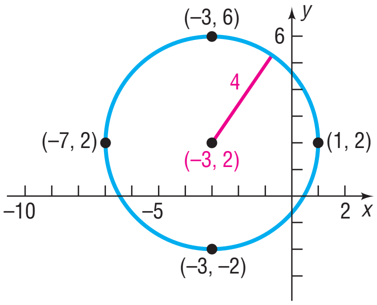


Figure 37
$(x + 3)^2 + (y - 2)^2 = 16$

In Words

The symbol ± is read "plus or minus." It means to add and subtract the quantity following the ± symbol. For example, 5 ± 2 means "5 − 2 = 3 or 5 + 2 = 7."

To graph a circle on a graphing utility, we must write the equation in the form $y = \{\text{expression involving } x\}$.* We must solve for y in the equation

$$\begin{aligned}(x+3)^2 + (y-2)^2 &= 16 \\ (y-2)^2 &= 16 - (x+3)^2 && \textbf{Subtract } (x+3)^2 \textbf{ from both sides.} \\ y - 2 &= \pm\sqrt{16-(x+3)^2} && \textbf{Use the Square Root Method.} \\ y &= 2 \pm \sqrt{16-(x+3)^2} && \textbf{Add 2 to both sides.}\end{aligned}$$

To graph the circle, graph the top half

$$Y_1 = 2 + \sqrt{16-(x+3)^2}$$

and the bottom half

$$Y_2 = 2 - \sqrt{16-(x+3)^2}$$

Also, be sure to use a square screen. Otherwise, the circle will appear distorted. Figure 38 shows the graph on a TI-84 Plus C. The graph is "disconnected" because of the resolution of the calculator. ■

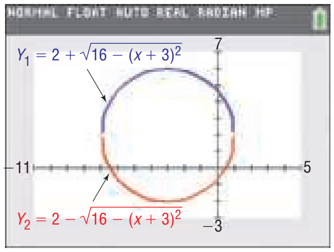


Figure 38 $(x+3)^2 + (y-2)^2 = 16$

Now Work PROBLEMS 25(a) AND (b)

EXAMPLE 3

Finding the Intercepts of a Circle

For the circle $(x+3)^2 + (y-2)^2 = 16$, find the intercepts, if any, of its graph.

Solution This is the equation discussed and graphed in Example 2. To find the x-intercepts, if any, let $y = 0$ and solve for x. Then

$$\begin{aligned}(x+3)^2 + (y-2)^2 &= 16 \\ (x+3)^2 + (0-2)^2 &= 16 && y = 0 \\ (x+3)^2 + 4 &= 16 && \textbf{Simplify.} \\ (x+3)^2 &= 12 && \textbf{Subtract 4 from both sides.} \\ x + 3 &= \pm\sqrt{12} && \textbf{Use the Square Root Method.} \\ x &= -3 \pm 2\sqrt{3} && \textbf{Solve for } x.\end{aligned}$$

The x-intercepts are $-3 - 2\sqrt{3} \approx -6.46$ and $-3 + 2\sqrt{3} \approx 0.46$.

To find the y-intercepts, if any, let $x = 0$ and solve for y. Then

$$\begin{aligned}(x+3)^2 + (y-2)^2 &= 16 \\ (0+3)^2 + (y-2)^2 &= 16 && x = 0 \\ 9 + (y-2)^2 &= 16 \\ (y-2)^2 &= 7 \\ y - 2 &= \pm\sqrt{7} && \textbf{Use the Square Root Method.} \\ y &= 2 \pm \sqrt{7} && \textbf{Solve for } y.\end{aligned}$$

The y-intercepts are $2 - \sqrt{7} \approx -0.65$ and $2 + \sqrt{7} \approx 4.65$.

Look back at Figure 37 or 38 to verify the approximate locations of the intercepts. ■

Now Work PROBLEM 25 (c)

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*Some graphing utilities (e.g., TI-83, TI-84, and TI-86) have a CIRCLE function that enables the user to enter only the coordinates of the center of the circle and its radius to graph the circle.

3 Work with the General Form of the Equation of a Circle

If we eliminate the parentheses from the standard form of the equation of the circle given in Example 3, we get

$$\begin{aligned}(x+3)^2+(y-2)^2&=16\\ x^2+6x+9+y^2-4y+4&=16\end{aligned}$$

which simplifies to

$$x^2+y^2+6x-4y-3=0$$

It can be shown that any equation of the form

$$x^2+y^2+ax+by+c=0$$

has a graph that is a circle or a point, or has no graph at all. For example, the graph of the equation $x^2+y^2=0$ is the single point $(0,0)$. The equation $x^2+y^2+5=0$, or $x^2+y^2=-5$, has no graph, because sums of squares of real numbers are never negative.

DEFINITION

When its graph is a circle, the equation

$$\boxed{x^2+y^2+ax+by+c=0}$$

is the **general form of the equation of a circle**.

If an equation of a circle is in the general form, we use the method of completing the square to put the equation in standard form so that we can identify its center and radius.

EXAMPLE 4 **Graphing a Circle Whose Equation Is in General Form**

Graph the equation $x^2+y^2+4x-6y+12=0$.

Solution Group the expression involving x, group the expression involving y, and put the constant on the right side of the equation. The result is

$$(x^2+4x)+(y^2-6y)=-12$$

Next, complete the square of each expression in parentheses. Remember that any number added on the left side of the equation must be added on the right.

$$\begin{aligned}(\underbrace{x^2+4x+4}_{\left(\frac{4}{2}\right)^2=4})+(\underbrace{y^2-6y+9}_{\left(\frac{-6}{2}\right)^2=9})&=-12+4+9\\ (x+2)^2+(y-3)^2&=1 \quad \text{Factor.}\end{aligned}$$

This equation is the standard form of the equation of a circle with radius 1 and center $(-2,3)$. To graph the equation by hand, use the center $(-2,3)$ and the radius 1. See Figure 39(a).

To graph the equation using a graphing utility, solve for y.

$$\begin{aligned}(y-3)^2&=1-(x+2)^2\\ y-3&=\pm\sqrt{1-(x+2)^2} \quad \text{Use the Square Root Method.}\\ y&=3\pm\sqrt{1-(x+2)^2} \quad \text{Add 3 to both sides.}\end{aligned}$$

Figure 39(b) illustrates the graph on a TI-84 Plus C graphing calculator.

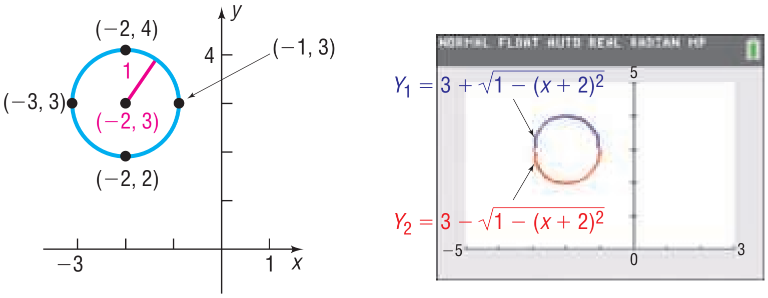


Figure 39 $(x + 2)^2 + (y - 3)^2 = 1$

(a) (b)

■

Now Work PROBLEM 29

EXAMPLE 5

Finding the General Equation of a Circle

Find the general equation of the circle whose center is $(1, -2)$ and whose graph contains the point $(4, -2)$.

Solution To find the equation of a circle, we need to know its center and its radius. Here, the center is $(1, -2)$. Since the point $(4, -2)$ is on the graph, the radius r will equal the distance from $(4, -2)$ to the center $(1, -2)$. See Figure 40. Thus,

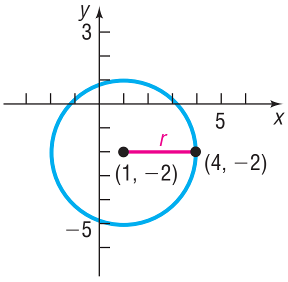


Figure 40 $(x - 1)^2 + (y + 2)^2 = 9$

$$
\begin{aligned}
r &= \sqrt{(4 - 1)^2 + [-2 - (-2)]^2} \\
&= \sqrt{9} = 3
\end{aligned}
$$

The standard form of the equation of the circle is

$$(x - 1)^2 + (y + 2)^2 = 9$$

Eliminate the parentheses and rearrange the terms to get the general equation

$$x^2 + y^2 - 2x + 4y - 4 = 0$$

■

Now Work PROBLEM 15

Overview

The discussion in Sections 2.2 and 2.3 about lines and circles dealt with two main types of problems that can be generalized as follows:

1. Given an equation, classify it and graph it.
2. Given a graph, or information about a graph, find its equation.

This text deals with both types of problems. We shall study various equations, classify them, and graph them. The second type of problem is usually more difficult to solve than the first. In many instances a graphing utility can be used to solve problems when information about the problem (such as data) is given.

2.3 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. To complete the square of $x^2 + 10x$, you would ______ (*add*/*subtract*) the number ________. (p. 57)

2. Use the Square Root Method to solve the equation $(x - 2)^2 = 9$. (p. 112)

Concepts and Vocabulary

3. ***True or False*** Every equation of the form

$$x^2 + y^2 + ax + by + c = 0$$

has a circle as its graph.

4. For a circle, the ______ is the distance from the center to any point on the circle.

5. ***True or False*** The radius of the circle $x^2 + y^2 = 9$ is 3.

6. ***True or False*** The center of the circle

$$(x + 3)^2 + (y - 2)^2 = 13$$

is $(3, -2)$.

7. Choose the equation of a circle with radius 6 and center $(3, -5)$.

(a) $(x - 3)^2 + (y + 5)^2 = 6$
(b) $(x + 3)^2 + (y - 5)^2 = 36$
(c) $(x + 3)^2 + (y - 5)^2 = 6$
(d) $(x - 3)^2 + (y + 5)^2 = 36$

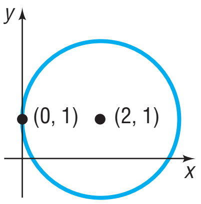
8. The equation of a circle can be changed from general form to standard form by doing which of the following?

(a) completing the squares
(b) solving for x
(c) solving for y
(d) squaring both sides

Skill Building

In Problems 9–12, find the center and radius of each circle. Write the standard form of the equation.

9.



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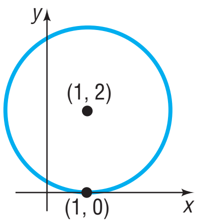

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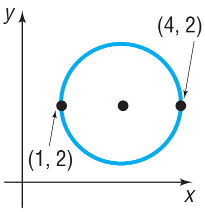

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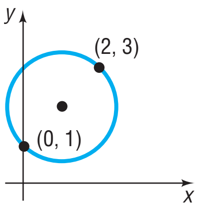


In Problems 13–22, write the standard form of the equation and the general form of the equation of each circle of radius r and center (h, k). Graph each circle.

13. $r = 2;\quad (h, k) = (0, 0)$
14. $r = 3;\quad (h, k) = (0, 0)$
15. $r = 2;\quad (h, k) = (0, 2)$
16. $r = 3;\quad (h, k) = (1, 0)$
17. $r = 5;\quad (h, k) = (4, -3)$
18. $r = 4;\quad (h, k) = (2, -3)$
19. $r = 4;\quad (h, k) = (-2, 1)$
20. $r = 7;\quad (h, k) = (-5, -2)$
21. $r = \frac{1}{2};\quad (h, k) = \left(\frac{1}{2}, 0\right)$
22. $r = \frac{1}{2};\quad (h, k) = \left(0, -\frac{1}{2}\right)$

In Problems 23–36, (a) find the center (h, k) and radius r of each circle; (b) graph each circle; (c) find the intercepts, if any.

23. $x^2 + y^2 = 4$
24. $x^2 + (y - 1)^2 = 1$
25. $2(x - 3)^2 + 2y^2 = 8$
26. $3(x + 1)^2 + 3(y - 1)^2 = 6$
27. $x^2 + y^2 - 2x - 4y - 4 = 0$
28. $x^2 + y^2 + 4x + 2y - 20 = 0$
29. $x^2 + y^2 + 4x - 4y - 1 = 0$
30. $x^2 + y^2 - 6x + 2y + 9 = 0$
31. $x^2 + y^2 - x + 2y + 1 = 0$
32. $x^2 + y^2 + x + y - \frac{1}{2} = 0$
33. $2x^2 + 2y^2 - 12x + 8y - 24 = 0$
34. $2x^2 + 2y^2 + 8x + 7 = 0$
35. $2x^2 + 8x + 2y^2 = 0$
36. $3x^2 + 3y^2 - 12y = 0$

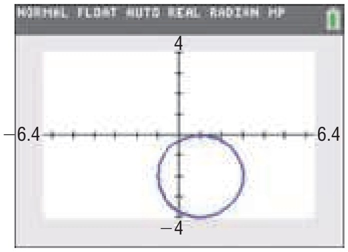
In Problems 37–44, find the standard form of the equation of each circle.

37. Center at the origin and containing the point $(-2, 3)$
38. Center $(1, 0)$ and containing the point $(-3, 2)$
39. Center $(2, 3)$ and tangent to the x-axis
40. Center $(-3, 1)$ and tangent to the y-axis
41. With endpoints of a diameter at $(1, 4)$ and $(-3, 2)$
42. With endpoints of a diameter at $(4, 3)$ and $(0, 1)$
43. Center $(-1, 3)$ and tangent to the line $y = 2$
44. Center $(4, -2)$ and tangent to the line $x = 1$

In Problems 45–48, match each graph with the correct equation.

(a) $(x - 3)^2 + (y + 3)^2 = 9$ (b) $(x + 1)^2 + (y - 2)^2 = 4$ (c) $(x - 1)^2 + (y + 2)^2 = 4$ (d) $(x + 3)^2 + (y - 3)^2 = 9$

45.



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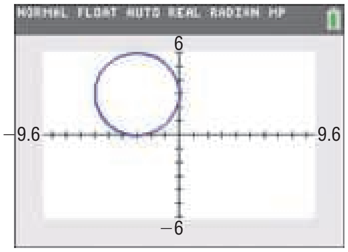
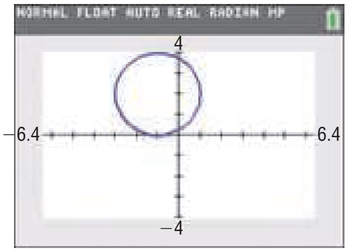
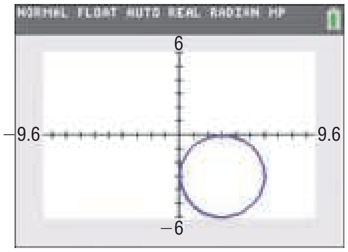


47.

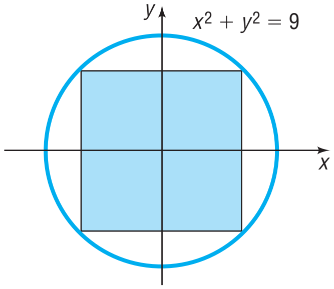


48.

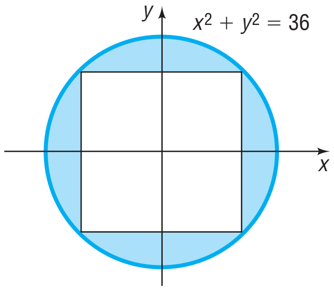


Applications and Extensions

49. Find the area of the square in the figure.



50. Find the area of the blue shaded region in the figure, assuming the quadrilateral inside the circle is a square.



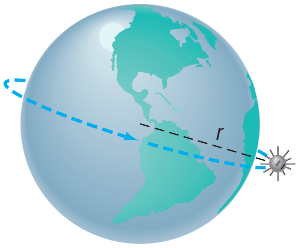
51. Ferris Wheel The original Ferris wheel was built in 1893 by Pittsburgh, Pennsylvania, bridge builder George W. Ferris. The Ferris wheel was originally built for the 1893 World's Fair in Chicago and was later reconstructed for the 1904 World's Fair in St. Louis. It had a maximum height of 264 feet and a wheel diameter of 250 feet. Find an equation for the wheel if the center of the wheel is on the y-axis.

Source: *guinnessworldrecords.com*

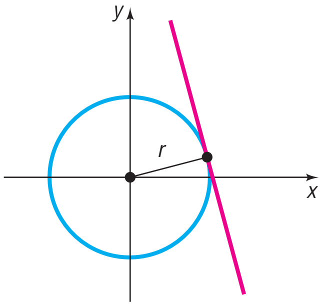
52. Ferris Wheel The High Roller observation wheel in Las Vegas has a maximum height of 550 feet and a diameter of 520 feet, with one full rotation taking approximately 30 minutes. Find an equation for the wheel if the center of the wheel is on the y-axis.

Source: *Las Vegas Review Journal*

53. Weather Satellites Earth is represented on a map of a portion of the solar system so that its surface is the circle with equation $x^2 + y^2 + 2x + 4y - 4091 = 0$. A weather satellite circles 0.6 unit above Earth with the center of its circular orbit at the center of Earth. Find the equation for the orbit of the satellite on this map.



54. The **tangent line** to a circle may be defined as the line that intersects the circle in a single point, called the **point of tangency**. See the figure.



If the equation of the circle is $x^2 + y^2 = r^2$ and the equation of the tangent line is $y = mx + b$, show that:

(a) $r^2(1 + m^2) = b^2$

[**Hint:** The quadratic equation $x^2 + (mx + b)^2 = r^2$ has exactly one solution.]

(b) The point of tangency is $\left(\dfrac{-r^2m}{b}, \dfrac{r^2}{b}\right)$.

(c) The tangent line is perpendicular to the line containing the center of the circle and the point of tangency.

55. The Greek Method The Greek method for finding the equation of the tangent line to a circle uses the fact that at any point on a circle the lines containing the center and the tangent line are perpendicular (see Problem 54). Use this method to find an equation of the tangent line to the circle $x^2 + y^2 = 9$ at the point $(1, 2\sqrt{2})$.

56. Use the Greek method described in Problem 55 to find an equation of the tangent line to the circle $x^2 + y^2 - 4x + 6y + 4 = 0$ at the point $(3, 2\sqrt{2} - 3)$.

57. Refer to Problem 54. The line $x - 2y + 4 = 0$ is tangent to a circle at $(0, 2)$. The line $y = 2x - 7$ is tangent to the same circle at $(3, -1)$. Find the center of the circle.

58. Find an equation of the line containing the centers of the two circles

$$x^2 + y^2 - 4x + 6y + 4 = 0$$

and

$$x^2 + y^2 + 6x + 4y + 9 = 0$$

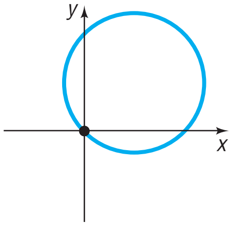
59. If a circle of radius 2 is made to roll along the x-axis, what is an equation for the path of the center of the circle?

60. If the circumference of a circle is 6π, what is its radius?

Explaining Concepts: Discussion and Writing

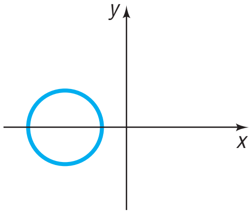
61. Which of the following equations might have the graph shown? (More than one answer is possible.)

(a) $(x-2)^2+(y+3)^2=13$
(b) $(x-2)^2+(y-2)^2=8$
(c) $(x-2)^2+(y-3)^2=13$
(d) $(x+2)^2+(y-2)^2=8$
(e) $x^2+y^2-4x-9y=0$
(f) $x^2+y^2+4x-2y=0$
(g) $x^2+y^2-9x-4y=0$
(h) $x^2+y^2-4x-4y=4$



62. Which of the following equations might have the graph shown? (More than one answer is possible.)

(a) $(x-2)^2+y^2=3$
(b) $(x+2)^2+y^2=3$
(c) $x^2+(y-2)^2=3$
(d) $(x+2)^2+y^2=4$
(e) $x^2+y^2+10x+16=0$
(f) $x^2+y^2+10x-2y=1$
(g) $x^2+y^2+9x+10=0$
(h) $x^2+y^2-9x-10=0$



63. Explain how the center and radius of a circle can be used to graph the circle.

64. **What Went Wrong?** A student stated that the center and radius of the graph whose equation is $(x+3)^2+(y-2)^2=16$ are $(3,-2)$ and 4, respectively. Why is this incorrect?

Retain Your Knowledge

Problems 65–68 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

65. Find the area and circumference of a circle of radius 13 cm.

66. Multiply $(3x-2)(x^2-2x+3)$. Express the answer as a polynomial in standard form.

67. Solve the equation: $\sqrt{2x^2+3x-1}=x+1$

68. Aaron can load a delivery van in 22 minutes. Elizabeth can load the same van in 28 minutes. How long would it take them to load the van if they worked together?

'Are You Prepared?' Answers

1. add; 25

2. $\{-1,5\}$

2.4 Variation

OBJECTIVES
1 Construct a Model Using Direct Variation (p. 197)
2 Construct a Model Using Inverse Variation (p. 197)
3 Construct a Model Using Joint Variation or Combined Variation (p. 198)

When a mathematical model is developed for a real-world problem, it often involves relationships between quantities that are expressed in terms of proportionality:

Force is proportional to acceleration.

When an ideal gas is held at a constant temperature, pressure and volume are inversely proportional.

The force of attraction between two heavenly bodies is inversely proportional to the square of the distance between them.

Revenue is directly proportional to sales.

Each of the preceding statements illustrates the idea of **variation**, or how one quantity varies in relation to another quantity. Quantities may vary *directly, inversely,* or *jointly*.

1 Construct a Model Using Direct Variation

DEFINITION

Let x and y denote two quantities. Then y **varies directly** with x, or y is **directly proportional to** x, if there is a nonzero number k such that

$$\boxed{y = kx}$$

The number k is called the **constant of proportionality**.

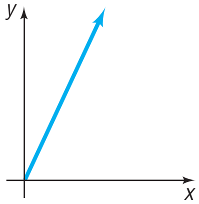


Figure 41
$y = kx$; $k > 0$, $x \geq 0$

The graph in Figure 41 illustrates the relationship between y and x if y varies directly with x and $k > 0$, $x \geq 0$. Note that the constant of proportionality is, in fact, the slope of the line.

If two quantities vary directly, then knowing the value of each quantity in one instance enables us to write a formula that is true in all cases.

EXAMPLE 1

Mortgage Payments

The monthly payment p on a mortgage varies directly with the amount borrowed B. If the monthly payment on a 30-year mortgage is \$6.65 for every \$1000 borrowed, find a formula that relates the monthly payment p to the amount borrowed B for a mortgage with these terms. Then find the monthly payment p when the amount borrowed B is \$120,000.

Solution

Because p varies directly with B, we know that

$$p = kB$$

for some constant k. Because $p = 6.65$ when $B = 1000$, it follows that

$$6.65 = k(1000)$$

$$k = 0.00665 \quad \textbf{Solve for } \boldsymbol{k}\textbf{.}$$

Since $p = kB$,

$$p = 0.00665B$$

In particular, when $B = \$120{,}000$,

$$p = 0.00665(\$120{,}000) = \$798$$

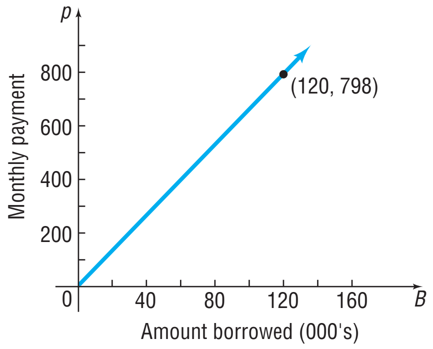


Figure 42

Figure 42 illustrates the relationship between the monthly payment p and the amount borrowed B.

Now Work PROBLEMS 5 AND 23

2 Construct a Model Using Inverse Variation

DEFINITION

Let x and y denote two quantities. Then y **varies inversely** with x, or y is **inversely proportional to** x, if there is a nonzero constant k such that

$$\boxed{y = \frac{k}{x}}$$

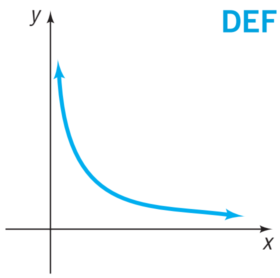


Figure 43 $y = \dfrac{k}{x}$; $k > 0$, $x > 0$

The graph in Figure 43 illustrates the relationship between y and x if y varies inversely with x and $k > 0$, $x > 0$.

EXAMPLE 2 Maximum Weight That Can Be Supported by a Piece of Pine

See Figure 44. The maximum weight W that can be safely supported by a 2-inch by 4-inch piece of pine varies inversely with its length l. Experiments indicate that the maximum weight that a 10-foot-long 2-by-4 piece of pine can support is 500 pounds. Write a general formula relating the maximum weight W (in pounds) to length l (in feet). Find the maximum weight W that can be safely supported by a length of 25 feet.

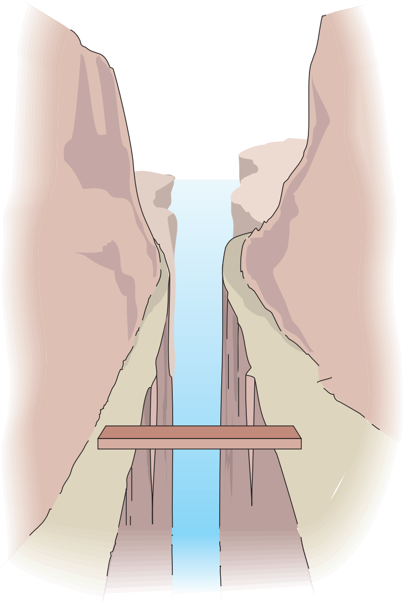


Figure 44

Solution Because W varies inversely with l, we know that

$$W = \frac{k}{l}$$

for some constant k. Because $W = 500$ when $l = 10$, we have

$$500 = \frac{k}{10}$$
$$k = 5000$$

Since $W = \frac{k}{l}$,

$$W = \frac{5000}{l}$$

In particular, the maximum weight W that can be safely supported by a piece of pine 25 feet in length is

$$W = \frac{5000}{25} = 200 \text{ pounds}$$

Figure 45 illustrates the relationship between the weight W and the length l. ■

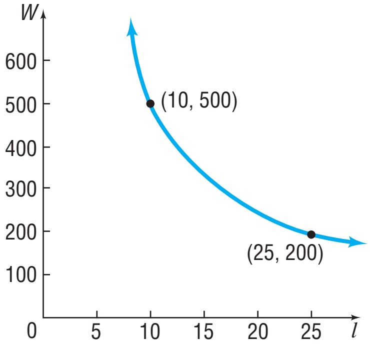


Figure 45 $W = \frac{5000}{l}$

Now Work PROBLEM 33

3 Construct a Model Using Joint Variation or Combined Variation

When a variable quantity Q is proportional to the product of two or more other variables, we say that Q **varies jointly** with these quantities. Finally, combinations of direct and/or inverse variation may occur. This is usually referred to as **combined variation**.

EXAMPLE 3 Loss of Heat through a Wall

The loss of heat through a wall varies jointly with the area of the wall and the difference between the inside and outside temperatures and varies inversely with the thickness of the wall. Write an equation that relates these quantities.

Solution Begin by assigning symbols to represent the quantities:

L = Heat loss $\quad$ T = Temperature difference

A = Area of wall $\quad$ d = Thickness of wall

Then

$$L = k\frac{AT}{d}$$

where k is the constant of proportionality. ■

In direct or inverse variation, the quantities that vary may be raised to powers. For example, in the early seventeenth century, Johannes Kepler (1571–1630) discovered that the square of the period of revolution T of a planet around the Sun varies directly with the cube of its mean distance a from the Sun. That is, $T^2 = ka^3$, where k is the constant of proportionality.

EXAMPLE 4 Force of the Wind on a Window

The force F of the wind on a flat surface positioned at a right angle to the direction of the wind varies jointly with the area A of the surface and the square of the speed v of the wind. A wind of 30 miles per hour blowing on a window measuring 4 feet by 5 feet has a force of 150 pounds. See Figure 46. What force does a wind of 50 miles per hour exert on a window measuring 3 feet by 4 feet?

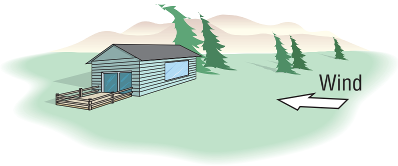


Figure 46

Solution Since F varies jointly with A and v^2, we have

$$F = kAv^2$$

where k is the constant of proportionality. We are told that $F = 150$ when $A = 4 \cdot 5 = 20$ and $v = 30$. Then

$$150 = k(20)(900) \qquad F = kAv^2, F = 150, A = 20, v = 30$$

$$k = \frac{1}{120}$$

Since $F = kAv^2$,

$$F = \frac{1}{120}Av^2$$

For a wind of 50 miles per hour blowing on a window whose area is $A = 3 \cdot 4 = 12$ square feet, the force F is

$$F = \frac{1}{120}(12)(2500) = 250 \text{ pounds}$$ ■

Now Work PROBLEM 41

2.4 Assess Your Understanding

Concepts and Vocabulary

1. If x and y are two quantities, then y is directly proportional to x if there is a nonzero number k such that ________.

2. ***True or False*** If y varies directly with x, then $y = \frac{k}{x}$, where k is a constant.

3. Which equation represents a joint variation model?
(a) $y = 5x$ (b) $y = 5xzw$ (c) $y = \frac{5}{x}$ (d) $y = \frac{5xz}{w}$

4. Choose the best description for the model $y = \frac{kx}{z}$, if k is a nonzero constant.
(a) y varies jointly with x and z.
(b) y is inversely proportional to x and z.
(c) y varies directly with x and inversely with z.
(d) y is directly proportion to z and inversely proportional to x.

Skill Building

In Problems 5–16, write a general formula to describe each variation.

5. y varies directly with x; $y = 2$ when $x = 10$

6. v varies directly with t; $v = 16$ when $t = 2$

7. A varies directly with x^2; $A = 4\pi$ when $x = 2$

8. V varies directly with x^3; $V = 36\pi$ when $x = 3$

9. F varies inversely with d^2; $F = 10$ when $d = 5$

10. y varies inversely with $\sqrt{x}$; $y = 4$ when $x = 9$

11. z varies directly with the sum of the squares of x and y; $z = 5$ when $x = 3$ and $y = 4$

12. T varies jointly with the cube root of x and the square of d; $T = 18$ when $x = 8$ and $d = 3$

13. M varies directly with the square of d and inversely with the square root of x; $M = 24$ when $x = 9$ and $d = 4$

14. z varies directly with the sum of the cube of x and the square of y; $z = 1$ when $x = 2$ and $y = 3$

15. The square of T varies directly with the cube of a and inversely with the square of d; $T = 2$ when $a = 2$ and $d = 4$

16. The cube of z varies directly with the sum of the squares of x and y; $z = 2$ when $x = 9$ and $y = 4$

Applications and Extensions

In Problems 17–22, write an equation that relates the quantities.

17. Geometry The volume V of a sphere varies directly with the cube of its radius r. The constant of proportionality is $\frac{4\pi}{3}$.

18. Geometry The square of the length of the hypotenuse c of a right triangle varies jointly with the sum of the squares of the lengths of its legs a and b. The constant of proportionality is 1.

19. Geometry The area A of a triangle varies jointly with the lengths of the base b and the height h. The constant of proportionality is $\frac{1}{2}$.

20. Geometry The perimeter p of a rectangle varies jointly with the sum of the lengths of its sides l and w. The constant of proportionality is 2.

21. Physics: Newton's Law The force F (in newtons) of attraction between two bodies varies jointly with their masses m and M (in kilograms) and inversely with the square of the distance d (in meters) between them. The constant of proportionality is $G = 6.67 \times 10^{-11}$.

22. Physics: Simple Pendulum The **period** of a pendulum is the time required for one oscillation; the pendulum is usually referred to as **simple** when the angle made to the vertical is less than 5°. The period T of a simple pendulum (in seconds) varies directly with the square root of its length l (in feet). The constant of proportionality is $\frac{2\pi}{\sqrt{32}}$.

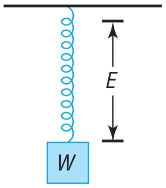
23. Mortgage Payments The monthly payment p on a mortgage varies directly with the amount borrowed B. If the monthly payment on a 30-year mortgage is \$6.49 for every \$1000 borrowed, find a linear equation that relates the monthly payment p to the amount borrowed B for a mortgage with the same terms. Then find the monthly payment p when the amount borrowed B is \$145,000.

24. Mortgage Payments The monthly payment p on a mortgage varies directly with the amount borrowed B. If the monthly payment on a 15-year mortgage is \$8.99 for every \$1000 borrowed, find a linear equation that relates the monthly payment p to the amount borrowed B for a mortgage with the same terms. Then find the monthly payment p when the amount borrowed B is \$175,000.

25. Physics: Falling Objects The distance s that an object falls is directly proportional to the square of the time t of the fall. If an object falls 16 feet in 1 second, how far will it fall in 3 seconds? How long will it take an object to fall 64 feet?

26. Physics: Falling Objects The velocity v of a falling object is directly proportional to the time t of the fall. If, after 2 seconds, the velocity of the object is 64 feet per second, what will its velocity be after 3 seconds?

27. Physics: Stretching a Spring The elongation E of a spring balance varies directly with the applied weight W (see the figure). If $E = 3$ when $W = 20$, find E when $W = 15$.



28. Physics: Vibrating String The rate of vibration of a string under constant tension varies inversely with the length of the string. If a string is 48 inches long and vibrates 256 times per second, what is the length of a string that vibrates 576 times per second?

29. Revenue Equation At the corner Shell station, the revenue R varies directly with the number g of gallons of gasoline sold. If the revenue is \$47.40 when the number of gallons sold is 12, find a linear equation that relates revenue R to the number g of gallons of gasoline. Then find the revenue R when the number of gallons of gasoline sold is 10.5.

30. Cost Equation The cost C of roasted almonds varies directly with the number A of pounds of almonds purchased. If the cost is \$23.75 when the number of pounds of roasted almonds purchased is 5, find a linear equation that relates the cost C to the number A of pounds of almonds purchased. Then find the cost C when the number of pounds of almonds purchased is 3.5.

31. Demand Suppose that the demand D for candy at the movie theater is inversely related to the price p.
- (a) When the price of candy is \$2.75 per bag, the theater sells 156 bags of candy. Express the demand for candy in terms of its price.
- (b) Determine the number of bags of candy that will be sold if the price is raised to \$3 a bag.

32. Driving to School The time t that it takes to get to school varies inversely with your average speed s.
- (a) Suppose that it takes you 40 minutes to get to school when your average speed is 30 miles per hour. Express the driving time to school in terms of average speed.
- (b) Suppose that your average speed to school is 40 miles per hour. How long will it take you to get to school?

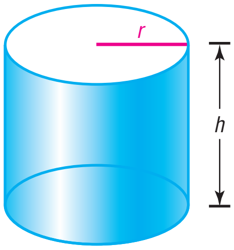
33. Pressure The volume of a gas V held at a constant temperature in a closed container varies inversely with its pressure P. If the volume of a gas is 600 cubic centimeters (cm^3) when the pressure is 150 millimeters of mercury (mm Hg), find the volume when the pressure is 200 mm Hg.

34. **Resistance** The current i in a circuit is inversely proportional to its resistance Z measured in ohms. Suppose that when the current in a circuit is 30 amperes, the resistance is 8 ohms. Find the current in the same circuit when the resistance is 10 ohms.

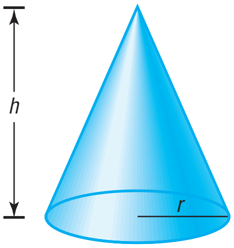
35. **Weight** The weight of an object above the surface of Earth varies inversely with the square of the distance from the center of Earth. If Maria weighs 125 pounds when she is on the surface of Earth (3960 miles from the center), determine Maria's weight when she is at the top of Mount McKinley (3.8 miles from the surface of Earth).

36. **Weight of a Body** The weight of a body above the surface of Earth varies inversely with the square of the distance from the center of Earth. If a certain body weighs 55 pounds when it is 3960 miles from the center of Earth, how much will it weigh when it is 3965 miles from the center?

37. **Geometry** The volume V of a right circular cylinder varies jointly with the square of its radius r and its height h. The constant of proportionality is π. See the figure. Write an equation for V.



38. **Geometry** The volume V of a right circular cone varies jointly with the square of its radius r and its height h. The constant of proportionality is $\frac{\pi}{3}$. See the figure. Write an equation for V.



39. **Intensity of Light** The intensity I of light (measured in foot-candles) varies inversely with the square of the distance from the bulb. Suppose that the intensity of a 100-watt light bulb at a distance of 2 meters is 0.075 foot-candle. Determine the intensity of the bulb at a distance of 5 meters.

40. **Force of the Wind on a Window** The force exerted by the wind on a plane surface varies jointly with the area of the surface and the square of the velocity of the wind. If the force on an area of 20 square feet is 11 pounds when the wind velocity is 22 miles per hour, find the force on a surface area of 47.125 square feet when the wind velocity is 36.5 miles per hour.

41. **Horsepower** The horsepower (hp) that a shaft can safely transmit varies jointly with its speed (in revolutions per minute, rpm) and the cube of its diameter. If a shaft of a certain material 2 inches in diameter can transmit 36 hp at 75 rpm, what diameter must the shaft have in order to transmit 45 hp at 125 rpm?

42. **Chemistry: Gas Laws** The volume V of an ideal gas varies directly with the temperature T and inversely with the pressure P. Write an equation relating V, T, and P using k as the constant of proportionality. If a cylinder contains oxygen at a temperature of 300 K and a pressure of 15 atmospheres in a volume of 100 liters, what is the constant of proportionality k? If a piston is lowered into the cylinder, decreasing the volume occupied by the gas to 80 liters and raising the temperature to 310 K, what is the gas pressure?

43. **Physics: Kinetic Energy** The kinetic energy K of a moving object varies jointly with its mass m and the square of its velocity v. If an object weighing 25 kilograms and moving with a velocity of 10 meters per second has a kinetic energy of 1250 joules, find its kinetic energy when the velocity is 15 meters per second.

44. **Electrical Resistance of a Wire** The electrical resistance of a wire varies directly with the length of the wire and inversely with the square of the diameter of the wire. If a wire 432 feet long and 4 millimeters in diameter has a resistance of 1.24 ohms, find the length of a wire of the same material whose resistance is 1.44 ohms and whose diameter is 3 millimeters.

45. **Measuring the Stress of Materials** The stress in the material of a pipe subject to internal pressure varies jointly with the internal pressure and the internal diameter of the pipe and inversely with the thickness of the pipe. The stress is 100 pounds per square inch when the diameter is 5 inches, the thickness is 0.75 inch, and the internal pressure is 25 pounds per square inch. Find the stress when the internal pressure is 40 pounds per square inch if the diameter is 8 inches and the thickness is 0.50 inch.

46. **Safe Load for a Beam** The maximum safe load for a horizontal rectangular beam varies jointly with the width of the beam and the square of the thickness of the beam and inversely with its length. If an 8-foot beam will support up to 750 pounds when the beam is 4 inches wide and 2 inches thick, what is the maximum safe load in a similar beam 10 feet long, 6 inches wide, and 2 inches thick?

Explaining Concepts: Discussion and Writing

47. In the early 17th century, Johannes Kepler discovered that the square of the period T of the revolution of a planet around the Sun varies directly with the cube of its mean distance a from the Sun. Go to the library and research this law and Kepler's other two laws. Write a brief paper about these laws and Kepler's place in history.

48. Using a situation that has not been discussed in the text, write a real-world problem that you think involves two variables that vary directly. Exchange your problem with another student's to solve and critique.

49. Using a situation that has not been discussed in the text, write a real-world problem that you think involves two variables that vary inversely. Exchange your problem with another student's to solve and critique.

50. Using a situation that has not been discussed in the text, write a real-world problem that you think involves three variables that vary jointly. Exchange your problem with another student's to solve and critique.

Retain Your Knowledge

Problems 51–54 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

51. Factor $3x^3 + 25x^2 - 12x - 100$ completely.

52. Add $\frac{5}{x+3} + \frac{x-2}{x^2+7x+12}$ and simplify the result.

53. Simplify: $\left(\frac{4}{25}\right)^{3/2}$

54. Rationalize the denominator of $\frac{3}{\sqrt{7}-2}$.

Chapter Review

Things to Know

Formulas

Slope (p. 174)	$m = \frac{y_2 - y_1}{x_2 - x_1}$ if $x_1 \neq x_2$; undefined if $x_1 = x_2$
Parallel lines (p. 181)	Equal slopes $(m_1 = m_2)$ and different y-intercepts $(b_1 \neq b_2)$
Perpendicular lines (p. 182)	Product of slopes is -1 $(m_1 \cdot m_2 = -1)$
Direct variation (p. 197)	$y = kx, k \neq 0$
Inverse variation (p. 197)	$y = \frac{k}{x}, k \neq 0$

Equations of Lines and Circles

Vertical line (p. 177)	$x = a$; a is the x-intercept
Point–slope form of the equation of a line (p. 177)	$y - y_1 = m(x - x_1)$; m is the slope of the line, (x_1, y_1) is a point on the line
Horizontal line (p. 178)	$y = b$; b is the y-intercept
Slope–intercept form of the equation of a line (p. 178)	$y = mx + b$; m is the slope of the line, b is the y-intercept
General form of the equation of a line (p. 180)	$Ax + By = C$; A, B not both 0
Standard form of the equation of a circle (p. 189)	$(x - h)^2 + (y - k)^2 = r^2$; r is the radius of the circle, (h, k) is the center of the circle
Equation of the unit circle (p. 190)	$x^2 + y^2 = 1$
General form of the equation of a circle (p. 192)	$x^2 + y^2 + ax + by + c = 0$, with restrictions on a, b, and c

Objectives

Section		You should be able to ...	Examples	Review Exercises
2.1	1	Find intercepts algebraically from an equation (p. 165)	1	5–9
	2	Test an equation for symmetry (p. 166)	2, 3	5–9
	3	Know how to graph key equations (p. 168)	4–6	27
2.2	1	Calculate and interpret the slope of a line (p. 173)	1	1–4, 29, 31
	2	Graph lines given a point and the slope (p. 176)	2	28
	3	Find the equation of a vertical line (p. 176)	3	17
	4	Use the point–slope form of a line; identify horizontal lines (p. 177)	4, 5	15, 16
	5	Write the equation of a line in slope–intercept form (p. 178)	6	15, 16, 18–24
	6	Find the equation of a line given two points (p. 179)	7	18–20
	7	Graph lines written in general form using intercepts (p. 180)	8	25, 26
	8	Find equations of parallel lines (p. 181)	9, 10	21
	9	Find equations of perpendicular lines (p. 182)	11, 12	22
2.3	1	Write the standard form of the equation of a circle (p. 189)	1	10, 11, 30
	2	Graph a circle by hand and by using a graphing utility (p. 190)	2, 3	12–14
	3	Work with the general form of the equation of a circle (p. 192)	4, 5	13, 14
2.4	1	Construct a model using direct variation (p. 197)	1	32
	2	Construct a model using inverse variation (p. 197)	2	33
	3	Construct a model using joint variation or combined variation (p. 198)	3, 4	34

Review Exercises

In Problems 1– 4, find the following for each pair of points:
  (a) The slope of the line containing the points
  (b) Interpret the slope found in part (a)

1. $(0, 0); (4, 2)$

2. $(1, -1); (-2, 3)$

3. $(4, -4); (4, 8)$

4. $(-2, -1); (3, -1)$

In Problems 5–9, list the intercepts and test for symmetry with respect to the x-axis, the y-axis, and the origin.

5. $2x = 3y^2$

6. $x^2 + 4y^2 = 16$

7. $y = x^4 - 3x^2 - 4$

8. $y = x^3 - x$

9. $x^2 + x + y^2 + 2y = 0$

In Problems 10 and 11, find the standard form of the equation of the circle whose center and radius are given.

10. $(h, k) = (-2, 3); r = 4$

11. $(h, k) = (-1, -2); r = 1$

In Problems 12–14, find the center and radius of each circle. Graph each circle by hand. Find the intercepts, if any, of each circle.

12. $x^2 + (y - 1)^2 = 4$

13. $x^2 + y^2 - 2x + 4y - 4 = 0$

14. $3x^2 + 3y^2 - 6x + 12y = 0$

In Problems 15–22, find an equation of the line having the given characteristics. Express your answer using either the general form or the slope–intercept form of the equation of a line, whichever you prefer.

15. Slope $= -2$; containing the point $(3, -1)$

16. Slope $= 0$; containing the point $(-5, 4)$

17. Vertical; containing the point $(-3, 4)$

18. x-intercept $= 2$; containing the point $(4, -5)$

19. y-intercept $= -2$; containing the point $(5, -3)$

20. Containing the points $(3, -4)$ and $(2, 1)$

21. Parallel to the line $2x - 3y = -4$; containing the point $(-5, 3)$

22. Perpendicular to the line $3x - y = -4$; containing the point $(-2, 4)$

In Problems 23 and 24, find the slope and y-intercept of each line. Graph the line, labeling any intercepts.

23. $4x - 5y = -20$

24. $\frac{1}{2}x - \frac{1}{3}y = -\frac{1}{6}$

In Problems 25 and 26, find the intercepts and graph each line.

25. $2x - 3y = 12$

26. $\frac{1}{2}x + \frac{1}{3}y = 2$

27. Sketch a graph of $y = x^3$.

28. Graph the line with slope $\frac{2}{3}$ containing the point $(1, 2)$.

29. Show that the points $A = (-2, 0)$, $B = (-4, 4)$, and $C = (8, 5)$ are the vertices of a right triangle by using the slopes of the lines joining the vertices.

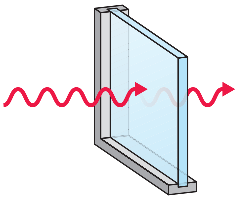
30. The endpoints of the diameter of a circle are $(-3, 2)$ and $(5, -6)$. Find the center and radius of the circle. Write the standard equation of this circle.

31. Show that the points $A = (2, 5)$, $B = (6, 1)$, and $C = (8, -1)$ lie on a line by using slopes.

32. **Mortgage Payments** The monthly payment p on a mortgage varies directly with the amount borrowed B. If the monthly payment on a 30-year mortgage is \$854.00 when \$130,000 is borrowed, find an equation that relates the monthly payment p to the amount borrowed B for a mortgage with the same terms. Then find the monthly payment p when the amount borrowed B is \$165,000.

33. **Weight of a Body** The weight of a body varies inversely with the square of its distance from the center of Earth. Assuming that the radius of Earth is 3960 miles, how much would a man weigh at an altitude of 1 mile above Earth's surface if he weighs 200 pounds on Earth's surface?

34. **Heat Loss** The amount of heat transferred per hour through a glass window varies jointly with the surface area of the window and the difference in temperature between the areas separated by the glass. A window with a surface area of 7.5 square feet loses 135 Btu per hour when the temperature difference is 40°F. How much heat is lost per hour for a similar window with a surface are of 12 square feet when the temperature difference is 35°F?



Chapter Test

CHAPTER
Test Prep
VIDEOS

The Chapter Test Prep Videos are step-by-step solutions available in MyMathLab®, or on this text's YouTube Channel. Flip back to the Resources for Success page for a link to this text's YouTube channel.

1. Use $P_1 = (-1, 3)$ and $P_2 = (5, -1)$.
  (a) Find the slope of the line containing P_1 and P_2.
  (b) Interpret this slope.

2. Graph $y = x^2 - 9$ by plotting points.

3. Sketch the graph of $y^2 = x$.

4. List the intercepts and test for symmetry: $x^2 + y = 9$.

5. Write the slope–intercept form of the line with slope -2 containing the point $(3, -4)$. Graph the line.

6. Find the slope and y-intercept: $2x + 3y = 9$.

7. Graph the line $3x - 4y = 24$ by finding the intercepts.

8. Write the general form of the circle with center $(4, -3)$ and radius 5.

9. Find the center and radius of the circle $x^2 + y^2 + 4x - 2y - 4 = 0$. Graph this circle.

10. For the line $2x + 3y = 6$, find a line parallel to it containing the point $(1, -1)$. Also find a line perpendicular to it containing the point $(0, 3)$.

11. Resistance Due to a Conductor The resistance (in ohms) of a circular conductor varies directly with the length of the conductor and inversely with the square of the radius of the conductor. If 50 feet of wire with a radius of 6×10^{-3} inch has a resistance of 10 ohms, what would be the resistance of 100 feet of the same wire if the radius is increased to 7×10^{-3} inch?

Cumulative Review

In Problems 1–8, find the real solution(s) of each equation.

1. $3x - 5 = 0$

2. $x^2 - x - 12 = 0$

3. $2x^2 - 5x - 3 = 0$

4. $x^2 - 2x - 2 = 0$

5. $x^2 + 2x + 5 = 0$

6. $\sqrt{2x + 1} = 3$

7. $|x - 2| = 1$

8. $\sqrt{x^2 + 4x} = 2$

In Problems 9 and 10, solve each equation in the complex number system.

9. $x^2 = -9$

10. $x^2 - 2x + 5 = 0$

In Problems 11–14, solve each inequality. Graph the solution set.

11. $2x - 3 \le 7$

12. $-1 < x + 4 < 5$

13. $|x - 2| \le 1$

14. $|2 + x| > 3$

15. Find the distance between the points $P = (-1, 3)$ and $Q = (4, -2)$. Find the midpoint of the line segment from P to Q.

16. Which of the following points are on the graph of $y = x^3 - 3x + 1$?
  (a) $(-2, -1)$ (b) $(2, 3)$ (c) $(3, 1)$

17. Sketch the graph of $y = x^3$.

18. Find the equation of the line containing the points $(-1, 4)$ and $(2, -2)$. Express your answer in slope–intercept form.

19. Find the equation of the line perpendicular to the line $y = 2x + 1$ and containing the point $(3, 5)$. Express your answer in slope–intercept form and graph the line.

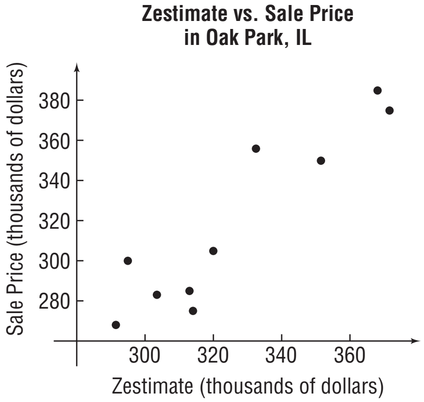
20. Graph the equation $x^2 + y^2 - 4x + 8y - 5 = 0$.

Chapter Project

Determining the Selling Price of a Home Determining how much to pay for a home is one of the more difficult decisions that must be made when purchasing a home. There are many factors that play a role in a home's value. Location, size, number of bedrooms, number of bathrooms, lot size, and building materials are just a few. Fortunately, the website Zillow.com has developed its own formula for predicting the selling price of a home. This information is a great tool for predicting the actual sale price. For example, the data below show the "zestimate"—the selling price of a home as predicted by the folks at Zillow and the actual selling price of the home for homes in Oak Park, Illinois.

Zestimate (000s of dollars)	Sale Price (000s of dollars)
291.5	268
320	305
371.5	375
303.5	283
351.5	350
314	275
332.5	356
295	300
313	285
368	385

The graph below, called a scatter diagram, shows the points (291.5, 268), (320, 305), . . . , (368, 385) in a Cartesian plane. From the graph, it appears that the data follow a linear relation.



1. Imagine drawing a line through the data that appears to fit the data well. Do you believe the slope of the line would be positive, negative, or close to zero? Why?
2. Pick two points from the scatter diagram. Treat the zestimate as the value of x and treat the sale price as the corresponding value of y. Find the equation of the line through the two points you selected.
3. Interpret the slope of the line.
4. Use your equation to predict the selling price of a home whose zestimate is \$335,000.
5. Do you believe it would be a good idea to use the equation you found in part 2 if the zestimate is \$950,000? Why or why not?
6. Choose a location in which you would like to live. Go to www.zillow.com and randomly select at least ten homes that have recently sold. Record the Zestimate and sale price for each home.
   (a) Draw a scatter diagram of your data.
   (b) Select two points from the scatter diagram and find the equation of the line through the points.
   (c) Interpret the slope.
   (d) Find a home from the Zillow website that interests you under the "Make Me Move" option for which a zestimate is available. Use your equation to predict the sale price based on the estimate.

3 Functions and Their Graphs

Choosing a Wireless Data Plan

Most consumers choose a cellular provider first and then select an appropriate data plan from that provider. The choice as to the type of plan selected depends on your use of the device. For example, is online gaming important? Do you want to stream audio or video? The mathematics learned in this chapter can help you decide what plan is best suited to your particular needs.

—*See the Internet-based Chapter Project*—

Outline

••• A Look Back

So far, our discussion has focused on techniques for graphing equations containing two variables.

A Look Ahead •••

In this chapter, we look at a special type of equation involving two variables called a *function*. This chapter deals with what a function is, how to graph functions, properties of functions, and how functions are used in applications. The word *function* apparently was introduced by René Descartes in 1637. For him, a function was simply any positive integral power of a variable *x*. Gottfried Wilhelm Leibniz (1646–1716), who always emphasized the geometric side of mathematics, used the word *function* to denote any quantity associated with a curve, such as the coordinates of a point on the curve. Leonhard Euler (1707–1783) employed the word to mean any equation or formula involving variables and constants. His idea of a function is similar to the one most often seen in courses that precede calculus. Later, the use of functions in investigating heat flow equations led to a very broad definition that originated with Lejeune Dirichlet (1805–1859), which describes a function as a correspondence between two sets. That is the definition used in this text.

3.1 Functions

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Intervals (Section 1.7, pp. 147–148)
- Solving Inequalities (Section 1.7, pp. 150–151)
- Evaluating Algebraic Expressions, Domain of a Variable (Chapter R, Section R.2, pp. 21–22)
- Rationalizing Denominators (Chapter R, Section R.8, p. 76)

Now Work the **'Are You Prepared?'** problems on page 218.

OBJECTIVES
1 Determine Whether a Relation Represents a Function (p. 207)
2 Find the Value of a Function (p. 210)
3 Find the Difference Quotient of a Function (p. 213)
4 Find the Domain of a Function Defined by an Equation (p. 214)
5 Form the Sum, Difference, Product, and Quotient of Two Functions (p. 216)

1 Determine Whether a Relation Represents a Function

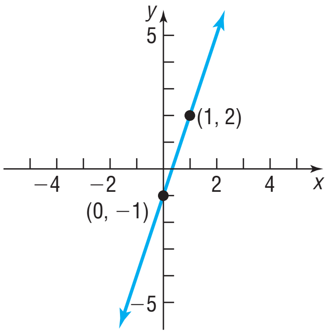


Figure 1 $y = 3x - 1$

Often there are situations where the value of one variable is somehow linked to the value of another variable. For example, an individual's level of education is linked to annual income. Engine size is linked to gas mileage. When the value of one variable is related to the value of a second variable, we have a *relation*. A **relation** is a correspondence between two sets. If x and y are two elements, one from each of these sets, and if a relation exists between x and y, then we say that x **corresponds** to y or that y **depends on** x, and we write $x \rightarrow y$.

There are a number of ways to express relations between two sets. For example, the equation $y = 3x - 1$ shows a relation between x and y. It says that if we take some number x, multiply it by 3, and then subtract 1, we obtain the corresponding value of y. In this sense, x serves as the **input** to the relation, and y is the **output** of the relation. This relation, expressed as a graph, is shown in Figure 1.

The set of all inputs for a relation is called the **domain** of the relation, and the set of all outputs is called the **range**.

In addition to being expressed as equations and graphs, relations can be expressed through a technique called *mapping*. A **map** illustrates a relation as a set of inputs with an arrow drawn from each element in the set of inputs to the corresponding element in the set of outputs. **Ordered pairs** can be used to represent $x \rightarrow y$ as (x, y).

EXAMPLE 1

Maps and Ordered Pairs as Relations

Figure 2 shows a relation between states and the number of representatives each state has in the House of Representatives. (***Source:*** *www.house.gov*). The relation might be named "number of representatives."

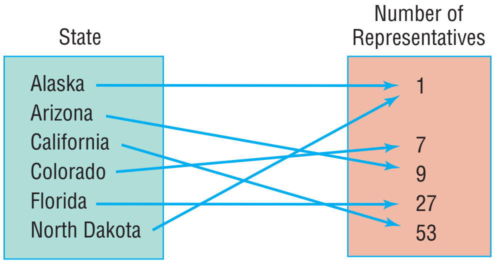


Figure 2 Number of representatives

In this relation, Alaska corresponds to 1, Arizona corresponds to 9, and so on. Using ordered pairs, this relation would be expressed as

$$\{(\text{Alaska}, 1), (\text{Arizona}, 9), (\text{California}, 53), (\text{Colorado}, 7), (\text{Florida}, 27), (\text{North Dakota}, 1)\}$$

The domain of the relation is {Alaska, Arizona, California, Colorado, Florida, North Dakota}, and the range is $\{1, 7, 9, 27, 53\}$. Since the range is a set, the output "1" is listed only once. ■

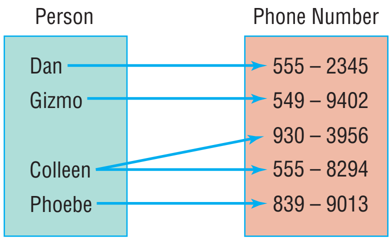


Figure 3 Phone numbers

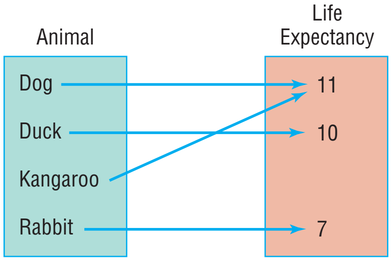


Figure 4 Animal life expectancy

One of the most important concepts in algebra is the *function*. A function is a special type of relation. To understand the idea behind a function, let's revisit the relation presented in Example 1. If we were to ask, "How many representatives does Alaska have?" you would respond "1." In fact, each input *state* corresponds to a single output *number of representatives*.

Let's consider a second relation, one that involves a correspondence between four people and their phone numbers. See Figure 3. Notice that Colleen has two telephone numbers. There is no single answer to the question "What is Colleen's phone number?"

Let's look at one more relation. Figure 4 is a relation that shows a correspondence between type of *animal* and *life expectancy*. If asked to determine the life expectancy of a dog, we would all respond, "11 years." If asked to determine the life expectancy of a rabbit, we would all respond, "7 years."

Notice that the relations presented in Figures 2 and 4 have something in common. What is it? In both of these relations, each input corresponds to exactly one output. This leads to the definition of a *function*.

DEFINITION

Let X and Y be two nonempty sets.* A **function** from X into Y is a relation that associates with each element of X exactly one element of Y. ■

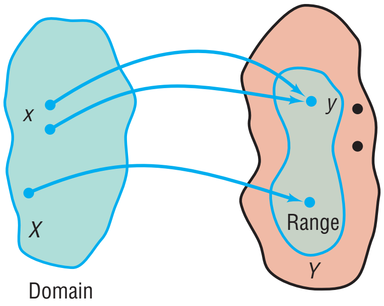


Figure 5

The set X is called the **domain** of the function. For each element x in X, the corresponding element y in Y is called the **value** of the function at x, or the **image** of x. The set of all images of the elements in the domain is called the **range** of the function. See Figure 5.

Since there may be some elements in Y that are not the image of some x in X, it follows that the range of a function may be a subset of Y, as shown in Figure 5. For example, consider the function $y = x^2$, where the domain is the set of all real numbers. Since $x^2 \geq 0$ for all real numbers x, the range of $y = x^2$ is $\{y \mid y \geq 0\}$, which is a subset of the set of all real numbers, Y.

Not all relations between two sets are functions. The next example shows how to determine whether a relation is a function.

EXAMPLE 2 Determining Whether a Relation Is a Function

For each relation in Figures 6, 7, and 8, state the domain and range. Then determine whether the relation is a function.

(a) See Figure 6. For this relation, the input is the number of calories in a fast-food sandwich, and the output is the fat content (in grams).

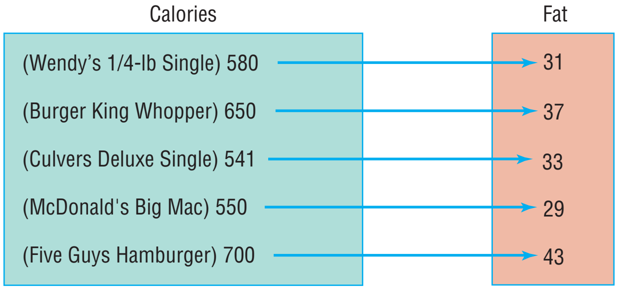


Figure 6 Fat content
Source: *Each company's website*

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*The sets X and Y will usually be sets of real numbers, in which case a (real) function results. The two sets can also be sets of complex numbers, and then we have defined a complex function. In the broad definition (proposed by Lejeune Dirichlet), X and Y can be any two sets.

(b) See Figure 7. For this relation, the inputs are gasoline stations in Harris County, Texas, and the outputs are the price per gallon of unleaded regular in March 2015.

(c) See Figure 8. For this relation, the inputs are the weight (in carats) of pear-cut diamonds and the outputs are the price (in dollars).

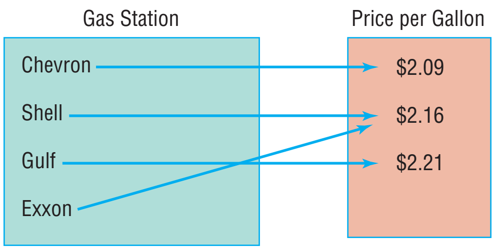


Figure 7 Unleaded price per gallon

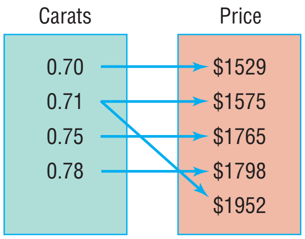


Figure 8 Diamond price

Source: *Used with permission of Diamonds.com*

Solution

(a) The domain of the relation is $\{541, 550, 580, 650, 700\}$, and the range of the relation is $\{29, 31, 33, 37, 43\}$. The relation in Figure 6 is a function because each element in the domain corresponds to exactly one element in the range.

(b) The domain of the relation is {Chevron, Exxon, Gulf, Shell}. The range of the relation is $\{\$2.09, \$2.16, \$2.21\}$. The relation in Figure 7 is a function because each element in the domain corresponds to exactly one element in the range. Notice that it is okay for more than one element in the domain to correspond to the same element in the range (Shell and Exxon both sold gas for $2.16 a gallon).

(c) The domain of the relation is $\{0.70, 0.71, 0.75, 0.78\}$, and the range is $\{\$1529, \$1575, \$1765, \$1798, \$1952\}$. The relation in Figure 8 is not a function because not every element in the domain corresponds to exactly one element in the range. If a 0.71-carat diamond is chosen from the domain, a single price cannot be assigned to it.

Now Work PROBLEM 19

The idea behind a function is its predictability. If the input is known, we can use the function to determine the output. With "nonfunctions," we don't have this predictability. Look back at Figure 6. If asked, "How many grams of fat are in a 580-calorie sandwich?" we could use the correspondence to answer, "31." Now consider Figure 8. If asked, "What is the price of a 0.71-carat diamond?" we could not give a single response because two outputs result from the single input "0.71." For this reason, the relation in Figure 8 is not a function.

We may also think of a function as a set of ordered pairs (x, y) in which no ordered pairs have the same first element and different second elements. The set of all first elements x is the domain of the function, and the set of all second elements y is its range. Each element x in the domain corresponds to exactly one element y in the range.

In Words
For a function, no input has more than one output. The domain of a function is the set of all inputs; the range is the set of all outputs.

EXAMPLE 3

Determining Whether a Relation Is a Function

For each relation, state the domain and range. Then determine whether the relation is a function.

(a) $\{(1, 4), (2, 5), (3, 6), (4, 7)\}$

(b) $\{(1, 4), (2, 4), (3, 5), (6, 10)\}$

(c) $\{(-3, 9), (-2, 4), (0, 0), (1, 1), (-3, 8)\}$

Solution (a) The domain of this relation is $\{1, 2, 3, 4\}$, and its range is $\{4, 5, 6, 7\}$. This relation is a function because there are no ordered pairs with the same first element and different second elements.

(b) The domain of this relation is $\{1, 2, 3, 6\}$, and its range is $\{4, 5, 10\}$. This relation is a function because there are no ordered pairs with the same first element and different second elements.

(c) The domain of this relation is $\{-3, -2, 0, 1\}$, and its range is $\{0, 1, 4, 8, 9\}$. This relation is not a function because there are two ordered pairs, $(-3, 9)$ and $(-3, 8)$, that have the same first element and different second elements. ■

In Example 3(b), notice that 1 and 2 in the domain both have the same image in the range. This does not violate the definition of a function; two different first elements can have the same second element. A violation of the definition occurs when two ordered pairs have the same first element and different second elements, as in Example 3(c).

Now Work PROBLEM 23

Up to now we have shown how to identify when a relation is a function for relations defined by mappings (Example 2) and ordered pairs (Example 3). But relations can also be expressed as equations. The circumstances under which equations are functions are discussed next.

To determine whether an equation, where y depends on x, is a function, it is often easiest to solve the equation for y. If any value of x in the domain corresponds to more than one y, the equation does not define a function; otherwise, it does define a function.

EXAMPLE 4 Determining Whether an Equation Is a Function

Determine whether the equation $y = 2x - 5$ defines y as a function of x.

Solution The equation tells us to take an input x, multiply it by 2, and then subtract 5. For any input x, these operations yield only one output y, so the equation is a function. For example, if $x = 1$, then $y = 2(1) - 5 = -3$. If $x = 3$, then $y = 2(3) - 5 = 1$. The graph of the equation $y = 2x - 5$ is a line with slope 2 and y-intercept -5. The function is called a *linear function*. ■

EXAMPLE 5 Determining Whether an Equation Is a Function

Determine whether the equation $x^2 + y^2 = 1$ defines y as a function of x.

Solution To determine whether the equation $x^2 + y^2 = 1$, which defines the unit circle, is a function, solve the equation for y.

$$\begin{aligned} x^2 + y^2 &= 1 \\ y^2 &= 1 - x^2 \\ y &= \pm\sqrt{1 - x^2} \end{aligned}$$

For values of x for which $-1 < x < 1$, two values of y result. For example, if $x = 0$, then $y = \pm 1$, so two different outputs result from the same input. This means that the equation $x^2 + y^2 = 1$ does not define a function. ■

Now Work PROBLEM 37

2 Find the Value of a Function

Functions are often denoted by letters such as f, F, g, G, and others. If f is a function, then for each number x in its domain, the corresponding image in the range is designated by the symbol $f(x)$, read as "f of x" or "f at x." We refer to $f(x)$ as the **value of f at the number x**; $f(x)$ is the number that results when x is given and the function f is applied; $f(x)$ is the output corresponding to x, or $f(x)$ is the image of x;

$f(x)$ does *not* mean "f times x." For example, the function given in Example 4 may be written as $y = f(x) = 2x - 5$. Then $f(1) = -3$ and $f(3) = 1$.

Figure 9 illustrates some other functions. Notice that in every function, for each x in the domain, there is one value in the range.

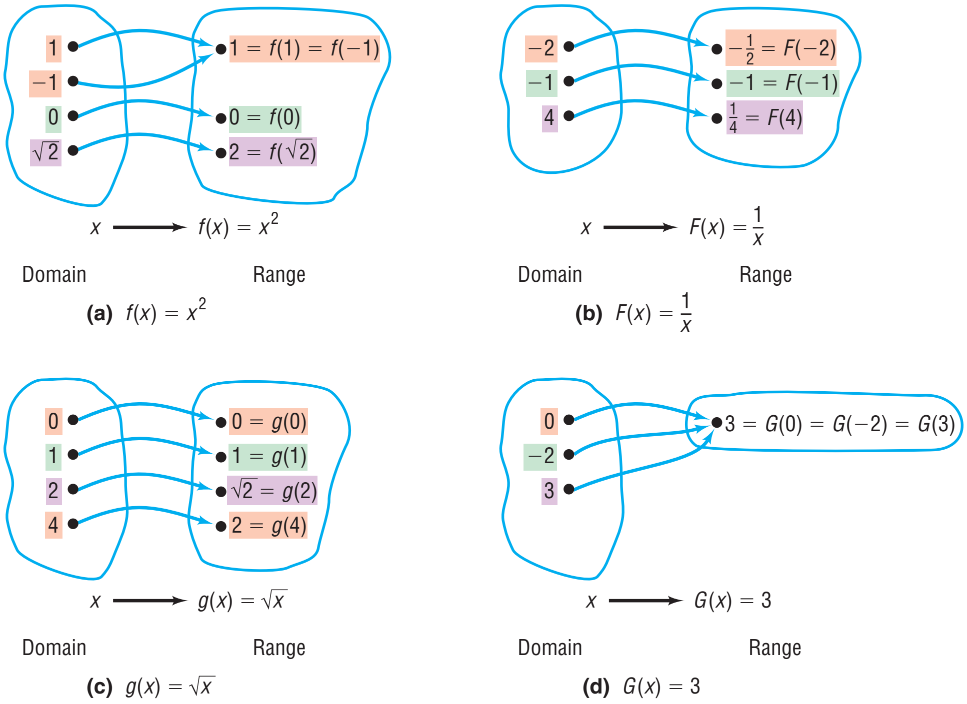


Figure 9

Sometimes it is helpful to think of a function f as a machine that receives as input a number from the domain, manipulates it, and outputs a value. See Figure 10.

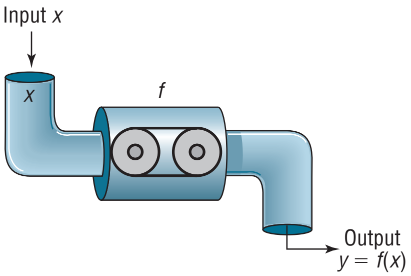


Figure 10 Input/output machine

The restrictions on this input/output machine are as follows:

1. It accepts only numbers from the domain of the function.
2. For each input, there is exactly one output (which may be repeated for different inputs).

For a function $y = f(x)$, the variable x is called the **independent variable**, because it can be assigned any of the permissible numbers from the domain. The variable y is called the **dependent variable**, because its value depends on x.

Any symbols can be used to represent the independent and dependent variables. For example, if f is the *cube function*, then f can be given by $f(x) = x^3$ or $f(t) = t^3$ or $f(z) = z^3$. All three functions are the same. Each says to cube the independent variable to get the output. In practice, the symbols used for the independent and dependent variables are based on common usage, such as using C for cost in business.

The independent variable is also called the **argument** of the function. Thinking of the independent variable as an argument can sometimes make it easier to find the value of a function. For example, if f is the function defined by $f(x) = x^3$, then f tells us to cube the argument. Thus $f(2)$ means to cube 2, $f(a)$ means to cube the number a, and $f(x + h)$ means to cube the quantity $x + h$.

EXAMPLE 6

Finding Values of a Function

For the function f defined by $f(x) = 2x^2 - 3x$, evaluate

(a) $f(3)$ (b) $f(x) + f(3)$ (c) $3f(x)$ (d) $f(-x)$

(e) $-f(x)$ (f) $f(3x)$ (g) $f(x + 3)$

Solution (a) Substitute 3 for x in the equation for f, $f(x) = 2x^2 - 3x$, to get

$$f(3) = 2(3)^2 - 3(3) = 18 - 9 = 9$$

The image of 3 is 9.

(b) $f(x) + f(3) = (2x^2 - 3x) + (9) = 2x^2 - 3x + 9$

(c) Multiply the equation for f by 3.

$$3f(x) = 3(2x^2 - 3x) = 6x^2 - 9x$$

(d) Substitute $-x$ for x in the equation for f and simplify.

$$f(-x) = 2(-x)^2 - 3(-x) = 2x^2 + 3x \quad \text{Notice the use of parentheses here.}$$

(e) $-f(x) = -(2x^2 - 3x) = -2x^2 + 3x$

(f) Substitute $3x$ for x in the equation for f and simplify.

$$f(3x) = 2(3x)^2 - 3(3x) = 2(9x^2) - 9x = 18x^2 - 9x$$

(g) Substitute $x + 3$ for x in the equation for f and simplify.

$$\begin{aligned} f(x + 3) &= 2(x + 3)^2 - 3(x + 3) \\ &= 2(x^2 + 6x + 9) - 3x - 9 \\ &= 2x^2 + 12x + 18 - 3x - 9 \\ &= 2x^2 + 9x + 9 \end{aligned}$$

■

Notice in this example that $f(x + 3) \neq f(x) + f(3)$, $f(-x) \neq -f(x)$, and $3f(x) \neq f(3x)$.

Now Work PROBLEM 43

Most calculators have special keys that allow you to find the value of certain commonly used functions. For example, you should be able to find the square function $f(x) = x^2$, the square root function $f(x) = \sqrt{x}$, the reciprocal function $f(x) = \dfrac{1}{x} = x^{-1}$, and many others that will be discussed later in this text (such as ln x and log x). Verify the results of the following example on your calculator.

EXAMPLE 7

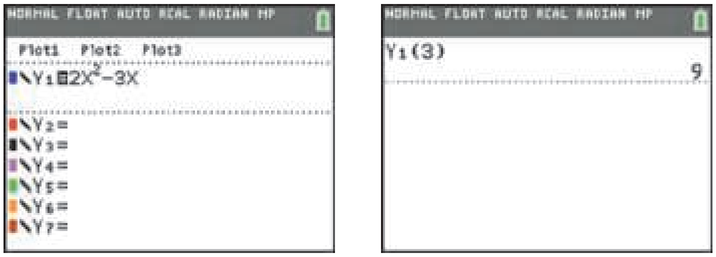
Finding Values of a Function on a Calculator

(a) $f(x) = x^2$ $\qquad f(1.234) = 1.234^2 = 1.522756$

(b) $F(x) = \dfrac{1}{x}$ $\qquad F(1.234) = \dfrac{1}{1.234} \approx 0.8103727715$

(c) $g(x) = \sqrt{x}$ $\qquad g(1.234) = \sqrt{1.234} \approx 1.110855526$ ■

COMMENT Graphing calculators can be used to evaluate any function. Figure 11 shows the result obtained in Example 6(a) on a TI-84 Plus C graphing calculator with the function to be evaluated, $f(x) = 2x^2 - 3x$, in Y_1.



```
NORMAL FLOAT AUTO REAL RADIAN MP
 Plot1  Plot2  Plot3
■\Y1■2X²-3X

■\Y2=
■\Y3=
■\Y4=
■\Y5=
■\Y6=
■\Y7=
```

```
NORMAL FLOAT AUTO REAL RADIAN MP
Y1(3)
                               9
```

Figure 11 Evaluating $f(x) = 2x^2 - 3x$ for $x = 3$ ■

Implicit Form of a Function

COMMENT The explicit form of a function is the form required by a graphing calculator. ■

In general, when a function f is defined by an equation in x and y, we say that the function f is given **implicitly**. If it is possible to solve the equation for y in terms of x, then we write $y = f(x)$ and say that the function is given **explicitly**. For example,

Implicit Form	**Explicit Form**
$3x + y = 5$	$y = f(x) = -3x + 5$
$x^2 - y = 6$	$y = f(x) = x^2 - 6$
$xy = 4$	$y = f(x) = \dfrac{4}{x}$

SUMMARY

Important Facts about Functions

(a) For each x in the domain of a function f, there is exactly one image $f(x)$ in the range; however, an element in the range can result from more than one x in the domain.

(b) f is the symbol that we use to denote the function. It is symbolic of the equation (rule) that we use to get from an x in the domain to $f(x)$ in the range.

(c) If $y = f(x)$, then x is called the independent variable or argument of f, and y is called the dependent variable or the value of f at x.

3 Find the Difference Quotient of a Function

An important concept in calculus involves looking at a certain quotient. For a given function $y = f(x)$, the inputs x and $x + h, h \neq 0$, result in the images $f(x)$ and $f(x + h)$. The quotient of their differences

$$\frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

with $h \neq 0$, is called the *difference quotient of f* at x.

DEFINITION

The **difference quotient** of a function f at x is given by

$$\frac{f(x+h) - f(x)}{h} \qquad h \neq 0 \qquad \textbf{(1)}$$

■

The difference quotient is used in calculus to define the derivative, which leads to applications such as the velocity of an object and optimization of resources.

When finding a difference quotient, it is necessary to simplify the expression in order to divide out the h in the denominator, as illustrated in the following example.

EXAMPLE 8 Finding the Difference Quotient of a Function

Find the difference quotient of each function.

(a) $f(x) = 2x^2 - 3x$

(b) $f(x) = \dfrac{4}{x}$

(c) $f(x) = \sqrt{x}$

Solution

(a)
$$\frac{f(x+h)-f(x)}{h} \underset{\uparrow}{=} \frac{[2(x+h)^2-3(x+h)]-[2x^2-3x]}{h}$$

$f(x+h) = 2(x+h)^2 - 3(x+h)$

$$\begin{aligned}
&= \frac{2(x^2+2xh+h^2)-3x-3h-2x^2+3x}{h} && \text{Simplify.}\\
&= \frac{2x^2+4xh+2h^2-3h-2x^2}{h} && \text{Distribute and combine like terms.}\\
&= \frac{4xh+2h^2-3h}{h} && \text{Combine like terms.}\\
&= \frac{h(4x+2h-3)}{h} && \text{Factor out } h.\\
&= 4x+2h-3 && \text{Divide out the factor } h.
\end{aligned}$$

(b)
$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\dfrac{4}{x+h}-\dfrac{4}{x}}{h} && f(x+h)=\frac{4}{x+h}\\
&= \frac{\dfrac{4x-4(x+h)}{x(x+h)}}{h} && \text{Subtract.}\\
&= \frac{4x-4x-4h}{x(x+h)h} && \text{Divide and distribute.}\\
&= \frac{-4h}{x(x+h)h} && \text{Simplify.}\\
&= -\frac{4}{x(x+h)} && \text{Divide out the factor } h.
\end{aligned}$$

(c)
$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\sqrt{x+h}-\sqrt{x}}{h} && f(x+h)=\sqrt{x+h}\\
&= \frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} && \text{Rationalize the numerator.}\\
&= \frac{(\sqrt{x+h})^2-(\sqrt{x})^2}{h(\sqrt{x+h}+\sqrt{x})} && (A-B)(A+B)=A^2-B^2\\
&= \frac{h}{h(\sqrt{x+h}+\sqrt{x})} && (\sqrt{x+h})^2-(\sqrt{x})^2 = x+h-x = h\\
&= \frac{1}{\sqrt{x+h}+\sqrt{x}} && \text{Divide out the factor } h.
\end{aligned}$$

■

Now Work PROBLEM 79

4 Find the Domain of a Function Defined by an Equation

Often the domain of a function f is not specified; instead, only the equation defining the function is given. In such cases, we agree that the **domain of f** is the largest set of real numbers for which the value $f(x)$ is a real number. The domain of a function f is the same as the domain of the variable x in the expression $f(x)$.

EXAMPLE 9 Finding the Domain of a Function

Find the domain of each of the following functions.

(a) $f(x) = x^2 + 5x$

(b) $g(x) = \dfrac{3x}{x^2 - 4}$

(c) $h(t) = \sqrt{4 - 3t}$

(d) $F(x) = \dfrac{\sqrt{3x + 12}}{x - 5}$

In Words

The domain of g found in Example 9(b) is $\{x \mid x \neq -2, x \neq 2\}$. This notation is read, "The domain of the function g is the set of all real numbers x such that x does not equal -2 and x does not equal 2."

Solution

(a) The function says to square a number and then add five times the number. Since these operations can be performed on any real number, the domain of f is the set of all real numbers.

(b) The function g says to divide $3x$ by $x^2 - 4$. Since division by 0 is not defined, the denominator $x^2 - 4$ can never be 0, so x can never equal -2 or 2. The domain of the function g is $\{x \mid x \neq -2, x \neq 2\}$.

(c) The function h says to take the square root of $4 - 3t$. But only nonnegative numbers have real square roots, so the expression under the square root (the radicand) must be nonnegative (greater than or equal to zero). This requires that

$$\begin{aligned} 4 - 3t &\geq 0 \\ -3t &\geq -4 \\ t &\leq \frac{4}{3} \end{aligned}$$

The domain of h is $\left\{t \middle| t \leq \frac{4}{3}\right\}$, or the interval $\left(-\infty, \frac{4}{3}\right]$.

(d) The function F says to take the square root of $3x + 12$ and divide this result by $x - 5$. This requires that $3x + 12 \geq 0$, so $x \geq -4$, and also that $x - 5 \neq 0$, so $x \neq 5$. Combining these two restrictions, the domain of F is

$$\{x \mid x \geq -4, \quad x \neq 5\}.$$ ■

The following steps may prove helpful for finding the domain of a function that is defined by an equation and whose domain is a subset of the real numbers.

Finding the Domain of a Function Defined by an Equation

1. Start with the domain as the set of all real numbers.
2. If the equation has a denominator, exclude any numbers that give a zero denominator.
3. If the equation has a radical of even index, exclude any numbers that cause the expression inside the radical (the radicand) to be negative.

Now Work PROBLEM 55

If x is in the domain of a function f, we shall say that **f is defined at x**, or **$f(x)$ exists**. If x is not in the domain of f, we say that **f is not defined at x**, or **$f(x)$ does not exist**. For example, if $f(x) = \frac{x}{x^2 - 1}$, then $f(0)$ exists, but $f(1)$ and $f(-1)$ do not exist. (Do you see why?)

We have not said much about finding the range of a function. We will say more about finding the range when we look at the graph of a function in the next section. When a function is defined by an equation, it can be difficult to find the range. Therefore, we shall usually be content to find just the domain of a function when the function is defined by an equation. We shall express the domain of a function using inequalities, interval notation, set notation, or words, whichever is most convenient.

When we use functions in applications, the domain may be restricted by physical or geometric considerations. For example, the domain of the function f defined by $f(x) = x^2$ is the set of all real numbers. However, if f is used to obtain the area of a square when the length x of a side is known, then we must restrict the domain of f to the positive real numbers, since the length of a side can never be 0 or negative.

EXAMPLE 10 Finding the Domain in an Application

Express the area of a circle as a function of its radius. Find the domain.

Solution See Figure 12. The formula for the area A of a circle of radius r is $A = \pi r^2$. Using r to represent the independent variable and A to represent the dependent variable, the function expressing this relationship is

$$A(r) = \pi r^2$$

In this setting, the domain is $\{r | r > 0\}$. (Do you see why?) ■

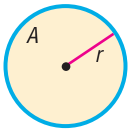


Figure 12 Circle of radius r

Observe, in the solution to Example 10, that the symbol A is used in two ways: It is used to name the function, and it is used to symbolize the dependent variable. This double use is common in applications and should not cause any difficulty.

Now Work PROBLEM 97

5 Form the Sum, Difference, Product, and Quotient of Two Functions

Next we introduce some operations on functions. Functions, like numbers, can be added, subtracted, multiplied, and divided. For example, if $f(x) = x^2 + 9$ and $g(x) = 3x + 5$, then

$$f(x) + g(x) = (x^2 + 9) + (3x + 5) = x^2 + 3x + 14$$

The new function $y = x^2 + 3x + 14$ is called the *sum function* $f + g$. Similarly,

$$f(x) \cdot g(x) = (x^2 + 9)(3x + 5) = 3x^3 + 5x^2 + 27x + 45$$

The new function $y = 3x^3 + 5x^2 + 27x + 45$ is called the *product function* $f \cdot g$.

The general definitions are given next.

DEFINITION

If f and g are functions:
The **sum** $f + g$ is the function defined by

$$\boxed{(f + g)(x) = f(x) + g(x)}$$

In Words
Remember, the symbol ∩ stands for intersection. It means you should find the elements that are common to two sets.

The domain of $f + g$ consists of the numbers x that are in the domains of both f and g. That is, domain of $f + g$ = domain of f ∩ domain of g.

DEFINITION

The **difference** $f - g$ is the function defined by

$$\boxed{(f - g)(x) = f(x) - g(x)}$$

The domain of $f - g$ consists of the numbers x that are in the domains of both f and g. That is, domain of $f - g$ = domain of f ∩ domain of g.

DEFINITION

The **product** $f \cdot g$ is the function defined by

$$\boxed{(f \cdot g)(x) = f(x) \cdot g(x)}$$

The domain of $f \cdot g$ consists of the numbers x that are in the domains of both f and g. That is, domain of $f \cdot g$ = domain of f ∩ domain of g.

DEFINITION

The **quotient** $\frac{f}{g}$ is the function defined by

$$\boxed{\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad g(x) \neq 0}$$

The domain of $\frac{f}{g}$ consists of the numbers x for which $g(x) \neq 0$ and that are in the domains of both f and g. That is,

$$\text{domain of } \frac{f}{g} = \{x \mid g(x) \neq 0\} \cap \text{domain of } f \cap \text{domain of } g$$

EXAMPLE 11 Operations on Functions

Let f and g be two functions defined as

$$f(x) = \frac{1}{x+2} \quad \text{and} \quad g(x) = \frac{x}{x-1}$$

Find the following functions, and determine the domain in each case.

(a) $(f+g)(x)$ (b) $(f-g)(x)$ (c) $(f \cdot g)(x)$ (d) $\left(\frac{f}{g}\right)(x)$

Solution The domain of f is $\{x \mid x \neq -2\}$ and the domain of g is $\{x \mid x \neq 1\}$.

(a) $$(f+g)(x) = f(x) + g(x) = \frac{1}{x+2} + \frac{x}{x-1}$$
$$= \frac{x-1}{(x+2)(x-1)} + \frac{x(x+2)}{(x+2)(x-1)} = \frac{x^2+3x-1}{(x+2)(x-1)}$$

The domain of $f+g$ consists of those numbers x that are in the domains of both f and g. Therefore, the domain of $f+g$ is $\{x \mid x \neq -2, x \neq 1\}$.

(b) $$(f-g)(x) = f(x) - g(x) = \frac{1}{x+2} - \frac{x}{x-1}$$
$$= \frac{x-1}{(x+2)(x-1)} - \frac{x(x+2)}{(x+2)(x-1)} = \frac{-(x^2+x+1)}{(x+2)(x-1)}$$

The domain of $f-g$ consists of those numbers x that are in the domains of both f and g. Therefore, the domain of $f-g$ is $\{x \mid x \neq -2, x \neq 1\}$.

(c) $$(f \cdot g)(x) = f(x) \cdot g(x) = \frac{1}{x+2} \cdot \frac{x}{x-1} = \frac{x}{(x+2)(x-1)}$$

The domain of $f \cdot g$ consists of those numbers x that are in the domains of both f and g. Therefore, the domain of $f \cdot g$ is $\{x \mid x \neq -2, x \neq 1\}$.

(d) $$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\frac{1}{x+2}}{\frac{x}{x-1}} = \frac{1}{x+2} \cdot \frac{x-1}{x} = \frac{x-1}{x(x+2)}$$

The domain of $\frac{f}{g}$ consists of the numbers x for which $g(x) \neq 0$ and that are in the domains of both f and g. Since $g(x) = 0$ when $x = 0$, we exclude 0 as well as -2 and 1 from the domain. The domain of $\frac{f}{g}$ is $\{x \mid x \neq -2, x \neq 0, x \neq 1\}$. ■

Now Work PROBLEM 67

In calculus, it is sometimes helpful to view a complicated function as the sum, difference, product, or quotient of simpler functions. For example,

$$F(x) = x^2 + \sqrt{x} \text{ is the sum of } f(x) = x^2 \text{ and } g(x) = \sqrt{x}.$$

$$H(x) = \frac{x^2 - 1}{x^2 + 1} \text{ is the quotient of } f(x) = x^2 - 1 \text{ and } g(x) = x^2 + 1.$$

SUMMARY

Function	A relation between two sets of real numbers so that each number x in the first set, the domain, has corresponding to it exactly one number y in the second set, the range. A set of ordered pairs (x, y) or $(x, f(x))$ in which no first element is paired with two different second elements. The range is the set of y-values of the function that are the images of the x-values in the domain. A function f may be defined implicitly by an equation involving x and y or explicitly by writing $y = f(x)$.
Unspecified domain	If a function f is defined by an equation and no domain is specified, then the domain will be taken to be the largest set of real numbers for which the equation defines a real number.
Function notation	$y = f(x)$ f is a symbol for the function. x is the independent variable, or argument. y is the dependent variable. $f(x)$ is the value of the function at x, or the image of x.

3.1 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. The inequality $-1 < x < 3$ can be written in interval notation as ________. (pp. 147–148)
2. If $x = -2$, the value of the expression $3x^2 - 5x + \frac{1}{x}$ is _____. (pp. 21–22)
3. The domain of the variable in the expression $\frac{x-3}{x+4}$ is ___________. (p. 22)
4. Solve the inequality: $3 - 2x > 5$. Graph the solution set. (pp. 150–151)
5. To rationalize the denominator of $\frac{3}{\sqrt{5} - 2}$, multiply the numerator and denominator by ________ (p. 76)
6. A quotient is considered rationalized if its denominator contains no ________. (p. 76)

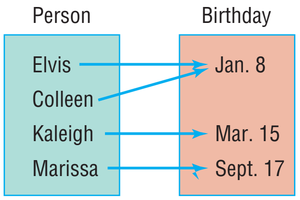
Concepts and Vocabulary

7. If f is a function defined by the equation $y = f(x)$, then x is called the ___________ variable, and y is the __________ variable.
8. If the domain of f is all real numbers in the interval $[0, 7]$, and the domain of g is all real numbers in the interval $[-2, 5]$, then the domain of $f + g$ is all real numbers in the interval _______.
9. The domain of $\frac{f}{g}$ consists of numbers x for which $g(x)$ ___ 0 that are in the domains of both ___ and ___.
10. If $f(x) = x + 1$ and $g(x) = x^3$, then ___________ $= x^3 - (x + 1)$.
11. ***True or False*** Every relation is a function.
12. ***True or False*** The domain of $(f \cdot g)(x)$ consists of the numbers x that are in the domains of both f and g.
13. ***True or False*** If no domain is specified for a function f, then the domain of f is taken to be the set of real numbers.
14. ***True or False*** The domain of the function $f(x) = \frac{x^2 - 4}{x}$ is $\{x \mid x \neq \pm 2\}$.
15. The set of all images of the elements in the domain of a function is called the ______.
    (a) range (b) domain (c) solution set (d) function
16. The independent variable is sometimes referred to as the ______ of the function.
    (a) range (b) value (c) argument (d) definition
17. The expression $\frac{f(x+h) - f(x)}{h}$ is called the ______ of f.
    (a) radicand (b) image
    (c) correspondence (d) difference quotient
18. When written as $y = f(x)$, a function is said to be defined ______.
    (a) explicitly (b) consistently
    (c) implicitly (d) rationally

Skill Building

In Problems 19–30, state the domain and range for each relation. Then determine whether each relation represents a function.

19.



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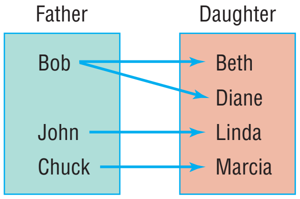

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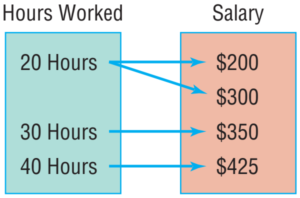

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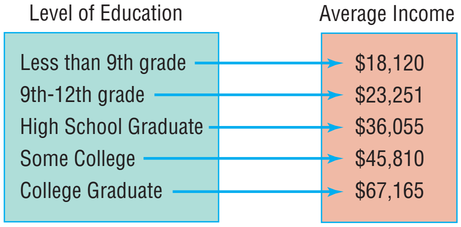


23. $\{(2,6),(-3,6),(4,9),(2,10)\}$ **24.** $\{(-2,5),(-1,3),(3,7),(4,12)\}$ **25.** $\{(1,3),(2,3),(3,3),(4,3)\}$

26. $\{(0,-2),(1,3),(2,3),(3,7)\}$ **27.** $\{(-2,4),(-2,6),(0,3),(3,7)\}$ **28.** $\{(-4,4),(-3,3),(-2,2),(-1,1),(-4,0)\}$

29. $\{(-2,4),(-1,1),(0,0),(1,1)\}$ **30.** $\{(-2,16),(-1,4),(0,3),(1,4)\}$

In Problems 31–42, determine whether the equation defines y as a function of x.

31. $y = 2x^2 - 3x + 4$ **32.** $y = x^3$ **33.** $y = \frac{1}{x}$ **34.** $y = |x|$

35. $y^2 = 4 - x^2$ **36.** $y = \pm\sqrt{1-2x}$ **37.** $x = y^2$ **38.** $x + y^2 = 1$

39. $y = \sqrt[3]{x}$ **40.** $y = \frac{3x-1}{x+2}$ **41.** $2x^2 + 3y^2 = 1$ **42.** $x^2 - 4y^2 = 1$

In Problems 43–50, find the following for each function:

(a) $f(0)$ *(b)* $f(1)$ *(c)* $f(-1)$ *(d)* $f(-x)$ *(e)* $-f(x)$ *(f)* $f(x+1)$ *(g)* $f(2x)$ *(h)* $f(x+h)$

43. $f(x) = 3x^2 + 2x - 4$ **44.** $f(x) = -2x^2 + x - 1$ **45.** $f(x) = \frac{x}{x^2+1}$ **46.** $f(x) = \frac{x^2-1}{x+4}$

47. $f(x) = |x| + 4$ **48.** $f(x) = \sqrt{x^2+x}$ **49.** $f(x) = \frac{2x+1}{3x-5}$ **50.** $f(x) = 1 - \frac{1}{(x+2)^2}$

In Problems 51–66, find the domain of each function.

51. $f(x) = -5x + 4$ **52.** $f(x) = x^2 + 2$ **53.** $f(x) = \frac{x}{x^2+1}$ **54.** $f(x) = \frac{x^2}{x^2+1}$

55. $g(x) = \frac{x}{x^2-16}$ **56.** $h(x) = \frac{2x}{x^2-4}$ **57.** $F(x) = \frac{x-2}{x^3+x}$ **58.** $G(x) = \frac{x+4}{x^3-4x}$

59. $h(x) = \sqrt{3x-12}$ **60.** $G(x) = \sqrt{1-x}$ **61.** $p(x) = \sqrt{\frac{2}{x-1}}$ **62.** $f(x) = \frac{4}{\sqrt{x-9}}$

63. $f(x) = \frac{x}{\sqrt{x-4}}$ **64.** $f(x) = \frac{-x}{\sqrt{-x-2}}$ **65.** $P(t) = \frac{\sqrt{t-4}}{3t-21}$ **66.** $h(z) = \frac{\sqrt{z+3}}{z-2}$

In Problems 67–76, for the given functions f and g, find the following. For parts (a)–(d), also find the domain.

(a) $(f+g)(x)$ *(b)* $(f-g)(x)$ *(c)* $(f \cdot g)(x)$ *(d)* $\left(\frac{f}{g}\right)(x)$

(e) $(f+g)(3)$ *(f)* $(f-g)(4)$ *(g)* $(f \cdot g)(2)$ *(h)* $\left(\frac{f}{g}\right)(1)$

67. $f(x) = 3x + 4;\quad g(x) = 2x - 3$ **68.** $f(x) = 2x + 1;\quad g(x) = 3x - 2$

69. $f(x) = x - 1;\quad g(x) = 2x^2$ **70.** $f(x) = 2x^2 + 3;\quad g(x) = 4x^3 + 1$

71. $f(x) = \sqrt{x}; \quad g(x) = 3x - 5$

72. $f(x) = |x|; \quad g(x) = x$

73. $f(x) = 1 + \frac{1}{x}; \quad g(x) = \frac{1}{x}$

74. $f(x) = \sqrt{x - 1}; \quad g(x) = \sqrt{4 - x}$

75. $f(x) = \frac{2x + 3}{3x - 2}; \quad g(x) = \frac{4x}{3x - 2}$

76. $f(x) = \sqrt{x + 1}; \quad g(x) = \frac{2}{x}$

77. Given $f(x) = 3x + 1$ and $(f + g)(x) = 6 - \frac{1}{2}x$, find the function g.

78. Given $f(x) = \frac{1}{x}$ and $\left(\frac{f}{g}\right)(x) = \frac{x + 1}{x^2 - x}$, find the function g.

In Problems 79–90, find the difference quotient of f; that is, find $\frac{f(x + h) - f(x)}{h}$, $h \neq 0$, *for each function. Be sure to simplify.*

79. $f(x) = 4x + 3$

80. $f(x) = -3x + 1$

81. $f(x) = x^2 - 4$

82. $f(x) = 3x^2 + 2$

83. $f(x) = x^2 - x + 4$

84. $f(x) = 3x^2 - 2x + 6$

85. $f(x) = \frac{1}{x^2}$

86. $f(x) = \frac{1}{x + 3}$

87. $f(x) = \frac{2x}{x + 3}$

88. $f(x) = \frac{5x}{x - 4}$

89. $f(x) = \sqrt{x - 2}$
[**Hint:** Rationalize the numerator.]

90. $f(x) = \sqrt{x + 1}$

Applications and Extensions

91. Given $f(x) = x^2 - 2x + 3$, find the value(s) for x such that $f(x) = 11$.

92. Given $f(x) = \frac{5}{6}x - \frac{3}{4}$, find the value(s) for x such that $f(x) = -\frac{7}{16}$.

93. If $f(x) = 2x^3 + Ax^2 + 4x - 5$ and $f(2) = 5$, what is the value of A?

94. If $f(x) = 3x^2 - Bx + 4$ and $f(-1) = 12$, what is the value of B?

95. If $f(x) = \frac{3x + 8}{2x - A}$ and $f(0) = 2$, what is the value of A?

96. If $f(x) = \frac{2x - B}{3x + 4}$ and $f(2) = \frac{1}{2}$, what is the value of B?

97. Geometry Express the area A of a rectangle as a function of the length x if the length of the rectangle is twice its width.

98. Geometry Express the area A of an isosceles right triangle as a function of the length x of one of the two equal sides.

99. Constructing Functions Express the gross salary G of a person who earns \$14 per hour as a function of the number x of hours worked.

100. Constructing Functions Tiffany, a commissioned salesperson, earns \$100 base pay plus \$10 per item sold. Express her gross salary G as a function of the number x of items sold.

101. Population as a Function of Age The function

$$P(a) = 0.014a^2 - 5.073a + 327.287$$

represents the population P (in millions) of Americans that are a years of age or older in 2012.

(a) Identify the dependent and independent variables.
(b) Evaluate $P(20)$. Provide a verbal explanation of the meaning of $P(20)$.
(c) Evaluate $P(0)$. Provide a verbal explanation of the meaning of $P(0)$.

Source: *U.S. Census Bureau*

102. Number of Rooms The function

$$N(r) = -1.35r^2 + 15.45r - 20.71$$

represents the number N of housing units (in millions) in 2012 that had r rooms, where r is an integer and $2 \le r \le 9$.

(a) Identify the dependent and independent variables.
(b) Evaluate $N(3)$. Provide a verbal explanation of the meaning of $N(3)$.

103. Effect of Gravity on Earth If a rock falls from a height of 20 meters on Earth, the height H (in meters) after x seconds is approximately

$$H(x) = 20 - 4.9x^2$$

(a) What is the height of the rock when $x = 1$ second? $x = 1.1$ seconds? $x = 1.2$ seconds? $x = 1.3$ seconds?
(b) When is the height of the rock 15 meters? When is it 10 meters? When is it 5 meters?
(c) When does the rock strike the ground?

104. Effect of Gravity on Jupiter If a rock falls from a height of 20 meters on the planet Jupiter, its height H (in meters) after x seconds is approximately

$$H(x) = 20 - 13x^2$$

(a) What is the height of the rock when $x = 1$ second? $x = 1.1$ seconds? $x = 1.2$ seconds?
(b) When is the height of the rock 15 meters? When is it 10 meters? When is it 5 meters?
(c) When does the rock strike the ground?

105. Cost of Transatlantic Travel A Boeing 747 crosses the Atlantic Ocean (3000 miles) with an airspeed of 500 miles per hour. The cost C (in dollars) per passenger is given by

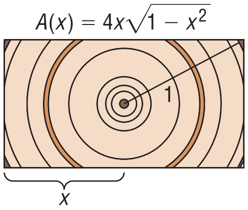
$$C(x) = 100 + \frac{x}{10} + \frac{36{,}000}{x}$$

where x is the ground speed (airspeed ± wind).

(a) What is the cost per passenger for quiescent (no wind) conditions?
(b) What is the cost per passenger with a head wind of 50 miles per hour?
(c) What is the cost per passenger with a tail wind of 100 miles per hour?
(d) What is the cost per passenger with a head wind of 100 miles per hour?

106. Cross-sectional Area The cross-sectional area of a beam cut from a log with radius 1 foot is given by the function $A(x) = 4x\sqrt{1 - x^2}$, where x represents the length, in feet, of half the base of the beam. See the figure. Determine the cross-sectional area of the beam if the length of half the base of the beam is as follows:

(a) One-third of a foot
(b) One-half of a foot
(c) Two-thirds of a foot



107. Economics The **participation rate** is the number of people in the labor force divided by the civilian population (excludes military). Let $L(x)$ represent the size of the labor force in year x, and $P(x)$ represent the civilian population in year x. Determine a function that represents the participation rate R as a function of x.

108. Crimes Suppose that $V(x)$ represents the number of violent crimes committed in year x and $P(x)$ represents the number of property crimes committed in year x. Determine a function T that represents the combined total of violent crimes and property crimes in year x.

109. Health Care Suppose that $P(x)$ represents the percentage of income spent on health care in year x and $I(x)$ represents income in year x. Determine a function H that represents total health care expenditures in year x.

110. Income Tax Suppose that $I(x)$ represents the income of an individual in year x before taxes and $T(x)$ represents the individual's tax bill in year x. Determine a function N that represents the individual's net income (income after taxes) in year x.

111. Profit Function Suppose that the revenue R, in dollars, from selling x cell phones, in hundreds, is $R(x) = -1.2x^2 + 220x$. The cost C, in dollars, of selling x cell phones, in hundreds, is $C(x) = 0.05x^3 - 2x^2 + 65x + 500$.

(a) Find the profit function, $P(x) = R(x) - C(x)$.
(b) Find the profit if $x = 15$ hundred cell phones are sold.
(c) Interpret $P(15)$.

112. Profit Function Suppose that the revenue R, in dollars, from selling x clocks is $R(x) = 30x$. The cost C, in dollars, of selling x clocks is $C(x) = 0.1x^2 + 7x + 400$.

(a) Find the profit function, $P(x) = R(x) - C(x)$.
(b) Find the profit if $x = 30$ clocks are sold.
(c) Interpret $P(30)$.

113. Stopping Distance When the driver of a vehicle observes an impediment, the total stopping distance involves both the reaction distance (the distance the vehicle travels while the driver moves his or her foot to the brake pedal) and the braking distance (the distance the vehicle travels once the brakes are applied). For a car traveling at a speed of v miles per hour, the reaction distance R, in feet, can be estimated by $R(v) = 2.2v$. Suppose that the braking distance B, in feet, for a car is given by $B(v) = 0.05v^2 + 0.4v - 15$.

(a) Find the stopping distance function

$$D(v) = R(v) + B(v).$$

(b) Find the stopping distance if the car is traveling at a speed of 60 mph.
(c) Interpret $D(60)$.

114. Some functions f have the property that $f(a + b) = f(a) + f(b)$ for all real numbers a and b. Which of the following functions have this property?

(a) $h(x) = 2x$
(b) $g(x) = x^2$
(c) $F(x) = 5x - 2$
(d) $G(x) = \dfrac{1}{x}$

Explaining Concepts: Discussion and Writing

115. Are the functions $f(x) = x - 1$ and $g(x) = \dfrac{x^2 - 1}{x + 1}$ the same? Explain.

116. Investigate when, historically, the use of the function notation $y = f(x)$ first appeared.

117. Find a function H that multiplies a number x by 3 and then subtracts the cube of x and divides the result by your age.

Retain Your Knowledge

Problems 118–121 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

118. List the intercepts and test for symmetry: $(x + 12)^2 + y^2 = 16$

119. Determine which of the given points are on the graph of the equation $y = 3x^2 - 8\sqrt{x}$.

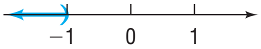
Points: $(-1, -5)$, $(4, 32)$, $(9, 171)$

120. How many pounds of lean hamburger that is 7% fat must be mixed with 12 pounds of ground chuck that is 20% fat to have a hamburger mixture that is 15% fat?

121. Solve $x^3 - 9x = 2x^2 - 18$.

'Are You Prepared?' Answers

1. $(-1, 3)$ **2.** 21.5 **3.** $\{x \mid x \neq -4\}$ **4.** $\{x \mid x < -1\}$



5. $\sqrt{5} + 2$ **6.** radicals

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Graphs of Equations (Section 1.1, pp. 88–92)
- Intercepts (Section 2.1, pp. 165–166)

Now Work the **'Are You Prepared?'** problems on page 226.

OBJECTIVES **1** Identify the Graph of a Function (p. 222)
2 Obtain Information from or about the Graph of a Function (p. 223)

In applications, a graph often demonstrates more clearly the relationship between two variables than, say, an equation or table. For example, Table 1 shows the average price of gasoline in the United States for the years 1985–2014 (adjusted for inflation, based on 2014 dollars). If we plot these data and then connect the points, we obtain Figure 13.

Table 1

Year	Price	Year	Price	Year	Price
1985	2.55	1995	1.72	2005	2.74
1986	1.90	1996	1.80	2006	3.01
1987	1.89	1997	1.76	2007	3.19
1988	1.81	1998	1.49	2008	3.56
1989	1.87	1999	1.61	2009	2.58
1990	2.03	2000	2.03	2010	3.00
1991	1.91	2001	1.90	2011	3.69
1992	1.82	2002	1.76	2012	3.72
1993	1.74	2003	1.99	2013	3.54
1994	1.71	2004	2.31	2014	3.43

Source: *U.S. Energy Information Administration*

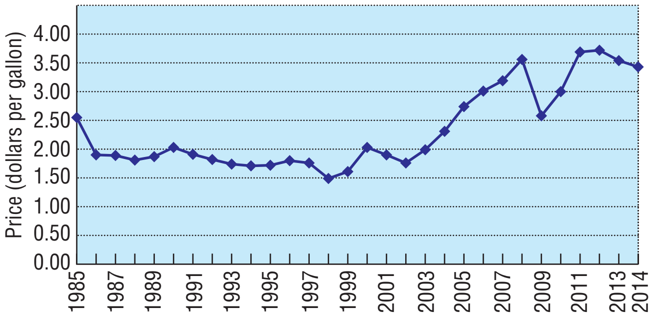


Figure 13 Average retail price of gasoline (2014 dollars)

Source: *U.S. Energy Information Administration*

We can see from the graph that the price of gasoline (adjusted for inflation) stayed roughly the same from 1986 to 1991 and rose rapidly from 2002 to 2008. The graph also shows that the lowest price occurred in 1998. To learn information such as this from an equation requires that some calculations be made.

Look again at Figure 13. The graph shows that for each date on the horizontal axis, there is only one price on the vertical axis. The graph represents a function, although an exact rule for getting from date to price is not given.

When a function is defined by an equation in x and y, the **graph of the function** is the graph of the equation; that is, it is the set of points (x, y) in the xy-plane that satisfy the equation.

1 Identify the Graph of a Function

Not every collection of points in the xy-plane represents the graph of a function. Remember, for a function, each number x in the domain has exactly one image y in the range. This means that the graph of a function cannot contain two points with the same x-coordinate and different y-coordinates. Therefore, the graph of a function must satisfy the following **vertical-line test**.

In Words
If any vertical line intersects a graph at more than one point, the graph is not the graph of a function.

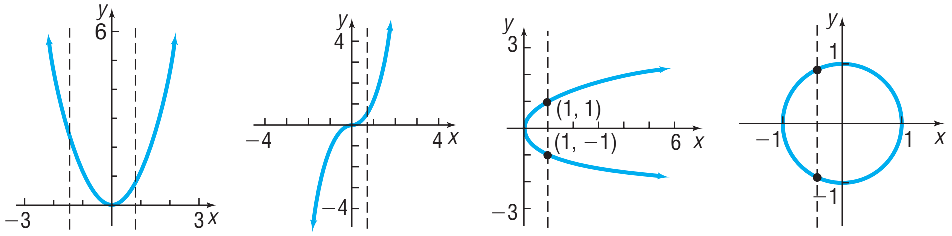
THEOREM

Vertical-Line Test

A set of points in the xy-plane is the graph of a function if and only if every vertical line intersects the graph in at most one point.

EXAMPLE 1 Identifying the Graph of a Function

Which of the graphs in Figure 14 are graphs of functions?



(a) $y = x^2$ **(b)** $y = x^3$ **(c)** $x = y^2$ **(d)** $x^2 + y^2 = 1$

Figure 14

Solution The graphs in Figures 14(a) and 14(b) are graphs of functions, because every vertical line intersects each graph in at most one point. The graphs in Figures 14(c) and 14(d) are not graphs of functions, because there is a vertical line that intersects each graph in more than one point. Notice in Figure 14(c) that the input 1 corresponds to two outputs, -1 and 1. This is why the graph does not represent a function. ■

Now Work PROBLEMS 15 AND 17

2 Obtain Information from or about the Graph of a Function

If (x, y) is a point on the graph of a function f, then y is the value of f at x; that is, $y = f(x)$. Also if $y = f(x)$, then (x, y) is a point on the graph of f. For example, if $(-2, 7)$ is on the graph of f, then $f(-2) = 7$, and if $f(5) = 8$, then the point $(5, 8)$ is on the graph of $y = f(x)$. The next example illustrates how to obtain information about a function if its graph is given.

EXAMPLE 2 Obtaining Information from the Graph of a Function

Let f be the function whose graph is given in Figure 15.

(a) What are $f(-5)$, $f(0)$, and $f(3)$?

(b) What is the domain of f?

(c) What is the range of f?

(d) List the intercepts. (Recall that these are the points, if any, where the graph crosses or touches the coordinate axes.)

(e) How many times does the line $y = 2$ intersect the graph?

(f) For what values of x does $f(x) = -8$?

(g) For what values of x is $f(x) > 0$?

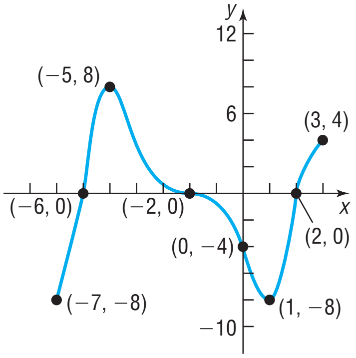


Figure 15

Solution

(a) Since $(-5, 8)$ is on the graph of f, the y-coordinate is the value of f at the x-coordinate -5; that is, $f(-5) = 8$. In a similar way, when $x = 0$, then $y = -4$, so $f(0) = -4$. When $x = 3$, then $y = 4$, so $f(3) = 4$.

(b) To determine the domain of f, notice that the points on the graph of f have x-coordinates between -7 and 3, inclusive; and for each number x between -7 and 3, there is a point $(x, f(x))$ on the graph. The domain of f is $\{x \mid -7 \le x \le 3\}$ or the interval $[-7, 3]$.

(c) The points on the graph all have y-coordinates between -8 and 8, inclusive; and for each number y, there is at least one number x in the domain. The range of f is $\{y \mid -8 \le y \le 8\}$ or the interval $[-8, 8]$.

(d) The intercepts are the points $(0, -4)$, $(-6, 0)$, $(-2, 0)$, and $(2, 0)$.

(e) Draw the horizontal line $y = 2$ on the graph in Figure 15. Notice that the line intersects the graph three times.

(f) Since $(-7, -8)$ and $(1, -8)$ are the only points on the graph for which $y = f(x) = -8$, we have $f(x) = -8$ when $x = -7$ and $x = 1$.

(g) To determine where $f(x) > 0$, look at Figure 15 and determine the x-values from -7 to 3 for which the y-coordinate is positive. This occurs on $(-6, -2) \cup (2, 3]$. Using inequality notation, $f(x) > 0$ for $-6 < x < -2$ or $2 < x \leq 3$. ■

When the graph of a function is given, its domain may be viewed as the shadow created by the graph on the x-axis by vertical beams of light. Its range can be viewed as the shadow created by the graph on the y-axis by horizontal beams of light. Try this technique with the graph given in Figure 15.

Now Work PROBLEM 11

EXAMPLE 3 Obtaining Information about the Graph of a Function

Consider the function: $f(x) = \dfrac{x+1}{x+2}$

(a) Find the domain of f.

(b) Is the point $\left(1, \dfrac{1}{2}\right)$ on the graph of f?

(c) If $x = 2$, what is $f(x)$? What point is on the graph of f?

(d) If $f(x) = 2$, what is x? What point is on the graph of f?

(e) What are the x-intercepts of the graph of f (if any)? What point(s) are on the graph of f?

Solution

(a) The domain of f is $\{x \mid x \neq -2\}$, since $x = -2$ results in division by 0.

(b) When $x = 1$, then

$$f(1) = \frac{1+1}{1+2} = \frac{2}{3} \qquad f(x) = \frac{x+1}{x+2}$$

The point $\left(1, \dfrac{2}{3}\right)$ is on the graph of f; the point $\left(1, \dfrac{1}{2}\right)$ is not.

(c) If $x = 2$, then

$$f(2) = \frac{2+1}{2+2} = \frac{3}{4}$$

The point $\left(2, \dfrac{3}{4}\right)$ is on the graph of f.

(d) If $f(x) = 2$, then

$$\begin{aligned} \frac{x+1}{x+2} &= 2 && f(x) = 2 \\ x + 1 &= 2(x+2) && \textbf{Multiply both sides by } x + 2. \\ x + 1 &= 2x + 4 && \textbf{Distribute.} \\ x &= -3 && \textbf{Solve for } x. \end{aligned}$$

If $f(x) = 2$, then $x = -3$. The point $(-3, 2)$ is on the graph of f.

(e) The x-intercepts of the graph of f are the real solutions of the equation $f(x) = 0$ that are in the domain of f.

$$\frac{x+1}{x+2} = 0$$

$$x + 1 = 0 \quad \text{Multiply both sides by } x + 2.$$

$$x = -1 \quad \text{Subtract 1 from both sides.}$$

The only real solution of the equation $f(x) = \frac{x+1}{x+2} = 0$ is $x = -1$, so -1 is the only x-intercept. Since $f(-1) = 0$, the point $(-1, 0)$ is on the graph of f. ■

Now Work PROBLEM 27

EXAMPLE 4 Average Cost Function

The average cost $\overline{C}$ per computer of manufacturing x computers per day is given by the function

$$\overline{C}(x) = 0.56x^2 - 34.39x + 1212.57 + \frac{20{,}000}{x}$$

Determine the average cost of manufacturing:

(a) 30 computers in a day
(b) 40 computers in a day
(c) 50 computers in a day
(d) Graph the function $\overline{C} = \overline{C}(x)$, $0 < x \le 80$.
(e) Create a TABLE with TblStart = 1 and ΔTbl = 1. Which value of x minimizes the average cost?

Solution (a) The average cost per computer of manufacturing $x = 30$ computers is

$$\overline{C}(30) = 0.56(30)^2 - 34.39(30) + 1212.57 + \frac{20{,}000}{30} = \$1351.54$$

(b) The average cost per computer of manufacturing $x = 40$ computers is

$$\overline{C}(40) = 0.56(40)^2 - 34.39(40) + 1212.57 + \frac{20{,}000}{40} = \$1232.97$$

(c) The average cost per computer of manufacturing $x = 50$ computers is

$$\overline{C}(50) = 0.56(50)^2 - 34.39(50) + 1212.57 + \frac{20{,}000}{50} = \$1293.07$$

(d) See Figure 16 for the graph of $\overline{C} = \overline{C}(x)$.

(e) With the function $\overline{C} = \overline{C}(x)$ in Y_1, we create Table 2. We scroll down until we find a value of x for which Y_1 is smallest. Table 3 shows that manufacturing $x = 41$ computers minimizes the average cost at \$1231.74 per computer.

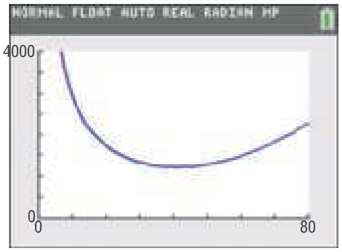
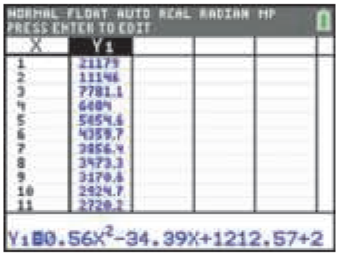


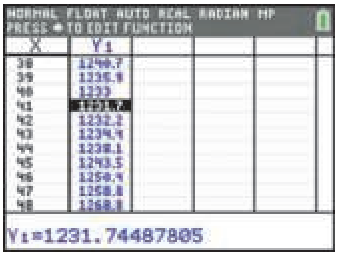
Figure 16 $\overline{C}(x) = 0.56x^2 - 34.39x + 1212.57 + \frac{20{,}000}{x}$



X	Y₁
1	21179
2	11146
3	7781.1
4	6084
5	5054.6
6	4359.7
7	3856.4
8	3473.3
9	3170.6
10	2924.7
11	2720.2

```
Y1=0.56X²-34.39X+1212.57+2
```

Table 2



X	Y₁
38	1240.7
39	1235.9
40	1233
41	1231.7
42	1232.2
43	1234.4
44	1238.1
45	1243.5
46	1250.4
47	1258.8
48	1268.8

```
Y1=1231.74487805
```

Table 3 ■

Now Work PROBLEM 35

SUMMARY

Graph of a Function	The collection of points (x, y) that satisfy the equation $y = f(x)$.
Vertical-Line Test	A collection of points is the graph of a function if and only if every vertical line intersects the graph in at most one point.

3.2 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. The intercepts of the equation $x^2 + 4y^2 = 16$ are ________. (pp. 165–166)

2. ***True or False*** The point $(-2, -6)$ is on the graph of the equation $x = 2y - 2$. (pp. 88–92)

Concepts and Vocabulary

3. A set of points in the xy-plane is the graph of a function if and only if every ________ line intersects the graph in at most one point.

4. If the point $(5, -3)$ is a point on the graph of f, then $f(\_\_\_) = \_\_\_\_$.

5. Find a so that the point $(-1, 2)$ is on the graph of $f(x) = ax^2 + 4$.

6. ***True or False*** Every graph represents a function.

7. ***True or False*** The graph of a function $y = f(x)$ always crosses the y-axis.

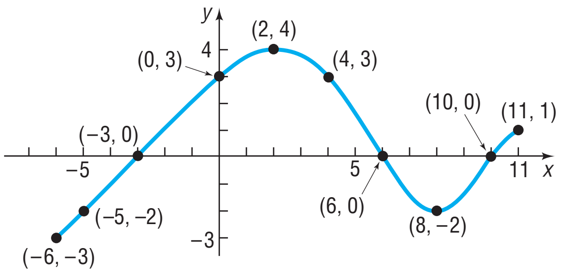
8. ***True or False*** The y-intercept of the graph of the function $y = f(x)$, whose domain is all real numbers, is $f(0)$.

9. If a function is defined by an equation in x and y, then the collection of points (x, y) in the xy-plane that satisfy the equation is called ________.
(a) the domain of the function (b) the range of the function
(c) the graph of the function (d) the relation of the function

10. The graph of a function $y = f(x)$ can have more than one of which type of intercept?
(a) x-intercept (b) y-intercept (c) both (d) neither

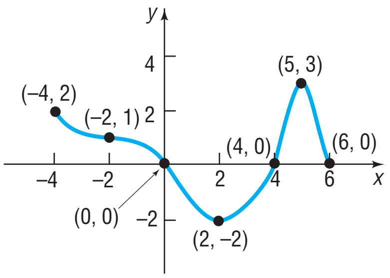
Skill Building

11. Use the given graph of the function f to answer parts (a)–(n).



(a) Find $f(0)$ and $f(-6)$.
(b) Find $f(6)$ and $f(11)$.
(c) Is $f(3)$ positive or negative?
(d) Is $f(-4)$ positive or negative?
(e) For what values of x is $f(x) = 0$?
(f) For what values of x is $f(x) > 0$?
(g) What is the domain of f?
(h) What is the range of f?
(i) What are the x-intercepts?
(j) What is the y-intercept?
(k) How often does the line $y = \frac{1}{2}$ intersect the graph?
(l) How often does the line $x = 5$ intersect the graph?
(m) For what values of x does $f(x) = 3$?
(n) For what values of x does $f(x) = -2$?

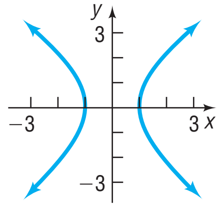
12. Use the given graph of the function f to answer parts (a)–(n).



(a) Find $f(0)$ and $f(6)$.
(b) Find $f(2)$ and $f(-2)$.
(c) Is $f(3)$ positive or negative?
(d) Is $f(-1)$ positive or negative?
(e) For what values of x is $f(x) = 0$?
(f) For what values of x is $f(x) < 0$?
(g) What is the domain of f?
(h) What is the range of f?
(i) What are the x-intercepts?
(j) What is the y-intercept?
(k) How often does the line $y = -1$ intersect the graph?
(l) How often does the line $x = 1$ intersect the graph?
(m) For what value of x does $f(x) = 3$?
(n) For what value of x does $f(x) = -2$?

In Problems 13–24, determine whether the graph is that of a function by using the vertical-line test. If it is, use the graph to find:
(a) The domain and range *(b) The intercepts, if any* *(c) Any symmetry with respect to the x-axis, the y-axis, or the origin*

13.



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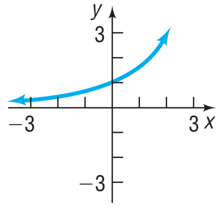

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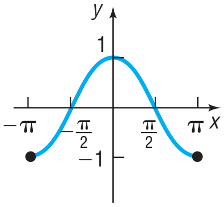

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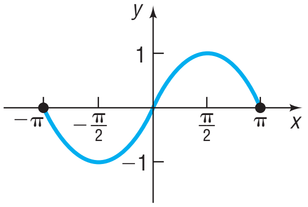

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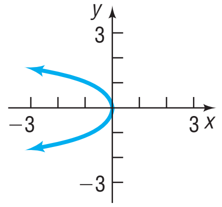

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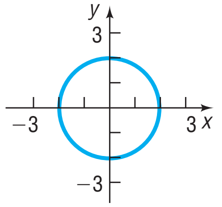

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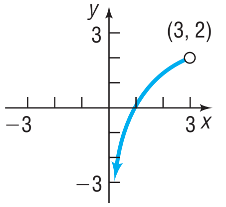

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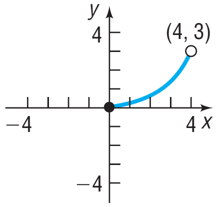

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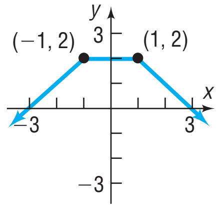

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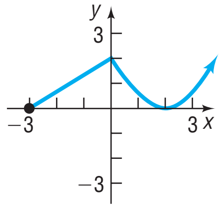

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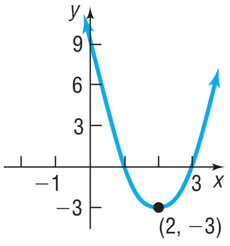

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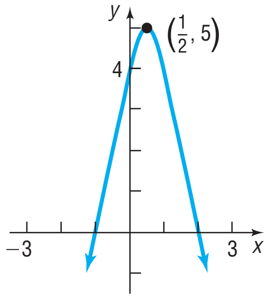


In Problems 25–30, answer the questions about the given function.

25. $f(x) = 2x^2 - x - 1$

(a) Is the point $(-1, 2)$ on the graph of f?
(b) If $x = -2$, what is $f(x)$? What point is on the graph of f?
(c) If $f(x) = -1$, what is x? What point(s) are on the graph of f?
(d) What is the domain of f?
(e) List the x-intercepts, if any, of the graph of f.
(f) List the y-intercept, if there is one, of the graph of f.

26. $f(x) = -3x^2 + 5x$

(a) Is the point $(-1, 2)$ on the graph of f?
(b) If $x = -2$, what is $f(x)$? What point is on the graph of f?
(c) If $f(x) = -2$, what is x? What point(s) are on the graph of f?
(d) What is the domain of f?
(e) List the x-intercepts, if any, of the graph of f.
(f) List the y-intercept, if there is one, of the graph of f.

27. $f(x) = \dfrac{x + 2}{x - 6}$

(a) Is the point $(3, 14)$ on the graph of f?
(b) If $x = 4$, what is $f(x)$? What point is on the graph of f?
(c) If $f(x) = 2$, what is x? What point(s) are on the graph of f?
(d) What is the domain of f?
(e) List the x-intercepts, if any, of the graph of f.
(f) List the y-intercept, if there is one, of the graph of f.

28. $f(x) = \dfrac{x^2 + 2}{x + 4}$

(a) Is the point $\left(1, \dfrac{3}{5}\right)$ on the graph of f?
(b) If $x = 0$, what is $f(x)$? What point is on the graph of f?
(c) If $f(x) = \dfrac{1}{2}$, what is x? What point(s) are on the graph of f?
(d) What is the domain of f?
(e) List the x-intercepts, if any, of the graph of f.
(f) List the y-intercept, if there is one, of the graph of f.

29. $f(x) = \dfrac{2x^2}{x^4 + 1}$

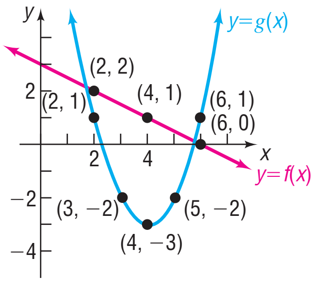
(a) Is the point $(-1, 1)$ on the graph of f?
(b) If $x = 2$, what is $f(x)$? What point is on the graph of f?
(c) If $f(x) = 1$, what is x? What point(s) are on the graph of f?
(d) What is the domain of f?
(e) List the x-intercepts, if any, of the graph of f.
(f) List the y-intercept, if there is one, of the graph of f.

30. $f(x) = \dfrac{2x}{x - 2}$

(a) Is the point $\left(\dfrac{1}{2}, -\dfrac{2}{3}\right)$ on the graph of f?
(b) If $x = 4$, what is $f(x)$? What point is on the graph of f?
(c) If $f(x) = 1$, what is x? What point(s) are on the graph of f?
(d) What is the domain of f?
(e) List the x-intercepts, if any, of the graph of f.
(f) List the y-intercept, if there is one, of the graph of f.

Applications and Extensions

31. The graphs of two functions, f and g, are illustrated. Use the graphs to answer parts (a)–(f).



(a) $(f+g)(2)$

(b) $(f+g)(4)$

(c) $(f-g)(6)$

(d) $(g-f)(6)$

(e) $(f\cdot g)(2)$

(f) $\left(\dfrac{f}{g}\right)(4)$

32. Granny Shots The last player in the NBA to use an underhand foul shot (a "granny" shot) was Hall of Fame forward Rick Barry, who retired in 1980. Barry believes that current NBA players could increase their free-throw percentage if they were to use an underhand shot. Since underhand shots are released from a lower position, the angle of the shot must be increased. If a player shoots an underhand foul shot, releasing the ball at a 70-degree angle from a position 3.5 feet above the floor, then the path of the ball can be modeled by the function $h(x) = -\dfrac{136x^2}{v^2} + 2.7x + 3.5$, where h is the height of the ball above the floor, x is the forward distance of the ball in front of the foul line, and v is the initial velocity with which the ball is shot in feet per second.

(a) The center of the hoop is 10 feet above the floor and 15 feet in front of the foul line. Determine the initial velocity with which the ball must be shot in order for the ball to go through the hoop.

(b) Write the function for the path of the ball using the velocity found in part (a).

(c) Determine the height of the ball after it has traveled 9 feet in front of the foul line.

(d) Find additional points and graph the path of the basketball.

Source: *The Physics of Foul Shots,* Discover, *Vol. 21, No. 10, October 2000*

33. Free-throw Shots According to physicist Peter Brancazio, the key to a successful foul shot in basketball lies in the arc of the shot. Brancazio determined the optimal angle of the arc from the free-throw line to be 45 degrees. The arc also depends on the velocity with which the ball is shot. If a player shoots a foul shot, releasing the ball at a 45-degree angle from a position 6 feet above the floor, then the path of the ball can be modeled by the function

$$h(x) = -\frac{44x^2}{v^2} + x + 6$$

where h is the height of the ball above the floor, x is the forward distance of the ball in front of the foul line, and v is the initial velocity with which the ball is shot in feet per second. Suppose a player shoots a ball with an initial velocity of 28 feet per second.

(a) Determine the height of the ball after it has traveled 8 feet in front of the foul line.

(b) Determine the height of the ball after it has traveled 12 feet in front of the foul line.

(c) Find additional points and graph the path of the basketball.

(d) The center of the hoop is 10 feet above the floor and 15 feet in front of the foul line. Will the ball go through the hoop? Why or why not? If not, with what initial velocity must the ball be shot in order for the ball to go through the hoop?

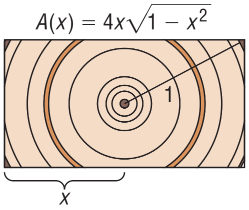
Source: *The Physics of Foul Shots,* Discover, *Vol. 21, No. 10, October 2000*

34. Cross-sectional Area The cross-sectional area of a beam cut from a log with radius 1 foot is given by the function $A(x) = 4x\sqrt{1-x^2}$, where x represents the length, in feet, of half the base of the beam. See the figure.

(a) Find the domain of A.

(b) Use a graphing utility to graph the function $A = A(x)$.

(c) Create a TABLE with TblStart $= 0$ and ΔTbl $= 0.1$ for $0 \le x \le 1$. Which value of x maximizes the cross-sectional area? What should be the length of the base of the beam to maximize the cross-sectional area?



35. Motion of a Golf Ball A golf ball is hit with an initial velocity of 130 feet per second at an inclination of 45° to the horizontal. In physics, it is established that the height h of the golf ball is given by the function

$$h(x) = \frac{-32x^2}{130^2} + x$$

where x is the horizontal distance that the golf ball has traveled.

(a) Determine the height of the golf ball after it has traveled 100 feet.
(b) What is the height after it has traveled 300 feet?
(c) What is the height after it has traveled 500 feet?
(d) How far was the golf ball hit?
(e) Use a graphing utility to graph the function $h = h(x)$.
(f) Use a graphing utility to determine the distance that the ball has traveled when the height of the ball is 90 feet.
(g) Create a TABLE with TblStart = 0 and ΔTbl = 25. To the nearest 25 feet, how far does the ball travel before it reaches a maximum height? What is the maximum height?
(h) Adjust the value of ΔTbl until you determine the distance, to within 1 foot, that the ball travels before it reaches its maximum height.

36. Effect of Elevation on Weight If an object weighs m pounds at sea level, then its weight W (in pounds) at a height of h miles above sea level is given approximately by

$$W(h) = m\left(\frac{4000}{4000 + h}\right)^2$$

(a) If Amy weighs 120 pounds at sea level, how much will she weigh on Pikes Peak, which is 14,110 feet above sea level?
(b) Use a graphing utility to graph the function $W = W(h)$. Use $m = 120$ pounds.
(c) Create a TABLE with TblStart = 0 and ΔTbl = 0.5 to see how the weight W varies as h changes from 0 to 5 miles.
(d) At what height will Amy weigh 119.95 pounds?
(e) Does your answer to part (d) seem reasonable? Explain.

37. Cost of Transatlantic Travel A Boeing 747 crosses the Atlantic Ocean (3000 miles) with an airspeed of 500 miles per hour. The cost C (in dollars) per passenger is given by

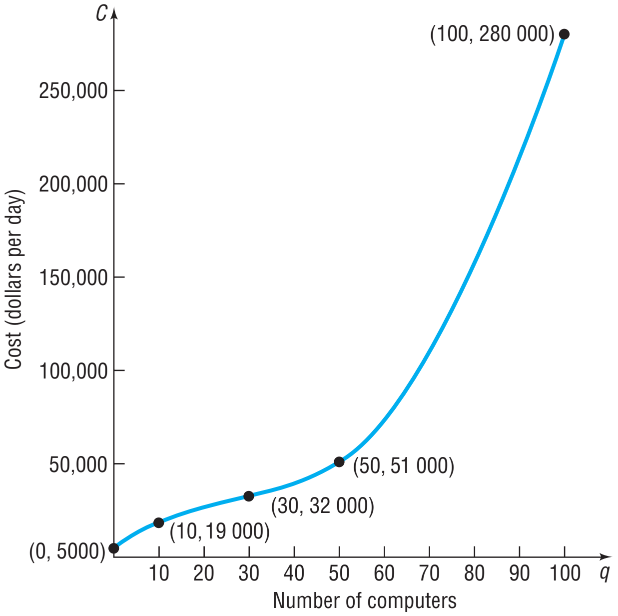
$$C(x) = 100 + \frac{x}{10} + \frac{36{,}000}{x}$$

where x is the groundspeed (airspeed ± wind).

(a) What is the cost when the groundspeed is 480 miles per hour? 600 miles per hour?
(b) Find the domain of C.
(c) Use a graphing utility to graph the function $C = C(x)$.
(d) Create a TABLE with TblStart = 0 and ΔTbl = 50.
(e) To the nearest 50 miles per hour, what groundspeed minimizes the cost per passenger?

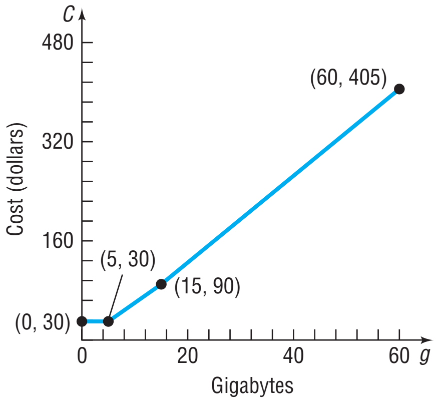
38. Reading and Interpreting Graphs Let C be the function whose graph is given in the next column. This graph represents the cost C of manufacturing q computers in a day.

(a) Determine $C(0)$. Interpret this value.
(b) Determine $C(10)$. Interpret this value.
(c) Determine $C(50)$. Interpret this value.
(d) What is the domain of C? What does this domain imply in terms of daily production?
(e) Describe the shape of the graph.
(f) The point $(30, 32\,000)$ is called an *inflection point.* Describe the behavior of the graph around the inflection point.



39. Reading and Interpreting Graphs Let C be the function whose graph is given below. This graph represents the cost C of using g gigabytes of data in a month for a shared-data family plan.

(a) Determine $C(0)$. Interpret this value.
(b) Determine $C(5)$. Interpret this value.
(c) Determine $C(15)$. Interpret this value.
(d) What is the domain of C? What does this domain imply in terms of the number of gigabytes?
(e) Describe the shape of the graph.



Explaining Concepts: Discussion and Writing

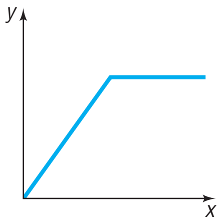
40. Describe how you would find the domain and range of a function if you were given its graph. How would your strategy change if you were given the equation defining the function instead of its graph?

41. How many x-intercepts can the graph of a function have? How many y-intercepts can the graph of a function have?

42. Is a graph that consists of a single point the graph of a function? Can you write the equation of such a function?

43. Match each of the following functions with the graph that best describes the situation.

(a) The cost of building a house as a function of its square footage
(b) The height of an egg dropped from a 300-foot building as a function of time
(c) The height of a human as a function of time
(d) The demand for Big Macs as a function of price
(e) The height of a child on a swing as a function of time



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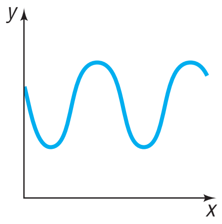

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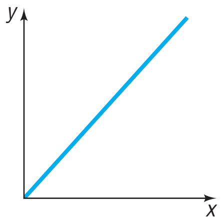

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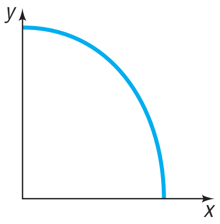

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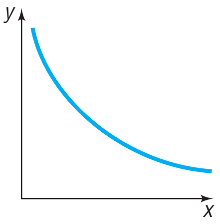
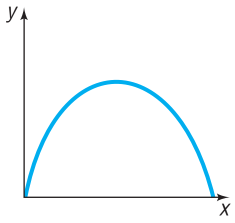


(V)

44. Match each of the following functions with the graph that best describes the situation.

(a) The temperature of a bowl of soup as a function of time
(b) The number of hours of daylight per day over a 2-year period
(c) The population of Florida as a function of time
(d) The distance traveled by a car going at a constant velocity as a function of time
(e) The height of a golf ball hit with a 7-iron as a function of time



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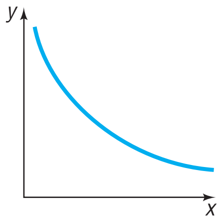

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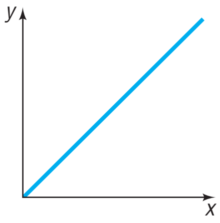

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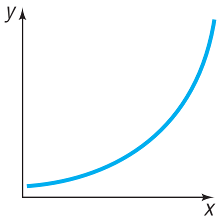

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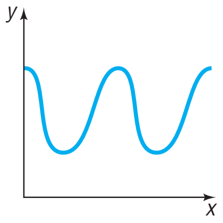

(V)

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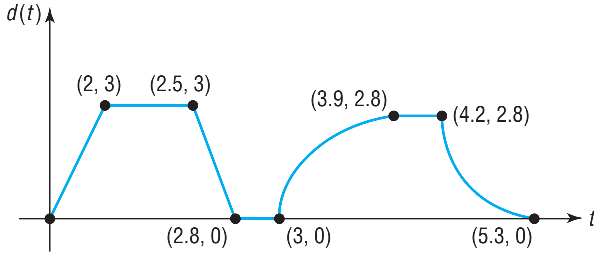
45. Consider the following scenario: Barbara decides to take a walk. She leaves home, walks 2 blocks in 5 minutes at a constant speed, and realizes that she forgot to lock the door. So Barbara runs home in 1 minute. While at her doorstep, it takes her 1 minute to find her keys and lock the door. Barbara walks 5 blocks in 15 minutes and then decides to jog home. It takes her 7 minutes to get home. Draw a graph of Barbara's distance from home (in blocks) as a function of time.

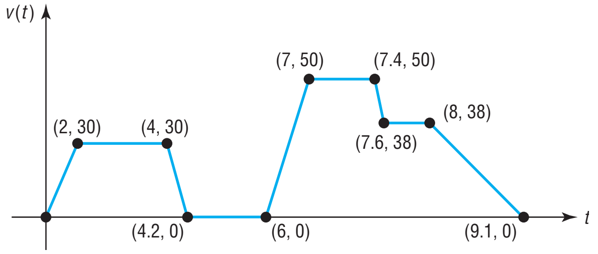
46. Consider the following scenario: Jayne enjoys riding her bicycle through the woods. At the forest preserve, she gets on her bicycle and rides up a 2000-foot incline in 10 minutes. She then travels down the incline in 3 minutes. The next 5000 feet is level terrain, and she covers the distance in 20 minutes. She rests for 15 minutes. Jayne then travels 10,000 feet in 30 minutes. Draw a graph of Jayne's distance traveled (in feet) as a function of time.

47. The following sketch represents the distance d (in miles) that Kevin was from home as a function of time t (in hours). Answer the questions by referring to the graph. In parts (a)–(g), how many hours elapsed and how far was Kevin from home during this time?



(a) From $t = 0$ to $t = 2$
(b) From $t = 2$ to $t = 2.5$
(c) From $t = 2.5$ to $t = 2.8$
(d) From $t = 2.8$ to $t = 3$
(e) From $t = 3$ to $t = 3.9$
(f) From $t = 3.9$ to $t = 4.2$
(g) From $t = 4.2$ to $t = 5.3$
(h) What is the farthest distance that Kevin was from home?
(i) How many times did Kevin return home?

48. The following sketch represents the speed v (in miles per hour) of Michael's car as a function of time t (in minutes).



(a) Over what interval of time was Michael traveling fastest?
(b) Over what interval(s) of time was Michael's speed zero?
(c) What was Michael's speed between 0 and 2 minutes?
(d) What was Michael's speed between 4.2 and 6 minutes?
(e) What was Michael's speed between 7 and 7.4 minutes?
(f) When was Michael's speed constant?

49. Draw the graph of a function whose domain is $\{x \mid -3 \le x \le 8,\ x \ne 5\}$ and whose range is $\{y \mid -1 \le y \le 2,\ y \ne 0\}$. What point(s) in the rectangle $-3 \le x \le 8, -1 \le y \le 2$ cannot be on the graph? Compare your graph with those of other students. What differences do you see?

50. Is there a function whose graph is symmetric with respect to the x-axis? Explain.

51. Explain why the vertical-line test works.

Retain Your Knowledge

Problems 52–55 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

52. Factor completely: $8x^2 + 24x + 18$

53. Find the distance between the points $(3, -6)$ and $(1, 0)$.

54. Write the equation of the line with slope $\frac{2}{3}$ that passes through the point $(-6, 4)$.

55. Subtract: $(4x^3 - 5x^2 + 2) - (3x^2 + 5x - 2)$

'Are You Prepared?' Answers

1. $(-4, 0), (4, 0), (0, -2), (0, 2)$

2. False

3.3 Properties of Functions

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Intervals (Section 1.7, pp. 147–148)
- Intercepts (Section 2.1, pp. 165–166)
- Slope of a Line (Section 2.2, pp. 173–176)
- Point–Slope Form of a Line (Section 2.2, p. 177)
- Symmetry (Section 2.1, pp. 166–168)

Now Work the **'Are You Prepared?'** problems on page 240.

OBJECTIVES

1 Determine Even and Odd Functions from a Graph (p. 231)
2 Identify Even and Odd Functions from an Equation (p. 233)
3 Use a Graph to Determine Where a Function Is Increasing, Decreasing, or Constant (p. 234)
4 Use a Graph to Locate Local Maxima and Local Minima (p. 235)
5 Use a Graph to Locate the Absolute Maximum and the Absolute Minimum (p. 236)
6 Use a Graphing Utility to Approximate Local Maxima and Local Minima and to Determine Where a Function Is Increasing or Decreasing (p. 237)
7 Find the Average Rate of Change of a Function (p. 238)

To obtain the graph of a function $y = f(x)$, it is often helpful to know certain properties that the function has and the impact of these properties on the way the graph will look.

1 Determine Even and Odd Functions from a Graph

The words *even* and *odd*, when applied to a function *f*, describe the symmetry that exists for the graph of the function.

A function f is even if and only if, whenever the point (x, y) is on the graph of f, the point $(-x, y)$ is also on the graph. Using function notation, we define an even function as follows:

DEFINITION

A function f is **even** if, for every number x in its domain, the number $-x$ is also in the domain and

$$\boxed{f(-x) = f(x)}$$

For the even function shown in Figure 17(a), notice that $f(x_1) = f(-x_1)$ and that $f(x_2) = f(-x_2)$.

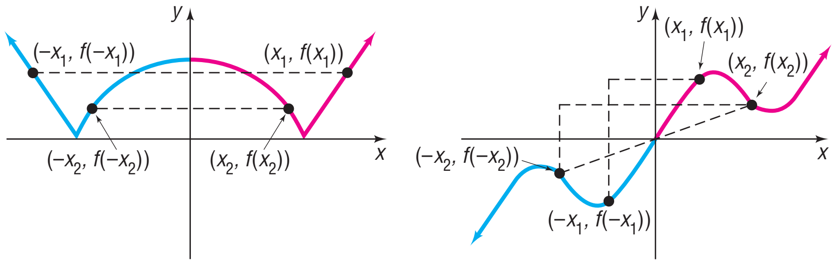
A function f is odd if and only if, whenever the point (x, y) is on the graph of f, the point $(-x, -y)$ is also on the graph. Using function notation, we define an odd function as follows:

DEFINITION

A function f is **odd** if, for every number x in its domain, the number $-x$ is also in the domain and

$$\boxed{f(-x) = -f(x)}$$

For the odd function shown in Figure 17(b), notice that $f(x_1) = -f(-x_1)$, which is equivalent to $f(-x_1) = -f(x_1)$, and that $f(x_2) = -f(-x_2)$, which is equivalent to $f(-x_2) = -f(x_2)$.



(a) Even function **(b)** Odd function

Figure 17

Refer to page 167, where the tests for symmetry are listed. The following results are then evident.

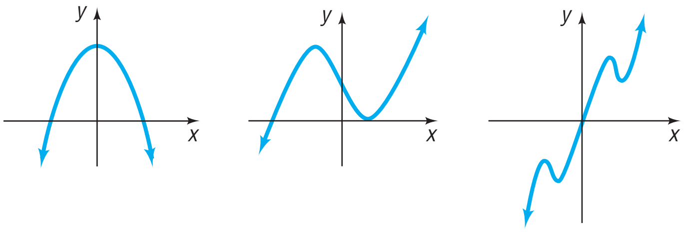
THEOREM

A function is even if and only if its graph is symmetric with respect to the y-axis. A function is odd if and only if its graph is symmetric with respect to the origin.

EXAMPLE 1

Determining Even and Odd Functions from the Graph

Determine whether each graph given in Figure 18 is the graph of an even function, an odd function, or a function that is neither even nor odd.



(a) **(b)** **(c)**

Figure 18

Solution (a) The graph in Figure 18(a) is that of an even function, because the graph is symmetric with respect to the y-axis.

(b) The function whose graph is given in Figure 18(b) is neither even nor odd, because the graph is neither symmetric with respect to the y-axis nor symmetric with respect to the origin.

(c) The function whose graph is given in Figure 18(c) is odd, because its graph is symmetric with respect to the origin. ■

Now Work PROBLEMS 25(a), (b), AND (d)

2 Identify Even and Odd Functions from an Equation

A graphing utility can be used to conjecture whether a function is even, odd, or neither. Remember that when the graph of an even function contains the point (x, y), it must also contain the point $(-x, y)$. Therefore, if the graph shows evidence of symmetry with respect to the y-axis, we would conjecture that the function is even. In addition, if the graph shows evidence of symmetry with respect to the origin, we would conjecture that the function is odd.

EXAMPLE 2 Identifying Even and Odd Functions

Use a graphing utility to conjecture whether each of the following functions is even, odd, or neither. Then algebraically determine whether the graph is symmetric with respect to the y-axis or with respect to the origin.

(a) $f(x) = x^2 - 5$ (b) $g(x) = x^3 - 1$ (c) $h(x) = 5x^3 - x$

Solution (a) Graph the function. See Figure 19. It appears that the graph is symmetric with respect to the y-axis. We conjecture that the function is even.

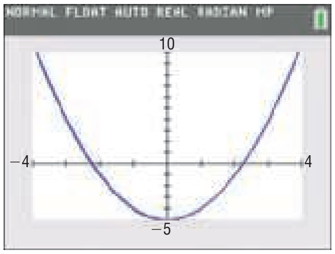


Figure 19

To algebraically verify the conjecture, replace x by $-x$ in $f(x) = x^2 - 5$. Then

$$f(-x) = (-x)^2 - 5 = x^2 - 5 = f(x)$$

Since $f(-x) = f(x)$, we conclude that f is an even function and that the graph is symmetric with respect to the y-axis.

(b) Graph the function. See Figure 20. It appears that there is no symmetry. We conjecture that the function is neither even nor odd.

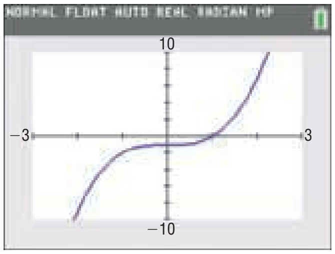


Figure 20

To algebraically verify that the function is not even, find $g(-x)$ and compare the result with $g(x)$.

$$g(-x) = (-x)^3 - 1 = -x^3 - 1; \quad g(x) = x^3 - 1$$

Since $g(-x) \neq g(x)$, the function is not even.

To algebraically verify that the function is not odd, find $-g(x)$ and compare the result with $g(-x)$.

$$-g(x) = -(x^3 - 1) = -x^3 + 1; \quad g(-x) = -x^3 - 1$$

Since $g(-x) \neq -g(x)$, the function is not odd. The graph is not symmetric with respect to the y-axis nor is it symmetric with respect to the origin.

(c) Graph the function. See Figure 21. It appears that there is symmetry with respect to the origin. We conjecture that the function is odd.

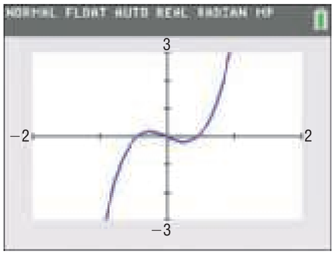


Figure 21

To algebraically verify the conjecture, replace x by $-x$ in $h(x) = 5x^3 - x$. Then

$$h(-x) = 5(-x)^3 - (-x) = -5x^3 + x = -(5x^3 - x) = -h(x)$$

Since $h(-x) = -h(x)$, h is an odd function and the graph of h is symmetric with respect to the origin. ■

Now Work PROBLEM 37

3 Use a Graph to Determine Where a Function Is Increasing, Decreasing, or Constant

Consider the graph given in Figure 22. If you look from left to right along the graph of the function, you will notice that parts of the graph are going up, parts are going down, and parts are horizontal. In such cases, the function is described as *increasing, decreasing,* or *constant,* respectively.

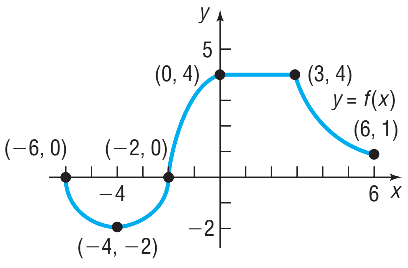


Figure 22

EXAMPLE 3 Determining Where a Function Is Increasing, Decreasing, or Constant from Its Graph

Determine the values of x for which the function in Figure 22 is increasing. Where is it decreasing? Where is it constant?

Solution When determining where a function is increasing, where it is decreasing, and where it is constant, we use inequalities involving the independent variable x, or we use intervals of x-coordinates. The function whose graph is given in Figure 22 is increasing on the interval $[-4, 0]$, or for $-4 \le x \le 0$. The function is decreasing on the intervals $[-6, -4]$ and $[3, 6]$, or for $-6 \le x \le -4$ and $3 \le x \le 6$. The function is constant on the closed interval $[0, 3]$, or for $0 \le x \le 3$. ■

WARNING Describe the behavior of a graph in terms of its x-values. Do not say the graph in Figure 22 is increasing from the point $(-4, -2)$ to the point $(0, 4)$. Rather, say it is increasing on the interval $[-4, 0]$. ■

In Words

If a function is decreasing, then as the values of x get bigger, the values of the function get smaller. If a function is increasing, then as the values of x get bigger, the values of the function also get bigger. If a function is constant, then as the values of x get bigger, the values of the function remain unchanged.

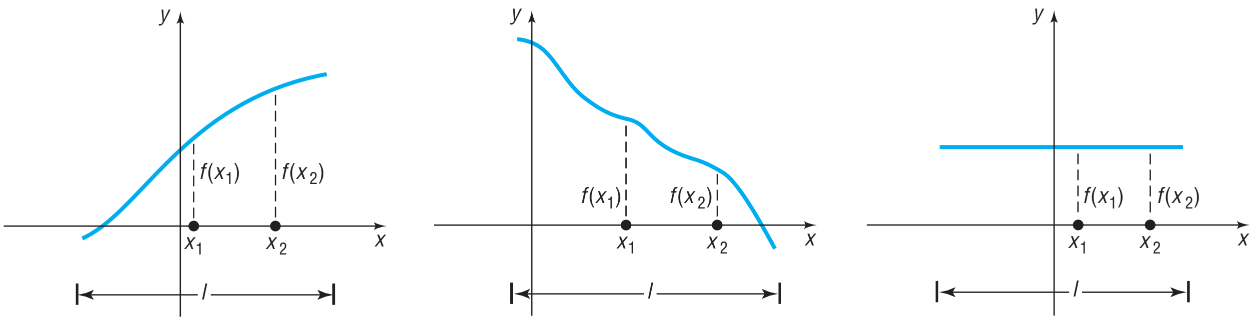
More precise definitions follow:

DEFINITIONS A function f is **increasing** on an interval I if, for any choice of x_1 and x_2 in I, with $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

A function f is **decreasing** on an interval I if, for any choice of x_1 and x_2 in I, with $x_1 < x_2$, we have $f(x_1) > f(x_2)$.

A function f is **constant** on an interval I if, for all choices of x in I, the values $f(x)$ are equal. ■

Figure 23 illustrates the definitions. The graph of an increasing function goes up from left to right, the graph of a decreasing function goes down from left to right, and the graph of a constant function remains at a fixed height.



(a) For $x_1 < x_2$ in I, $f(x_1) < f(x_2)$; f is increasing on I.

(b) For $x_1 < x_2$ in I, $f(x_1) > f(x_2)$; f is decreasing on I.

(c) For all x in I, the values of f are equal; f is constant on I.

Figure 23

Now Work PROBLEMS 13, 15, 17, AND 25(c)

4 Use a Graph to Locate Local Maxima and Local Minima

Suppose f is a function defined on an open interval I containing c. If the value of f at c is greater than or equal to the values of f on I, then f has a *local maximum* at c.* See Figure 24(a).

If the value of f at c is less than or equal to the values of f on I, then f has a *local minimum* at c. See Figure 24(b).

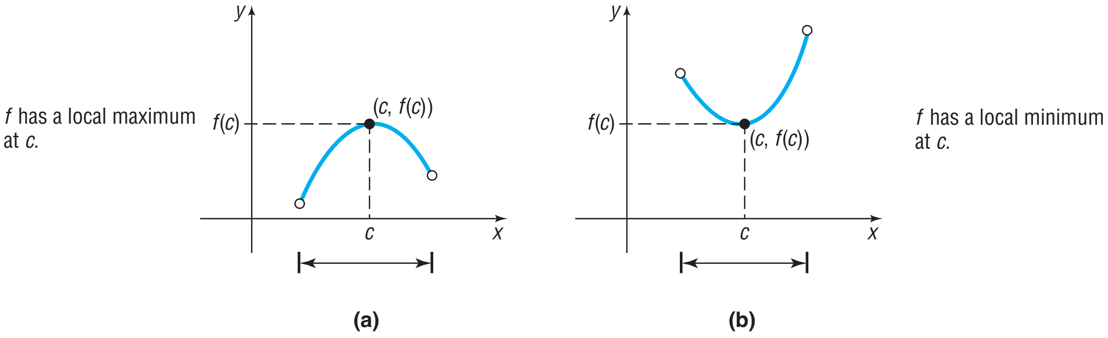


Figure 24 Local maximum and local minimum

DEFINITIONS

Let f be a function defined on some interval I.

A function f has a **local maximum** at c if there is an open interval in I containing c so that, for all x in this open interval, we have $f(x) \leq f(c)$. We call $f(c)$ a **local maximum value of *f***.

A function f has a **local minimum** at c if there is an open interval in I containing c so that, for all x in this open interval, we have $f(x) \geq f(c)$. We call $f(c)$ a **local minimum value of *f***.

If f has a local maximum at c, then the value of f at c is greater than or equal to the values of f near c. If f has a local minimum at c, then the value of f at c is less than or equal to the values of f near c. The word *local* is used to suggest that it is only near c, not necessarily over the entire domain, that the value $f(c)$ has these properties.

EXAMPLE 4

Finding Local Maxima and Local Minima from the Graph of a Function and Determining Where the Function Is Increasing, Decreasing, or Constant

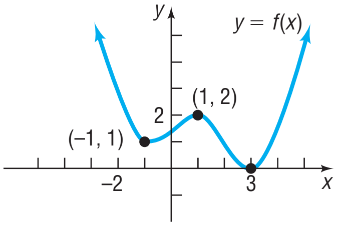


Figure 25

Figure 25 shows the graph of a function f.

(a) At what value(s) of x, if any, does f have a local maximum? List the local maximum value(s).

(b) At what value(s) of x, if any, does f have a local minimum? List the local minimum value(s).

(c) Find the intervals on which f is increasing. Find the intervals on which f is decreasing.

Solution

The domain of f is the set of real numbers.

(a) f has a local maximum at 1, since for all x close to 1, we have $f(x) \leq f(1)$. The local maximum value is $f(1) = 2$.

(b) f has local minima at -1 and at 3. The local minimum values are $f(-1) = 1$ and $f(3) = 0$.

(c) The function whose graph is given in Figure 25 is increasing on the intervals $[-1, 1]$ and $[3, \infty)$, or for $-1 \leq x \leq 1$ and $x \geq 3$. The function is decreasing on the intervals $(-\infty, -1]$ and $[1, 3]$, or for $x \leq -1$ and $1 \leq x \leq 3$. ■

WARNING The y-value is the local maximum value or local minimum value, and it occurs at some x-value. For example, in Figure 25, we say f has a local maximum at 1 and the local maximum value is 2. ■

Now Work PROBLEMS 19 AND 21

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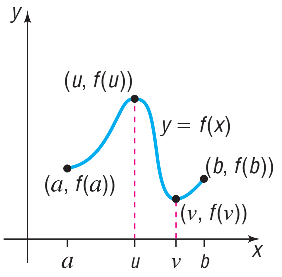
*Some texts use the term *relative* instead of *local*.



domain: $[a, b]$
for all x in $[a, b]$, $f(x) \le f(u)$
for all x in $[a, b]$, $f(x) \ge f(v)$
absolute maximum: $f(u)$
absolute minimum: $f(v)$

Figure 26

5 Use a Graph to Locate the Absolute Maximum and the Absolute Minimum

Look at the graph of the function f given in Figure 26. The domain of f is the closed interval $[a, b]$. Also, the largest value of f is $f(u)$ and the smallest value of f is $f(v)$. These are called, respectively, the *absolute maximum* and the *absolute minimum* of f on $[a, b]$.

DEFINITION Let f be a function defined on some interval I. If there is a number u in I for which $f(x) \le f(u)$ for all x in I, then f has an **absolute maximum at u**, and the number $f(u)$ is the **absolute maximum of f on I**.

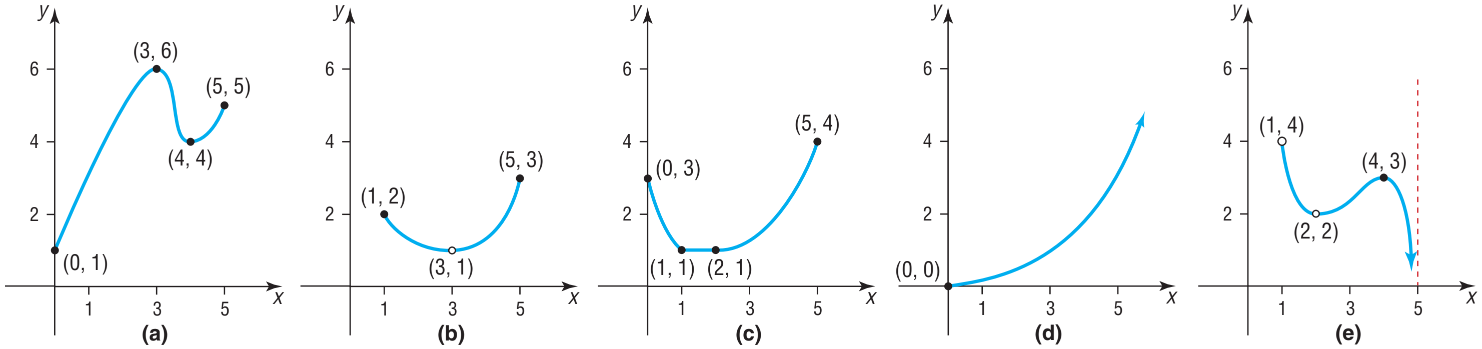
If there is a number v in I for which $f(x) \ge f(v)$ for all x in I, then f has an **absolute minimum at v**, and the number $f(v)$ is the **absolute minimum of f on I**.

The absolute maximum and absolute minimum of a function f are sometimes called the **absolute extrema** or **extreme values** of f on I.

The absolute maximum or absolute minimum of a function f may not exist. Let's look at some examples.

EXAMPLE 5 Finding the Absolute Maximum and the Absolute Minimum from the Graph of a Function

For each graph of a function $y = f(x)$ in Figure 27, find the absolute maximum and the absolute minimum, if they exist. Also, find any local maxima or local minima.



(a) (b) (c) (d) (e)

Figure 27

Solution

(a) The function f whose graph is given in Figure 27(a) has the closed interval $[0, 5]$ as its domain. The largest value of f is $f(3) = 6$, the absolute maximum. The smallest value of f is $f(0) = 1$, the absolute minimum. The function has a local maximum value of 6 at $x = 3$ and a local minimum value of 4 at $x = 4$.

(b) The function f whose graph is given in Figure 27(b) has the domain $\{x | 1 \le x \le 5, x \ne 3\}$. Note that we exclude 3 from the domain because of the "hole" at $(3, 1)$. The largest value of f on its domain is $f(5) = 3$, the absolute maximum. There is no absolute minimum. Do you see why? As you trace the graph, getting closer to the point $(3, 1)$, there is no single smallest value. [As soon as you claim a smallest value, we can trace closer to $(3, 1)$ and get a smaller value!] The function has no local maximum or local minimum.

(c) The function f whose graph is given in Figure 27(c) has the interval $[0, 5]$ as its domain. The absolute maximum of f is $f(5) = 4$. The absolute minimum is 1. Notice that the absolute minimum 1 occurs at any number in the interval $[1, 2]$. The function has a local minimum value of 1 at every x in the interval $[1, 2]$, but it has no local maximum value.

(d) The function f given in Figure 27(d) has the interval $[0, \infty)$ as its domain. The function has no absolute maximum; the absolute minimum is $f(0) = 0$. The function has no local maximum or local minimum.

(e) The function f in Figure 27(e) has the domain $\{x | 1 < x < 5, x \ne 2\}$. The function has no absolute maximum and no absolute minimum. Do you see why? The function has a local maximum value of 3 at $x = 4$, but no local minimum value. ■

WARNING A function may have an absolute maximum or an absolute minimum at an endpoint but not a local maximum or a local minimum. Why? Local maxima and local minima are found over some open interval I, and this interval cannot be created around an endpoint. ■

In calculus, there is a theorem with conditions that guarantee a function will have an absolute maximum and an absolute minimum.

THEOREM

Extreme Value Theorem

If f is a continuous function* whose domain is a closed interval $[a, b]$, then f has an absolute maximum and an absolute minimum on $[a, b]$.

The absolute maximum (minimum) can be found by selecting the largest (smallest) value of f from the following list:

1. The values of f at any local maxima or local minima of f in $[a, b]$.
2. The value of f at each endpoint of $[a, b]$—that is, $f(a)$ and $f(b)$.

For example, the graph of the function f given in Figure 27(a) is continuous on the closed interval $[0, 5]$. The Extreme Value Theorem guarantees that f has extreme values on $[0, 5]$. To find them, we list

1. The value of f at the local extrema: $f(3) = 6, f(4) = 4$
2. The value of f at the endpoints: $f(0) = 1, f(5) = 5$

The largest of these, 6, is the absolute maximum; the smallest of these, 1, is the absolute minimum.

Notice that absolute extrema may occur at the endpoints of a function defined on a closed interval. However, local extrema cannot occur at the endpoints because an open interval cannot be constructed around the endpoint. So, in Figure 27(b), for example, $f(1) = 2$ is not a local maximum.

Now Work PROBLEM 49

6 Use a Graphing Utility to Approximate Local Maxima and Local Minima and to Determine Where a Function Is Increasing or Decreasing

To locate the exact value at which a function f has a local maximum or a local minimum usually requires calculus. However, a graphing utility may be used to approximate these values using the MAXIMUM and MINIMUM features.

EXAMPLE 6 Using a Graphing Utility to Approximate Local Maxima and Minima and to Determine Where a Function Is Increasing or Decreasing

(a) Use a graphing utility to graph $f(x) = 6x^3 - 12x + 5$ for $-2 \le x \le 2$. Approximate where f has a local maximum and where f has a local minimum.

(b) Determine where f is increasing and where it is decreasing.

Solution (a) Graphing utilities have a feature that finds the maximum or minimum point of a graph within a given interval. Graph the function f for $-2 \le x \le 2$. The MAXIMUM and MINIMUM commands require us to first determine the open interval I. The graphing utility will then approximate the maximum or minimum value in the interval. Using MAXIMUM, we find that the local maximum value is 11.53 and that it occurs at $x = -0.82$, rounded to two decimal places. See Figure 28(a). Using MINIMUM, we find that the local minimum value is -1.53 and that it occurs at $x = 0.82$, rounded to two decimal places. See Figure 28(b).

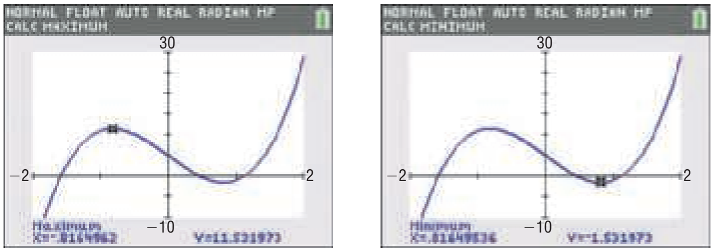


Figure 28 **(a)** Local maximum **(b)** Local minimum

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*Although a precise definition requires calculus, we'll agree for now that a continuous function is one whose graph has no gaps or holes and can be traced without lifting the pencil from the paper.

(b) Looking at Figures 28(a) and (b), we see that the graph of f is increasing from $x = -2$ to $x = -0.82$ and from $x = 0.82$ to $x = 2$, so f is increasing on the intervals $[-2, -0.82]$ and $[0.82, 2]$, or for $-2 \le x \le -0.82$ and $0.82 \le x \le 2$. The graph is decreasing from $x = -0.82$ to $x = 0.82$, so f is decreasing on the interval $[-0.82, 0.82]$, or for $-0.82 \le x \le 0.82$. ■

Now Work PROBLEM 57

7 Find the Average Rate of Change of a Function

In Section 2.2, we said that the slope of a line can be interpreted as the average rate of change. To find the average rate of change of a function between any two points on its graph, calculate the slope of the line containing the two points.

DEFINITION

If a and b, $a \neq b$, are in the domain of a function $y = f(x)$, the **average rate of change of f** from a to b is defined as

$$\boxed{\text{Average rate of change} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a} \qquad a \neq b \qquad \textbf{(1)}}$$

In Words

The symbol Δ is the Greek capital letter delta and is read "change in."

The symbol Δy in equation (1) is the "change in y," and Δx is the "change in x." The average rate of change of f is the change in y divided by the change in x.

EXAMPLE 7 Finding the Average Rate of Change

Find the average rate of change of $f(x) = 3x^2$:

(a) From 1 to 3 (b) From 1 to 5 (c) From 1 to 7

Solution

(a) The average rate of change of $f(x) = 3x^2$ from 1 to 3 is

$$\frac{\Delta y}{\Delta x} = \frac{f(3) - f(1)}{3 - 1} = \frac{27 - 3}{3 - 1} = \frac{24}{2} = 12$$

(b) The average rate of change of $f(x) = 3x^2$ from 1 to 5 is

$$\frac{\Delta y}{\Delta x} = \frac{f(5) - f(1)}{5 - 1} = \frac{75 - 3}{5 - 1} = \frac{72}{4} = 18$$

(c) The average rate of change of $f(x) = 3x^2$ from 1 to 7 is

$$\frac{\Delta y}{\Delta x} = \frac{f(7) - f(1)}{7 - 1} = \frac{147 - 3}{7 - 1} = \frac{144}{6} = 24$$

■

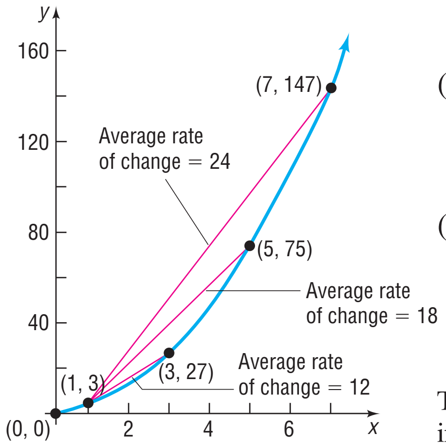


Figure 29 $f(x) = 3x^2$

See Figure 29 for a graph of $f(x) = 3x^2$. The function f is increasing for $x \ge 0$. The fact that the average rate of change is positive for any $x_1, x_2, x_1 \neq x_2$, in the interval $[1, 7]$ indicates that the graph is increasing on $1 \le x \le 7$. Further, the average rate of change is consistently getting larger for $1 \le x \le 7$, which indicates that the graph is increasing at an increasing rate.

Now Work PROBLEM 65

The Secant Line

The average rate of change of a function has an important geometric interpretation. Look at the graph of $y = f(x)$ in Figure 30. Two points are labeled on the graph: $(a, f(a))$ and $(b, f(b))$. The line containing these two points is called the **secant line**; its slope is

$$m_{\text{sec}} = \frac{f(b) - f(a)}{b - a} = \frac{f(a + h) - f(a)}{h}$$

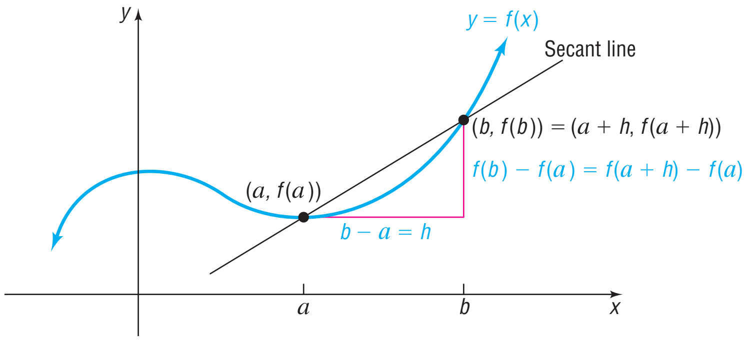


Figure 30 Secant line

THEOREM

Slope of the Secant Line

The average rate of change of a function from a to b equals the slope of the secant line containing the two points $(a, f(a))$ and $(b, f(b))$ on its graph.

EXAMPLE 8 Finding the Equation of a Secant Line

Suppose that $g(x) = 3x^2 - 2x + 3$.

(a) Find the average rate of change of g from -2 to 1.

(b) Find an equation of the secant line containing $(-2, g(-2))$ and $(1, g(1))$.

(c) Using a graphing utility, draw the graph of g and the secant line obtained in part (b) on the same screen.

Solution (a) The average rate of change of $g(x) = 3x^2 - 2x + 3$ from -2 to 1 is

$$\text{Average rate of change} = \frac{g(1) - g(-2)}{1 - (-2)}$$

$$= \frac{4 - 19}{3} \qquad \boldsymbol{g(1) = 3(1)^2 - 2(1) + 3 = 4}$$

$$\boldsymbol{g(-2) = 3(-2)^2 - 2(-2) + 3 = 19}$$

$$= -\frac{15}{3} = -5$$

(b) The slope of the secant line containing $(-2, g(-2)) = (-2, 19)$ and $(1, g(1)) = (1, 4)$ is $m_{\text{sec}} = -5$. Use the point–slope form to find an equation of the secant line.

$$y - y_1 = m_{\text{sec}}(x - x_1) \qquad \textbf{Point–slope form of the secant line}$$

$$y - 19 = -5(x - (-2)) \qquad \boldsymbol{x_1 = -2, y_1 = g(-2) = 19, m_{\text{sec}} = -5}$$

$$y - 19 = -5x - 10 \qquad \textbf{Distribute.}$$

$$y = -5x + 9 \qquad \textbf{Slope–intercept form of the secant line}$$

(c) Figure 31 shows the graph of g along with the secant line $y = -5x + 9$. ■

Now Work PROBLEM 71

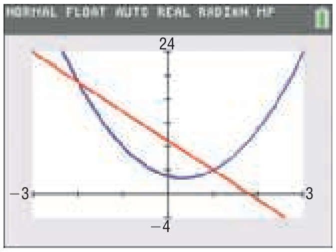


Figure 31 Graph of g and the secant line

3.3 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. The interval $(2, 5)$ can be written as the inequality ______. (pp. 147–148)

2. The slope of the line containing the points $(-2, 3)$ and $(3, 8)$ is ___. (pp. 173–176)

3. Test the equation $y = 5x^2 - 1$ for symmetry with respect to the x-axis, the y-axis, and the origin. (pp. 166–168)

4. Write the point–slope form of the line with slope 5 containing the point $(3, -2)$. (p. 177)

5. The intercepts of the equation $y = x^2 - 9$ are ______. (pp. 165–166)

Concepts and Vocabulary

6. A function f is _________ on an interval I if, for any choice of x_1 and x_2 in I, with $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

7. A(n) _____ function f is one for which $f(-x) = f(x)$ for every x in the domain of f; a(n) ____ function f is one for which $f(-x) = -f(x)$ for every x in the domain of f.

8. ***True or False*** A function f is decreasing on an interval I if, for any choice of x_1 and x_2 in I, with $x_1 < x_2$, we have $f(x_1) > f(x_2)$.

9. ***True or False*** A function f has a local maximum at c if there is an open interval I containing c such that for all x in I, $f(x) \leq f(c)$.

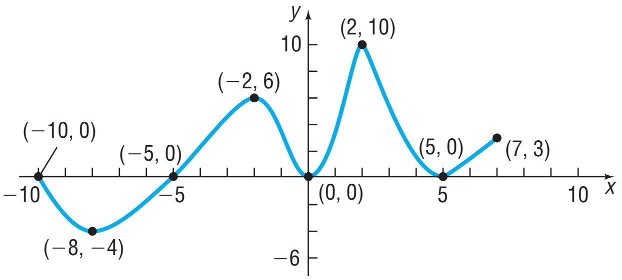
10. ***True or False*** Even functions have graphs that are symmetric with respect to the origin.

11. An odd function is symmetric with respect to ______.
(a) the x-axis (b) the y-axis
(c) the origin (d) the line $y = x$

12. Which of the following intervals is required to guarantee a continuous function will have both an absolute maximum and an absolute minimum?
(a) (a, b) (b) $(a, b]$
(c) $[a, b)$ (d) $[a, b]$

Skill Building

In Problems 13–24, use the graph of the function f given.



13. Is f increasing on the interval $[-8, -2]$?

14. Is f decreasing on the interval $[-8, -4]$?

15. Is f increasing on the interval $[-2, 6]$?

16. Is f decreasing on the interval $[2, 5]$?

17. List the interval(s) on which f is increasing.

18. List the interval(s) on which f is decreasing.

19. Is there a local maximum at 2? If yes, what is it?

20. Is there a local maximum at 5? If yes, what is it?

21. List the number(s) at which f has a local maximum. What are the local maximum values?

22. List the number(s) at which f has a local minimum. What are the local minimum values?

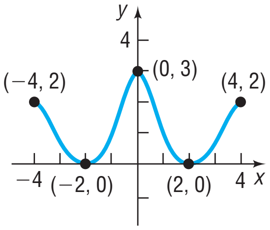
23. Find the absolute minimum of f on $[-10, 7]$.

24. Find the absolute maximum of f on $[-10, 7]$.

In Problems 25–32, the graph of a function is given. Use the graph to find:

(a) The intercepts, if any
(b) The domain and range
(c) The intervals on which the function is increasing, decreasing, or constant
(d) Whether the function is even, odd, or neither

25.



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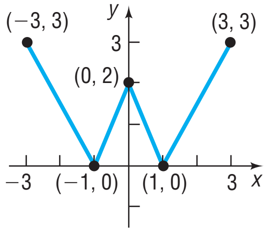

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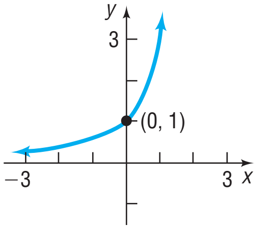

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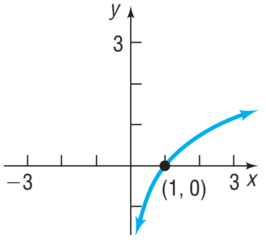

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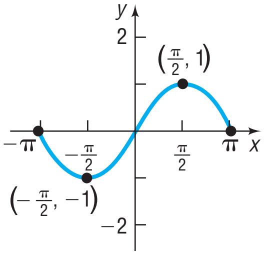

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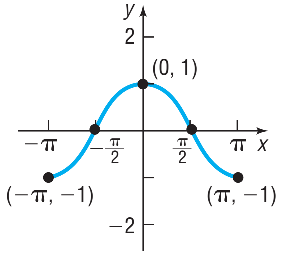

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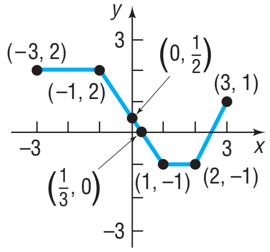

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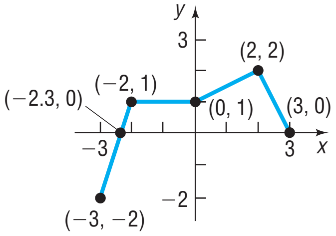
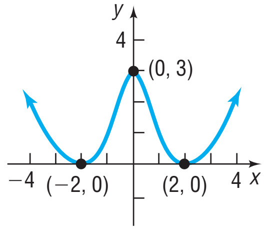


In Problems 33–36, the graph of a function f is given. Use the graph to find:

(a) The numbers, if any, at which f has a local maximum. What are the local maximum values?

(b) The numbers, if any, at which f has a local minimum. What are the local minimum values?

33.



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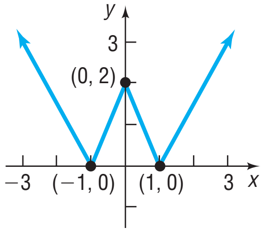

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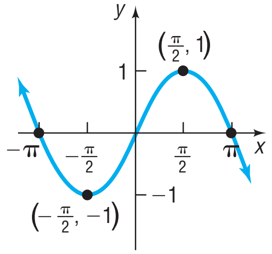

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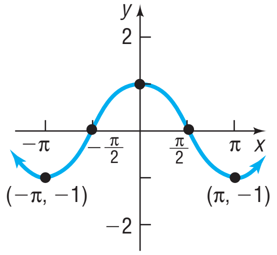


In Problems 37–48, determine algebraically whether each function is even, odd, or neither.

37. $f(x) = 4x^3$

38. $f(x) = 2x^4 - x^2$

39. $g(x) = -3x^2 - 5$

40. $h(x) = 3x^3 + 5$

41. $F(x) = \sqrt[3]{x}$

42. $G(x) = \sqrt{x}$

43. $f(x) = x + |x|$

44. $f(x) = \sqrt[3]{2x^2 + 1}$

45. $g(x) = \dfrac{1}{x^2}$

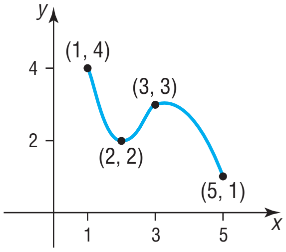
46. $h(x) = \dfrac{x}{x^2 - 1}$

47. $h(x) = \dfrac{-x^3}{3x^2 - 9}$

48. $F(x) = \dfrac{2x}{|x|}$

In Problems 49–56, for each graph of a function $y = f(x)$, find the absolute maximum and the absolute minimum, if they exist. Identify any local maximum values or local minimum values.

49.



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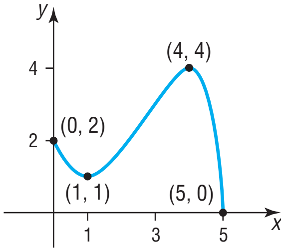

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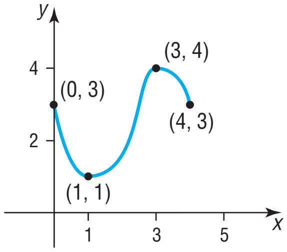

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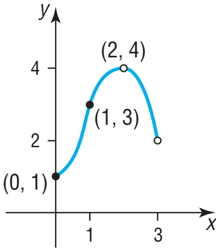

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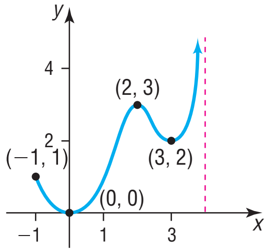

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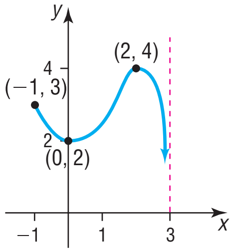

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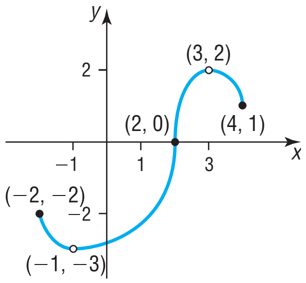

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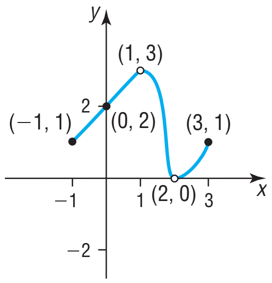


In Problems 57–64, use a graphing utility to graph each function over the indicated interval and approximate any local maximum values and local minimum values. Determine where the function is increasing and where it is decreasing. Round answers to two decimal places.

57. $f(x) = x^3 - 3x + 2 \quad [-2, 2]$

58. $f(x) = x^3 - 3x^2 + 5 \quad [-1, 3]$

59. $f(x) = x^5 - x^3 \quad [-2, 2]$

60. $f(x) = x^4 - x^2 \quad [-2, 2]$

61. $f(x) = -0.2x^3 - 0.6x^2 + 4x - 6 \quad [-6, 4]$

62. $f(x) = -0.4x^3 + 0.6x^2 + 3x - 2 \quad [-4, 5]$

63. $f(x) = 0.25x^4 + 0.3x^3 - 0.9x^2 + 3 \quad [-3, 2]$

64. $f(x) = -0.4x^4 - 0.5x^3 + 0.8x^2 - 2 \quad [-3, 2]$

65. Find the average rate of change of $f(x) = -2x^2 + 4$:
(a) From 0 to 2
(b) From 1 to 3
(c) From 1 to 4
(d) By hand, graph f and illustrate the average rate of change found in parts (a), (b), and (c).

66. Find the average rate of change of $f(x) = -x^3 + 1$:
(a) From 0 to 2
(b) From 1 to 3
(c) From -1 to 1
(d) By hand, graph f and illustrate the average rate of change found in parts (a), (b), and (c).

67. Find the average rate of change of $g(x) = x^3 - 2x + 1$:
(a) From -3 to -2
(b) From -1 to 1
(c) From 1 to 3

68. Find the average rate of change of $h(x) = x^2 - 2x + 3$:
(a) From -1 to 1
(b) From 0 to 2
(c) From 2 to 5

69. $f(x) = 5x - 2$
(a) Find the average rate of change from 1 to 3.
(b) Find an equation of the secant line containing $(1, f(1))$ and $(3, f(3))$.

70. $f(x) = -4x + 1$
(a) Find the average rate of change from 2 to 5.
(b) Find an equation of the secant line containing $(2, f(2))$ and $(5, f(5))$.

71. $g(x) = x^2 - 2$
(a) Find the average rate of change from -2 to 1.
(b) Find an equation of the secant line containing $(-2, g(-2))$ and $(1, g(1))$.
(c) Using a graphing utility, draw the graph of g and the secant line obtained in part (b) on the same screen.

72. $g(x) = x^2 + 1$
(a) Find the average rate of change from -1 to 2.
(b) Find an equation of the secant line containing $(-1, g(-1))$ and $(2, g(2))$.
(c) Using a graphing utility, draw the graph of g and the secant line obtained in part (b) on the same screen.

73. $h(x) = x^2 - 2x$
(a) Find the average rate of change from 2 to 4.
(b) Find an equation of the secant line containing $(2, h(2))$ and $(4, h(4))$.
(c) Using a graphing utility, draw the graph of h and the secant line obtained in part (b) on the same screen.

74. $h(x) = -2x^2 + x$
(a) Find the average rate of change from 0 to 3.
(b) Find an equation of the secant line containing $(0, h(0))$ and $(3, h(3))$.
(c) Using a graphing utility, draw the graph of h and the secant line obtained in part (b) on the same screen.

Mixed Practice

75. $g(x) = x^3 - 27x$
(a) Determine whether g is even, odd, or neither.
(b) There is a local minimum value of -54 at 3. Determine the local maximum value.

77. $F(x) = -x^4 + 8x^2 + 9$
(a) Determine whether F is even, odd, or neither.
(b) There is a local maximum value of 25 at $x = 2$. Determine a second local maximum value.
(c) Suppose the area under the graph of F between $x = 0$ and $x = 3$ that is bounded from below by the x-axis is 50.4 square units. Using the result from part (a), determine the area under the graph of F between $x = -3$ and $x = 0$ that is bounded from below by the x-axis.

76. $f(x) = -x^3 + 12x$
(a) Determine whether f is even, odd, or neither.
(b) There is a local maximum value of 16 at 2. Determine the local minimum value.

78. $G(x) = -x^4 + 32x^2 + 144$
(a) Determine whether G is even, odd, or neither.
(b) There is a local maximum value of 400 at $x = 4$. Determine a second local maximum value.
(c) Suppose the area under the graph of G between $x = 0$ and $x = 6$ that is bounded from below by the x-axis is 1612.8 square units. Using the result from part (a), determine the area under the graph of G between $x = -6$ and $x = 0$ that is bounded from below by the x-axis.

Applications and Extensions

79. Minimum Average Cost The average cost per hour in dollars, $\overline{C}$, of producing x riding lawn mowers can be modeled by the function

$$\overline{C}(x) = 0.3x^2 + 21x - 251 + \frac{2500}{x}$$

(a) Use a graphing utility to graph $\overline{C} = \overline{C}(x)$.
(b) Determine the number of riding lawn mowers to produce in order to minimize average cost.
(c) What is the minimum average cost?

80. Medicine Concentration The concentration C of a medication in the bloodstream t hours after being administered is modeled by the function

$$C(t) = -0.002x^4 + 0.039t^3 - 0.285t^2 + 0.766t + 0.085$$

(a) After how many hours will the concentration be highest?
(b) A woman nursing a child must wait until the concentration is below 0.5 before she can feed him. After taking the medication, how long must she wait before feeding her child?

81. Data Plan Cost The monthly cost C, in dollars, for wireless data plans with x gigabytes of data included is shown in the table on the top left column of the next page. Since each input value for x corresponds to exactly one output value for C, the plan cost is a function of the number of data gigabytes. Thus $C(x)$ represents the monthly cost for a wireless data plan with x gigabytes included.

GB	Cost ($)	GB	Cost ($)
4	70	20	150
6	80	30	225
10	100	40	300
15	130	50	375

(a) Plot the points $(4, 70)$, $(6, 80)$, $(10, 100)$, and so on in a Cartesian plane.
(b) Draw a line segment from the point $(10, 100)$ to $(30, 225)$. What does the slope of this line segment represent?
(c) Find the average rate of change of the monthly cost from 4 to 10 gigabytes.
(d) Find the average rate of change of the monthly cost from 10 to 30 gigabytes.
(e) Find the average rate of change of the monthly cost from 30 to 50 gigabytes.
(f) What is happening to the average rate of change as the gigabytes of data increase?

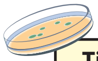
82. National Debt The size of the total debt owed by the United States federal government continues to grow. In fact, according to the Department of the Treasury, the debt per person living in the United States is approximately \$53,000 (or over \$140,000 per U.S. household). The following data represent the U.S. debt for the years 2001–2014. Since the debt D depends on the year y, and each input corresponds to exactly one output, the debt is a function of the year. So $D(y)$ represents the debt for each year y.

Year	Debt (billions of dollars)	Year	Debt (billions of dollars)
2001	5807	2008	10,025
2002	6228	2009	11,910
2003	6783	2010	13,562
2004	7379	2011	14,790
2005	7933	2012	16,066
2006	8507	2013	16,738
2007	9008	2014	17,824

Source: *www.treasurydirect.gov*

(a) Plot the points $(2001, 5807)$, $(2002, 6228)$, and so on in a Cartesian plane.
(b) Draw a line segment from the point $(2001, 5807)$ to $(2006, 8507)$. What does the slope of this line segment represent?
(c) Find the average rate of change of the debt from 2002 to 2004.
(d) Find the average rate of change of the debt from 2006 to 2008.
(e) Find the average rate of change of the debt from 2010 to 2012.
(f) What appears to be happening to the average rate of change as time passes?

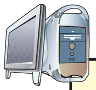
83. *E. coli* Growth A strain of *E. coli* Beu 397-recA441 is placed into a nutrient broth at 30° Celsius and allowed to grow. The data shown in the table are collected. The population is measured in grams and the time in hours. Since population P depends on time t, and each input corresponds to exactly one output, we can say that population is a function of time. Thus $P(t)$ represents the population at time t.



Time (hours), t	Population (grams), P
0	0.09
2.5	0.18
3.5	0.26
4.5	0.35
6	0.50

(a) Find the average rate of change of the population from 0 to 2.5 hours.
(b) Find the average rate of change of the population from 4.5 to 6 hours.
(c) What is happening to the average rate of change as time passes?

84. e-Filing Tax Returns The Internal Revenue Service Restructuring and Reform Act (RRA) was signed into law by President Bill Clinton in 1998. A major objective of the RRA was to promote electronic filing of tax returns. The data in the table that follows show the percentage of individual income tax returns filed electronically for filing years 2004–2013. Since the percentage P of returns filed electronically depends on the filing year y, and each input corresponds to exactly one output, the percentage of returns filed electronically is a function of the filing year; so $P(y)$ represents the percentage of returns filed electronically for filing year y.
(a) Find the average rate of change of the percentage of e-filed returns from 2004 to 2006.
(b) Find the average rate of change of the percentage of e-filed returns from 2007 to 2009.
(c) Find the average rate of change of the percentage of e-filed returns from 2010 to 2012.
(d) What is happening to the average rate of change as time passes?



Year	Percentage of returns e-filed
2004	46.5
2005	51.1
2006	53.8
2007	57.1
2008	58.5
2009	67.2
2010	69.8
2011	77.2
2012	82.7
2013	84.7

Source: *Internal Revenue Service*

85. For the function $f(x) = x^2$, compute the average rate of change:
- (a) From 0 to 1
- (b) From 0 to 0.5
- (c) From 0 to 0.1
- (d) From 0 to 0.01
- (e) From 0 to 0.001
- (f) Use a graphing utility to graph each of the secant lines along with f.
- (g) What do you think is happening to the secant lines?
- (h) What is happening to the slopes of the secant lines? Is there some number that they are getting closer to? What is that number?

86. For the function $f(x) = x^2$, compute the average rate of change:
- (a) From 1 to 2
- (b) From 1 to 1.5
- (c) From 1 to 1.1
- (d) From 1 to 1.01
- (e) From 1 to 1.001
- (f) Use a graphing utility to graph each of the secant lines along with f.
- (g) What do you think is happening to the secant lines?
- (h) What is happening to the slopes of the secant lines? Is there some number that they are getting closer to? What is that number?

Problems 87–94 require the following discussion of a secant line. The slope of the secant line containing the two points $(x, f(x))$ *and* $(x + h, f(x + h))$ *on the graph of a function* $y = f(x)$ *may be given as*

$$m_{\text{sec}} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}, \qquad h \neq 0$$

In calculus, this expression is called the **difference quotient of *f*.**
- (a) *Express the slope of the secant line of each function in terms of x and h. Be sure to simplify your answer.*
- (b) *Find* m_{sec} *for* $h = 0.5, 0.1,$ *and* 0.01 *at* $x = 1$. *What value does* m_{sec} *approach as h approaches* 0?
- (c) *Find an equation for the secant line at* $x = 1$ *with* $h = 0.01$.
- (d) *Use a graphing utility to graph f and the secant line found in part (c) in the same viewing window.*

87. $f(x) = 2x + 5$

88. $f(x) = -3x + 2$

89. $f(x) = x^2 + 2x$

90. $f(x) = 2x^2 + x$

91. $f(x) = 2x^2 - 3x + 1$

92. $f(x) = -x^2 + 3x - 2$

93. $f(x) = \dfrac{1}{x}$

94. $f(x) = \dfrac{1}{x^2}$

Explaining Concepts: Discussion and Writing

95. Draw the graph of a function that has the following properties: domain: all real numbers; range: all real numbers; intercepts: $(0, -3)$ and $(3, 0)$; a local maximum value of -2 is at -1; a local minimum value of -6 is at 2. Compare your graph with those of others. Comment on any differences.

96. Redo Problem 95 with the following additional information: increasing on $(-\infty, -1]$, $[2, \infty)$; decreasing on $[-1, 2]$. Again compare your graph with others and comment on any differences.

97. How many x-intercepts can a function defined on an interval have if it is increasing on that interval? Explain.

98. Suppose that a friend of yours does not understand the idea of increasing and decreasing functions. Provide an explanation, complete with graphs, that clarifies the idea.

99. Can a function be both even and odd? Explain.

100. Using a graphing utility, graph $y = 5$ on the interval $(-3, 3)$. Use MAXIMUM to find the local maximum values on $(-3, 3)$. Comment on the result provided by the calculator.

101. A function f has a positive average rate of change on the interval $[2, 5]$. Is f increasing on $[2, 5]$? Explain.

102. Show that a constant function $f(x) = b$ has an average rate of change of 0. Compute the average rate of change of $y = \sqrt{4 - x^2}$ on the interval $[-2, 2]$. Explain how this can happen.

Retain Your Knowledge

Problems 103–106 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

103. Write each number in scientific notation.
- (a) 0.00000701
- (b) 2,305,000,000

104. Simplify: $\sqrt{540}$

105. Solve: $14 < 5 - 3x \leq 29$

106. The shelf life of a perishable commodity varies inversely with the storage temperature. If the shelf life at 10°C is 33 days, what is the shelf life at 40°C?

'Are You Prepared?' Answers

1. $2 < x < 5$ **2.** 1 **3.** symmetric with respect to the y-axis **4.** $y + 2 = 5(x - 3)$ **5.** $(-3, 0), (3, 0), (0, -9)$

3.4 Library of Functions; Piecewise-defined Functions

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Intercepts (Section 2.1, pp. 165–166)
- Graphs of Key Equations (Section 2.1: Example 4, p. 168; Example 5, p. 169; Example 6, pp. 169–170)

Now Work the **'Are You Prepared?'** problems on page 252.

OBJECTIVES 1 Graph the Functions Listed in the Library of Functions (p. 245)
2 Graph Piecewise-defined Functions (p. 250)

1 Graph the Functions Listed in the Library of Functions

First we introduce a few more functions, beginning with the *square root function*.

On page 169, we graphed the equation $y = \sqrt{x}$. Figure 32 shows a graph of the function $f(x) = \sqrt{x}$. Based on the graph, we have the following properties:

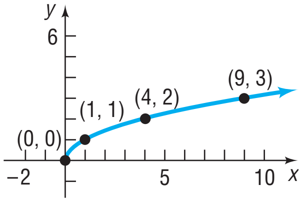


Figure 32 Square root function

Properties of $f(x) = \sqrt{x}$

1. The domain and the range are the set of nonnegative real numbers.
2. The x-intercept of the graph of $f(x) = \sqrt{x}$ is 0. The y-intercept of the graph of $f(x) = \sqrt{x}$ is also 0.
3. The function is neither even nor odd.
4. The function is increasing on the interval $[0, \infty)$.
5. The function has an absolute minimum of 0 at $x = 0$.

EXAMPLE 1 **Graphing the Cube Root Function**

(a) Determine whether $f(x) = \sqrt[3]{x}$ is even, odd, or neither. State whether the graph of f is symmetric with respect to the y-axis or symmetric with respect to the origin.

(b) Determine the intercepts, if any, of the graph of $f(x) = \sqrt[3]{x}$.

(c) Graph $f(x) = \sqrt[3]{x}$.

Solution (a) Because

$$f(-x) = \sqrt[3]{-x} = -\sqrt[3]{x} = -f(x)$$

the function is odd. The graph of f is symmetric with respect to the origin.

(b) The y-intercept is $f(0) = \sqrt[3]{0} = 0$. The x-intercept is found by solving the equation $f(x) = 0$.

$$f(x) = 0$$
$$\sqrt[3]{x} = 0 \quad f(x) = \sqrt[3]{x}$$
$$x = 0 \quad \text{Cube both sides of the equation.}$$

The x-intercept is also 0.

(c) Use the function to form Table 4 (on page 246) and obtain some points on the graph. Because of the symmetry with respect to the origin, we find only points (x, y) for which $x \geq 0$. Figure 33 shows the graph of $f(x) = \sqrt[3]{x}$.

Table 4

x	$y = f(x) = \sqrt[3]{x}$	(x, y)
0	0	$(0, 0)$
$\frac{1}{8}$	$\frac{1}{2}$	$\left(\frac{1}{8}, \frac{1}{2}\right)$
1	1	$(1, 1)$
2	$\sqrt[3]{2} \approx 1.26$	$(2, \sqrt[3]{2})$
8	2	$(8, 2)$

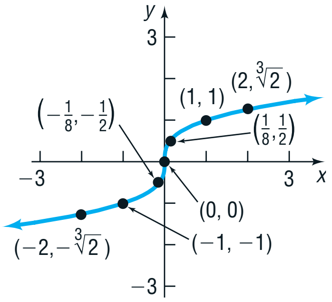


Figure 33 Cube root function

From the results of Example 1 and Figure 33, we have the following properties of the cube root function.

Properties of $f(x) = \sqrt[3]{x}$

1. The domain and the range are the set of all real numbers.
2. The x-intercept of the graph of $f(x) = \sqrt[3]{x}$ is 0. The y-intercept of the graph of $f(x) = \sqrt[3]{x}$ is also 0.
3. The function is odd. The graph is symmetric with respect to the origin.
4. The function is increasing on the interval $(-\infty, \infty)$.
5. The function does not have any local minima or any local maxima.

EXAMPLE 2 Graphing the Absolute Value Function

(a) Determine whether $f(x) = |x|$ is even, odd, or neither. State whether the graph of f is symmetric with respect to the y-axis, symmetric with respect to the origin, or neither.

(b) Determine the intercepts, if any, of the graph of $f(x) = |x|$.

(c) Graph $f(x) = |x|$.

Solution (a) Because

$$\begin{aligned} f(-x) &= |-x| \\ &= |x| = f(x) \end{aligned}$$

the function is even. The graph of f is symmetric with respect to the y-axis.

(b) The y-intercept is $f(0) = |0| = 0$. The x-intercept is found by solving the equation $f(x) = 0$, or $|x| = 0$. The x-intercept is 0.

(c) Use the function to form Table 5 and obtain some points on the graph. Because of the symmetry with respect to the y-axis, we only need to find points (x, y) for which $x \geq 0$. Figure 34 shows the graph of $f(x) = |x|$.

Table 5

x	$y = f(x) = \lvert x \rvert$	(x, y)
0	0	$(0, 0)$
1	1	$(1, 1)$
2	2	$(2, 2)$
3	3	$(3, 3)$

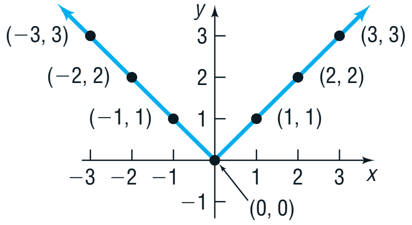


Figure 34 Absolute value function

From the results of Example 2 and Figure 34, we have the following properties of the absolute value function.

Properties of $f(x) = |x|$

1. The domain is the set of all real numbers. The range of f is $\{y|y \geq 0\}$.
2. The x-intercept of the graph of $f(x) = |x|$ is 0. The y-intercept of the graph of $f(x) = |x|$ is also 0.
3. The function is even. The graph is symmetric with respect to the y-axis.
4. The function is decreasing on the interval $(-\infty, 0]$. It is increasing on the interval $[0, \infty)$.
5. The function has an absolute minimum of 0 at $x = 0$.

Seeing the Concept

Graph $y = |x|$ on a square screen and compare what you see with Figure 34. Note that some graphing calculators use abs(x) for absolute value. ■

Below is a list of the key functions that we have discussed. In going through this list, pay special attention to the properties of each function, particularly to the shape of each graph. Knowing these graphs, along with key points on each graph, will lay the foundation for further graphing techniques.

Constant Function

$$f(x) = b \qquad b \text{ is a real number}$$

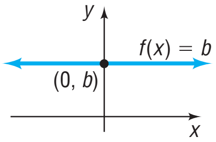


Figure 35 Constant function

See Figure 35.

The domain of a **constant function** is the set of all real numbers; its range is the set consisting of a single number b. Its graph is a horizontal line whose y-intercept is b. The constant function is an even function.

Identity Function

$$f(x) = x$$

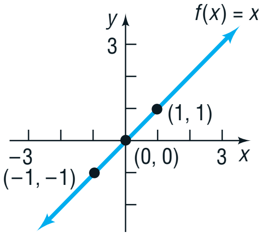


Figure 36 Identity function

See Figure 36.

The domain and the range of the **identity function** are the set of all real numbers. Its graph is a line whose slope is 1 and whose y-intercept is 0. The line consists of all points for which the x-coordinate equals the y-coordinate. The identity function is an odd function that is increasing over its domain. Note that the graph bisects quadrants I and III.

Square Function

$$f(x) = x^2$$

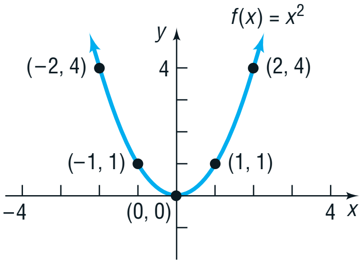


Figure 37 Square function

See Figure 37.

The domain of the **square function** is the set of all real numbers; its range is the set of nonnegative real numbers. The graph of this function is a parabola whose intercept is at $(0, 0)$. The square function is an even function that is decreasing on the interval $(-\infty, 0]$ and increasing on the interval $[0, \infty)$.

Cube Function

$$f(x) = x^3$$

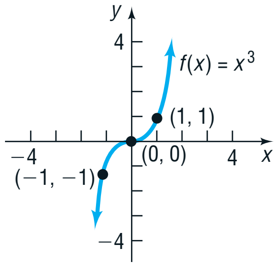


Figure 38 Cube function

See Figure 38.

The domain and the range of the **cube function** are the set of all real numbers. The intercept of the graph is at $(0, 0)$. The cube function is odd and is increasing on the interval $(-\infty, \infty)$.

Square Root Function

$$f(x) = \sqrt{x}$$

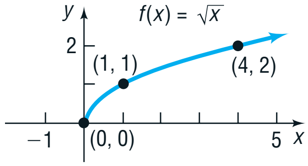


Figure 39 Square root function

See Figure 39.

The domain and the range of the **square root function** are the set of nonnegative real numbers. The intercept of the graph is at $(0, 0)$. The square root function is neither even nor odd and is increasing on the interval $[0, \infty)$.

Cube Root Function

$$f(x) = \sqrt[3]{x}$$

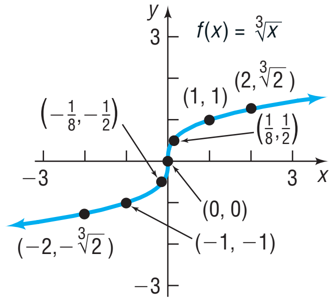


Figure 40 Cube root function

See Figure 40.

The domain and the range of the **cube root function** are the set of all real numbers. The intercept of the graph is at $(0, 0)$. The cube root function is an odd function that is increasing on the interval $(-\infty, \infty)$.

Reciprocal Function

$$f(x) = \frac{1}{x}$$

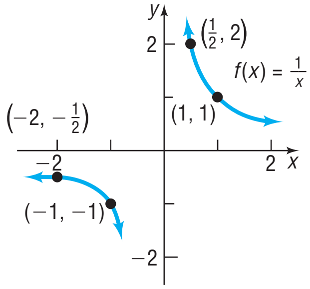


Figure 41 Reciprocal function

Refer to Example 6, page 169, for a discussion of the equation $y = \frac{1}{x}$. See Figure 41.

The domain and the range of the **reciprocal function** are the set of all nonzero real numbers. The graph has no intercepts. The reciprocal function is decreasing on the intervals $(-\infty, 0)$ and $(0, \infty)$ and is an odd function.

Absolute Value Function

$$f(x) = |x|$$

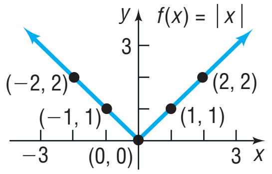


Figure 42 Absolute value function

See Figure 42.

The domain of the **absolute value function** is the set of all real numbers; its range is the set of nonnegative real numbers. The intercept of the graph is at $(0, 0)$. If $x \geq 0$, then $f(x) = x$, and the graph of f is part of the line $y = x$; if $x < 0$, then $f(x) = -x$, and the graph of f is part of the line $y = -x$. The absolute value function is an even function; it is decreasing on the interval $(-\infty, 0]$ and increasing on the interval $[0, \infty)$.

The notation $\text{int}(x)$ stands for the largest integer less than or equal to x. For example,

$$\text{int}(1) = 1, \quad \text{int}(2.5) = 2, \quad \text{int}\left(\frac{1}{2}\right) = 0, \quad \text{int}\left(-\frac{3}{4}\right) = -1, \quad \text{int}(\pi) = 3$$

This type of correspondence occurs frequently enough in mathematics that we give it a name.

Table 6

x	$y = f(x)$ $= \text{int}(x)$	(x, y)
-1	-1	$(-1, -1)$
$-\frac{1}{2}$	-1	$\left(-\frac{1}{2}, -1\right)$
$-\frac{1}{4}$	-1	$\left(-\frac{1}{4}, -1\right)$
0	0	$(0, 0)$
$\frac{1}{4}$	0	$\left(\frac{1}{4}, 0\right)$
$\frac{1}{2}$	0	$\left(\frac{1}{2}, 0\right)$
$\frac{3}{4}$	0	$\left(\frac{3}{4}, 0\right)$

DEFINITION Greatest Integer Function

$$f(x) = \text{int}(x)^* = \text{ greatest integer less than or equal to } x$$

We obtain the graph of $f(x) = \text{int}(x)$ by plotting several points. See Table 6. For values of x, $-1 \le x < 0$, the value of $f(x) = \text{int}(x)$ is -1; for values of x, $0 \le x < 1$, the value of f is 0. See Figure 43 for the graph.

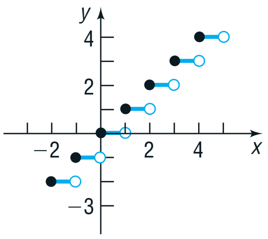


Figure 43 Greatest integer function

The domain of the **greatest integer function** is the set of all real numbers; its range is the set of integers. The y-intercept of the graph is 0. The x-intercepts lie in the interval $[0, 1)$. The greatest integer function is neither even nor odd. It is constant on every interval of the form $[k, k + 1)$, for k an integer. In Figure 43, a solid dot is used to indicate, for example, that at $x = 1$ the value of f is $f(1) = 1$; an open circle is used to illustrate that the function does not assume the value of 0 at $x = 1$.

Although a precise definition requires the idea of a limit (discussed in calculus), in a rough sense, a function is said to be *continuous* if its graph has no gaps or holes and can be drawn without lifting a pencil from the paper on which the graph is drawn. We contrast this with a *discontinuous* function. A function is discontinuous if its graph has gaps or holes and so cannot be drawn without lifting a pencil from the paper.

From the graph of the greatest integer function, we can see why it is also called a **step function**. At $x = 0$, $x = \pm 1$, $x = \pm 2$, and so on, this function is discontinuous because, at integer values, the graph suddenly "steps" from one value to another without taking on any of the intermediate values. For example, to the immediate left of $x = 3$, the y-coordinates of the points on the graph are 2, and at $x = 3$ and to the immediate right of $x = 3$, the y-coordinates of the points on the graph are 3. Consequently, the graph has gaps in it.

COMMENT When graphing a function using a graphing utility, typically you can choose either **connected mode**, in which points plotted on the screen are connected, making the graph appear without any breaks, or **dot mode**, in which only the points plotted appear. When graphing the greatest integer function with a graphing utility, it may be necessary to be in **dot mode**. This is to prevent the utility from "connecting the dots" when $f(x)$ changes from one integer value to the next. However, some utilities will display the gaps even when in "connected" mode. See Figure 44.

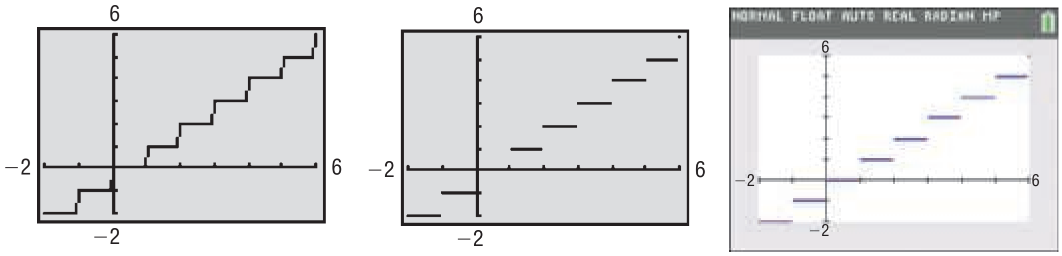


Figure 44 $f(x) = \text{int}(x)$

(a) TI-83 Plus, connected mode

(b) TI-83 Plus, dot mode

(c) TI-84 Plus C

---

* Some texts use the notation $f(x) = [x]$ instead of $\text{int}(x)$.

The functions discussed so far are basic. Whenever you encounter one of them, you should see a mental picture of its graph. For example, if you encounter the function $f(x) = x^2$, you should see in your mind's eye a picture like Figure 37.

Now Work PROBLEMS 11 THROUGH 18

2 Graph Piecewise-defined Functions

Sometimes a function is defined using different equations on different parts of its domain. For example, the absolute value function $f(x) = |x|$ is actually defined by two equations: $f(x) = x$ if $x \geq 0$ and $f(x) = -x$ if $x < 0$. For convenience, these equations are generally combined into one expression as

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

When a function is defined by different equations on different parts of its domain, it is called a **piecewise-defined** function.

EXAMPLE 3 Graphing a Piecewise-defined Function

The function f is defined as

$$f(x) = \begin{cases} 2x + 3 & \text{if } x < 0 \\ -\dfrac{1}{2}x + 3 & \text{if } x \geq 0 \end{cases}$$

(a) Find $f(-2)$, $f(0)$, and $f(2)$.
(b) Graph f.

Solution (a) To find $f(-2)$, observe that when $x = -2$, the equation for f is given by $f(x) = 2x + 3$, so

$$f(-2) = 2(-2) + 3 = -1$$

When $x = 0$, the equation for f is $f(x) = -\dfrac{1}{2}x + 3$, so

$$f(0) = -\frac{1}{2}(0) + 3 = 3$$

When $x = 2$, the equation for f is $f(x) = -\dfrac{1}{2}x + 3$, so

$$f(2) = -\frac{1}{2}(2) + 3 = 2$$

(b) To graph f, graph each "piece." First graph the line $y = 2x + 3$ and keep only the part for which $x < 0$. Next, graph the line $y = -\dfrac{1}{2}x + 3$ and keep only the part for which $x \geq 0$. See Figure 45. ■

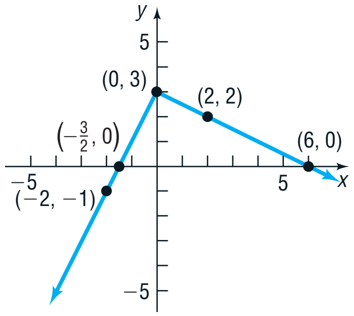


Figure 45

EXAMPLE 4 Analyzing a Piecewise-defined Function

The function f is defined as

$$f(x) = \begin{cases} -2x + 1 & \text{if } -3 \leq x < 1 \\ 2 & \text{if } x = 1 \\ x^2 & \text{if } x > 1 \end{cases}$$

(a) Determine the domain of f.
(b) Locate any intercepts.
(c) Graph f.
(d) Use the graph to find the range of f.

Solution (a) To find the domain of f, look at its definition. Since f is defined for all x greater than or equal to -3, the domain of f is $\{x | x \ge -3\}$, or the interval $[-3, \infty)$.

(b) The y-intercept of the graph of the function is $f(0)$. Because the equation for f when $x = 0$ is $f(x) = -2x + 1$, the y-intercept is $f(0) = -2(0) + 1 = 1$. The x-intercepts of the graph of a function f are the real solutions to the equation $f(x) = 0$. To find the x-intercepts of f, solve $f(x) = 0$ for each "piece" of the function, and then determine what values of x, if any, satisfy the condition that defines the piece.

$$\begin{array}{rlcrlcrl}
f(x) &= 0 & & f(x) &= 0 & & f(x) &= 0 \\
-2x + 1 &= 0 \quad -3 \le x < 1 & & 2 &= 0 \quad x = 1 & & x^2 &= 0 \quad x > 1 \\
-2x &= -1 & & \text{No solution} & & & x &= 0 \\
x &= \frac{1}{2} & & & & & &
\end{array}$$

The first potential x-intercept, $x = \frac{1}{2}$, satisfies the condition $-3 \le x < 1$, so $x = \frac{1}{2}$ is an x-intercept. The second potential x-intercept, $x = 0$, does not satisfy the condition $x > 1$, so $x = 0$ is not an x-intercept. The only x-intercept is $\frac{1}{2}$. The intercepts are $(0, 1)$ and $\left(\frac{1}{2}, 0\right)$.

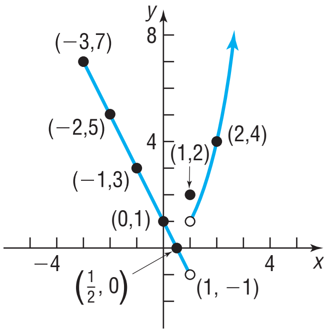


Figure 46

(c) To graph f, graph each "piece." First graph the line $y = -2x + 1$ and keep only the part for which $-3 \le x < 1$. Then plot the point $(1, 2)$ because, when $x = 1$, $f(x) = 2$. Finally, graph the parabola $y = x^2$ and keep only the part for which $x > 1$. See Figure 46.

(d) From the graph, we conclude that the range of f is $\{y | y > -1\}$, or the interval $(-1, \infty)$. ■

Now Work PROBLEM 31

EXAMPLE 5 Cost of Electricity

In the spring of 2015, Duke Energy Progress supplied electricity to residences in South Carolina for a monthly customer charge of \$6.50 plus 9.997¢ per kilowatt-hour (kWh) for the first 800 kWh supplied in the month and 8.997¢ per kWh for all usage over 800 kWh in the month.

(a) What is the charge for using 300 kWh in a month?

(b) What is the charge for using 1500 kWh in a month?

(c) If C is the monthly charge for x kWh, develop a model relating the monthly charge and kilowatt-hours used. That is, express C as a function of x.

Source: *Duke Energy Progress, 2015*

Solution (a) For 300 kWh, the charge is \$6.50 plus (9.997¢ = \$0.09997) per kWh. That is,

$$\text{Charge} = \$6.50 + \$0.09997(300) = \$36.49$$

(b) For 1500 kWh, the charge is \$6.50 plus 9.997¢ per kWh for the first 800 kWh plus 8.997¢ per kWh for the 700 in excess of 800. That is,

$$\text{Charge} = \$6.50 + \$0.09997(800) + \$0.08997(700) = \$149.46$$

(c) Let x represent the number of kilowatt-hours used. If $0 \le x \le 800$, then the monthly charge C (in dollars) can be found by multiplying x times \$0.09997 and adding the monthly customer charge of \$6.50. So if $0 \le x \le 800$, then

$$C(x) = 0.09997x + 6.50$$

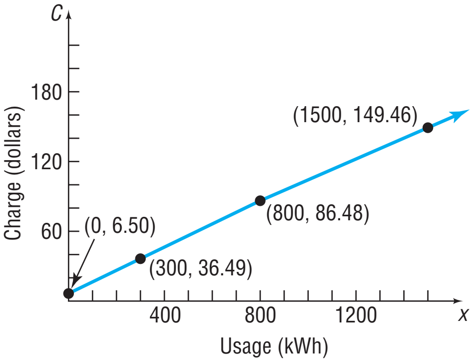


Figure 47

For $x > 800$, the charge is $0.09997(800) + 6.50 + 0.08997(x - 800)$, since $(x - 800)$ equals the usage in excess of 800 kWh, which costs \$0.08997 per kWh. That is, if $x > 800$, then

$$\begin{aligned} C(x) &= 0.09997(800) + 6.50 + 0.08997(x - 800) \\ &= 79.976 + 6.50 + 0.08997x - 71.976 \\ &= 0.08997x + 14.50 \end{aligned}$$

The rule for computing C follows two equations:

$$C(x) = \begin{cases} 0.09997x + 6.50 & \text{if } 0 \le x \le 800 \\ 0.08997x + 14.50 & \text{if } x > 800 \end{cases}$$

The Model

See Figure 47 for the graph. Note that the two "pieces" are linear, but they have different slopes (rates), and meet at the point $(800, 86.48)$. ■

3.4 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Sketch the graph of $y = \sqrt{x}$. (p. 169)

2. Sketch the graph of $y = \dfrac{1}{x}$. (p. 169)

3. List the intercepts of the equation $y = x^3 - 8$. (pp. 165–166)

Concepts and Vocabulary

4. The function $f(x) = x^2$ is decreasing on the interval ________.

5. When functions are defined by more than one equation, they are called ______________ functions.

6. ***True or False*** The cube function is odd and is increasing on the interval $(-\infty, \infty)$.

7. ***True or False*** The cube root function is odd and is decreasing on the interval $(-\infty, \infty)$.

8. ***True or False*** The domain and the range of the reciprocal function are the set of all real numbers.

9. Which of the following functions has a graph that is symmetric about the y-axis?
(a) $y = \sqrt{x}$ (b) $y = |x|$ (c) $y = x^3$ (d) $y = \dfrac{1}{x}$

10. Consider the following function.

$$f(x) = \begin{cases} 3x - 2 & \text{if} \quad x < 2 \\ x^2 + 5 & \text{if} \quad 2 \le x < 10 \\ 3 & \text{if} \quad x \ge 10 \end{cases}$$

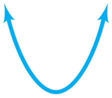
Which "piece(s)" should be used to find the y-intercept?
(a) $3x - 2$ (b) $x^2 + 5$ (c) 3 (d) all three

Skill Building

In Problems 11–18, match each graph to its function.

A. Constant function
B. Identity function
C. Square function
D. Cube function
E. Square root function
F. Reciprocal function
G. Absolute value function
H. Cube root function

11.

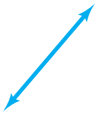


12.

13.

14.

15.



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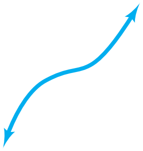
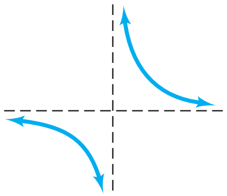


17.



18.

In Problems 19–26, sketch the graph of each function. Be sure to label three points on the graph.

19. $f(x) = x$

20. $f(x) = x^2$

21. $f(x) = x^3$

22. $f(x) = \sqrt{x}$

23. $f(x) = \dfrac{1}{x}$

24. $f(x) = |x|$

25. $f(x) = \sqrt[3]{x}$

26. $f(x) = 3$

27. If $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 2 & \text{if } x = 0 \\ 2x + 1 & \text{if } x > 0 \end{cases}$

find: (a) $f(-2)$ (b) $f(0)$ (c) $f(2)$

28. If $f(x) = \begin{cases} -3x & \text{if } x < -1 \\ 0 & \text{if } x = -1 \\ 2x^2 + 1 & \text{if } x > -1 \end{cases}$

find: (a) $f(-2)$ (b) $f(-1)$ (c) $f(0)$

29. If $f(x) = \begin{cases} 2x - 4 & \text{if } -1 \le x \le 2 \\ x^3 - 2 & \text{if } 2 < x \le 3 \end{cases}$

find: (a) $f(0)$ (b) $f(1)$ (c) $f(2)$ (d) $f(3)$

30. If $f(x) = \begin{cases} x^3 & \text{if } -2 \le x < 1 \\ 3x + 2 & \text{if } 1 \le x \le 4 \end{cases}$

find: (a) $f(-1)$ (b) $f(0)$ (c) $f(1)$ (d) $f(3)$

In Problems 31–42:

(a) Find the domain of each function. *(b) Locate any intercepts.*

(c) Graph each function. *(d) Based on the graph, find the range.*

31. $f(x) = \begin{cases} 2x & \text{if } x \ne 0 \\ 1 & \text{if } x = 0 \end{cases}$

32. $f(x) = \begin{cases} 3x & \text{if } x \ne 0 \\ 4 & \text{if } x = 0 \end{cases}$

33. $f(x) = \begin{cases} -2x + 3 & \text{if } x < 1 \\ 3x - 2 & \text{if } x \ge 1 \end{cases}$

34. $f(x) = \begin{cases} x + 3 & \text{if } x < -2 \\ -2x - 3 & \text{if } x \ge -2 \end{cases}$

35. $f(x) = \begin{cases} x + 3 & \text{if } -2 \le x < 1 \\ 5 & \text{if } x = 1 \\ -x + 2 & \text{if } x > 1 \end{cases}$

36. $f(x) = \begin{cases} 2x + 5 & \text{if } -3 \le x < 0 \\ -3 & \text{if } x = 0 \\ -5x & \text{if } x > 0 \end{cases}$

37. $f(x) = \begin{cases} 1 + x & \text{if } x < 0 \\ x^2 & \text{if } x \ge 0 \end{cases}$

38. $f(x) = \begin{cases} \dfrac{1}{x} & \text{if } x < 0 \\ \sqrt[3]{x} & \text{if } x \ge 0 \end{cases}$

39. $f(x) = \begin{cases} |x| & \text{if } -2 \le x < 0 \\ x^3 & \text{if } x > 0 \end{cases}$

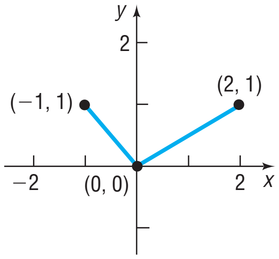
40. $f(x) = \begin{cases} 2 - x & \text{if } -3 \le x < 1 \\ \sqrt{x} & \text{if } x > 1 \end{cases}$

41. $f(x) = 2\,\text{int}(x)$

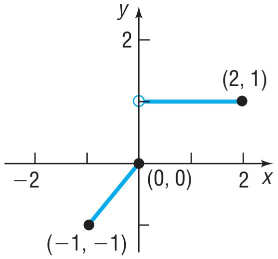
42. $f(x) = \text{int}(2x)$

In Problems 43–46, the graph of a piecewise-defined function is given. Write a definition for each function.

43.



44.



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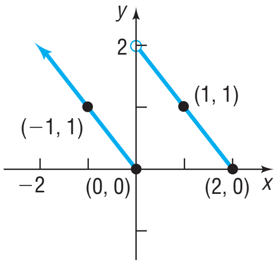
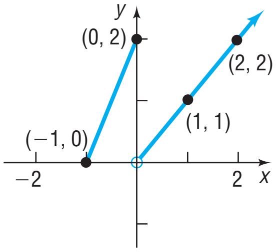


46.



47. If $f(x) = \text{int}(2x)$, find

(a) $f(1.2)$ (b) $f(1.6)$ (c) $f(-1.8)$

48. If $f(x) = \text{int}\left(\dfrac{x}{2}\right)$, find

(a) $f(1.2)$ (b) $f(1.6)$ (c) $f(-1.8)$

49. (a) Graph $f(x) = \begin{cases} (x - 1)^2 & \text{if } 0 \le x < 2 \\ -2x + 10 & \text{if } 2 \le x \le 6 \end{cases}$

(b) Find the domain of *f*.

(c) Find the absolute maximum and the absolute minimum, if they exist.

(d) Find the local maximum and the local minimum values, if they exist.

50. (a) Graph $f(x) = \begin{cases} -x + 1 & \text{if } -2 \le x < 0 \\ 2 & \text{if } \quad x = 0 \\ x + 1 & \text{if } 0 < x \le 2 \end{cases}$

(b) Find the domain of *f*.

(c) Find the absolute maximum and the absolute minimum, if they exist.

(d) Find the local maximum and the local minimum values, if they exist.

Applications and Extensions

51. Tablet Service A monthly tablet plan costs \$34.99. It includes 3 gigabytes of data and charges \$15 per gigabyte for additional gigabytes. The following function is used to compute the monthly cost for a subscriber.

$$C(x) = \begin{cases} 34.99 & \text{if } 0 \le x \le 3 \\ 15x - 10.01 & \text{if } x > 3 \end{cases}$$

Compute the monthly cost for each of the following gigabytes of use.

(a) 2 (b) 5 (c) 13

52. Parking at O'Hare International Airport The short-term (no more than 24 hours) parking fee *F* (in dollars) for parking *x* hours on a weekday at O'Hare International Airport's main parking garage can be modeled by the function

$$F(x) = \begin{cases} 4 & \text{if } \ 0 < x \le 3 \\ 2\,\text{int}(x + 1) + 3 & \text{if } \ 3 < x \le 11 \\ 35 & \text{if } 11 < x \le 24 \end{cases}$$

Determine the fee for parking in the short-term parking garage for

(a) 2 hours (b) 7 hours

(c) 15 hours (d) 8 hours and 24 minutes

Source: *O'Hare International Airport*

53. Cost of Natural Gas In March 2015, Laclede Gas had the following rate schedule for natural gas usage in single-family residences.

Monthly service charge	\$19.50
Delivery charge	
First 30 therms	\$0.91686/therm
Over 30 therms	\$0
Natural gas cost	
First 30 therms	\$0.348/therm
Over 30 therms	\$0.5922/therm

(a) What is the charge for using 20 therms in a month?
(b) What is the charge for using 150 therms in a month?
(c) Develop a function that models the monthly charge C for x therms of gas.
(d) Graph the function found in part (c).

Source: *Laclede Gas*

54. Cost of Natural Gas In April 2015, Nicor Gas had the following rate schedule for natural gas usage in small businesses.

Monthly customer charge	\$72.60
Distribution charge	
1st 150 therms	\$0.1201/therm
Next 4850 therms	\$0.0549/therm
Over 5000 therms	\$0.0482/therm
Gas supply charge	\$0.35/therm

(a) What is the charge for using 1000 therms in a month?
(b) What is the charge for using 6000 therms in a month?
(c) Develop a function that models the monthly charge C for x therms of gas.
(d) Graph the function found in part (c).

Source: *Nicor Gas, 2015*

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55. Federal Income Tax Two 2015 Tax Rate Schedules are given in the accompanying table. If x equals taxable income and y equals the tax due, construct a function $y = f(x)$ for Schedule X.

2015 Tax Rate Schedules											
Schedule X—Single						Schedule Y-1—Married Filing Jointly or Qualified Widow(er)					
If Taxable Income is Over	But Not Over	The Tax is This Amount		Plus This %	Of the Excess Over	If Taxable Income is Over	But Not Over	The Tax is This Amount		Plus This %	Of the Excess Over
\$0	\$9,225	\$0	+	10%	\$0	\$0	\$18,450	\$0	+	10%	\$0
9,225	37,450	922.50	+	15%	9,225	18,450	74,900	1,845	+	15%	18,450
37,450	90,750	5,156.25	+	25%	37,450	74,900	151,200	10,312.50	+	25%	74,900
90,750	189,300	18,481.25	+	28%	90,750	151,200	230,450	29,387.50	+	28%	151,200
189,300	411,500	46,075.25	+	33%	189,300	230,450	411,500	51,577.50	+	33%	230,450
411,500	413,200	119,401.25	+	35%	411,500	411,500	464,850	111,324.00	+	35%	411,500
413,200	–	119,996.25	+	39.6%	413,200	464,850	–	129,996.50	+	39.6%	464,850

56. Federal Income Tax Refer to the 2015 tax rate schedules. If x equals taxable income and y equals the tax due, construct a function $y = f(x)$ for Schedule Y-1.

57. Cost of Transporting Goods A trucking company transports goods between Chicago and New York, a distance of 960 miles. The company's policy is to charge, for each pound, \$0.50 per mile for the first 100 miles, \$0.40 per mile for the next 300 miles, \$0.25 per mile for the next 400 miles, and no charge for the remaining 160 miles.

(a) Graph the relationship between the cost of transportation in dollars and mileage over the entire 960-mile route.
(b) Find the cost as a function of mileage for hauls between 100 and 400 miles from Chicago.
(c) Find the cost as a function of mileage for hauls between 400 and 800 miles from Chicago.

58. Car Rental Costs An economy car rented in Florida from Enterprise® on a weekly basis costs \$185 per week. Extra days cost \$37 per day until the day rate exceeds the weekly rate, in which case the weekly rate applies. Also, any part of a day used counts as a full day. Find the cost C of renting an economy car as a function of the number x of days used, where $7 \le x \le 14$. Graph this function.

59. Mortgage Fees Fannie Mae charges a loan-level price adjustment (LLPA) on all mortgages, which represents a fee homebuyers seeking a loan must pay. The rate paid depends on the credit score of the borrower, the amount borrowed, and the loan-to-value (LTV) ratio. The LTV ratio is the ratio of amount borrowed to appraised value of the home. For example, a homebuyer who wishes to borrow \$250,000 with a credit score of 730 and an LTV ratio of 80% will pay 0.5% (0.005) of \$250,000, or \$1250. The table shows the LLPA for various credit scores and an LTV ratio of 80%.

Credit Score	Loan-Level Price Adjustment Rate
≤659	3.00%
660–679	2.50%
680–699	1.75%
700–719	1%
720–739	0.5%
≥740	0.25%

Source: *Fannie Mae*

(a) Construct a function $C = C(s)$, where C is the loan-level price adjustment (LLPA) and s is the credit score of an individual who wishes to borrow \$300,000 with an 80% LTV ratio.

(b) What is the LLPA on a \$300,000 loan with an 80% LTV ratio for a borrower whose credit score is 725?

(c) What is the LLPA on a \$300,000 loan with an 80% LTV ratio for a borrower whose credit score is 670?

60. Minimum Payments for Credit Cards Holders of credit cards issued by banks, department stores, oil companies, and so on receive bills each month that state minimum amounts that must be paid by a certain due date. The minimum due depends on the total amount owed. One such credit card company uses the following rules: For a bill of less than \$10, the entire amount is due. For a bill of at least \$10 but less than \$500, the minimum due is \$10. A minimum of \$30 is due on a bill of at least \$500 but less than \$1000, a minimum of \$50 is due on a bill of at least \$1000 but less than \$1500, and a minimum of \$70 is due on bills of \$1500 or more. Find the function f that describes the minimum payment due on a bill of x dollars. Graph f.

61. Wind Chill The wind chill factor represents the air temperature at a standard wind speed that would produce the same heat loss as the given temperature and wind speed. One formula for computing the equivalent temperature is

$$W = \begin{cases} t & 0 \le v < 1.79 \\ 33 - \dfrac{(10.45 + 10\sqrt{v} - v)(33 - t)}{22.04} & 1.79 \le v \le 20 \\ 33 - 1.5958(33 - t) & v > 20 \end{cases}$$

where v represents the wind speed (in meters per second) and t represents the air temperature (°C). Compute the wind chill for the following:

(a) An air temperature of 10°C and a wind speed of 1 meter per second (m/sec)

(b) An air temperature of 10°C and a wind speed of 5 m/sec

(c) An air temperature of 10°C and a wind speed of 15 m/sec

(d) An air temperature of 10°C and a wind speed of 25 m/sec

(e) Explain the physical meaning of the equation corresponding to $0 \le v < 1.79$.

(f) Explain the physical meaning of the equation corresponding to $v > 20$.

62. Wind Chill Redo Problem 61(a)–(d) for an air temperature of −10°C.

63. First-class Mail In 2015 the U.S. Postal Service charged \$0.98 postage for first-class mail retail flats (such as an 8.5" by 11" envelope) weighing up to 1 ounce, plus \$0.22 for each additional ounce up to 13 ounces. First-class rates do not apply to flats weighing more than 13 ounces. Develop a model that relates C, the first-class postage charged, for a flat weighing x ounces. Graph the function.

Source: *United States Postal Service*

Explaining Concepts: Discussion and Writing

In Problems 64–71, use a graphing utility.

64. Exploration Graph $y = x^2$. Then on the same screen graph $y = x^2 + 2$, followed by $y = x^2 + 4$, followed by $y = x^2 - 2$. What pattern do you observe? Can you predict the graph of $y = x^2 - 4$? Of $y = x^2 + 5$?

65. Exploration Graph $y = x^2$. Then on the same screen graph $y = (x - 2)^2$, followed by $y = (x - 4)^2$, followed by $y = (x + 2)^2$. What pattern do you observe? Can you predict the graph of $y = (x + 4)^2$? Of $y = (x - 5)^2$?

66. Exploration Graph $y = |x|$. Then on the same screen graph $y = 2|x|$, followed by $y = 4|x|$, followed by $y = \frac{1}{2}|x|$. What pattern do you observe? Can you predict the graph of $y = \frac{1}{4}|x|$? Of $y = 5|x|$?

67. Exploration Graph $y = x^2$. Then on the same screen graph $y = -x^2$. Now try $y = |x|$ and $y = -|x|$. What do you conclude?

68. Exploration Graph $y = \sqrt{x}$. Then on the same screen graph $y = \sqrt{-x}$. Now try $y = 2x + 1$ and $y = 2(-x) + 1$. What do you conclude?

69. Exploration Graph $y = x^3$. Then on the same screen graph $y = (x - 1)^3 + 2$. Could you have predicted the result?

70. Exploration Graph $y = x^2$, $y = x^4$, and $y = x^6$ on the same screen. What do you notice is the same about each graph? What do you notice is different?

71. Exploration Graph $y = x^3$, $y = x^5$, and $y = x^7$ on the same screen. What do you notice is the same about each graph? What do you notice is different?

72. Consider the equation

$$y = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Is this a function? What is its domain? What is its range? What is its y-intercept, if any? What are its x-intercepts, if any? Is it even, odd, or neither? How would you describe its graph?

73. Define some functions that pass through $(0, 0)$ and $(1, 1)$ and are increasing for $x \ge 0$. Begin your list with $y = \sqrt{x}$, $y = x$, and $y = x^2$. Can you propose a general result about such functions?

Retain Your Knowledge

Problems 74–77 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

74. Simplify: $(3 + 2i)(4 - 5i)$

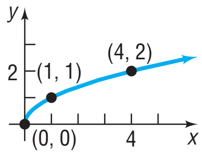
75. Find the center and radius of the circle $x^2 + y^2 = 6y + 16$.

76. Solve: $4x - 5(2x - 1) = 4 - 7(x + 1)$

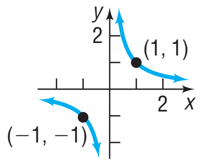
77. Ethan has \$60,000 to invest. He puts part of the money in a CD that earns 3% simple interest per year and the rest in a mutual fund that earns 8% simple interest per year. How much did he invest in each if his earned interest the first year was \$3700?

‘Are You Prepared?’ Answers

1.



2.



3. $(0, -8), (2, 0)$

3.5 Graphing Techniques: Transformations

OBJECTIVES
1 Graph Functions Using Vertical and Horizontal Shifts (p. 256)
2 Graph Functions Using Compressions and Stretches (p. 258)
3 Graph Functions Using Reflections about the *x*-Axis or *y*-Axis (p. 260)

At this stage, if you were asked to graph any of the functions defined by $y = x, y = x^2, y = x^3, y = \sqrt{x}, y = \sqrt[3]{x}, y = |x|$, or $y = \frac{1}{x}$, your response should be, “Yes, I recognize these functions and know the general shapes of their graphs.” (If this is not your answer, review the previous section, Figures 36 through 42.)

Sometimes we are asked to graph a function that is “almost” like one that we already know how to graph. In this section, we develop techniques for graphing such functions. Collectively, these techniques are referred to as **transformations**. We introduce the method of transformations because it is a more efficient method of graphing than point-plotting.

1 Graph Functions Using Vertical and Horizontal Shifts

Exploration On the same screen, graph each of the following functions:

$$Y_1 = x^2,\ Y_2 = x^2 + 2,\ Y_3 = x^2 - 2$$

What do you observe? Now create a table of values for Y_1, Y_2, and Y_3. What do you observe?

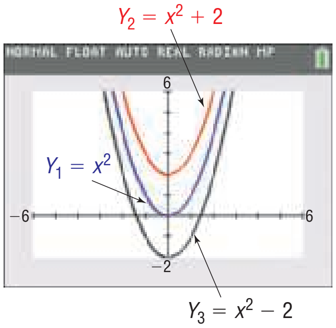
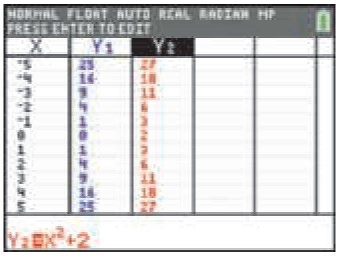


Figure 48 Vertical shift

Result Figure 48 illustrates the graphs. You should have observed a general pattern. With $Y_1 = x^2$ on the screen, the graph of $Y_2 = x^2 + 2$ is identical to that of $Y_1 = x^2$, except that it is shifted vertically up 2 units. The graph of $Y_3 = x^2 - 2$ is identical to that of $Y_1 = x^2$, except that it is shifted vertically down 2 units. From Table 7(a), we see that the *y*-coordinates on $Y_2 = x^2 + 2$ are 2 units larger than the *y*-coordinates on $Y_1 = x^2$ for any given *x*-coordinate. From Table 7(b), we see that the *y*-coordinates on $Y_3 = x^2 - 2$ are 2 units smaller than the *y*-coordinates on $Y_1 = x^2$ for any given *x*-coordinate.

Notice a vertical shift only affects the range of a function, not the domain. For example, the range of Y_1 is $[0, \infty)$ while the range of Y_2 is $[2, \infty)$. The domain of both functions is all real numbers.

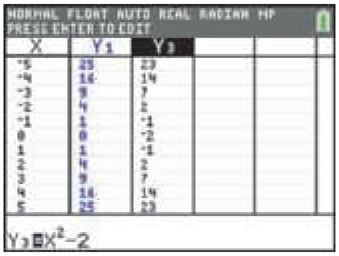
Table 7



X	Y1	Y2
-5	25	27
-4	16	18
-3	9	11
-2	4	6
-1	1	3
0	0	2
1	1	3
2	4	6
3	9	11
4	16	18
5	25	27

Y2=X²+2

(a)



X	Y1	Y3
-5	25	23
-4	16	14
-3	9	7
-2	4	2
-1	1	-1
0	0	-2
1	1	-1
2	4	2
3	9	7
4	16	14
5	25	23

Y3=X²-2

(b)

■

We are led to the following conclusions:

If a positive real number k is added to the outputs of a function $y = f(x)$, the graph of the new function $y = f(x) + k$ is the graph of f **shifted vertically up** k units.

If a positive real number k is subtracted from the outputs of a function $y = f(x)$, the graph of the new function $y = f(x) - k$ is the graph of f **shifted vertically down** k units.

In Words

For $y = f(x) + k, k > 0$, add k to each *y*-coordinate on the graph of $y = f(x)$ to shift the graph up k units.

For $y = f(x) - k, k > 0$, subtract k from each *y*-coordinate to shift the graph down k units.

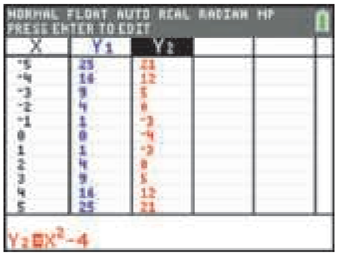
EXAMPLE 1 Vertical Shift Down

Use the graph of $f(x) = x^2$ to obtain the graph of $h(x) = x^2 - 4$. Find the domain and range of h.

Solution Table 8 lists some points on the graphs of $Y_1 = f(x) = x^2$ and $Y_2 = h(x) = f(x) - 4 = x^2 - 4$. Notice that each y-coordinate of h is 4 units less than the corresponding y-coordinate of f.

To obtain the graph of h from the graph of f, subtract 4 from each y-coordinate on the graph of f. The graph of h is identical to that of f, except that it is shifted down 4 units. See Figure 49.

Table 8



X	Y_1	Y_2
-5	25	21
-4	16	12
-3	9	5
-2	4	0
-1	1	-3
0	0	-4
1	1	-3
2	4	0
3	9	5
4	16	12
5	25	21

Y_2■X^2-4

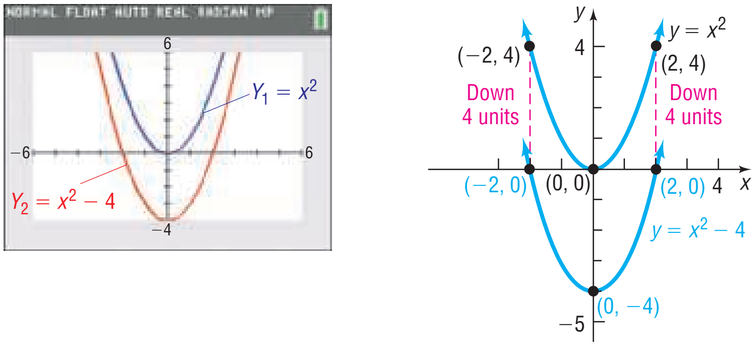


Figure 49

The domain of h is the set of all real numbers. The range of h is $[-4, \infty)$. ■

Now Work PROBLEM 39

Exploration

On the same screen, graph each of the following functions:

$$Y_1 = x^2$$
$$Y_2 = (x - 3)^2$$
$$Y_3 = (x + 2)^2$$

What do you observe?

Result Figure 50 illustrates the graphs. You should have observed the following pattern. With the graph of $Y_1 = x^2$ on the screen, the graph of $Y_2 = (x - 3)^2$ is identical to that of $Y_1 = x^2$, except it is shifted horizontally to the right 3 units. The graph of $Y_3 = (x + 2)^2$ is identical to that of $Y_1 = x^2$, except it is shifted horizontally to the left 2 units.

From Table 9(a), we see the x-coordinates on $Y_2 = (x - 3)^2$ are 3 units larger than they are for $Y_1 = x^2$ for any given y-coordinate. For example, when $Y_1 = 0$, then $x = 0$, and when $Y_2 = 0$, then $x = 3$. Also, when $Y_1 = 1$, then $x = -1$ or 1, and when $Y_2 = 1$, then $x = 2$ or 4. From Table 9(b), we see the x-coordinates on $Y_3 = (x + 2)^2$ are 2 units smaller than they are for $Y_1 = x^2$ for any given y-coordinate. For example, when $Y_1 = 0$, then $x = 0$, and when $Y_3 = 0$, then $x = -2$. Also, when $Y_1 = 4$, then $x = -2$ or 2, and when $Y_3 = 4$, then $x = -4$ or 0.

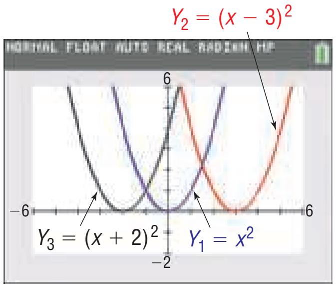
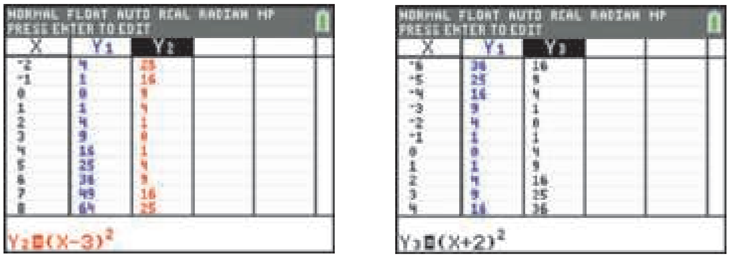


Figure 50 Horizontal shift

Table 9



X	Y_1	Y_2
-2	4	25
-1	1	16
0	0	9
1	1	4
2	4	1
3	9	0
4	16	1
5	25	4
6	36	9
7	49	16
8	64	25

Y_2■$(X-3)^2$

(a)

X	Y_1	Y_3
-6	36	16
-5	25	9
-4	16	4
-3	9	1
-2	4	0
-1	1	1
0	0	4
1	1	9
2	4	16
3	9	25
4	16	36

Y_3■$(X+2)^2$

(b) ■

We are led to the following conclusions:

If the argument x of a function f is replaced by $x - h, h > 0$, the graph of the new function $y = f(x - h)$ is the graph of f **shifted horizontally right** h units.

If the argument x of a function f is replaced by $x + h, h > 0$, the graph of the new function $y = f(x + h)$ is the graph of f **shifted horizontally left** h units.

In Words
For $y = f(x - h)$, $h > 0$, add h to each x-coordinate on the graph of $y = f(x)$ to shift the graph right h units. For $y = f(x + h)$, $h > 0$, subtract h from each x-coordinate on the graph of $y = f(x)$ to shift the graph left h units.

Now Work PROBLEM 43

Note: Vertical shifts result when adding or subtracting a real number k after performing the operation suggested by the basic function, while horizontal shifts result when adding or subtracting a real number h to or from x before performing the operation suggested by the basic function. For example, the graph of $f(x) = \sqrt{x} + 3$ is obtained by shifting the graph of $y = \sqrt{x}$ up 3 units, because we evaluate the square root function first and then add 3. The graph of $g(x) = \sqrt{x + 3}$ is obtained by shifting the graph of $y = \sqrt{x}$ left 3 units, because we first add 3 to x before we evaluate the square root function. ■

Vertical and horizontal shifts are sometimes combined.

EXAMPLE 2 Combining Vertical and Horizontal Shifts

Graph the function $f(x) = |x + 3| - 5$. Find the domain and range of f.

Solution We graph f in steps. First, note that the rule for f is basically an absolute value function, so begin with the graph of $y = |x|$ as shown in Figure 51(a). Next, to get the graph of $y = |x + 3|$, shift the graph of $y = |x|$ horizontally 3 units to the left. See Figure 51(b). Finally, to get the graph of $y = |x + 3| - 5$, shift the graph of $y = |x + 3|$ vertically down 5 units. See Figure 51(c). Note the points plotted on each graph. Using key points can be helpful in keeping track of the transformation that has taken place.

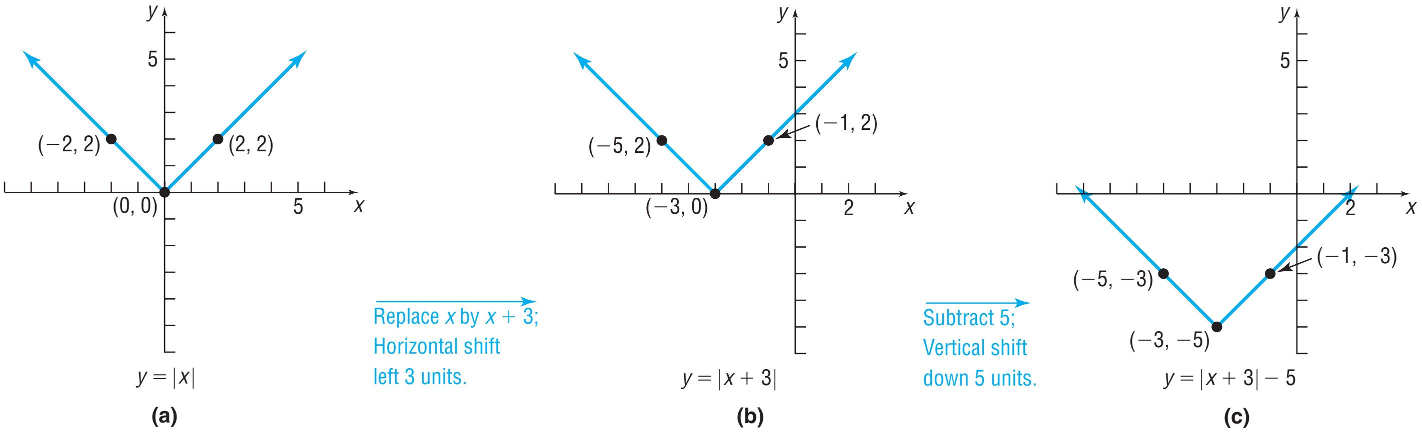


Figure 51

The domain of f is all real numbers, or $(-\infty, \infty)$. The range of f is $[-5, \infty)$. ■

✓**Check:** Graph $Y_1 = f(x) = |x + 3| - 5$ and compare the graph to Figure 51(c).

In Example 2, if the vertical shift had been done first, followed by the horizontal shift, the final graph would have been the same. (Try it for yourself.)

Now Work PROBLEMS 45 AND 71

2 Graph Functions Using Compressions and Stretches

Exploration On the same screen, graph each of the following functions:

$$Y_1 = |x|$$
$$Y_2 = 2|x|$$
$$Y_3 = \frac{1}{2}|x|$$

Then create a table of values and compare the y-coordinates for any given x-coordinate.

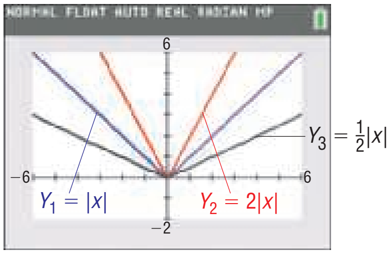


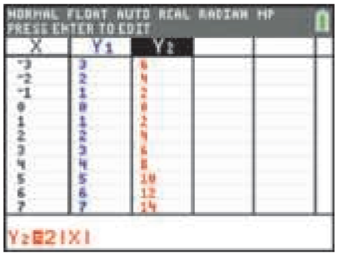
Figure 52 Vertical stretch or compression

Result Figure 52 illustrates the graphs.

Now look at Table 10, where $Y_1 = |x|$ and $Y_2 = 2|x|$. Notice that the values for Y_2 are two times the values of Y_1 for each x-value. This means that the graph of $Y_2 = 2|x|$ can be obtained from the graph of $Y_1 = |x|$ by multiplying each y-coordinate of $Y_1 = |x|$ by 2. Therefore, the graph of Y_2 will be the graph of Y_1 vertically *stretched* by a factor of 2.

Look at Table 11, where $Y_1 = |x|$ and $Y_3 = \frac{1}{2}|x|$. The values of Y_3 are half the values of Y_1 for each x-value. So, the graph of $Y_3 = \frac{1}{2}|x|$ can be obtained from the graph of $Y_1 = |x|$ by multiplying each y-coordinate by $\frac{1}{2}$. Therefore, the graph of Y_3 will be the graph of Y_1 vertically *compressed* by a factor of $\frac{1}{2}$.

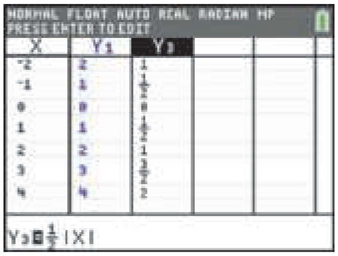
Table 10



X	Y1	Y2
−3	3	6
−2	2	4
−1	1	2
0	0	0
1	1	2
2	2	4
3	3	6
4	4	8
5	5	10
6	6	12
7	7	14

Y2=2|X|

Table 11



X	Y1	Y3
−2	2	1
−1	1	$\frac{1}{2}$
0	0	0
1	1	$\frac{1}{2}$
2	2	1
3	3	$\frac{3}{2}$
4	4	2

$Y_3 = \frac{1}{2}|X|$

■

In Words

For $y = af(x)$, $a > 0$, the factor a is "outside" the function, so it affects the y-coordinates. Multiply each y-coordinate on the graph of $y = f(x)$ by a.

Based on the Exploration, we have the following result:

When the right side of a function $y = f(x)$ is multiplied by a positive number a, the graph of the new function $y = af(x)$ is obtained by multiplying each y-coordinate on the graph of $y = f(x)$ by a. The new graph is a **vertically compressed** (if $0 < a < 1$) or a **vertically stretched** (if $a > 1$) version of the graph of $y = f(x)$.

Now Work PROBLEM 47

What happens if the argument x of a function $y = f(x)$ is multiplied by a positive number a, creating a new function $y = f(ax)$? To find the answer, look at the following Exploration.

Exploration

On the same screen, graph each of the following functions:

$$Y_1 = f(x) = \sqrt{x} \qquad Y_2 = f(2x) = \sqrt{2x} \qquad Y_3 = f\left(\frac{1}{2}x\right) = \sqrt{\frac{1}{2}x} = \sqrt{\frac{x}{2}}$$

Create a table of values to explore the relation between the x- and y-coordinates of each function.

Result You should have obtained the graphs in Figure 53. Look at Table 12(a). Note that (1, 1), (4, 2), and (9, 3) are points on the graph of $Y_1 = \sqrt{x}$. Also, (0.5, 1), (2, 2), and (4.5, 3) are points on the graph of $Y_2 = \sqrt{2x}$. For a given y-coordinate, the x-coordinate on the graph of Y_2 is $\frac{1}{2}$ of the x-coordinate on Y_1.

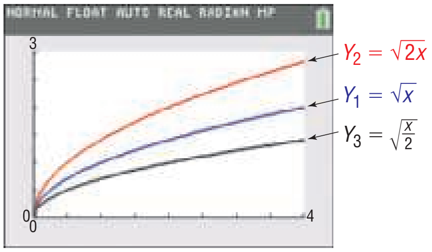
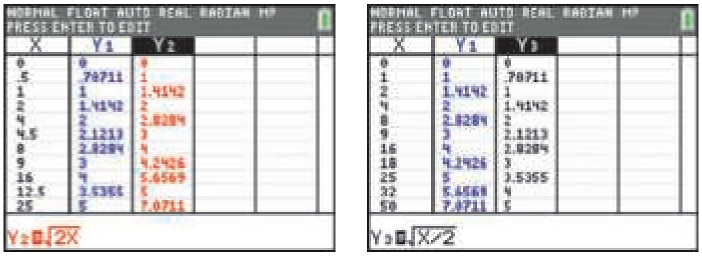


Figure 53 Horizontal stretch or compression

Table 12



X	Y1	Y2
0	0	0
.5	.70711	1
1	1	1.4142
2	1.4142	2
4	2	2.8284
4.5	2.1213	3
8	2.8284	4
9	3	4.2426
16	4	5.6569
12.5	3.5355	5
25	5	7.0711

$Y_2 = \sqrt{2X}$

(a)

X	Y1	Y3
0	0	0
1	1	.70711
2	1.4142	1
4	2	1.4142
8	2.8284	2
9	3	2.1213
16	4	2.8284
18	4.2426	3
25	5	3.5355
32	5.6569	4
50	7.0711	5

$Y_3 = \sqrt{X/2}$

(b)

We conclude that the graph of $Y_2 = \sqrt{2x}$ is obtained by multiplying the x-coordinate of each point on the graph of $Y_1 = \sqrt{x}$ by $\frac{1}{2}$. The graph of $Y_2 = \sqrt{2x}$ is the graph of $Y_1 = \sqrt{x}$ *compressed* horizontally.

Look at Table 12(b). Notice that (1, 1), (4, 2), and (9, 3) are points on the graph of $Y_1 = \sqrt{x}$. Also notice that (2, 1), (8, 2), and (18, 3) are points on the graph of $Y_3 = \sqrt{\frac{x}{2}}$. For a given y-coordinate, the x-coordinate on the graph of Y_3 is 2 times the x-coordinate on Y_1. We conclude that the graph of $Y_3 = \sqrt{\frac{x}{2}}$ is obtained by multiplying the x-coordinate of each point on the graph of $Y_1 = \sqrt{x}$ by 2. The graph of $Y_3 = \sqrt{\frac{x}{2}}$ is the graph of $Y_1 = \sqrt{x}$ *stretched* horizontally. ■

Based on the Exploration, we have the following result:

In Words

For $y = f(ax)$, $a > 0$, the factor a is "inside" the function, so it affects the x-coordinates. Multiply each x-coordinate on the graph of $y = f(x)$ by $\frac{1}{a}$.

If the argument x of a function $y = f(x)$ is multiplied by a positive number a, then the graph of the new function $y = f(ax)$ is obtained by multiplying each x-coordinate of $y = f(x)$ by $\frac{1}{a}$. A **horizontal compression** results if $a > 1$, and a **horizontal stretch** results if $0 < a < 1$.

Let's look at an example.

EXAMPLE 3 Graphing Using Stretches and Compressions

The graph of $y = f(x)$ is given in Figure 54. Use this graph to find the graphs of

(a) $y = 2f(x)$ (b) $y = f(3x)$

Solution

(a) The graph of $y = 2f(x)$ is obtained by multiplying each y-coordinate of $y = f(x)$ by 2. See Figure 55.

(b) The graph of $y = f(3x)$ is obtained from the graph of $y = f(x)$ by multiplying each x-coordinate of $y = f(x)$ by $\frac{1}{3}$. See Figure 56.

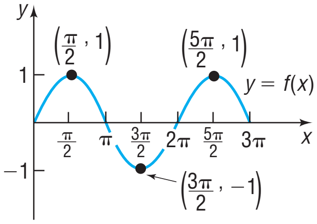


Figure 54 $y = f(x)$

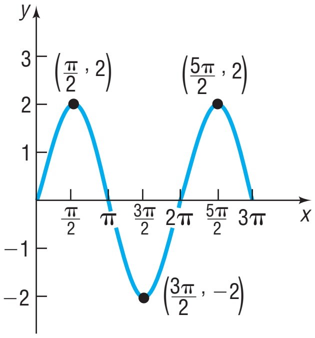


Figure 55 $y = 2f(x)$

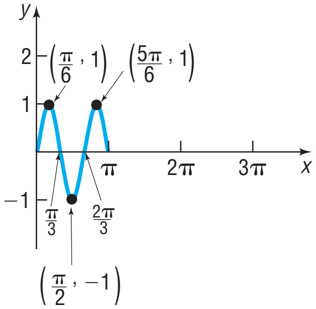


Figure 56 $y = f(3x)$ ■

Now Work PROBLEMS 63(e) AND (g)

3 Graph Functions Using Reflections about the x-Axis or y-Axis

Exploration

Reflection about the x-axis:

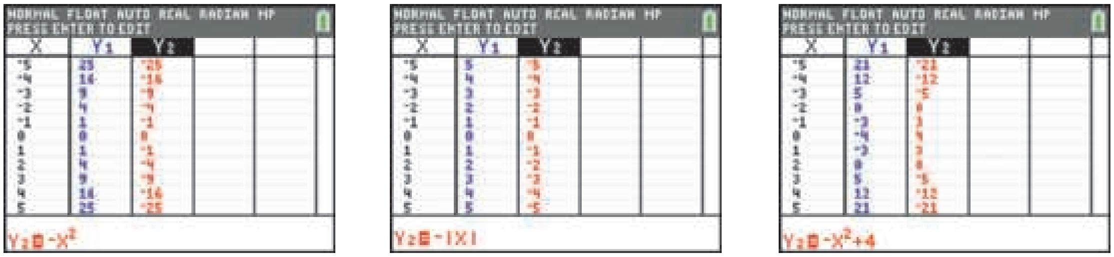
(a) Graph and create a table of $Y_1 = x^2$ and $Y_2 = -x^2$.

(b) Graph and create a table of $Y_1 = |x|$ and $Y_2 = -|x|$.

(c) Graph and create a table of $Y_1 = x^2 - 4$ and $Y_2 = -(x^2 - 4) = -x^2 + 4$.

Result See Tables 13(a), (b), and (c) and Figures 57(a), (b), and (c). For each point (x, y) on the graph of Y_1, the point $(x, -y)$ is on the graph of Y_2. Put another way, Y_2 is the reflection about the x-axis of Y_1.

Table 13



NORMAL FLOAT AUTO REAL RADIAN MP
PRESS ENTER TO EDIT

X	Y_1	Y_2
-5	25	-25
-4	16	-16
-3	9	-9
-2	4	-4
-1	1	-1
0	0	0
1	1	-1
2	4	-4
3	9	-9
4	16	-16
5	25	-25

$Y_2 = -X^2$

(a)

NORMAL FLOAT AUTO REAL RADIAN MP
PRESS ENTER TO EDIT

X	Y_1	Y_2
-5	5	-5
-4	4	-4
-3	3	-3
-2	2	-2
-1	1	-1
0	0	0
1	1	-1
2	2	-2
3	3	-3
4	4	-4
5	5	-5

$Y_2 = -|X|$

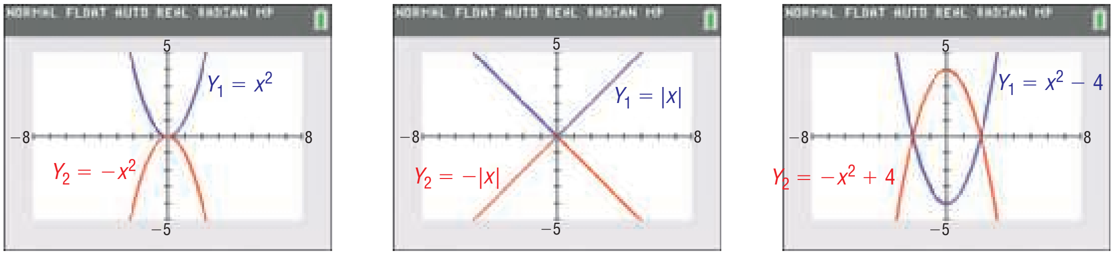
(b)

NORMAL FLOAT AUTO REAL RADIAN MP
PRESS ENTER TO EDIT

X	Y_1	Y_2
-5	21	-21
-4	12	-12
-3	5	-5
-2	0	0
-1	-3	3
0	-4	4
1	-3	3
2	0	0
3	5	-5
4	12	-12
5	21	-21

$Y_2 = -X^2+4$

(c)



(a) (b) (c)

Figure 57 Reflection about the *x*-axis ■

Based on the previous Exploration, we have the following result:

When the right side of the function $y = f(x)$ is multiplied by -1, the graph of the new function $y = -f(x)$ is the **reflection about the *x*-axis** of the graph of the function $y = f(x)$.

Now Work PROBLEM 49

Exploration

Reflection about the *y*-axis:

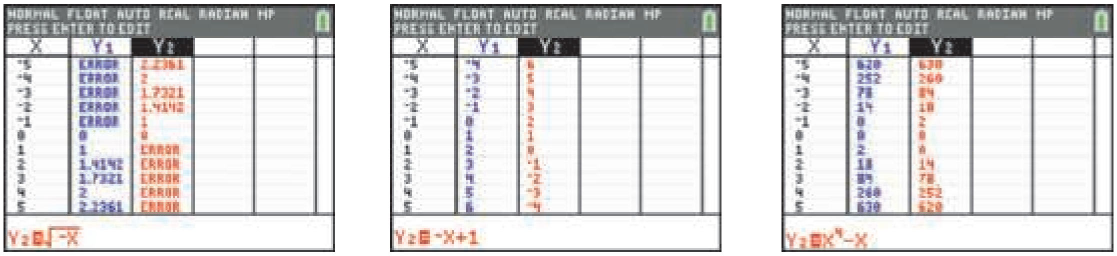
(a) Graph $Y_1 = \sqrt{x}$, followed by $Y_2 = \sqrt{-x}$.

(b) Graph $Y_1 = x + 1$, followed by $Y_2 = -x + 1$.

(c) Graph $Y_1 = x^4 + x$, followed by $Y_2 = (-x)^4 + (-x) = x^4 - x$.

Result See Tables 14(a), (b), and (c) and Figures 58(a), (b), and (c). For each point (x, y) on the graph of Y_1, the point $(-x, y)$ is on the graph of Y_2. Put another way, Y_2 is the reflection about the *y*-axis of Y_1.

Table 14



NORMAL FLOAT AUTO REAL RADIAN MP
PRESS ENTER TO EDIT

X	Y_1	Y_2
-5	ERROR	2.2361
-4	ERROR	2
-3	ERROR	1.7321
-2	ERROR	1.4142
-1	ERROR	1
0	0	0
1	1	ERROR
2	1.4142	ERROR
3	1.7321	ERROR
4	2	ERROR
5	2.2361	ERROR

$Y_2 = \sqrt{-X}$

(a)

NORMAL FLOAT AUTO REAL RADIAN MP
PRESS ENTER TO EDIT

X	Y_1	Y_2
-5	-4	6
-4	-3	5
-3	-2	4
-2	-1	3
-1	0	2
0	1	1
1	2	0
2	3	-1
3	4	-2
4	5	-3
5	6	-4

$Y_2 = -X+1$

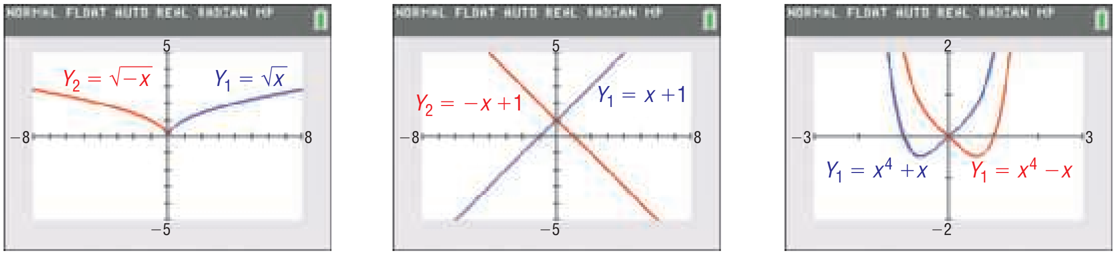
(b)

NORMAL FLOAT AUTO REAL RADIAN MP
PRESS ENTER TO EDIT

X	Y_1	Y_2
-5	620	630
-4	252	260
-3	78	84
-2	14	18
-1	0	2
0	0	0
1	2	0
2	18	14
3	84	78
4	260	252
5	630	620

$Y_2 = X^4-X$

(c)



(a) (b) (c)

Figure 58 Reflection about the *y*-axis ■

In Words

For $y = -f(x)$, multiply each y-coordinate on the graph of $y = f(x)$ by -1.
For $y = f(-x)$, multiply each x-coordinate by -1.

Based on the previous Exploration, we have the following result:

When the graph of the function $y = f(x)$ is known, the graph of the new function $y = f(-x)$ is the **reflection about the *y*-axis** of the graph of the function $y = f(x)$.

SUMMARY OF GRAPHING TECHNIQUES

To Graph:	**Draw the Graph of *f* and:**	**Functional Change to $f(x)$**
Vertical shifts		
$y = f(x) + k, \quad k > 0$	Raise the graph of *f* by *k* units.	Add *k* to $f(x)$.
$y = f(x) - k, \quad k > 0$	Lower the graph of *f* by *k* units.	Subtract *k* from $f(x)$.
Horizontal shifts		
$y = f(x + h), \quad h > 0$	Shift the graph of *f* to the left *h* units.	Replace *x* by $x + h$.
$y = f(x - h), \quad h > 0$	Shift the graph of *f* to the right *h* units.	Replace *x* by $x - h$.
Compressing or stretching		
$y = af(x), \quad a > 0$	Multiply each *y*-coordinate of $y = f(x)$ by *a*. Stretch the graph of *f* vertically if $a > 1$. Compress the graph of *f* vertically if $0 < a < 1$.	Multiply $f(x)$ by *a*.
$y = f(ax), \quad a > 0$	Multiply each *x*-coordinate of $y = f(x)$ by $\frac{1}{a}$. Stretch the graph of *f* horizontally if $0 < a < 1$. Compress the graph of *f* horizontally if $a > 1$.	Replace *x* by *ax*.
Reflection about the *x*-axis		
$y = -f(x)$	Reflect the graph of *f* about the *x*-axis.	Multiply $f(x)$ by -1.
Reflection about the *y*-axis		
$y = f(-x)$	Reflect the graph of *f* about the *y*-axis.	Replace *x* by $-x$.

EXAMPLE 4 Determining the Function Obtained from a Series of Transformations

Find the function that is finally graphed after the following three transformations are applied to the graph of $y = |x|$.

1. Shift left 2 units.
2. Shift up 3 units.
3. Reflect about the *y*-axis.

Solution

1. Shift left 2 units:	Replace *x* by $x + 2$.	$y = \lvert x + 2\rvert$
2. Shift up 3 units:	Add 3.	$y = \lvert x + 2\rvert + 3$
3. Reflect about the *y*-axis:	Replace *x* by $-x$.	$y = \lvert -x + 2\rvert + 3$ ■

Now Work PROBLEM 27

EXAMPLE 5 Combining Graphing Procedures

Graph the function $f(x) = \dfrac{3}{x - 2} + 1$. Find the domain and range of f.

Solution It is helpful to write f as $f(x) = 3\left(\frac{1}{x-2}\right) + 1$. Now use the following steps to obtain the graph of f.

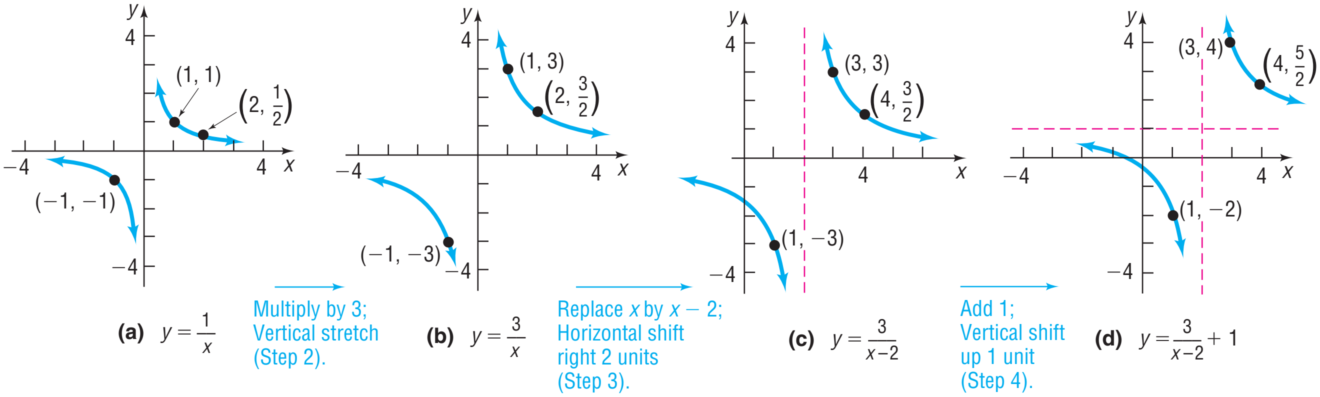
STEP 1: $y = \frac{1}{x}$ **Reciprocal function**

STEP 2: $y = 3 \cdot \left(\frac{1}{x}\right) = \frac{3}{x}$ **Multiply by 3; vertical stretch by a factor of 3.**

STEP 3: $y = \frac{3}{x-2}$ **Replace x by $x - 2$; horizontal shift to the right 2 units.**

STEP 4: $y = \frac{3}{x-2} + 1$ **Add 1; vertical shift up 1 unit.**

See Figure 59.



(a) $y = \frac{1}{x}$ (b) $y = \frac{3}{x}$ (c) $y = \frac{3}{x-2}$ (d) $y = \frac{3}{x-2} + 1$

Figure 59

The domain of $y = \frac{1}{x}$ is $\{x | x \neq 0\}$ and its range is $\{y | y \neq 0\}$. Because we shifted right 2 units and up 1 unit to obtain f, the domain of f is $\{x | x \neq 2\}$ and its range is $\{y | y \neq 1\}$. ■

Hint: Although the order in which transformations are performed can be altered, you may consider using the following order for consistency:

1. Reflections
2. Compressions and stretches
3. Shifts ■

Other orderings of the steps shown in Example 5 would also result in the graph of f. For example, try this one:

STEP 1: $y = \frac{1}{x}$ **Reciprocal function**

STEP 2: $y = \frac{1}{x-2}$ **Replace x by $x - 2$; horizontal shift to the right 2 units.**

STEP 3: $y = \frac{3}{x-2}$ **Multiply by 3; vertical stretch of the graph of $y = \frac{1}{x-2}$ by a factor of 3.**

STEP 4: $y = \frac{3}{x-2} + 1$ **Add 1; vertical shift up 1 unit.**

EXAMPLE 6 Combining Graphing Procedures

Graph the function $f(x) = \sqrt{1-x} + 2$. Find the domain and range of f.

Solution Because horizontal shifts require the form $x - h$, begin by rewriting $f(x)$ as $f(x) = \sqrt{1-x} + 2 = \sqrt{-(x-1)} + 2$. Now use the following steps:

STEP 1: $y = \sqrt{x}$ **Square root function**

STEP 2: $y = \sqrt{-x}$ **Replace x by $-x$; reflect about the y-axis.**

STEP 3: $y = \sqrt{-(x-1)} = \sqrt{1-x}$ **Replace x by $x - 1$; horizontal shift to the right 1 unit.**

STEP 4: $y = \sqrt{1-x} + 2$ **Add 2; vertical shift up 2 units.**

See Figure 60.

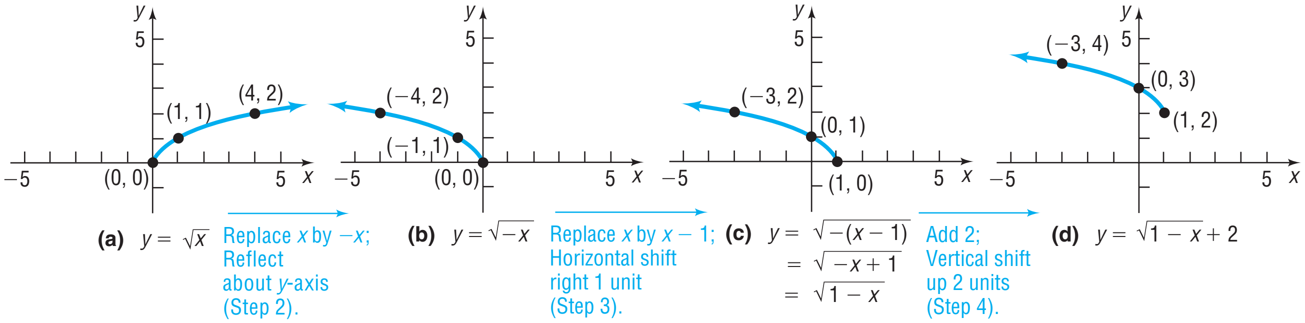


Figure 60

The domain of f is $(-\infty, 1]$ and the range is $[2, \infty)$. ■

Now Work PROBLEM 55

3.5 Assess Your Understanding

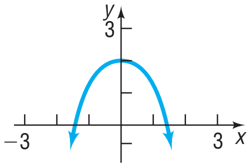
Concepts and Vocabulary

1. Suppose that the graph of a function f is known. Then the graph of $y = f(x-2)$ may be obtained by a(n) ________ shift of the graph of f to the ______ a distance of 2 units.
2. Suppose that the graph of a function f is known. Then the graph of $y = f(-x)$ may be obtained by a reflection about the ___-axis of the graph of the function $y = f(x)$.
3. ***True or False*** The graph of $y = \frac{1}{3}g(x)$ is the graph of $y = g(x)$ stretched by a factor of 3.
4. ***True or False*** The graph of $y = -f(x)$ is the reflection about the x-axis of the graph of $y = f(x)$.
5. Which of the following functions has a graph that is the graph of $y = \sqrt{x}$ shifted down 3 units?
   (a) $y = \sqrt{x+3}$ (b) $y = \sqrt{x-3}$
   (c) $y = \sqrt{x} + 3$ (d) $y = \sqrt{x} - 3$
6. Which of the following functions has a graph that is the graph of $y = f(x)$ compressed horizontally by a factor of 4?
   (a) $y = f(4x)$ (b) $y = f\left(\frac{1}{4}x\right)$
   (c) $y = 4f(x)$ (d) $y = \frac{1}{4}f(x)$

Skill Building

In Problems 7–18, match each graph to one of the following functions:

A. $y = x^2 + 2$ *B.* $y = -x^2 + 2$ *C.* $y = |x| + 2$ *D.* $y = -|x| + 2$
E. $y = (x-2)^2$ *F.* $y = -(x+2)^2$ *G.* $y = |x-2|$ *H.* $y = -|x+2|$
I. $y = 2x^2$ *J.* $y = -2x^2$ *K.* $y = 2|x|$ *L.* $y = -2|x|$

7.


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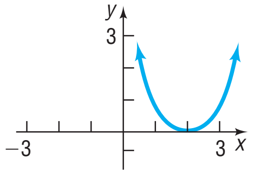
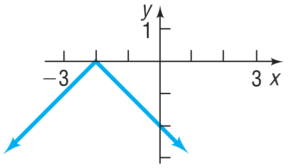


9.


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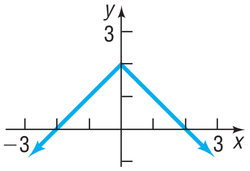

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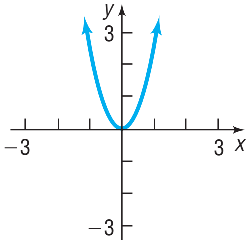

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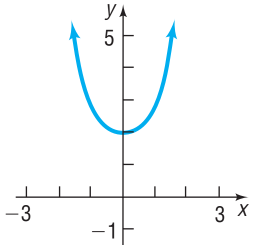

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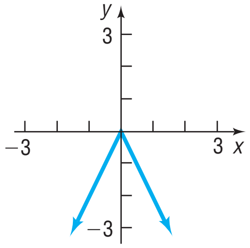

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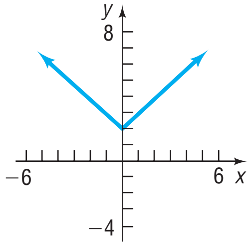

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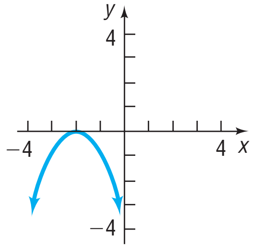

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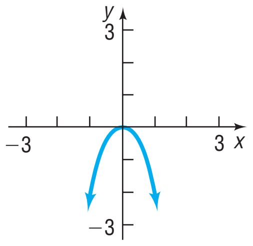

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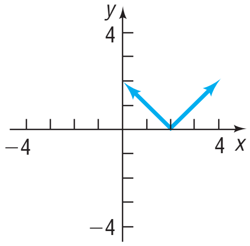

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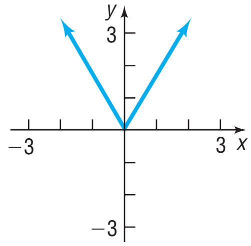


In Problems 19–26, write the function whose graph is the graph of $y = x^3$*, but is:*

19. Shifted to the right 4 units

20. Shifted to the left 4 units

21. Shifted up 4 units

22. Shifted down 4 units

23. Reflected about the y-axis

24. Reflected about the x-axis

25. Vertically stretched by a factor of 4

26. Horizontally stretched by a factor of 4

In Problems 27–30, find the function that is finally graphed after each of the following transformations is applied to the graph of $y = \sqrt{x}$ *in the order stated.*

27. (1) Shift up 2 units
(2) Reflect about the x-axis
(3) Reflect about the y-axis

28. (1) Reflect about the x-axis
(2) Shift right 3 units
(3) Shift down 2 units

29. (1) Reflect about the x-axis
(2) Shift up 2 units
(3) Shift left 3 units

30. (1) Shift up 2 units
(2) Reflect about the y-axis
(3) Shift left 3 units

31. If $(3, 6)$ is a point on the graph of $y = f(x)$, which of the following points must be on the graph of $y = -f(x)$?

(a) $(6, 3)$ (b) $(6, -3)$
(c) $(3, -6)$ (d) $(-3, 6)$

32. If $(3, 6)$ is a point on the graph of $y = f(x)$, which of the following points must be on the graph of $y = f(-x)$?

(a) $(6, 3)$ (b) $(6, -3)$
(c) $(3, -6)$ (d) $(-3, 6)$

33. If $(1, 3)$ is a point on the graph of $y = f(x)$, which of the following points must be on the graph of $y = 2f(x)$?

(a) $\left(1, \frac{3}{2}\right)$ (b) $(2, 3)$
(c) $(1, 6)$ (d) $\left(\frac{1}{2}, 3\right)$

34. If $(4, 2)$ is a point on the graph of $y = f(x)$, which of the following points must be on the graph of $y = f(2x)$?

(a) $(4, 1)$ (b) $(8, 2)$
(c) $(2, 2)$ (d) $(4, 4)$

35. Suppose that the x-intercepts of the graph of $y = f(x)$ are -5 and 3.

(a) What are the x-intercepts of the graph of $y = f(x + 2)$?
(b) What are the x-intercepts of the graph of $y = f(x - 2)$?
(c) What are the x-intercepts of the graph of $y = 4f(x)$?
(d) What are the x-intercepts of the graph of $y = f(-x)$?

36. Suppose that the x-intercepts of the graph of $y = f(x)$ are -8 and 1.

(a) What are the x-intercepts of the graph of $y = f(x + 4)$?
(b) What are the x-intercepts of the graph of $y = f(x - 3)$?
(c) What are the x-intercepts of the graph of $y = 2f(x)$?
(d) What are the x-intercepts of the graph of $y = f(-x)$?

37. Suppose that the function $y = f(x)$ is increasing on the interval $[-1, 5]$.

(a) Over what interval is the graph of $y = f(x + 2)$ increasing?
(b) Over what interval is the graph of $y = f(x - 5)$ increasing?
(c) What can be said about the graph of $y = -f(x)$?
(d) What can be said about the graph of $y = f(-x)$?

38. Suppose that the function $y = f(x)$ is decreasing on the interval $[-2, 7]$.

(a) Over what interval is the graph of $y = f(x + 2)$ decreasing?
(b) Over what interval is the graph of $y = f(x - 5)$ decreasing?
(c) What can be said about the graph of $y = -f(x)$?
(d) What can be said about the graph of $y = f(-x)$?

In Problems 39–62, graph each function using the techniques of shifting, compressing, stretching, and/or reflecting. Start with the graph of the basic function (for example, $y = x^2$*) and show all stages. Be sure to show at least three key points. Find the domain and the range of each function.*

39. $f(x) = x^2 - 1$

40. $f(x) = x^2 + 4$

41. $g(x) = x^3 + 1$

42. $g(x) = x^3 - 1$

43. $h(x) = \sqrt{x + 2}$

44. $h(x) = \sqrt{x + 1}$

45. $f(x) = (x - 1)^3 + 2$

46. $f(x) = (x + 2)^3 - 3$

47. $g(x) = 4\sqrt{x}$

48. $g(x) = \frac{1}{2}\sqrt{x}$

49. $f(x) = -\sqrt[3]{x}$

50. $f(x) = -\sqrt{x}$

51. $f(x) = 2(x + 1)^2 - 3$

52. $f(x) = 3(x - 2)^2 + 1$

53. $g(x) = 2\sqrt{x - 2} + 1$

54. $g(x) = 3|x + 1| - 3$

55. $h(x) = \sqrt{-x} - 2$

56. $h(x) = \frac{4}{x} + 2$

57. $f(x) = -(x + 1)^3 - 1$

58. $f(x) = -4\sqrt{x - 1}$

59. $g(x) = 2|1 - x|$

60. $g(x) = 4\sqrt{2 - x}$

61. $h(x) = \frac{1}{2x}$

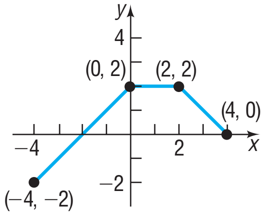
62. $h(x) = \sqrt[3]{x - 1} + 3$

In Problems 63–66, the graph of a function f is illustrated. Use the graph of f as the first step toward graphing each of the following functions:

(a) $F(x) = f(x) + 3$ (b) $G(x) = f(x + 2)$ (c) $P(x) = -f(x)$ (d) $H(x) = f(x + 1) - 2$

(e) $Q(x) = \frac{1}{2}f(x)$ (f) $g(x) = f(-x)$ (g) $h(x) = f(2x)$

63.



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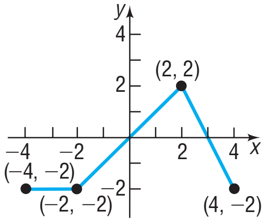

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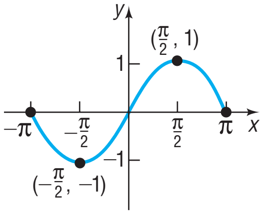

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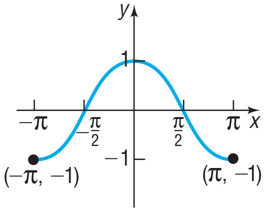


Mixed Practice

67. (a) Using a graphing utility, graph $f(x) = x^3 - 9x$ for $-4 \le x \le 4$.
(b) Find the x-intercepts of the graph of f.
(c) Approximate any local maxima and local minima.
(d) Determine where f is increasing and where it is decreasing.
(e) Without using a graphing utility, repeat parts (b)–(d) for $y = f(x + 2)$.
(f) Without using a graphing utility, repeat parts (b)–(d) for $y = 2f(x)$.
(g) Without using a graphing utility, repeat parts (b)–(d) for $y = f(-x)$.

68. (a) Using a graphing utility, graph $f(x) = x^3 - 4x$ for $-3 \le x \le 3$.
(b) Find the x-intercepts of the graph of f.
(c) Approximate any local maxima and local minima.
(d) Determine where f is increasing and where it is decreasing.
(e) Without using a graphing utility, repeat parts (b)–(d) for $y = f(x - 4)$.
(f) Without using a graphing utility, repeat parts (b)–(d) for $y = f(2x)$.
(g) Without using a graphing utility, repeat parts (b)–(d) for $y = -f(x)$.

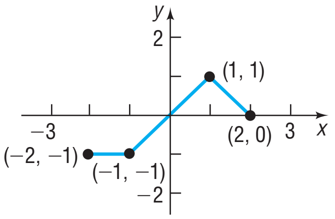
In Problems 69–76, complete the square of each quadratic expression. Then graph each function using the technique of shifting. (If necessary, refer to Chapter R, Section R.5 to review completing the square.)

69. $f(x) = x^2 + 2x$ **70.** $f(x) = x^2 - 6x$ **71.** $f(x) = x^2 - 8x + 1$ **72.** $f(x) = x^2 + 4x + 2$

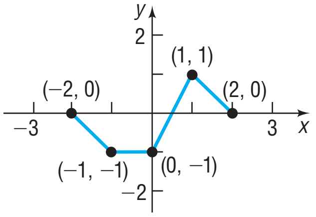
73. $f(x) = 2x^2 - 12x + 19$ **74.** $f(x) = 3x^2 + 6x + 1$ **75.** $f(x) = -3x^2 - 12x - 17$ **76.** $f(x) = -2x^2 - 12x - 13$

Applications and Extensions

77. The graph of a function f is illustrated in the figure.
(a) Draw the graph of $y = |f(x)|$.
(b) Draw the graph of $y = f(|x|)$.



78. The graph of a function f is illustrated in the figure.
(a) Draw the graph of $y = |f(x)|$.
(b) Draw the graph of $y = f(|x|)$.



79. Suppose $(1, 3)$ is a point on the graph of $y = f(x)$.
(a) What point is on the graph of $y = f(x + 3) - 5$?
(b) What point is on the graph of $y = -2f(x - 2) + 1$?
(c) What point is on the graph of $y = f(2x + 3)$?

80. Suppose $(-3, 5)$ is a point on the graph of $y = g(x)$.
(a) What point is on the graph of $y = g(x + 1) - 3$?
(b) What point is on the graph of $y = -3g(x - 4) + 3$?
(c) What point is on the graph of $y = g(3x + 9)$?

81. Graph the following functions using transformations.
(a) $f(x) = \text{int}(-x)$ (b) $g(x) = -\text{int}(x)$

82. Graph the following functions using transformations.
(a) $f(x) = \text{int}(x - 1)$ (b) $g(x) = \text{int}(1 - x)$

83. (a) Graph $f(x) = |x - 3| - 3$ using transformations.
(b) Find the area of the region that is bounded by f and the x-axis and lies below the x-axis.

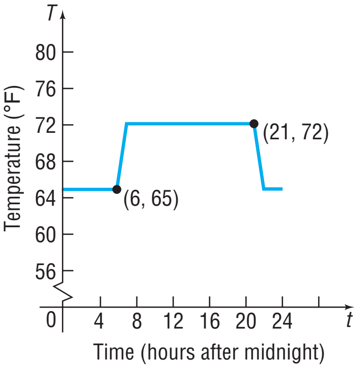
84. (a) Graph $f(x) = -2|x - 4| + 4$ using transformations.
(b) Find the area of the region that is bounded by f and the x-axis and lies above the x-axis.

85. Thermostat Control Energy conservation experts estimate that homeowners can save 5% to 10% on winter heating bills by programming their thermostats 5 to 10 degrees lower while sleeping. In the graph (next page), the temperature T (in degrees Fahrenheit) of a home is given as a function of time t (in hours after midnight) over a 24-hour period.
(a) At what temperature is the thermostat set during daytime hours? At what temperature is the thermostat set overnight?

(b) The homeowner reprograms the thermostat to $y = T(t) - 2$. Explain how this affects the temperature in the house. Graph this new function.

(c) The homeowner reprograms the thermostat to $y = T(t + 1)$. Explain how this affects the temperature in the house. Graph this new function.

Source: Roger Albright, *547 Ways to Be Fuel Smart*, 2000



86. Digital Music Revenues The total projected worldwide digital music revenues R, in millions of dollars, for the years 2012 through 2017 can be estimated by the function

$$R(x) = 28.6x^2 + 300x + 4843$$

where x is the number of years after 2012.

(a) Find $R(0), R(3)$, and $R(5)$ and explain what each value represents.

(b) Find $r(x) = R(x - 2)$.

(c) Find $r(2), r(5)$, and $r(7)$ and explain what each value represents.

(d) In the model $r = r(x)$, what does x represent?

(e) Would there be an advantage in using the model r when estimating the projected revenues for a given year instead of the model R?

Source: *IFPI Digital Music Report*

87. Temperature Measurements The relationship between the Celsius (°C) and Fahrenheit (°F) scales for measuring temperature is given by the equation

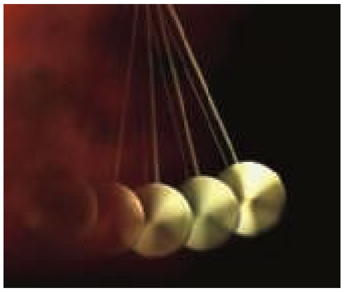
$$F = \frac{9}{5}C + 32$$

The relationship between the Celsius (°C) and Kelvin (K) scales is $K = C + 273$. Graph the equation $F = \frac{9}{5}C + 32$ using degrees Fahrenheit on the y-axis and degrees Celsius on the x-axis. Use the techniques introduced in this section to obtain the graph showing the relationship between Kelvin and Fahrenheit temperatures.

88. Period of a Pendulum The period T (in seconds) of a simple pendulum is a function of its length l (in feet) defined by the equation

$$T = 2\pi\sqrt{\frac{l}{g}}$$

where $g \approx 32.2$ feet per second per second is the acceleration due to gravity.



(a) Use a graphing utility to graph the function $T = T(l)$.

(b) Now graph the functions $T = T(l + 1)$, $T = T(l + 2)$, and $T = T(l + 3)$.

(c) Discuss how adding to the length l changes the period T.

(d) Now graph the functions $T = T(2l)$, $T = T(3l)$, and $T = T(4l)$.

(e) Discuss how multiplying the length l by factors of 2, 3, and 4 changes the period T.

89. The equation $y = (x - c)^2$ defines a *family of parabolas*, one parabola for each value of c. On one set of coordinate axes, graph the members of the family for $c = 0$, $c = 3$, and $c = -2$.

90. Repeat Problem 89 for the family of parabolas $y = x^2 + c$.

Explaining Concepts: Discussion and Writing

91. Suppose that the graph of a function f is known. Explain how the graph of $y = 4f(x)$ differs from the graph of $y = f(4x)$.

92. Suppose that the graph of a function f is known. Explain how the graph of $y = f(x) - 2$ differs from the graph of $y = f(x - 2)$.

93. The area under the curve $y = \sqrt{x}$ bounded from below by the x-axis and on the right by $x = 4$ is $\frac{16}{3}$ square units. Using the ideas presented in this section, what do you think is the area under the curve of $y = \sqrt{-x}$ bounded from below by the x-axis and on the left by $x = -4$? Justify your answer.

94. Explain how the range of the function $f(x) = x^2$ compares to the range of $g(x) = f(x) + k$.

95. Explain how the domain of $g(x) = \sqrt{x}$ compares to the domain of $g(x - k)$, where $k \geq 0$.

Retain Your Knowledge

Problems 96–99 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

96. Determine the slope and y-intercept of the graph of $3x - 5y = 30$.

97. Simplify $\dfrac{(x^{-2}y^3)^4}{(x^2y^{-5})^{-2}}$.

98. The amount of water used when taking a shower varies directly with the number of minutes the shower is run. If a 4-minute shower uses 7 gallons of water, how much water is used in a 9-minute shower?

99. List the intercepts and test for symmetry: $y^2 = x + 4$

3.6 Mathematical Models: Building Functions

OBJECTIVE 1 Build and Analyze Functions (p. 268)

1 Build and Analyze Functions

Real-world problems often result in mathematical models that involve functions. These functions need to be constructed or built based on the information given. In building functions, we must be able to translate the verbal description into the language of mathematics. This is done by assigning symbols to represent the independent and dependent variables and then by finding the function or rule that relates these variables.

EXAMPLE 1 Finding the Distance from the Origin to a Point on a Graph

Let $P = (x, y)$ be a point on the graph of $y = x^2 - 1$.

(a) Express the distance d from P to the origin O as a function of x.

(b) What is d if $x = 0$?

(c) What is d if $x = 1$?

(d) What is d if $x = \dfrac{\sqrt{2}}{2}$?

(e) Use a graphing utility to graph the function $d = d(x)$, $x \geq 0$. Rounding to two decimal places, find the value(s) of x at which d has a local minimum. [This gives the point(s) on the graph of $y = x^2 - 1$ closest to the origin.]

Solution (a) Figure 61 illustrates the graph of $y = x^2 - 1$. The distance d from P to O is

$$d = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$$

Since P is a point on the graph of $y = x^2 - 1$, substitute $x^2 - 1$ for y. Then

$$d(x) = \sqrt{x^2 + (x^2 - 1)^2} = \sqrt{x^4 - x^2 + 1}$$

The distance d is expressed as a function of x.

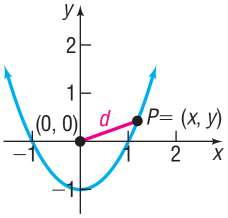


Figure 61 $y = x^2 - 1$

(b) If $x = 0$, the distance d is

$$d(0) = \sqrt{0^4 - 0^2 + 1} = \sqrt{1} = 1$$

(c) If $x = 1$, the distance d is

$$d(1) = \sqrt{1^4 - 1^2 + 1} = 1$$

(d) If $x = \dfrac{\sqrt{2}}{2}$, the distance d is

$$d\left(\frac{\sqrt{2}}{2}\right) = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^4 - \left(\frac{\sqrt{2}}{2}\right)^2 + 1} = \sqrt{\frac{1}{4} - \frac{1}{2} + 1} = \frac{\sqrt{3}}{2}$$

(e) Figure 62 shows the graph of $Y_1 = \sqrt{x^4 - x^2 + 1}$. Using the MINIMUM feature on a graphing utility, we find that when $x \approx 0.71$ the value of d is smallest. The local minimum value is $d \approx 0.87$ rounded to two decimal places. Since $d(x)$ is even, it follows by symmetry that when $x \approx -0.71$, the value of d is the same local minimum value. Since $(\pm 0.71)^2 - 1 \approx -0.50$, the points $(-0.71, -0.50)$ and $(0.71, -0.50)$ on the graph of $y = x^2 - 1$ are closest to the origin. ■

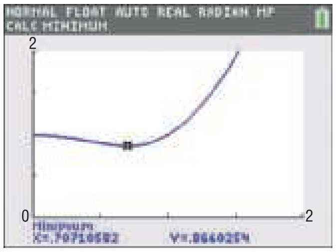


Figure 62 $d(x) = \sqrt{x^4 - x^2 + 1}$

Now Work PROBLEM 1

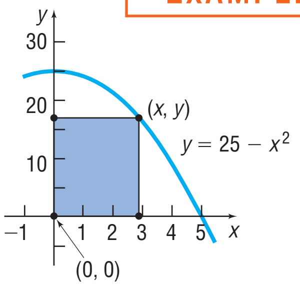


Figure 63

EXAMPLE 2 Area of a Rectangle

A rectangle has one corner in quadrant I on the graph of $y = 25 - x^2$, another at the origin, a third on the positive y-axis, and the fourth on the positive x-axis. See Figure 63.

(a) Express the area A of the rectangle as a function of x.
(b) What is the domain of A?
(c) Graph $A = A(x)$.
(d) For what value of x is the area largest?

Solution

(a) The area A of the rectangle is $A = xy$, where $y = 25 - x^2$. Substituting this expression for y, we obtain $A(x) = x(25 - x^2) = 25x - x^3$.

(b) Since (x, y) is in quadrant I, we have $x > 0$. Also, $y = 25 - x^2 > 0$, which implies that $x^2 < 25$, so $-5 < x < 5$. Combining these restrictions, we have the domain of A as $\{x | 0 < x < 5\}$, or $(0, 5)$ using interval notation.

(c) See Figure 64 for the graph of $A = A(x)$.

(d) Using MAXIMUM, we find that the maximum area is 48.11 square units at $x = 2.89$ units, each rounded to two decimal places. See Figure 65.

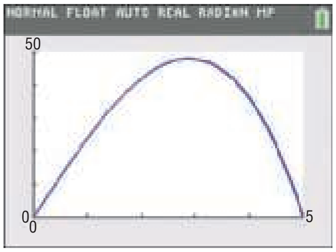


Figure 64 $A(x) = 25x - x^3$

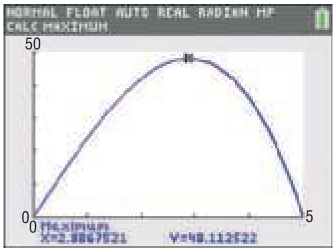


Figure 65

Now Work PROBLEM 7

EXAMPLE 3 Close Call?

Suppose two planes flying at the same altitude are headed toward each other. One plane is flying due south at a groundspeed of 400 miles per hour and is 600 miles from the potential intersection point of the planes. The other plane is flying due west with a groundspeed of 250 miles per hour and is 400 miles from the potential intersection point of the planes. See Figure 66.

(a) Build a model that expresses the distance d between the planes as a function of time t.
(b) Use a graphing utility to graph $d = d(t)$. How close do the planes come to each other? At what time are the planes closest?

Solution

(a) Refer to Figure 66. The distance d between the two planes is the hypotenuse of a right triangle. At any time t, the length of the north/south leg of the triangle is $600 - 400t$. At any time t, the length of the east/west leg of the triangle is $400 - 250t$. Use the Pythagorean Theorem to find that the square of the distance between the two planes is

$$d^2 = (600 - 400t)^2 + (400 - 250t)^2$$

Therefore, the distance between the two planes as a function of time is given by the model

$$d(t) = \sqrt{(600 - 400t)^2 + (400 - 250t)^2}$$

(b) Figure 67(a) on the next page shows the graph of $d = d(t)$. Using MINIMUM, the minimum distance between the planes is 21.20 miles, and the time at which the planes are closest is after 1.53 hours, each rounded to two decimal places. See Figure 67(b).

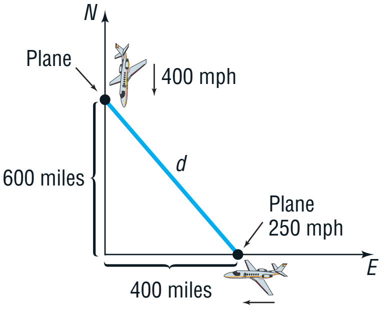
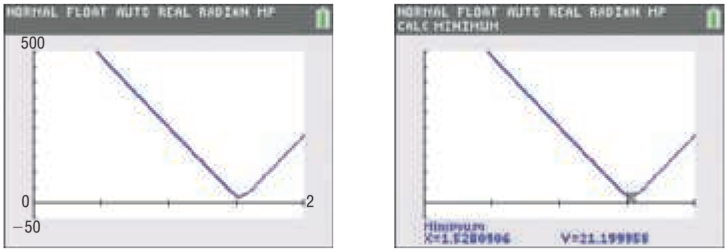


Figure 66



(a) (b)

Figure 67

Now Work PROBLEM 19

3.6 Assess Your Understanding

Applications and Extensions

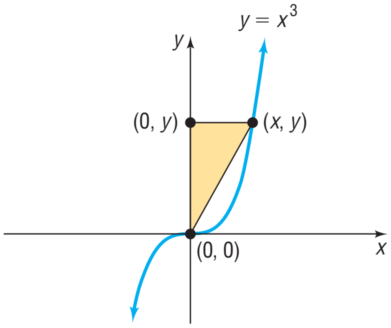
1. Let $P = (x, y)$ be a point on the graph of $y = x^2 - 8$.
   (a) Express the distance d from P to the origin as a function of x.
   (b) What is d if $x = 0$?
   (c) What is d if $x = 1$?
   (d) Use a graphing utility to graph $d = d(x)$.
   (e) For what values of x is d smallest?

2. Let $P = (x, y)$ be a point on the graph of $y = x^2 - 8$.
   (a) Express the distance d from P to the point $(0, -1)$ as a function of x.
   (b) What is d if $x = 0$?
   (c) What is d if $x = -1$?
   (d) Use a graphing utility to graph $d = d(x)$.
   (e) For what values of x is d smallest?

3. Let $P = (x, y)$ be a point on the graph of $y = \sqrt{x}$.
   (a) Express the distance d from P to the point $(1, 0)$ as a function of x.
   (b) Use a graphing utility to graph $d = d(x)$.
   (c) For what values of x is d smallest?

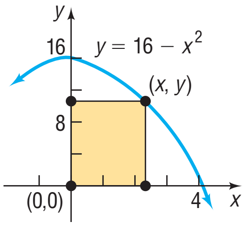
4. Let $P = (x, y)$ be a point on the graph of $y = \dfrac{1}{x}$.
   (a) Express the distance d from P to the origin as a function of x.
   (b) Use a graphing utility to graph $d = d(x)$.
   (c) For what values of x is d smallest?

5. A right triangle has one vertex on the graph of $y = x^3, x > 0$, at (x, y), another at the origin, and the third on the positive y-axis at $(0, y)$, as shown in the figure. Express the area A of the triangle as a function of x.



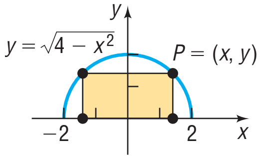
6. A right triangle has one vertex on the graph of $y = 9 - x^2, x > 0$, at (x, y), another at the origin, and the third on the positive x-axis at $(x, 0)$. Express the area A of the triangle as a function of x.

7. A rectangle has one corner in quadrant I on the graph of $y = 16 - x^2$, another at the origin, a third on the positive y-axis, and the fourth on the positive x-axis. See the figure below.



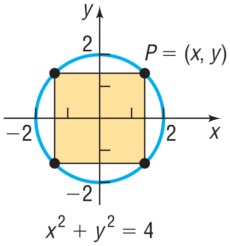
   (a) Express the area A of the rectangle as a function of x.
   (b) What is the domain of A?
   (c) Graph $A = A(x)$. For what value of x is A largest?

8. A rectangle is inscribed in a semicircle of radius 2. See the figure. Let $P = (x, y)$ be the point in quadrant I that is a vertex of the rectangle and is on the circle.



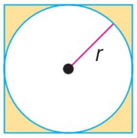
   (a) Express the area A of the rectangle as a function of x.
   (b) Express the perimeter p of the rectangle as a function of x.
   (c) Graph $A = A(x)$. For what value of x is A largest?
   (d) Graph $p = p(x)$. For what value of x is p largest?

9. A rectangle is inscribed in a circle of radius 2. See the figure. Let $P = (x, y)$ be the point in quadrant I that is a vertex of the rectangle and is on the circle.



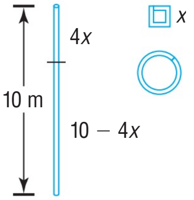
(a) Express the area A of the rectangle as a function of x.
(b) Express the perimeter p of the rectangle as a function of x.
(c) Graph $A = A(x)$. For what value of x is A largest?
(d) Graph $p = p(x)$. For what value of x is p largest?

10. A circle of radius r is inscribed in a square. See the figure.



(a) Express the area A of the square as a function of the radius r of the circle.
(b) Express the perimeter p of the square as a function of r.

11. Geometry A wire 10 meters long is to be cut into two pieces. One piece will be shaped as a square, and the other piece will be shaped as a circle. See the figure.



(a) Express the total area A enclosed by the pieces of wire as a function of the length x of a side of the square.
(b) What is the domain of A?
(c) Graph $A = A(x)$. For what value of x is A smallest?

12. Geometry A wire 10 meters long is to be cut into two pieces. One piece will be shaped as an equilateral triangle, and the other piece will be shaped as a circle.

(a) Express the total area A enclosed by the pieces of wire as a function of the length x of a side of the equilateral triangle.
(b) What is the domain of A ?
(c) Graph $A = A(x)$. For what value of x is A smallest?

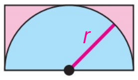
13. Geometry A wire of length x is bent into the shape of a circle.

(a) Express the circumference C of the circle as a function of x.
(b) Express the area A of the circle as a function of x.

14. Geometry A wire of length x is bent into the shape of a square.

(a) Express the perimeter p of the square as a function of x.
(b) Express the area A of the square as a function of x.

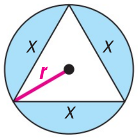
15. Geometry A semicircle of radius r is inscribed in a rectangle so that the diameter of the semicircle is the length of the rectangle. See the figure.



(a) Express the area A of the rectangle as a function of the radius r of the semicircle.
(b) Express the perimeter p of the rectangle as a function of r.

16. Geometry An equilateral triangle is inscribed in a circle of radius r. See the figure. Express the circumference C of the circle as a function of the length x of a side of the triangle.

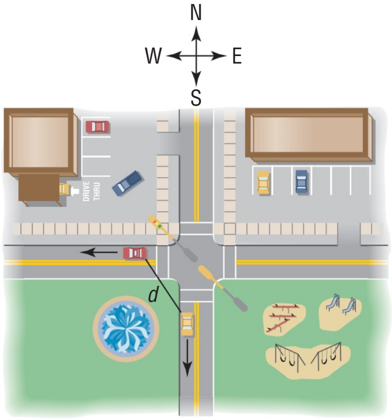
[**Hint:** First show that $r^2 = \frac{x^2}{3}$.]



17. Geometry An equilateral triangle is inscribed in a circle of radius r. See the figure in Problem 16. Express the area A within the circle, but outside the triangle, as a function of the length x of a side of the triangle.

18. Uniform Motion Two cars leave an intersection at the same time. One is headed south at a constant speed of 30 miles per hour, and the other is headed west at a constant speed of 40 miles per hour (see the figure). Build a model that expresses the distance d between the cars as a function of the time t.

[**Hint:** At $t = 0$, the cars leave the intersection.]

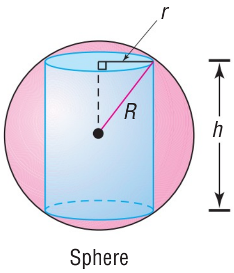


19. Uniform Motion Two cars are approaching an intersection. One is 2 miles south of the intersection and is moving at a constant speed of 30 miles per hour. At the same time, the other car is 3 miles east of the intersection and is moving at a constant speed of 40 miles per hour.

(a) Build a model that expresses the distance d between the cars as a function of time t.
[**Hint:** At $t = 0$, the cars are 2 miles south and 3 miles east of the intersection, respectively.]
(b) Use a graphing utility to graph $d = d(t)$. For what value of t is d smallest?

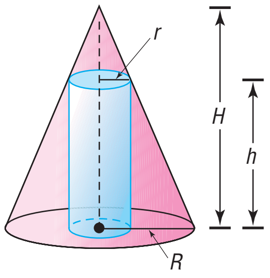
20. Inscribing a Cylinder in a Sphere Inscribe a right circular cylinder of height h and radius r in a sphere of fixed radius R. See the illustration. Express the volume V of the cylinder as a function of h.

[**Hint:** $V = \pi r^2 h$. Note also the right triangle.]



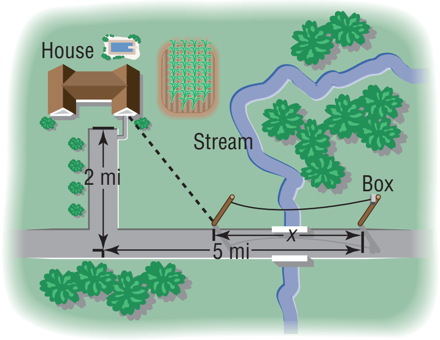
21. **Inscribing a Cylinder in a Cone** Inscribe a right circular cylinder of height h and radius r in a cone of fixed radius R and fixed height H. See the illustration. Express the volume V of the cylinder as a function of r.

[**Hint:** $V = \pi r^2 h$. Note also the similar triangles.]



Cone

22. **Installing Cable TV** MetroMedia Cable is asked to provide service to a customer whose house is located 2 miles from the road along which the cable is buried. The nearest connection box for the cable is located 5 miles down the road. See the figure.



(a) If the installation cost is \$500 per mile along the road and \$700 per mile off the road, build a model that expresses the total cost C of installation as a function of the distance x (in miles) from the connection box to the point where the cable installation turns off the road. Find the domain of $C = C(x)$.

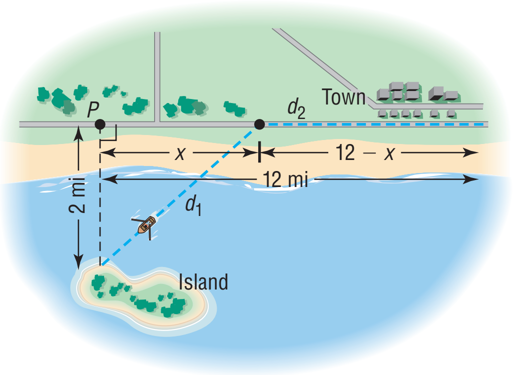
(b) Compute the cost if $x = 1$ mile.

(c) Compute the cost if $x = 3$ miles.

(d) Graph the function $C = C(x)$. Use TRACE to see how the cost C varies as x changes from 0 to 5.

(e) What value of x results in the least cost?

23. **Time Required to Go from an Island to a Town** An island is 2 miles from the nearest point P on a straight shoreline. A town is 12 miles down the shore from P. See the illustration.



(a) If a person can row a boat at an average speed of 3 miles per hour and the same person can walk 5 miles per hour, build a model that expresses the time T that it takes to go from the island to town as a function of the distance x from P to where the person lands the boat.

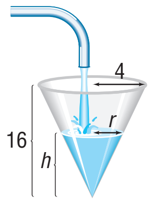
(b) What is the domain of T?

(c) How long will it take to travel from the island to town if the person lands the boat 4 miles from P?

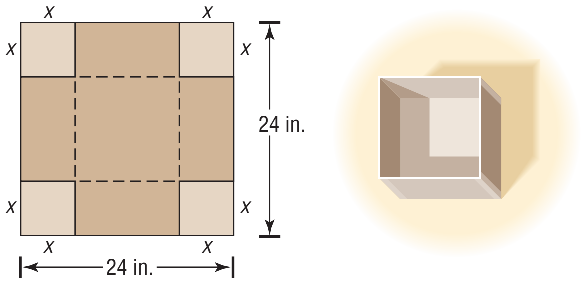
(d) How long will it take if the person lands the boat 8 miles from P?

24. **Filling a Conical Tank** Water is poured into a container in the shape of a right circular cone with radius 4 feet and height 16 feet. See the figure. Express the volume V of the water in the cone as a function of the height h of the water.

[**Hint:** The volume V of a cone of radius r and height h is $V = \frac{1}{3}\pi r^2 h$.]



25. **Constructing an Open Box** An open box with a square base is to be made from a square piece of cardboard 24 inches on a side by cutting out a square from each corner and turning up the sides. See the figure.



(a) Express the volume V of the box as a function of the length x of the side of the square cut from each corner.

(b) What is the volume if a 3-inch square is cut out?

(c) What is the volume if a 10-inch square is cut out?

(d) Graph $V = V(x)$. For what value of x is V largest?

26. **Constructing an Open Box** An open box with a square base is required to have a volume of 10 cubic feet.

(a) Express the amount A of material used to make such a box as a function of the length x of a side of the square base.

(b) How much material is required for a base 1 foot by 1 foot?

(c) How much material is required for a base 2 feet by 2 feet?

(d) Use a graphing utility to graph $A = A(x)$. For what value of x is A smallest?

Retain Your Knowledge

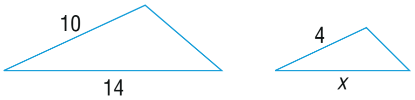
Problems 27–30 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

27. Solve: $|2x - 3| - 5 = -2$

28. A 16-foot long Ford Fusion wants to pass a 50-foot truck traveling at 55 mi/h. How fast must the car travel to completely pass the truck in 5 seconds?

29. Find the slope of the line containing the points $(3, -2)$ and $(1, 6)$.

30. Find the missing length x for the given pair of similar triangles.



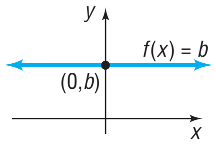
Chapter Review

Library of Functions

Constant function (p. 247)

$f(x) = b$

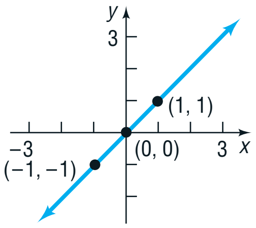
The graph is a horizontal line with y-intercept b.



Identity function (p. 247)

$f(x) = x$

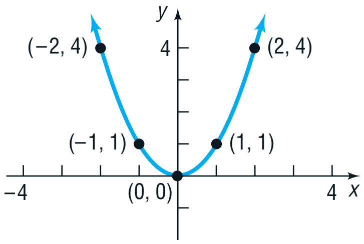
The graph is a line with slope 1 and y-intercept 0.



Square function (p. 247)

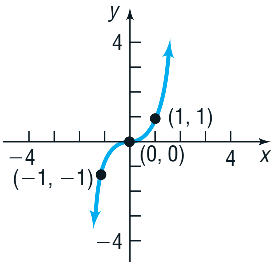
$f(x) = x^2$

The graph is a parabola with intercept at $(0, 0)$.



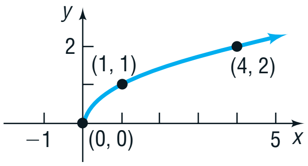
Cube function (p. 248)

$f(x) = x^3$



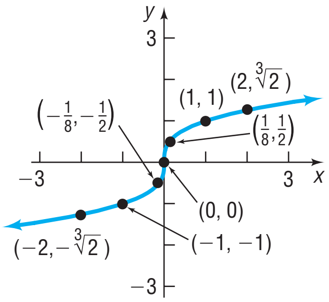
Square root function (p. 248)

$f(x) = \sqrt{x}$



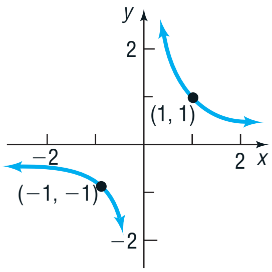
Cube root function (p. 248)

$f(x) = \sqrt[3]{x}$



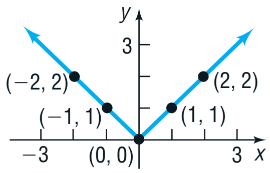
Reciprocal function (p. 248)

$f(x) = \frac{1}{x}$



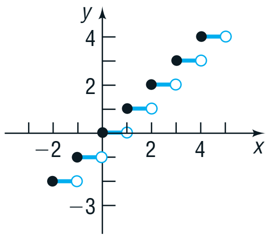
Absolute value function (p. 248)

$f(x) = |x|$



Greatest integer function (p. 249)

$f(x) = \text{int}(x)$



Things to Know

Function (pp. 207–210)	A relation between two sets so that each element x in the first set, the domain, has corresponding to it exactly one element y in the second set, the range. The range is the set of y-values of the function for the x-values in the domain. A function can also be described as a set of ordered pairs (x, y) in which no first element is paired with two different second elements.
Function notation (pp. 210–213)	$y = f(x)$ f is a symbol for the function. x is the argument, or independent variable. y is the dependent variable. $f(x)$ is the value of the function at x, or the image of x. A function f may be defined implicitly by an equation involving x and y or explicitly by writing $y = f(x)$.
Difference quotient of f (p. 213)	$\dfrac{f(x+h) - f(x)}{h} \quad h \neq 0$
Domain (pp. 214–216)	If unspecified, the domain of a function f defined by an equation is the largest set of real numbers for which $f(x)$ is a real number.
Vertical-line test (p. 222)	A set of points in the xy-plane is the graph of a function if and only if every vertical line intersects the graph in at most one point.
Even function f (p. 232)	$f(-x) = f(x)$ for every x in the domain ($-x$ must also be in the domain).
Odd function f (p. 232)	$f(-x) = -f(x)$ for every x in the domain ($-x$ must also be in the domain).
Increasing function (p. 234)	A function f is increasing on an interval I if, for any choice of x_1 and x_2 in I, with $x_1 < x_2$, we have $f(x_1) < f(x_2)$.
Decreasing function (p. 234)	A function f is decreasing on an interval I if, for any choice of x_1 and x_2 in I, with $x_1 < x_2$, we have $f(x_1) > f(x_2)$.
Constant function (p. 234)	A function f is constant on an interval I if, for all choices of x in I, the values of $f(x)$ are equal.
Local maximum (p. 235)	A function f, defined on some interval I, has a local maximum at c if there is an open interval in I containing c such that, for all x in this open interval, $f(x) \leq f(c)$.
Local minimum (p. 235)	A function f, defined on some interval I, has a local minimum at c if there is an open interval in I containing c such that, for all x in this open interval, $f(x) \geq f(c)$.
Absolute maximum and Absolute minimum (p. 236)	Let f denote a function defined on some interval I. If there is a number u in I for which $f(x) \leq f(u)$ for all x in I, then f has an absolute maximum at u, and the number $f(u)$ is the absolute maximum of f on I. If there is a number v in I for which $f(x) \geq f(v)$, for all x in I, then f has an absolute minimum at v, and the number $f(v)$ is the absolute minimum of f on I.
Average rate of change of a function (p. 238)	The average rate of change of f from a to b is $\dfrac{\Delta y}{\Delta x} = \dfrac{f(b) - f(a)}{b - a} \quad a \neq b$

Objectives

Section	You should be able to . . .	Examples	Review Exercises
3.1	1 Determine whether a relation represents a function (p. 207)	1–5	1, 2
	2 Find the value of a function (p. 210)	6, 7	3–5, 39
	3 Find the difference quotient of a function (p. 213)	8	15
	4 Find the domain of a function defined by an equation (p. 214)	9, 10	6–11
	5 Form the sum, difference, product, and quotient of two functions (p. 216)	11	12–14
3.2	1 Identify the graph of a function (p. 222)	1	27, 28
	2 Obtain information from or about the graph of a function (p. 223)	2–4	16(a)–(e), 17(a), 17(e), 17(g)

Section		You should be able to . . .	Examples	Review Exercises
3.3	1	Determine even and odd functions from a graph (p. 231)	1	17(f)
	2	Identify even and odd functions from an equation (p. 233)	2	18–21
	3	Use a graph to determine where a function is increasing, decreasing, or constant (p. 234)	3	17(b)
	4	Use a graph to locate local maxima and local minima (p. 235)	4	17(c)
	5	Use a graph to locate the absolute maximum and the absolute minimum (p. 236)	5	17(d)
	6	Use a graphing utility to approximate local maxima and local minima and to determine where a function is increasing or decreasing (p. 237)	6	22, 23, 40(d), 41(b)
	7	Find the average rate of change of a function (p. 238)	7, 8	24–26
3.4	1	Graph the functions listed in the library of functions (p. 245)	1, 2	29, 30
	2	Graph piecewise-defined functions (p. 250)	3, 4, 5	37, 38
3.5	1	Graph functions using vertical and horizontal shifts (p. 256)	1, 2, 4–6	16(f), 31, 33–36
	2	Graph functions using compressions and stretches (p. 258)	3, 5	16(g), 32, 36
	3	Graph functions using reflections about the x-axis or y-axis (p. 260)	4, 6	16(h), 32, 34, 36
3.6	1	Build and analyze functions (p. 268)	1–3	40, 41

Review Exercises

In Problems 1 and 2, determine whether each relation represents a function. For each function, state the domain and range.

1. $\{(-1, 0), (2, 3), (4, 0)\}$

2. $\{(4, -1), (2, 1), (4, 2)\}$

In Problems 3–5, find the following for each function:

(a) $f(2)$ (b) $f(-2)$ (c) $f(-x)$ (d) $-f(x)$ (e) $f(x-2)$ (f) $f(2x)$

3. $f(x) = \dfrac{3x}{x^2 - 1}$

4. $f(x) = \sqrt{x^2 - 4}$

5. $f(x) = \dfrac{x^2 - 4}{x^2}$

In Problems 6–11, find the domain of each function.

6. $f(x) = \dfrac{x}{x^2 - 9}$

7. $f(x) = \sqrt{2 - x}$

8. $g(x) = \dfrac{|x|}{x}$

9. $f(x) = \dfrac{x}{x^2 + 2x - 3}$

10. $f(x) = \dfrac{\sqrt{x + 1}}{x^2 - 4}$

11. $g(x) = \dfrac{x}{\sqrt{x + 8}}$

In Problems 12–14, find $f + g, f - g, f \cdot g$, and $\dfrac{f}{g}$ for each pair of functions. State the domain of each of these functions.

12. $f(x) = 2 - x;\quad g(x) = 3x + 1$

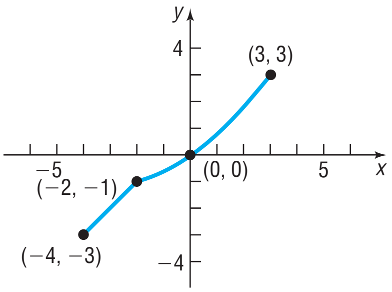
13. $f(x) = 3x^2 + x + 1;\quad g(x) = 3x$

14. $f(x) = \dfrac{x + 1}{x - 1};\quad g(x) = \dfrac{1}{x}$

15. Find the difference quotient of $f(x) = -2x^2 + x + 1$; that is, find $\dfrac{f(x + h) - f(x)}{h}$, $h \neq 0$.

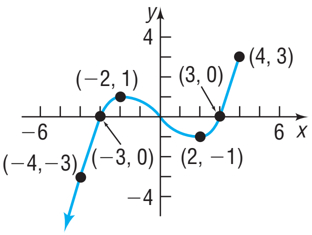
16. Consider the graph of the function f on the right.

(a) Find the domain and the range of f.
(b) List the intercepts.
(c) Find $f(-2)$.
(d) For what value of x does $f(x) = -3$?
(e) Solve $f(x) > 0$.
(f) Graph $y = f(x - 3)$.
(g) Graph $y = f\left(\dfrac{1}{2}x\right)$.
(h) Graph $y = -f(x)$.



17. Use the graph of the function f shown to find:

(a) The domain and the range of f.
(b) The intervals on which f is increasing, decreasing, or constant.
(c) The local minimum values and local maximum values.
(d) The absolute maximum and absolute minimum.
(e) Whether the graph is symmetric with respect to the x-axis, the y-axis, or the origin.
(f) Whether the function is even, odd, or neither.
(g) The intercepts, if any.



In Problems 18–21, determine (algebraically) whether the given function is even, odd, or neither.

18. $f(x) = x^3 - 4x$

19. $g(x) = \dfrac{4 + x^2}{1 + x^4}$

20. $G(x) = 1 - x + x^3$

21. $f(x) = \dfrac{x}{1 + x^2}$

In Problems 22 and 23, use a graphing utility to graph each function over the indicated interval. Approximate any local maximum values and local minimum values. Determine where the function is increasing and where it is decreasing.

22. $f(x) = 2x^3 - 5x + 1 \quad [-3, 3]$

23. $f(x) = 2x^4 - 5x^3 + 2x + 1 \quad [-2, 3]$

24. Find the average rate of change of $f(x) = 8x^2 - x$:

(a) From 1 to 2 (b) From 0 to 1 (c) From 2 to 4

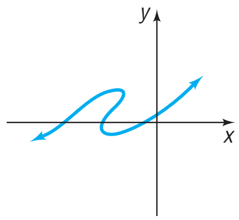
In Problems 25 and 26, find the average rate of change from 2 to 3 for each function f. Be sure to simplify.

25. $f(x) = 2 - 5x$

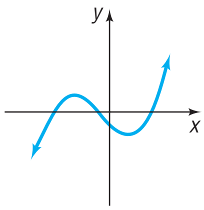
26. $f(x) = 3x - 4x^2$

In Problems 27 and 28, is the graph shown the graph of a function?

27.



28.



In Problems 29 and 30, graph each function. Be sure to label at least three points.

29. $f(x) = |x|$

30. $f(x) = \sqrt{x}$

In Problems 31–36, graph each function using the techniques of shifting, compressing or stretching, and reflections. Identify any intercepts of the graph. State the domain and, based on the graph, find the range.

31. $F(x) = |x| - 4$

32. $g(x) = -2|x|$

33. $h(x) = \sqrt{x - 1}$

34. $f(x) = \sqrt{1 - x}$

35. $h(x) = (x - 1)^2 + 2$

36. $g(x) = -2(x + 2)^3 - 8$

In Problems 37 and 38:

(a) Find the domain of each function. *(b) Locate any intercepts.*
(c) Graph each function. *(d) Based on the graph, find the range.*

37. $f(x) = \begin{cases} 3x & \text{if } -2 < x \le 1 \\ x + 1 & \text{if } x > 1 \end{cases}$

38. $f(x) = \begin{cases} x & \text{if } -4 \le x < 0 \\ 1 & \text{if } x = 0 \\ 3x & \text{if } x > 0 \end{cases}$

39. A function f is defined by

$$f(x) = \frac{Ax + 5}{6x - 2}$$

If $f(1) = 4$, find A.

40. Constructing a Closed Box A closed box with a square base is required to have a volume of 10 cubic feet.

(a) Build a model that expresses the amount A of material used to make such a box as a function of the length x of a side of the square base.
(b) How much material is required for a base 1 foot by 1 foot?
(c) How much material is required for a base 2 feet by 2 feet?
(d) Graph $A = A(x)$. For what value of x is A smallest?

41. Area of a Rectangle A rectangle has one vertex in quadrant I on the graph of $y = 10 - x^2$, another at the origin, one on the positive x-axis, and one on the positive y-axis.

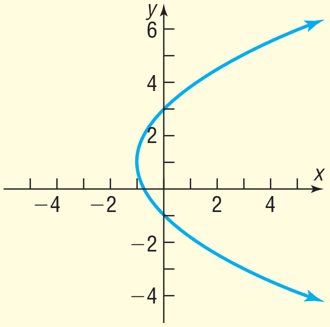
(a) Express the area A of the rectangle as a function of x.
(b) Find the largest area A that can be enclosed by the rectangle.

Chapter Test

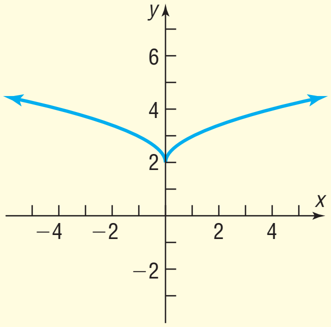
CHAPTER **Test Prep** VIDEOS The Chapter Test Prep Videos are step-by-step solutions available in MyMathLab®, or on this text's YouTube Channel. Flip back to the Resources for Success page for a link to this text's YouTube channel.

1. Determine whether each relation represents a function. For each function, state the domain and the range.

(a) $\{(2, 5), (4, 6), (6, 7), (8, 8)\}$
(b) $\{(1, 3), (4, -2), (-3, 5), (1, 7)\}$
(c)



(d)



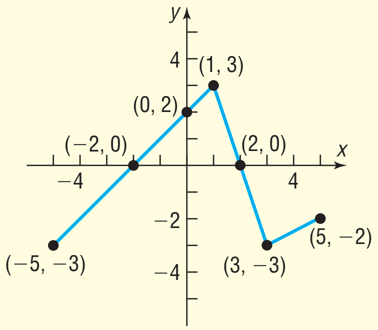
In Problems 2–4, find the domain of each function and evaluate each function at $x = -1$.

2. $f(x) = \sqrt{4 - 5x}$

3. $g(x) = \dfrac{x + 2}{|x + 2|}$

4. $h(x) = \dfrac{x - 4}{x^2 + 5x - 36}$

5. Consider the graph of the function f below.



(a) Find the domain and the range of f.
(b) List the intercepts.
(c) Find $f(1)$.
(d) For what value(s) of x does $f(x) = -3$?
(e) Solve $f(x) < 0$.

6. Use a graphing utility to graph the function $f(x) = -x^4 + 2x^3 + 4x^2 - 2$ on the interval $[-5, 5]$. Then approximate any local maximum values and local minimum values rounded to two decimal places. Determine where the function is increasing and where it is decreasing.

7. Consider the function $g(x) = \begin{cases} 2x + 1 & \text{if } x < -1 \\ x - 4 & \text{if } x \geq -1 \end{cases}$

(a) Graph the function.
(b) List the intercepts.
(c) Find $g(-5)$.
(d) Find $g(2)$.

8. For the function $f(x) = 3x^2 - 2x + 4$, find the average rate of change of f from 3 to 4.

9. For the functions $f(x) = 2x^2 + 1$ and $g(x) = 3x - 2$, find the following and simplify.

(a) $(f - g)(x)$
(b) $(f \cdot g)(x)$
(c) $f(x + h) - f(x)$

10. Graph each function using the techniques of shifting, compressing or stretching, and reflecting. Start with the graph of the basic function and show all stages.

(a) $h(x) = -2(x + 1)^3 + 3$
(b) $g(x) = |x + 4| + 2$

11. The variable interest rate on a student loan changes each July 1 based on the bank prime loan rate. For the years 1992–2007, this rate can be approximated by the model

$$r(x) = -0.115x^2 + 1.183x + 5.623$$

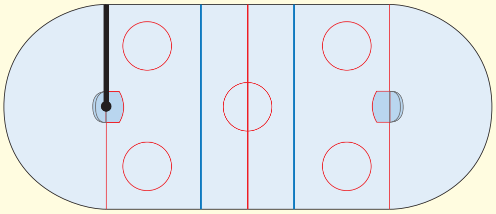
where x is the number of years since 1992 and r is the interest rate as a percent.

(a) Use a graphing utility to estimate the highest rate during this time period. During which year was the interest rate the highest?
(b) Use the model to estimate the rate in 2010. Does this value seem reasonable?

Source: *U.S. Federal Reserve*

12. A community skating rink is in the shape of a rectangle with semicircles attached at the ends. The length of the rectangle is 20 feet less than twice the width. The thickness of the ice is 0.75 inch.

(a) Build a model that expresses the ice volume, V, as a function of the width, x.
(b) How much ice is in the rink if the width is 90 feet?



Cumulative Review

In Problems 1–6, find the real solutions of each equation.

1. $3x - 8 = 10$

2. $3x^2 - x = 0$

3. $x^2 - 8x - 9 = 0$

4. $6x^2 - 5x + 1 = 0$

5. $|2x + 3| = 4$

6. $\sqrt{2x + 3} = 2$

In Problems 7–9, solve each inequality. Graph the solution set.

7. $2 - 3x > 6$

8. $|2x - 5| < 3$

9. $|4x + 1| \geq 7$

10. (a) Find the distance from $P_1 = (-2, -3)$ to $P_2 = (3, -5)$.
(b) What is the midpoint of the line segment from P_1 to P_2?
(c) What is the slope of the line containing the points P_1 and P_2?

In Problems 11–14, graph each equation.

11. $3x - 2y = 12$

12. $x = y^2$

13. $x^2 + (y - 3)^2 = 16$

14. $y = \sqrt{x}$

15. For the equation $3x^2 - 4y = 12$, find the intercepts and check for symmetry.

16. Find the slope–intercept form of the equation of the line containing the points $(-2, 4)$ and $(6, 8)$.

In Problems 17–19, graph each function.

17. $f(x) = (x + 2)^2 - 3$

18. $f(x) = \dfrac{1}{x}$

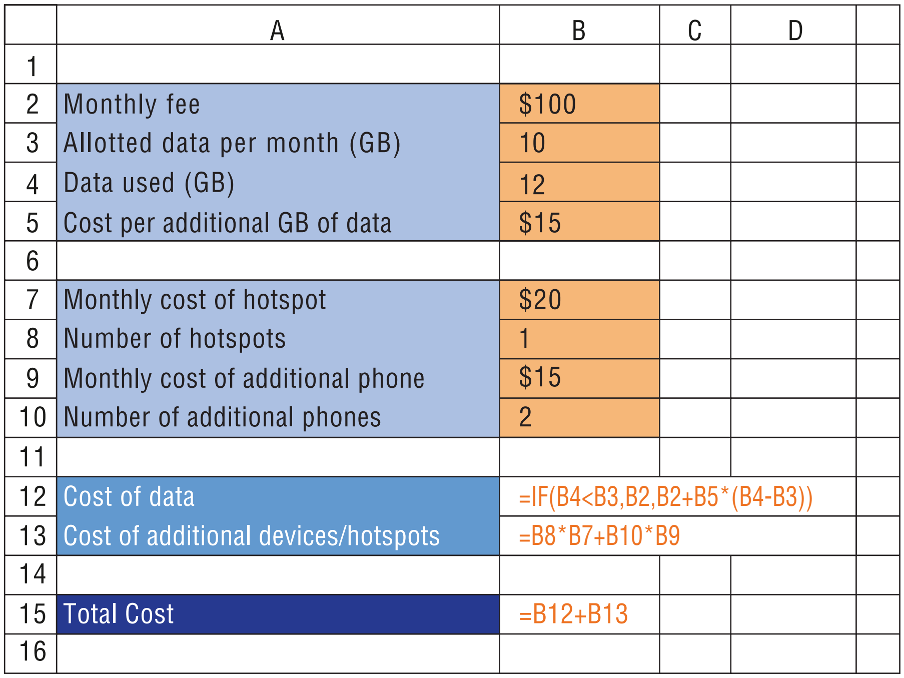
19. $f(x) = \begin{cases} 2 - x & \text{if } x \leq 2 \\ |x| & \text{if } x > 2 \end{cases}$

Chapter Projects

I. **Choosing a Wireless Data Plan** Collect information from your family, friends, or consumer agencies such as Consumer Reports. Then decide on a cellular provider, choosing the company that you feel offers the best service. Once you have selected a service provider, research the various types of individual plans offered by the company by visiting the provider's website. Many providers offer family plans that include unlimited talk and text. The monthly cost is primarily determined by the amount of data used and the number of devices.

1. Suppose you expect to use 10 gigabytes of data for a single smartphone. What would be the monthly cost of each plan you are considering?
2. Suppose you expect to use 30 gigabytes of data and want a personal hotspot, but you still have only a single smartphone. What would be the monthly cost of each plan you are considering?
3. Suppose you expect to use 20 gigabytes of data with three smartphones sharing the data. What would be the monthly cost of each plan you are considering?
4. Suppose you expect to use 20 gigabytes of data with a single smartphone and a personal hotspot. What would be the monthly cost of each plan you are considering?
5. Build a model that describes the monthly cost C, in dollars, as a function of the number of data gigabytes used, g, assuming a single smartphone and a personal hotspot for each plan you are considering.
6. Graph each function from Problem 5.
7. Based on your particular usage, which plan is best for you?
8. Now, develop an Excel spreadsheet to analyze the various plans you are considering. Suppose you want a family plan with unlimited talk and text that offers 10 gigabytes of shared data and costs \$100 per month. Additional gigabytes of data cost \$15 per gigabyte, extra phones can be added to the plan for \$15 each per month, and each hotspot costs \$20 per month. Because wireless

data plans have a cost structure based on piecewise-defined functions, we need an "if/then" statement within Excel to analyze the cost of the plan. Use the accompanying Excel spreadsheet as a guide in developing your spreadsheet. Enter into your spreadsheet a variety of possible amounts of data and various numbers of additional phones and hotspots.



	A	B	C	D	
1					
2	Monthly fee	$100			
3	Allotted data per month (GB)	10			
4	Data used (GB)	12			
5	Cost per additional GB of data	$15			
6					
7	Monthly cost of hotspot	$20			
8	Number of hotspots	1			
9	Monthly cost of additional phone	$15			
10	Number of additional phones	2			
11					
12	Cost of data	=IF(B4<B3,B2,B2+B5*(B4-B3))			
13	Cost of additional devices/hotspots	=B8*B7+B10*B9			
14					
15	Total Cost	=B12+B13			
16					

9. Write a paragraph supporting the choice in plans that best meets your needs.

10. How are "if/then" loops similar to a piecewise-defined function?

Citation: Excel © 2013 Microsoft Corporation. Used with permission from Microsoft.

The following projects are available on the Instructor's Resource Center (IRC).

II. **Project at Motorola:** ***Wireless Internet Service*** Use functions and their graphs to analyze the total cost of various wireless Internet service plans.

III. **Cost of Cable** When government regulations and customer preference influence the path of a new cable line, the Pythagorean Theorem can be used to assess the cost of installation.

IV. **Oil Spill** Functions are used to analyze the size and spread of an oil spill from a leaking tanker.

4 Linear and Quadratic Functions

The Beta of a Stock

Investing in the stock market can be rewarding and fun, but how does one go about selecting which stocks to purchase? Financial investment firms hire thousands of analysts who track individual stocks (equities) and assess the value of the underlying company. One measure the analysts consider is the *beta* of the stock. **Beta** measures the relative risk of an individual company's equity to that of a market basket of stocks, such as the Standard & Poor's 500. But how is beta computed?

—See the Internet-based Chapter Project I—

Outline

••• A Look Back

Up to now, our discussion has focused on graphs of equations and functions. We learned how to graph equations using the point-plotting method, intercepts, and the tests for symmetry. In addition, we learned what a function is and how to identify whether a relation represents a function. We also discussed properties of functions, such as domain/range, increasing/decreasing, even/odd, and average rate of change.

A Look Ahead •••

Going forward, we will look at classes of functions. This chapter focuses on linear and quadratic functions, their properties, and their applications.

4.1 Properties of Linear Functions and Linear Models

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Lines (Section 2.2, pp. 173–184)
- Graphs of Equations in Two Variables; Intercepts; Symmetry (Section 2.1, pp. 165–170)
- Linear Equations (Section 1.2, pp. 102–103)
- Functions (Section 3.1, pp. 207–218)
- The Graph of a Function (Section 3.2, pp. 222–226)
- Properties of Functions (Section 3.3, pp. 231–239)

Now Work the **'Are You Prepared?'** problems on page 287.

OBJECTIVES
1 Graph Linear Functions (p. 281)
2 Use Average Rate of Change to Identify Linear Functions (p. 281)
3 Determine Whether a Linear Function Is Increasing, Decreasing, or Constant (p. 284)
4 Build Linear Models from Verbal Descriptions (p. 285)

1 Graph Linear Functions

In Section 2.2 we discussed lines. In particular, for nonvertical lines we developed the slope–intercept form of the equation of a line $y = mx + b$. When the slope–intercept form of a line is written using function notation, the result is a *linear function*.

DEFINITION

A **linear function** is a function of the form

$$\boxed{f(x) = mx + b}$$

The graph of a linear function is a line with slope m and y-intercept b. Its domain is the set of all real numbers. ■

Functions that are not linear are said to be **nonlinear**.

EXAMPLE 1

Graphing a Linear Function

Graph the linear function $f(x) = -3x + 7$. What are the domain and the range of f?

Solution This is a linear function with slope $m = -3$ and y-intercept $b = 7$. To graph this function, plot the point $(0, 7)$, the y-intercept, and use the slope to find an additional point by moving right 1 unit and down 3 units. See Figure 1. The domain and range of f are each the set of all real numbers. ■

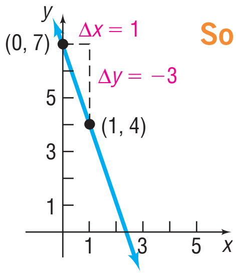


Figure 1 $f(x) = -3x + 7$

Alternatively, an additional point could have been found by evaluating the function at some $x \neq 0$. For $x = 1$, $f(1) = -3(1) + 7 = 4$ and the point $(1, 4)$ lies on the graph.

Now Work PROBLEMS 13(a) AND (b)

2 Use Average Rate of Change to Identify Linear Functions

Look at Table 1 on the next page, which shows certain values of the independent variable x and corresponding values of the dependent variable y for the function $f(x) = -3x + 7$. Notice that as the value of the independent variable, x, increases by 1, the value of the dependent variable y decreases by 3. That is, the average rate of change of y with respect to x is a constant, -3.

Table 1

x	$y = f(x) = -3x + 7$	Average Rate of Change $= \dfrac{\Delta y}{\Delta x}$
-2	13	
		$\dfrac{10 - 13}{-1 - (-2)} = \dfrac{-3}{1} = -3$
-1	10	
		$\dfrac{7 - 10}{0 - (-1)} = \dfrac{-3}{1} = -3$
0	7	
		-3
1	4	
		-3
2	1	
		-3
3	-2	

It is not a coincidence that the average rate of change of the linear function $f(x) = -3x + 7$ is the slope of the linear function. That is, $\dfrac{\Delta y}{\Delta x} = m = -3$. The following theorem states this fact.

THEOREM

Average Rate of Change of a Linear Function

Linear functions have a constant average rate of change. That is, the average rate of change of a linear function $f(x) = mx + b$ is

$$\boxed{\frac{\Delta y}{\Delta x} = m}$$

■

Proof The average rate of change of $f(x) = mx + b$ from x_1 to x_2, $x_1 \neq x_2$, is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1}$$
$$= \frac{mx_2 - mx_1}{x_2 - x_1} = \frac{m(x_2 - x_1)}{x_2 - x_1} = m$$

■

Based on the theorem just proved, the average rate of change of the function $g(x) = -\dfrac{2}{5}x + 5$ is $-\dfrac{2}{5}$.

Now Work PROBLEM 13(c)

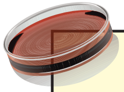
As it turns out, only linear functions have a constant average rate of change. Because of this, the average rate of change can be used to determine whether a function is linear. This is especially useful if the function is defined by a data set.

EXAMPLE 2 Using the Average Rate of Change to Identify Linear Functions

(a) A strain of *E. coli* known as Beu 397-recA441 is placed into a Petri dish at 30° Celsius and allowed to grow. The data shown in Table 2 are collected. The population is measured in grams and the time in hours. Plot the ordered pairs (x, y) in the Cartesian plane, and use the average rate of change to determine whether the function is linear.

(b) The data in Table 3 represent the maximum number of heartbeats that healthy individuals of different ages should have during a 15-second interval of time while exercising. Plot the ordered pairs (x, y) in the Cartesian plane, and use the average rate of change to determine whether the function is linear.

Table 2



Time (hours), x	Population (grams), y	(x, y)
0	0.09	(0, 0.09)
1	0.12	(1, 0.12)
2	0.16	(2, 0.16)
3	0.22	(3, 0.22)
4	0.29	(4, 0.29)
5	0.39	(5, 0.39)

Table 3

Age, x	Maximum Number of Heartbeats, y	(x, y)
20	50	(20, 50)
30	47.5	(30, 47.5)
40	45	(40, 45)
50	42.5	(50, 42.5)
60	40	(60, 40)
70	37.5	(70, 37.5)

Source: American Heart Association

Solution Compute the average rate of change of each function. If the average rate of change is constant, the function is linear. If the average rate of change is not constant, the function is nonlinear.

(a) Figure 2 shows the points listed in Table 2 plotted in the Cartesian plane. Note that it is impossible to draw a straight line that contains all the points. Table 4 displays the average rate of change of the population.

Table 4

Time (hours), x	Population (grams), y	Average Rate of Change $= \frac{\Delta y}{\Delta x}$
0	0.09	
		$\frac{0.12 - 0.09}{1 - 0} = 0.03$
1	0.12	
		0.04
2	0.16	
		0.06
3	0.22	
		0.07
	0.29	
4		0.10
5	0.39	

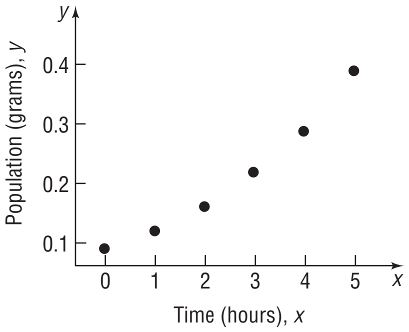


Figure 2

Because the average rate of change is not constant, the function is not linear. In fact, because the average rate of change is increasing as the value of the independent variable increases, the function is increasing at an increasing rate. So not only is the population increasing over time, but it is also growing more rapidly as time passes.

(b) Figure 3 on the next page shows the points listed in Table 3 plotted in the Cartesian plane. Note that the data in Figure 3 lie on a straight line. Table 5 on the next page contains the average rate of change of the maximum number of heartbeats. The average rate of change of the heartbeat data is constant, -0.25 beat per year, so the function is linear. To find the linear function, use the point-slope formula with $x_1 = 20$, $y_1 = 50$, and $m = -0.25$.

$$\begin{aligned} y - 50 &= -0.25(x - 20) \qquad y - y_1 = m(x - x_1) \\ y - 50 &= -0.25x + 5 \\ y &= -0.25x + 55 \end{aligned}$$

Table 5

Age, x	Maximum Number of Heartbeats, y	Average Rate of Change $= \dfrac{\Delta y}{\Delta x}$
20	50	
		$\dfrac{47.5 - 50}{30 - 20} = -0.25$
30	47.5	
		-0.25
40	45	
		-0.25
50	42.5	
		-0.25
60	40	
		-0.25
70	37.5	

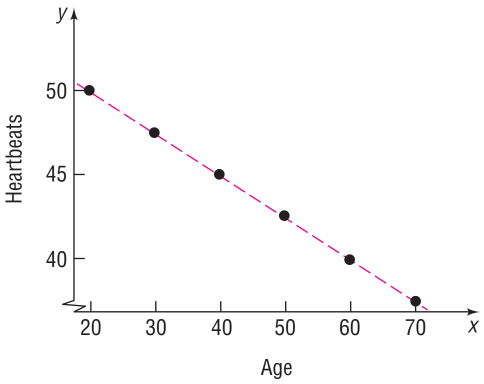


Figure 3

Now Work PROBLEM 21

3 Determine Whether a Linear Function Is Increasing, Decreasing, or Constant

Look back at the Seeing the Concept on page 178. When the slope m of a linear function is positive $(m > 0)$, the line slants upward from left to right. When the slope m of a linear function is negative $(m < 0)$, the line slants downward from left to right. When the slope m of a linear function is zero $(m = 0)$, the line is horizontal.

THEOREM

Increasing, Decreasing, and Constant Linear Functions

A linear function $f(x) = mx + b$ is increasing over its domain if its slope, m, is positive. It is decreasing over its domain if its slope, m, is negative. It is constant over its domain if its slope, m, is zero.

EXAMPLE 3 Determining Whether a Linear Function Is Increasing, Decreasing, or Constant

Determine whether the following linear functions are increasing, decreasing, or constant.

(a) $f(x) = 5x - 2$

(b) $g(x) = -2x + 8$

(c) $s(t) = \dfrac{3}{4}t - 4$

(d) $h(z) = 7$

Solution

(a) For the linear function $f(x) = 5x - 2$, the slope is 5, which is positive. The function f is increasing on the interval $(-\infty, \infty)$.

(b) For the linear function $g(x) = -2x + 8$, the slope is -2, which is negative. The function g is decreasing on the interval $(-\infty, \infty)$.

(c) For the linear function $s(t) = \dfrac{3}{4}t - 4$, the slope is $\dfrac{3}{4}$, which is positive. The function s is increasing on the interval $(-\infty, \infty)$.

(d) The linear function h can be written as $h(z) = 0z + 7$. Because the slope is 0, the function h is constant on the interval $(-\infty, \infty)$. ■

Now Work PROBLEM 13(d)

4 Build Linear Models from Verbal Descriptions

When the average rate of change of a function is constant, a linear function can model the relation between the two variables. For example, if a recycling company pays \$0.52 per pound for aluminum cans, then the relation between the price paid p and the pounds recycled x can be modeled as the linear function $p(x) = 0.52x$, with slope $m = \dfrac{0.52 \text{ dollar}}{1 \text{ pound}}$.

Modeling with a Linear Function

If the average rate of change of a function is a constant m, a linear function f can be used to model the relation between the two variables as follows:

$$\boxed{f(x) = mx + b}$$

where b is the value of f at 0; that is, $b = f(0)$.

EXAMPLE 4

Straight-line Depreciation

Book value is the value of an asset that a company uses to create its balance sheet. Some companies depreciate their assets using straight-line depreciation so that the value of the asset declines by a fixed amount each year. The amount of the decline depends on the useful life that the company assigns to the asset. Suppose that a company just purchased a fleet of new cars for its sales force at a cost of \$31,500 per car. The company chooses to depreciate each vehicle using the straight-line method over 7 years. This means that each car will depreciate by $\dfrac{\$31{,}500}{7} = \$4500$ per year.

(a) Write a linear function that expresses the book value V of each car as a function of its age, x.

(b) Graph the linear function.

(c) What is the book value of each car after 3 years?

(d) Interpret the slope.

(e) When will the book value of each car be \$9000?
[**Hint:** Solve the equation $V(x) = 9000$.]

Solution

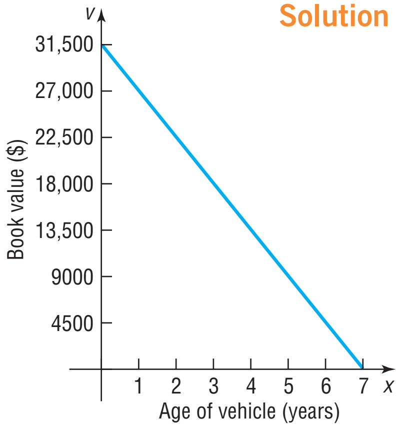


Figure 4
$V(x) = -4500x + 31{,}500$

(a) If we let $V(x)$ represent the value of each car after x years, then $V(0)$ represents the original value of each car, so $V(0) = \$31{,}500$. The y-intercept of the linear function is \$31,500. Because each car depreciates by \$4500 per year, the slope of the linear function is -4500. The linear function that represents the book value V of each car after x years is

$$V(x) = -4500x + 31{,}500$$

(b) Figure 4 shows the graph of V.

(c) The book value of each car after 3 years is

$$\begin{aligned} V(3) &= -4500(3) + 31{,}500 \\ &= \$18{,}000 \end{aligned}$$

(d) Since the slope of $V(x) = -4500x + 31{,}500$ is -4500, the average rate of change of the book value is –\$4500/year. So for each additional year that passes, the book value of the car decreases by \$4500.

(e) To find when the book value will be \$9000, solve the equation

$$\begin{aligned} V(x) &= 9000 \\ -4500x + 31{,}500 &= 9000 \\ -4500x &= -22{,}500 && \text{Subtract 31,500 from each side.} \\ x &= \frac{-22{,}500}{-4500} = 5 && \text{Divide by } -4500. \end{aligned}$$

Each car will have a book value of \$9000 when it is 5 years old. ■

Now Work PROBLEM 45

EXAMPLE 5 Supply and Demand

The **quantity supplied** of a good is the amount of a product that a company is willing to make available for sale at a given price. The **quantity demanded** of a good is the amount of a product that consumers are willing to purchase at a given price. Suppose that the quantity supplied, S, and quantity demanded, D, of cell phones each month are given by the following functions:

$$\begin{aligned} S(p) &= 30p - 900 \\ D(p) &= -7.5p + 2850 \end{aligned}$$

where p is the price (in dollars) of the cell phone.

(a) The **equilibrium price** of a product is defined as the price at which quantity supplied equals quantity demanded. That is, the equilibrium price is the price at which $S(p) = D(p)$. Find the equilibrium price of cell phones. What is the **equilibrium quantity**, the amount demanded (or supplied) at the equilibrium price?

(b) Determine the prices for which quantity supplied is greater than quantity demanded. That is, solve the inequality $S(p) > D(p)$.

(c) Graph $S = S(p)$, $D = D(p)$ and label the equilibrium point.

Solution (a) To find the equilibrium price, solve the equation $S(p) = D(p)$.

$$\begin{aligned} 30p - 900 &= -7.5p + 2850 && S(p) = 30p - 900;\ D(p) = -7.5p + 2850 \\ 30p &= -7.5p + 3750 && \text{Add 900 to each side.} \\ 37.5p &= 3750 && \text{Add } 7.5p \text{ to each side.} \\ p &= 100 && \text{Divide each side by 37.5.} \end{aligned}$$

The equilibrium price is \$100 per cell phone. To find the equilibrium quantity, evaluate either $S(p)$ or $D(p)$ at $p = 100$.

$$S(100) = 30(100) - 900 = 2100$$

The equilibrium quantity is 2100 cell phones. At a price of \$100 per phone, the company will produce and sell 2100 phones each month and have no shortages or excess inventory.

(b) The inequality $S(p) > D(p)$ is

$$\begin{aligned} 30p - 900 &> -7.5p + 2850 && S(p) > D(p) \\ 30p &> -7.5p + 3750 && \text{Add 900 to each side.} \\ 37.5p &> 3750 && \text{Add } 7.5p \text{ to each side.} \\ p &> 100 && \text{Divide each side by 37.5.} \end{aligned}$$

If the company charges more than \$100 per phone, quantity supplied will exceed quantity demanded. In this case the company will have excess phones in inventory.

(c) Figure 5 shows the graphs of $S = S(p)$ and $D = D(p)$ with the equilibrium point labeled.

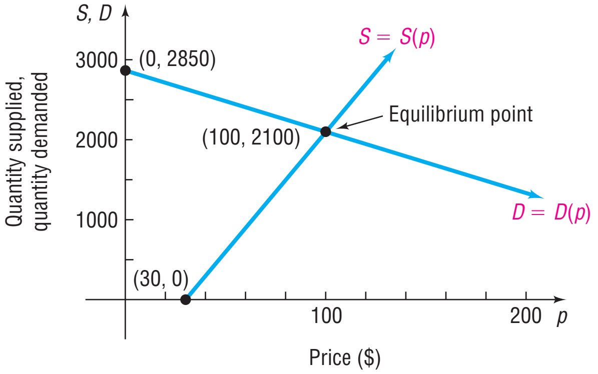


Figure 5 Supply and demand functions

Now Work PROBLEM 39

4.1 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. Graph $y = 2x - 3$. (pp. 176–179)

2. Find the slope of the line joining the points $(2, 5)$ and $(-1, 3)$. (pp. 173–175)

3. Find the average rate of change of $f(x) = 3x^2 - 2$, from 2 to 4. (pp. 238–239)

4. Solve: $60x - 900 = -15x + 2850$. (pp. 102–103)

5. If $f(x) = x^2 - 4$, find $f(-2)$. (pp. 210–212)

6. ***True or False*** The graph of the function $f(x) = x^2$ is increasing on the interval $[0, \infty)$. (p. 234)

Concepts and Vocabulary

7. For the graph of the linear function $f(x) = mx + b$, m is the _____ and b is the __________.

8. If the slope m of the graph of a linear function is _______, the function is increasing over its domain.

9. ***True or False*** The slope of a nonvertical line is the average rate of change of the linear function.

10. ***True or False*** The average rate of change of $f(x) = 2x + 8$ is 8.

11. What is the only type of function that has a constant average rate of change?
(a) linear function
(b) quadratic function
(c) step function
(d) absolute value function

12. A car has 12,500 miles on its odometer. Say the car is driven an average of 40 miles per day. Choose the model that expresses the number of miles N that will be on its odometer after x days.
(a) $N(x) = -40x + 12{,}500$
(b) $N(x) = 40x - 12{,}500$
(c) $N(x) = 12{,}500x + 40$
(d) $N(x) = 40x + 12{,}500$

Skill Building

In Problems 13–20, a linear function is given.
(a) Determine the slope and y-intercept of each function.
(b) Use the slope and y-intercept to graph the linear function.
(c) Determine the average rate of change of each function.
(d) Determine whether the linear function is increasing, decreasing, or constant.

13. $f(x) = 2x + 3$

14. $g(x) = 5x - 4$

15. $h(x) = -3x + 4$

16. $p(x) = -x + 6$

17. $f(x) = \frac{1}{4}x - 3$

18. $h(x) = -\frac{2}{3}x + 4$

19. $F(x) = 4$

20. $G(x) = -2$

In Problems 21–28, determine whether the given function is linear or nonlinear. If it is linear, determine the equation of the line.

21.

x	y = f(x)
−2	4
−1	1
0	−2
1	−5
2	−8

22.

x	y = f(x)
−2	1/4
−1	1/2
0	1
1	2
2	4

23.

x	y = f(x)
−2	−8
−1	−3
0	0
1	1
2	0

24.

x	y = f(x)
−2	−4
−1	0
0	4
1	8
2	12

25.

x	$y = f(x)$
−2	−26
−1	−4
0	2
1	−2
2	−10

26.

x	$y = f(x)$
−2	−4
−1	−3.5
0	−3
1	−2.5
2	−2

27.

x	$y = f(x)$
−2	8
−1	8
0	8
1	8
2	8

28.

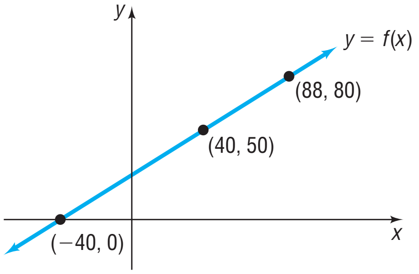
x	$y = f(x)$
−2	0
−1	1
0	4
1	9
2	16

Applications and Extensions

29. Suppose that $f(x) = 4x - 1$ and $g(x) = -2x + 5$.
(a) Solve $f(x) = 0$.
(b) Solve $f(x) > 0$.
(c) Solve $f(x) = g(x)$.
(d) Solve $f(x) \leq g(x)$.
(e) Graph $y = f(x)$ and $y = g(x)$ and label the point that represents the solution to the equation $f(x) = g(x)$.

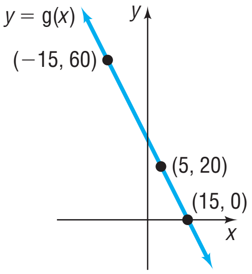
30. Suppose that $f(x) = 3x + 5$ and $g(x) = -2x + 15$.
(a) Solve $f(x) = 0$.
(b) Solve $f(x) < 0$.
(c) Solve $f(x) = g(x)$.
(d) Solve $f(x) \geq g(x)$.
(e) Graph $y = f(x)$ and $y = g(x)$ and label the point that represents the solution to the equation $f(x) = g(x)$.

31. In parts (a)–(f), use the following figure.



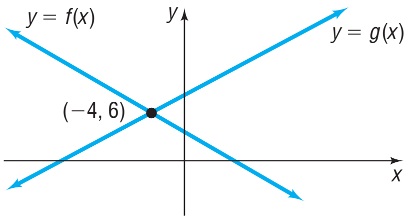
(a) Solve $f(x) = 50$.
(b) Solve $f(x) = 80$.
(c) Solve $f(x) = 0$.
(d) Solve $f(x) > 50$.
(e) Solve $f(x) \leq 80$.
(f) Solve $0 < f(x) < 80$.

32. In parts (a)–(f), use the following figure.



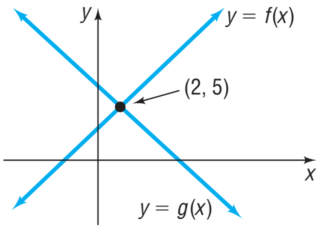
(a) Solve $g(x) = 20$.
(b) Solve $g(x) = 60$.
(c) Solve $g(x) = 0$.
(d) Solve $g(x) > 20$.
(e) Solve $g(x) \leq 60$.
(f) Solve $0 < g(x) < 60$.

33. In parts (a) and (b), use the following figure.



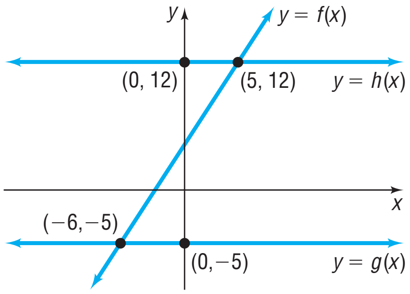
(a) Solve the equation: $f(x) = g(x)$.
(b) Solve the inequality: $f(x) > g(x)$.

34. In parts (a) and (b), use the following figure.



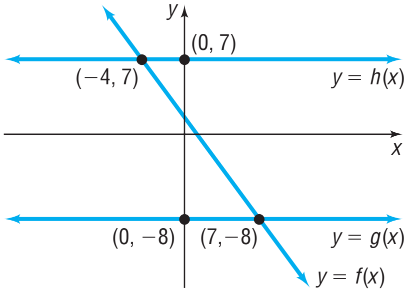
(a) Solve the equation: $f(x) = g(x)$.
(b) Solve the inequality: $f(x) \leq g(x)$.

35. In parts (a) and (b), use the following figure.



(a) Solve the equation: $f(x) = g(x)$.
(b) Solve the inequality: $g(x) \leq f(x) < h(x)$.

36. In parts (a) and (b), use the following figure.



(a) Solve the equation: $f(x) = g(x)$.
(b) Solve the inequality: $g(x) < f(x) \leq h(x)$.

37. Car Rentals The cost C, in dollars, of a one-day car rental is modeled by the function $C(x) = 0.35x + 45$, where x is the number of miles driven.
(a) What is the cost if you drive $x = 40$ miles?
(b) If the cost of renting the car is \$108, how many miles did you drive?
(c) Suppose that you want the cost to be no more than \$150. What is the maximum number of miles that you can drive?
(d) What is the implied domain of C?
(e) Interpret the slope.
(f) Interpret the y-intercept.

38. Phone Charges The monthly cost C, in dollars, for calls from the United States to Japan on a certain phone plan is modeled by the function $C(x) = 0.24x + 5$, where x is the number of minutes used.
(a) What is the cost if you talk on the phone for $x = 50$ minutes?
(b) Suppose that your monthly bill is \$20.36. How many minutes did you use the phone?
(c) Suppose that you budget yourself \$40 per month for the phone. What is the maximum number of minutes that you can talk?
(d) What is the implied domain of C if there are 30 days in the month?
(e) Interpret the slope.
(f) Interpret the y-intercept.

39. Supply and Demand Suppose that the quantity supplied S and the quantity demanded D of T-shirts at a concert are given by the following functions:

$$S(p) = -600 + 50p$$
$$D(p) = 1200 - 25p$$

where p is the price of a T-shirt.
(a) Find the equilibrium price for T-shirts at this concert. What is the equilibrium quantity?
(b) Determine the prices for which quantity demanded is greater than quantity supplied.
(c) What do you think will eventually happen to the price of T-shirts if quantity demanded is greater than quantity supplied?

40. Supply and Demand Suppose that the quantity supplied S and the quantity demanded D of hot dogs at a baseball game are given by the following functions:

$$S(p) = -2000 + 3000p$$
$$D(p) = 10{,}000 - 1000p$$

where p is the price of a hot dog.
(a) Find the equilibrium price for hot dogs at the baseball game. What is the equilibrium quantity?
(b) Determine the prices for which quantity demanded is less than quantity supplied.
(c) What do you think will eventually happen to the price of hot dogs if quantity demanded is less than quantity supplied?

41. Taxes The function $T(x) = 0.15(x - 9225) + 922.50$ represents the tax bill T of a single person whose adjusted gross income is x dollars for income between \$9225 and \$37,450, inclusive, in 2015.
Source: *Internal Revenue Service*
(a) What is the domain of this linear function?
(b) What is a single filer's tax bill if adjusted gross income is \$20,000?
(c) Which variable is independent and which is dependent?
(d) Graph the linear function over the domain specified in part (a).
(e) What is a single filer's adjusted gross income if the tax bill is \$3663.75?
(f) Interpret the slope.

42. Competitive Balance Tax In 2011, major league baseball signed a labor agreement with the players. In this agreement, any team whose payroll exceeded \$189 million in 2015 had to pay a competitive balance tax of 50% (for four or more consecutive offenses). The linear function $T(p) = 0.50(p - 189)$ describes the competitive balance tax T of a team whose payroll was p (in millions of dollars).
Source: *Major League Baseball*
(a) What is the implied domain of this linear function?
(b) What was the competitive balance tax for the New York Yankees whose 2015 payroll was \$214.2 million?
(c) Graph the linear function.
(d) What was the payroll of a team that paid a competitive balance tax of \$15.7 million?
(e) Interpret the slope.

The point at which a company's profits equal zero is called the company's ***break-even point****. For Problems 43 and 44, let R represent a company's revenue, let C represent the company's costs, and let x represent the number of units produced and sold each day.*
(a) *Find the firm's break-even point; that is, find x so that $R = C$.*
(b) *Find the values of x such that $R(x) > C(x)$. This represents the number of units that the company must sell to earn a profit.*

43. $R(x) = 8x$
$C(x) = 4.5x + 17{,}500$

44. $R(x) = 12x$
$C(x) = 10x + 15{,}000$

45. Straight-line Depreciation Suppose that a company has just purchased a new computer for \$3000. The company chooses to depreciate the computer using the straight-line method over 3 years.
(a) Write a linear model that expresses the book value V of the computer as a function of its age x.
(b) What is the implied domain of the function found in part (a)?
(c) Graph the linear function.
(d) What is the book value of the computer after 2 years?
(e) When will the computer have a book value of \$2000?

46. Straight-line Depreciation Suppose that a company has just purchased a new machine for its manufacturing facility for \$120,000. The company chooses to depreciate the machine using the straight-line method over 10 years.
(a) Write a linear model that expresses the book value V of the machine as a function of its age x.
(b) What is the implied domain of the function found in part (a)?
(c) Graph the linear function.
(d) What is the book value of the machine after 4 years?
(e) When will the machine have a book value of \$72,000?

47. Cost Function The simplest cost function is the linear cost function, $C(x) = mx + b$, where the y-intercept b represents the fixed costs of operating a business and the slope m represents the cost of each item produced. Suppose that a small bicycle manufacturer has daily fixed costs of \$1800, and each bicycle costs \$90 to manufacture.
(a) Write a linear model that expresses the cost C of manufacturing x bicycles in a day.
(b) Graph the model.
(c) What is the cost of manufacturing 14 bicycles in a day?
(d) How many bicycles could be manufactured for \$3780?

48. Cost Function Refer to Problem 47. Suppose that the landlord of the building increases the bicycle manufacturer's rent by \$100 per month.
(a) Assuming that the manufacturer is open for business 20 days per month, what are the new daily fixed costs?
(b) Write a linear model that expresses the cost C of manufacturing x bicycles in a day with the higher rent.
(c) Graph the model.
(d) What is the cost of manufacturing 14 bicycles in a day?
(e) How many bicycles can be manufactured for \$3780?

49. **Truck Rentals** A truck rental company rents a truck for one day by charging \$39.95 plus \$0.89 per mile.
 (a) Write a linear model that relates the cost C, in dollars, of renting the truck to the number x of miles driven.
 (b) What is the cost of renting the truck if the truck is driven 110 miles? 230 miles?

50. **International Calling** A cell phone company offers an international plan by charging \$30 for the first 80 minutes, plus \$0.50 for each minute over 80.
 (a) Write a linear model that relates the cost C, in dollars, of talking x minutes, assuming $x \geq 80$.
 (b) What is the cost of talking 105 minutes? 120 minutes?

Mixed Practice

51. **Developing a Linear Model from Data** How many songs can an iPod hold? The following data represent the memory m and the number of songs n.

Memory, m (gigabytes)	Number of songs, n
8	1750
16	3500
32	7000
64	14,000

 (a) Plot the ordered pairs (m, n) in a Cartesian plane.
 (b) Show that the number of songs n is a linear function of memory m.
 (c) Determine the linear function that describes the relation between m and n.
 (d) What is the implied domain of the linear function?
 (e) Graph the linear function in the Cartesian plane drawn in part (a).
 (f) Interpret the slope.

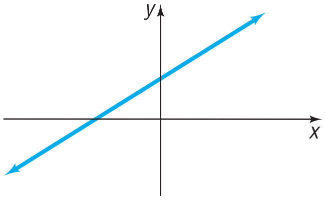
52. **Developing a Linear Model from Data** The following data represent the various combinations of soda and hot dogs that Yolanda can buy at a baseball game with \$60.

Soda, s	Hot Dogs, h
20	0
15	3
10	6
5	9

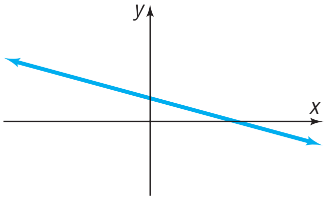
 (a) Plot the ordered pairs (s, h) in a Cartesian plane.
 (b) Show that the number of hot dogs purchased h is a linear function of the number of sodas purchased s.
 (c) Determine the linear function that describes the relation between s and h.
 (d) What is the implied domain of the linear function?
 (e) Graph the linear function in the Cartesian plane drawn in part (a).
 (f) Interpret the slope.
 (g) Interpret the values of the intercepts.

Explaining Concepts: Discussion and Writing

53. Which of the following functions might have the graph shown? (More than one answer is possible.)
 (a) $f(x) = 2x - 7$
 (b) $g(x) = -3x + 4$
 (c) $H(x) = 5$
 (d) $F(x) = 3x + 4$
 (e) $G(x) = \frac{1}{2}x + 2$



54. Which of the following functions might have the graph shown? (More than one answer is possible.)
 (a) $f(x) = 3x + 1$
 (b) $g(x) = -2x + 3$
 (c) $H(x) = 3$
 (d) $F(x) = -4x - 1$
 (e) $G(x) = -\frac{2}{3}x + 3$



55. Under what circumstances is a linear function $f(x) = mx + b$ odd? Can a linear function ever be even?

56. Explain how the graph of $f(x) = mx + b$ can be used to solve $mx + b > 0$.

Retain Your Knowledge

Problems 57–60 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

57. Graph $x^2 - 4x + y^2 + 10y - 7 = 0$.

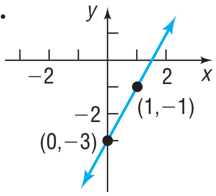
58. If $f(x) = \dfrac{2x + B}{x - 3}$ and $f(5) = 8$, what is the value of B?

59. Find the average rate of change of $f(x) = 3x^2 - 5x$ from 1 to 3.

60. Graph $g(x) = \begin{cases} x^2 & x \leq 0 \\ \sqrt{x} + 1 & x > 0 \end{cases}$

‘Are You Prepared?’ Answers

1.



2. $\frac{2}{3}$

3. 18

4. $\{50\}$

5. 0

6. True

4.2 Building Linear Models from Data

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Rectangular Coordinates (Section 1.1, pp. 83–85)
- Lines (Section 2.2, pp. 173–184)
- Functions (Section 3.1, pp. 270–218)

Now Work the **'Are You Prepared?'** problems on page 294.

OBJECTIVES
1 Draw and Interpret Scatter Diagrams (p. 291)
2 Distinguish between Linear and Nonlinear Relations (p. 292)
3 Use a Graphing Utility to Find the Line of Best Fit (p. 293)

1 Draw and Interpret Scatter Diagrams

In Section 4.1, we built linear models from verbal descriptions. Linear models can also be constructed by fitting a linear function to data. The first step is to plot the ordered pairs using rectangular coordinates. The resulting graph is a **scatter diagram**.

EXAMPLE 1 Drawing and Interpreting a Scatter Diagram

In baseball, the on-base percentage for a team represents the percentage of time that the players safely reach base. The data given in Table 6 represent the number of runs scored y and the on-base percentage x for teams in the National League during the 2014 baseball season.

Table 6

Team	On-base Percentage, x	Runs Scored, y	(x, y)
Arizona	30.2	615	(30.2, 615)
Atlanta	30.5	573	(30.5, 573)
Chicago Cubs	30.0	614	(30.0, 614)
Cincinnati	29.6	595	(29.6, 595)
Colorado	32.7	755	(32.7, 755)
LA Dodgers	33.3	718	(33.3, 718)
Miami	31.7	645	(31.7, 645)
Milwaukee	31.1	650	(31.1, 650)
NY Mets	30.8	629	(30.8, 629)
Philadelphia	30.2	619	(30.2, 619)
Pittsburgh	33.0	682	(33.0, 682)
San Diego	29.2	535	(29.2, 535)
San Francisco	31.1	665	(31.1, 665)
St. Louis	32.0	619	(32.0, 619)
Washington	32.1	686	(32.1, 686)

Source: *espn.go.com*

(a) Draw a scatter diagram of the data, treating on-base percentage as the independent variable.
(b) Use a graphing utility to draw a scatter diagram.
(c) Describe what happens to runs scored as the on-base percentage increases.

Solution

(a) To draw a scatter diagram, plot the ordered pairs listed in Table 6, with the on-base percentage as the x-coordinate and the runs scored as the y-coordinate. See Figure 6(a) on the next page. Notice that the points in the scatter diagram are not connected.

(b) Figure 6(b) shows a scatter diagram using a TI-84 Plus C graphing calculator.

(c) The scatter diagrams show that as the on-base percentage increases, the number of runs scored also increases.

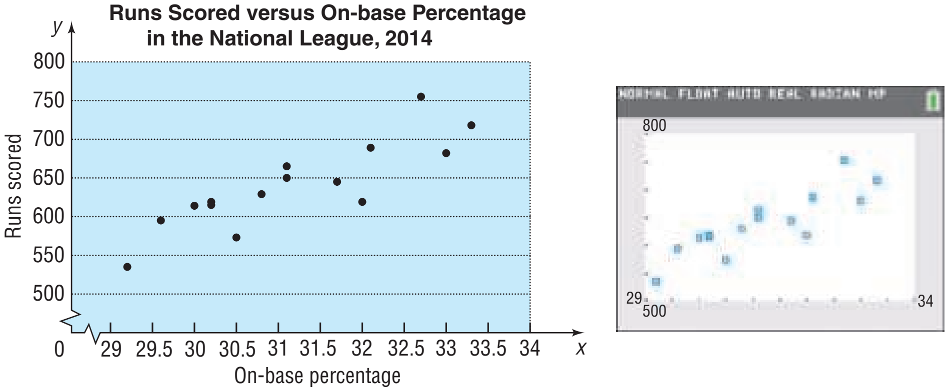


Figure 6 On-base percentage

(a)

(b) ■

Now Work PROBLEM 11(a)

2 Distinguish between Linear and Nonlinear Relations

Notice that the points in Figure 6 do not follow a perfect linear relation (as they do in Figure 3 in Section 4.1). However, the data do exhibit a linear pattern. There are numerous possible explanations why the data are not perfectly linear, but one easy explanation is the fact that other variables besides on-base percentage (such as number of home runs hit) play a role in determining runs scored.

Scatter diagrams are used to help us see the type of relation that exists between two variables. In this text, we will discuss a variety of different relations that may exist between two variables. For now, we concentrate on distinguishing between linear and nonlinear relations. See Figure 7.

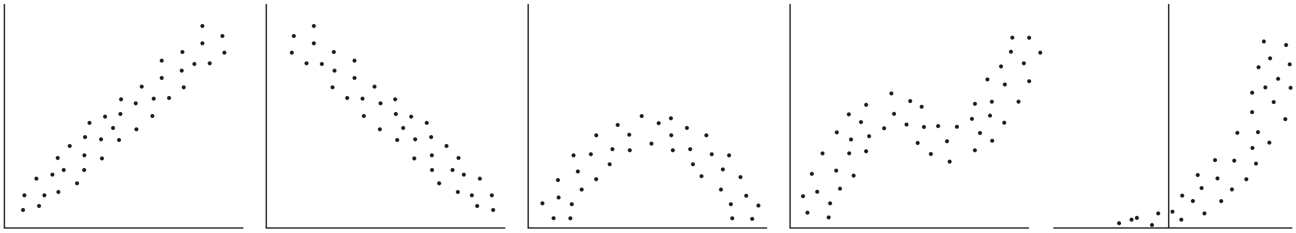


Figure 7 **(a)** Linear $y = mx + b, m > 0$ **(b)** Linear $y = mx + b, m < 0$ **(c)** Nonlinear **(d)** Nonlinear **(e)** Nonlinear

EXAMPLE 2 Distinguishing between Linear and Nonlinear Relations

Determine whether the relation between the two variables in each scatter diagram in Figure 8 is linear or nonlinear.

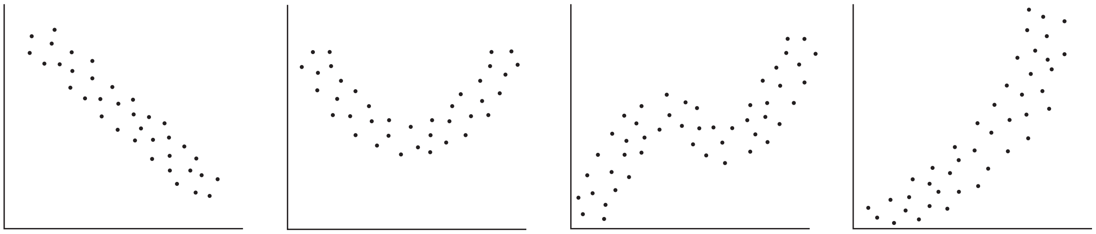


Figure 8 **(a)** **(b)** **(c)** **(d)**

Solution (a) Linear (b) Nonlinear (c) Nonlinear (d) Nonlinear ■

Now Work PROBLEM 5

This section considers data whose scatter diagrams suggest that a linear relation exists between the two variables.

Suppose that the scatter diagram of a set of data indicates a linear relationship, as in Figure 7(a) or (b). We might want to model the data by finding an equation of a line that relates the two variables. One way to obtain a model for such data is to draw a line through two points on the scatter diagram and determine the equation of the line.

EXAMPLE 3 Finding a Model for Linearly Related Data

Use the data in Table 6 from Example 1.

(a) Select two points and find an equation of the line containing the points.
(b) Graph the line on the scatter diagram obtained in Example 1(a).

Solution (a) Select two points, say $(30.8, 629)$ and $(32.1, 686)$. The slope of the line joining the points $(30.8, 629)$ and $(32.1, 686)$ is

$$m = \frac{686 - 629}{32.1 - 30.8} = \frac{57}{1.3} \approx 43.85$$

The equation of the line with slope 43.85 and passing through $(30.8, 629)$ is found using the point–slope form with $m = 43.85$, $x_1 = 30.8$, and $y_1 = 629$.

$$\begin{aligned} y - y_1 &= m(x - x_1) && \textbf{Point–slope form of a line} \\ y - 629 &= 43.85(x - 30.8) && x_1 = 30.8, y_1 = 629, m = 43.85 \\ y - 629 &= 43.85x - 1350.58 \\ y &= 43.85x - 721.58 \end{aligned}$$

(b) Figure 9 shows the scatter diagram with the graph of the line found in part (a).

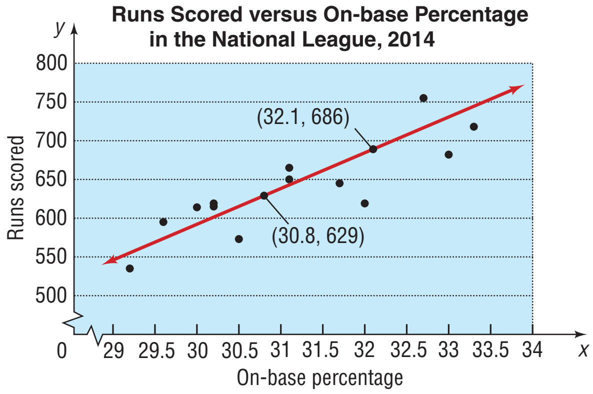


Figure 9

■

Select two other points and complete the solution. Graph the line on the scatter diagram obtained in Figure 6.

Now Work PROBLEM 11(b) AND (c)

3 Use a Graphing Utility to Find the Line of Best Fit

The model obtained in Example 3 depends on the selection of points, which will vary from person to person. So the model that we found might be different from the model you found. Although the model in Example 3 appears to fit the data well, there may be a model that "fits it better." Do you think your model fits the data better? Is there a *line of best fit*? As it turns out, there is a method for finding a model that best fits linearly related data (called the **line of best fit**).*

EXAMPLE 4 Finding a Model for Linearly Related Data

Use the data in Table 6 from Example 1.

(a) Use a graphing utility to find the line of best fit that models the relation between on-base percentage and runs scored.

---

*We shall not discuss the underlying mathematics of lines of best fit in this text.

(b) Graph the line of best fit on the scatter diagram obtained in Example 1(b).

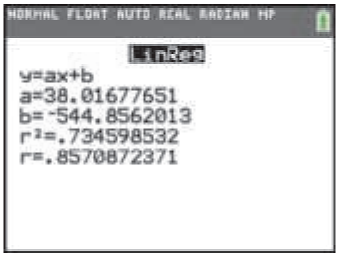
(c) Interpret the slope.

(d) Use the line of best fit to predict the number of runs a team will score if their on-base percentage is 31.5.

Solution

(a) Graphing utilities contain built-in programs that find the line of best fit for a collection of points in a scatter diagram. Executing the LINear REGression program provides the results shown in Figure 10. This output shows the equation $y = ax + b$, where a is the slope of the line and b is the y-intercept. The line of best fit that relates on-base percentage to runs scored may be expressed as the line

$$y = 38.02x - 544.86 \quad \textbf{The model}$$



```
NORMAL FLOAT AUTO REAL RADIAN MP
            LinReg
 y=ax+b
 a=38.01677651
 b=-544.8562013
 r²=.734598532
 r=.8570872371
```

Figure 10

(b) Figure 11 shows the graph of the line of best fit, along with the scatter diagram.

(c) The slope of the line of best fit is 38.02, which means that for every 1 percent increase in the on-base percentage, runs scored increase 38.02, on average.

(d) Letting $x = 31.5$ in the equation of the line of best fit, we obtain $y = 38.02(31.5) - 544.86 \approx 653$ runs. ■

Now Work PROBLEM 11(d) AND (e)

Does the line of best fit appear to be a good fit? In other words, does the line appear to accurately describe the relation between on-base percentage and runs scored?

And just how "good" is this line of best fit? Look again at Figure 10. The last line of output is $r = 0.857$. This number, called the **correlation coefficient**, $r, -1 \le r \le 1$, is a measure of the strength of the linear relation that exists between two variables. The closer $|r|$ is to 1, the more nearly perfect the linear relationship is. If r is close to 0, there is little or no linear relationship between the variables. A negative value of r, $r < 0$, indicates that as x increases, y decreases; a positive value of r, $r > 0$, indicates that as x increases, y does also. The data given in Table 6, having a correlation coefficient of 0.857, are indicative of a linear relationship with positive slope.

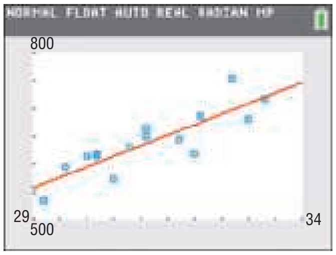


Figure 11

4.2 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Plot the points $(1, 5), (2, 6), (3, 9), (1, 12)$ in the Cartesian plane. Is the relation $\{(1, 5), (2, 6), (3, 9), (1, 12)\}$ a function? Why? (pp. 83 and 207–210)

2. Find an equation of the line containing the points $(1, 4)$ and $(3, 8)$. (pp. 179–180)

Concepts and Vocabulary

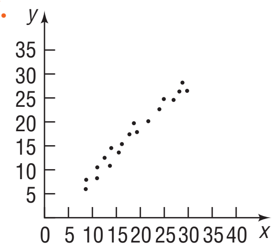
3. A ______________ is used to help us to see what type of relation, if any, may exist between two variables.

4. If the independent variable in a line of best fit $y = -0.008x + 14$ is credit score, and the dependent variable is the interest rate on a used-car loan, then the slope is interpreted as follows: "If credit score increases by 1 point, the interest rate will __________ (increase/decrease) by _____ percent, on average."

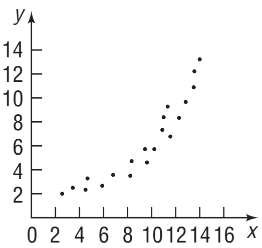
Skill Building

In Problems 5–10, examine the scatter diagram and determine whether the type of relation is linear or nonlinear.

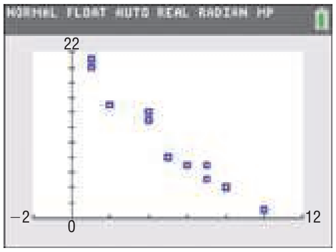
5.



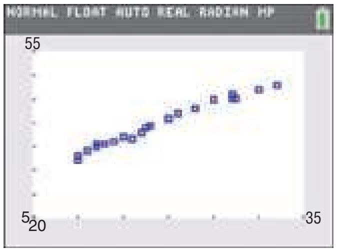
6.



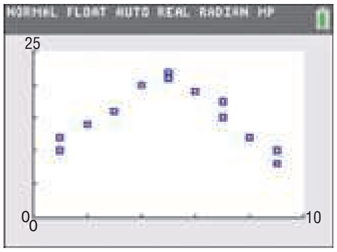
7.



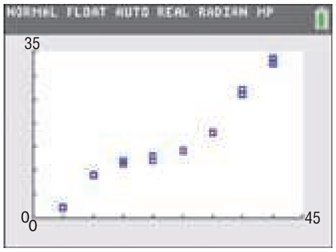
8.



9.



10.



In Problems 11–16,

(a) Draw a scatter diagram.
(b) Select two points from the scatter diagram and find the equation of the line containing the points selected.
(c) Graph the line found in part (b) on the scatter diagram.
(d) Use a graphing utility to find the line of best fit.
(e) Use a graphing utility to draw the scatter diagram and graph the line of best fit on it.

11.

x	3	4	5	6	7	8	9
y	4	6	7	10	12	14	16

12.

x	3	5	7	9	11	13
y	0	2	3	6	9	11

13.

x	−2	−1	0	1	2
y	−4	0	1	4	5

14.

x	−2	−1	0	1	2
y	7	6	3	2	0

15.

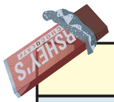
x	−20	−17	−15	−14	−10
y	100	120	118	130	140

16.

x	−30	−27	−25	−20	−14
y	10	12	13	13	18

Applications and Extensions

17. Candy The following data represent the weight (in grams) of various candy bars and the corresponding number of calories.



Candy Bar	Weight, *x*	Calories, *y*
Hershey's Milk Chocolate®	44.28	230
Nestle's Crunch®	44.84	230
Butterfinger®	61.30	270
Baby Ruth®	66.45	280
Almond Joy®	47.33	220
Twix® (with caramel)	58.00	280
Snickers®	61.12	280
Heath®	39.52	210

Source: Megan Pocius, student at Joliet Junior College

(a) Draw a scatter diagram of the data, treating weight as the independent variable.
(b) What type of relation appears to exist between the weight of a candy bar and the number of calories?
(c) Select two points and find a linear model that contains the points.
(d) Graph the line on the scatter diagram drawn in part (a).
(e) Use the linear model to predict the number of calories in a candy bar that weighs 62.3 grams.
(f) Interpret the slope of the line found in part (c).

18. Tornadoes The following data represent the width (in yards) and length (in miles) of tornadoes.

Width (yards), *w*	Length (miles), *L*
200	2.5
350	4.8
180	2.0
300	2.5
500	5.8
400	4.5
500	8.0
800	8.0
100	3.4
50	0.5
700	9.0
600	5.7

Source: NOAA

(a) Draw a scatter diagram of the data, treating width as the independent variable.
(b) What type of relation appears to exist between the width and the length of tornadoes?
(c) Select two points and find a linear model that contains the points.
(d) Graph the line on the scatter diagram drawn in part (b).
(e) Use the linear model to predict the length of a tornado that has a width of 450 yards.
(f) Interpret the slope of the line found in part (c).

19. Video Games and Grade-Point Average Professor Grant Alexander wanted to find a linear model that relates the number of hours a student plays video games each week, *h*, to the cumulative grade-point average, *G*, of the student. He obtained a random sample of 10 full-time students at his college and asked each student to disclose the number of hours spent playing video games and the student's cumulative grade-point average.

Hours of Video Games per Week, *h*	Grade-Point Average, *G*
0	3.49
0	3.05
2	3.24
3	2.82
3	3.19
5	2.78
8	2.31
8	2.54
10	2.03
12	2.51

(a) Explain why the number of hours spent playing video games is the independent variable and cumulative grade-point average is the dependent variable.
(b) Use a graphing utility to draw a scatter diagram.
(c) Use a graphing utility to find the line of best fit that models the relation between number of hours of video game playing each week and grade-point average. Express the model using function notation.
(d) Interpret the slope.
(e) Predict the grade-point average of a student who plays video games for 8 hours each week.
(f) How many hours of video game playing do you think a student plays whose grade-point average is 2.40?

20. Hurricanes The following data represent the atmospheric pressure *p* (in millibars) and the wind speed *w* (in knots) measured during various tropical systems in the Atlantic Ocean.

Atmospheric Pressure (millibars), *p*	Wind Speed (knots), *w*
993	50
994	60
997	45
1003	45
1004	40
1000	55
994	55
942	105
1006	40
942	120
986	50
983	70
940	120
966	100
982	55

Source: National Hurricane Center

(a) Use a graphing utility to draw a scatter diagram of the data, treating atmospheric pressure as the independent variable.
(b) Use a graphing utility to find the line of best fit that models the relation between atmospheric pressure and wind speed. Express the model using function notation.
(c) Interpret the slope.
(d) Predict the wind speed of a tropical storm if the atmospheric pressure measures 990 millibars.
(e) What is the atmospheric pressure of a hurricane if the wind speed is 85 knots?

Mixed Practice

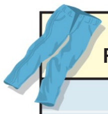
21. Homeruns A baseball analyst wishes to find a function that relates the distance, *d*, of a homerun and the speed, *s*, of the ball off the bat. Consider the data shown to the right.

(a) Does the relation defined by the set of orders pairs (*s*, *d*) represent a function?
(b) Draw a scatter diagram of the data, treating speed off bat as the independent variable.
(c) Using a graphing utility, find the line of best fit that models the relation between the speed off the bat and the distance of the homerun?
(d) Interpret the slope.
(e) Express the relationship found in part (c), using function notation.
(f) What is the domain of the function?
(g) Predict the homerun distance if the speed of the ball off the bat is 103 miles per hour.

Speed Off Bat (mph), *s*	Homerun Distance (feet), *d*
98	369
99	381
100	380
100	397
101	400
102	383
102	408
104	392
104	406
104	421
105	411
107	396
107	429
109	404
109	418

Source: Major League Baseball, 2014

22. **Demand for Jeans** The marketing manager at Levi-Strauss wishes to find a function that relates the demand D for men's jeans and p, the price of the jeans. The following data were obtained based on a price history of the jeans.



Price ($/Pair), p	Demand (Pairs of Jeans Sold per Day), D
20	60
22	57
23	56
23	53
27	52
29	49
30	44

(a) Does the relation defined by the set of ordered pairs (p, D) represent a function?
(b) Draw a scatter diagram of the data.
(c) Using a graphing utility, find the line of best fit that models the relation between price and quantity demanded.
(d) Interpret the slope.
(e) Express the relationship found in part (c) using function notation.
(f) What is the domain of the function?
(g) How many jeans will be demanded if the price is $28 a pair?

Explaining Concepts: Discussion and Writing

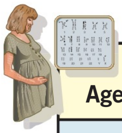
23. **Maternal Age versus Down Syndrome** A biologist would like to know how the age of the mother affects the incidence of Down syndrome. The data to the right represent the age of the mother and the incidence of Down syndrome per 1000 pregnancies. Draw a scatter diagram treating age of the mother as the independent variable. Would it make sense to find the line of best fit for these data? Why or why not?

24. Find the line of best fit for the ordered pairs $(1, 5)$ and $(3, 8)$. What is the correlation coefficient for these data? Why is this result reasonable?

25. What does a correlation coefficient of 0 imply?

26. Explain why it does not make sense to interpret the y-intercept in Problem 17.

27. Refer to Problem 19. Solve $G(h) = 0$. Provide an interpretation of this result. Find $G(0)$. Provide an interpretation of this result.



Age of Mother, x	Incidence of Down Syndrome, y
33	2.4
34	3.1
35	4
36	5
37	6.7
38	8.3
39	10
40	13.3
41	16.7
42	22.2
43	28.6
44	33.3
45	50

Source: Hook, E.B., *Journal of the American Medical Association*, 249, 2034-2038, 1983.

Retain Your Knowledge

Problems 28–31 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

28. Find an equation for the line containing the points $(-1, 5)$ and $(3, -3)$. Express your answer using either the general form or the slope-intercept form of the equation of a line, whichever you prefer.

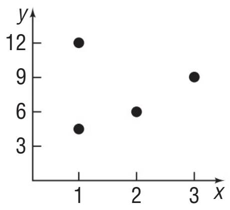
29. Find the domain of $f(x) = \dfrac{x-1}{x^2-25}$.

30. For $f(x) = 5x - 8$ and $g(x) = x^2 - 3x + 4$, find $(g - f)(x)$.

31. Write the function whose graph is the graph of $y = x^2$, but shifted to the left 3 units and shifted down 4 units.

'Are You Prepared?' Answers

1.



No, because the input, 1, corresponds to two different outputs.

2. $y = 2x + 2$

4.3 Quadratic Functions and Their Properties

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Intercepts (Section 2.1, pp. 165–166)
- Graphing Techniques: Transformations (Section 3.5, pp. 256–264)
- Completing the Square (Section R.5, p. 57)
- Quadratic Equations (Section 1.3, pp. 110–117)

Now Work the **'Are You Prepared?'** problems on page 306.

OBJECTIVES
1 Graph a Quadratic Function Using Transformations (p. 299)
2 Identify the Vertex and Axis of Symmetry of a Quadratic Function (p. 301)
3 Graph a Quadratic Function Using Its Vertex, Axis, and Intercepts (p. 301)
4 Find a Quadratic Function Given Its Vertex and One Other Point (p. 304)
5 Find the Maximum or Minimum Value of a Quadratic Function (p. 305)

Quadratic Functions

DEFINITION

A **quadratic function** is a function of the form

$$\boxed{f(x) = ax^2 + bx + c}$$

where a, b, and c are real numbers and $a \neq 0$. The domain of a quadratic function is the set of all real numbers.

In Words
A quadratic function is a function defined by a second-degree polynomial in one variable.

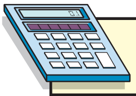
Some examples of quadratic functions are

$$F(x) = 3x^2 - 5x + 1 \quad g(x) = -6x^2 + 1 \quad H(x) = \frac{1}{2}x^2 + \frac{2}{3}x$$

Many applications require a knowledge of quadratic functions. For example, suppose that Texas Instruments collects the data shown in Table 7, which relate the number of calculators sold to the price p (in dollars) per calculator. Since the price of a product determines the quantity that will be purchased, we treat price as the independent variable. The relationship between the number x of calculators sold and the price p per calculator is given by the linear equation

$$x = 21{,}000 - 150p$$

Table 7



Price per Calculator, p (dollars)	Number of Calculators, x
60	12,000
65	11,250
70	10,500
75	9,750
80	9,000
85	8,250
90	7,500

Then the revenue R derived from selling x calculators at the price p per calculator is equal to the unit selling price p of the calculator times the number x of units actually sold. That is,

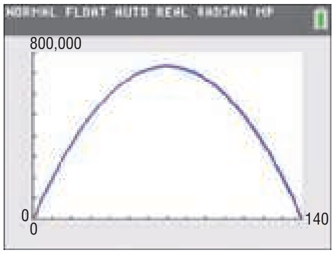


Figure 12
$R(p) = -150p^2 + 21{,}000p$

$$
\begin{aligned}
R &= xp \\
R(p) &= (21{,}000 - 150p)p \quad \mathbf{x = 21{,}000 - 150p} \\
&= -150p^2 + 21{,}000p
\end{aligned}
$$

So the revenue R is a quadratic function of the price p. Figure 12 illustrates the graph of this revenue function, whose domain is $0 \le p \le 140$, since both x and p must be nonnegative.

A second situation in which a quadratic function appears involves the motion of a projectile. Based on Newton's Second Law of Motion (force equals mass times acceleration, $F = ma$), it can be shown that, ignoring air resistance, the path of a projectile propelled upward at an inclination to the horizontal is the graph of a quadratic function. See Figure 13 for an illustration.

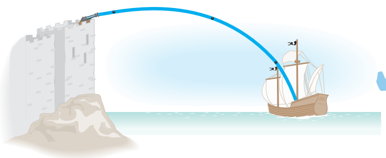


Figure 13
Path of a cannonball

1 Graph a Quadratic Function Using Transformations

We know how to graph the square function $f(x) = x^2$. Figure 14 shows the graph of three functions of the form $f(x) = ax^2$, $a > 0$, for $a = 1$, $a = \frac{1}{2}$, and $a = 3$. Notice that the larger the value of a, the "narrower" the graph is, and the smaller the value of a, the "wider" the graph is.

Figure 15 shows the graphs of $f(x) = ax^2$ for $a < 0$. Notice that these graphs are reflections about the x-axis of the graphs in Figure 14. Based on the results of these two figures, general conclusions can be drawn about the graph of $f(x) = ax^2$. First, as $|a|$ increases, the graph is stretched vertically (becomes "taller"), and as $|a|$ gets closer to zero, the graph is compressed vertically (becomes "shorter"). Second, if a is positive, the graph opens "up," and if a is negative, the graph opens "down."

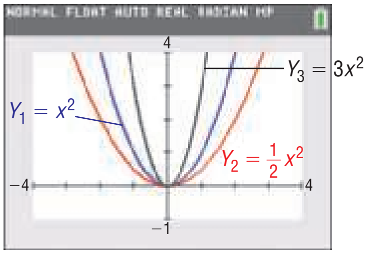


Figure 14

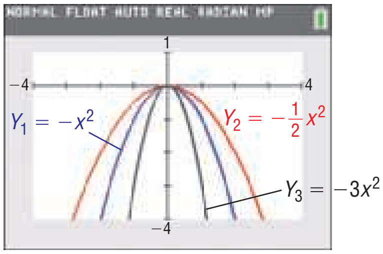
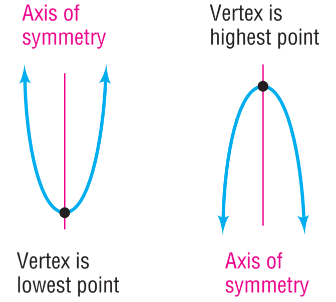


Figure 15

The graphs in Figures 14 and 15 are typical of the graphs of all quadratic functions, which are called **parabolas**.* Refer to Figure 16, where two parabolas are pictured. The one on the left **opens up** and has a lowest point; the one on the right **opens down** and has a highest point. The lowest or highest point of a parabola is called the **vertex**. The vertical line passing through the vertex in each parabola in Figure 16 is called the **axis of symmetry** (usually abbreviated to **axis**) of the parabola. Because the parabola is symmetric about its axis, the axis of symmetry of a parabola can be used to find additional points on the parabola.



(a) Opens up
$a > 0$

(b) Opens down
$a < 0$

Figure 16
Graphs of a quadratic function, $f(x) = ax^2 + bx + c$, $a \neq 0$

The parabolas shown in Figure 16 are the graphs of a quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$. Notice that the coordinate axes are not included in the figure. Depending on the values of a, b, and c, the axes could be placed anywhere. The important fact is that the shape of the graph of a quadratic function will look like one of the parabolas in Figure 16.

In the following example, techniques from Section 3.5 are used to graph a quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$. The method of completing the square is used to write the function f in the form $f(x) = a(x - h)^2 + k$.

---

*Parabolas will be studied using a geometric definition later in this text.

EXAMPLE 1 Graphing a Quadratic Function Using Transformations

Graph the function $f(x) = 2x^2 + 8x + 5$. Find the vertex and axis of symmetry.

Solution Begin by completing the square on the right side.

$$\begin{aligned} f(x) &= 2x^2 + 8x + 5 \\ &= 2(x^2 + 4x) + 5 && \text{Factor out the 2 from } 2x^2 + 8x. \\ &= 2(x^2 + 4x + 4) + 5 - 8 && \text{Complete the square of } x^2 + 4x \text{ by adding 4.} \\ &= 2(x + 2)^2 - 3 && \text{Notice that the factor of 2 requires that 8 be subtracted.} \end{aligned}$$

The graph of f can be obtained from the graph of $y = x^2$ in three stages, as shown in Figure 17. Now compare this graph to the graph in Figure 16(a). The graph of $f(x) = 2x^2 + 8x + 5$ is a parabola that opens up and has its vertex (lowest point) at $(-2, -3)$. Its axis of symmetry is the line $x = -2$.

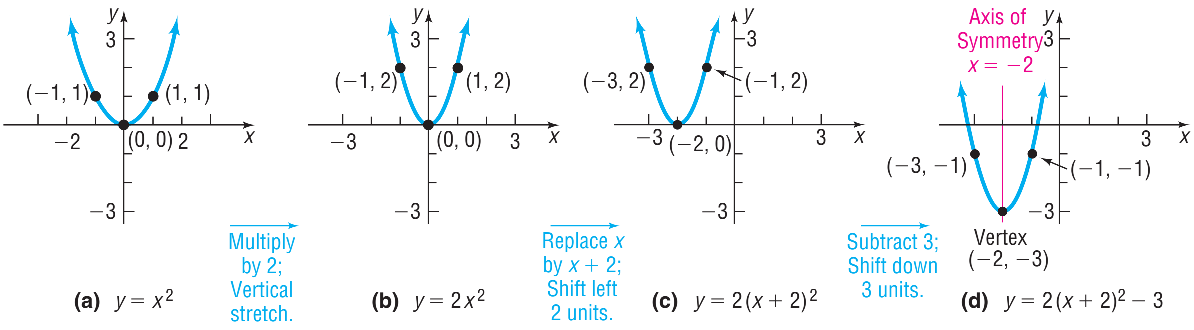


Figure 17 (a) $y = x^2$ (b) $y = 2x^2$ (c) $y = 2(x + 2)^2$ (d) $y = 2(x + 2)^2 - 3$

✓**Check:** Use a graphing utility to graph $Y_1 = f(x) = 2x^2 + 8x + 5$ and use the MINIMUM command to locate its vertex. ■

Now Work PROBLEM 25

The method used in Example 1 can be used to graph any quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$, as follows:

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ &= a\left(x^2 + \frac{b}{a}x\right) + c && \text{Factor out } a \text{ from } ax^2 + bx. \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - a\left(\frac{b^2}{4a^2}\right) && \text{Complete the square by adding } \frac{b^2}{4a^2}. \text{ Look closely at this step!} \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} && \text{Factor; simplify.} \\ &= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} && c - \frac{b^2}{4a} = c \cdot \frac{4a}{4a} - \frac{b^2}{4a} = \frac{4ac - b^2}{4a} \end{aligned}$$

These results lead to the following conclusion:

If $h = -\dfrac{b}{2a}$ and $k = \dfrac{4ac - b^2}{4a}$, then

$$\boxed{f(x) = ax^2 + bx + c = a(x - h)^2 + k \qquad (1)}$$

The graph of $f(x) = a(x - h)^2 + k$ is the parabola $y = ax^2$ shifted horizontally h units (replace x by $x - h$) and vertically k units (add k). As a result, the vertex is at (h, k), and the graph opens up if $a > 0$ and down if $a < 0$. The axis of symmetry is the vertical line $x = h$. ■

For example, compare equation (1) with the solution given in Example 1.

$$\begin{aligned} f(x) &= 2(x+2)^2 - 3 \\ &= 2(x-(-2))^2 + (-3) \\ &= a(x-h)^2 + k \end{aligned}$$

Because $a = 2$, the graph opens up. Also, because $h = -2$ and $k = -3$, its vertex is at $(-2, -3)$.

2 Identify the Vertex and Axis of Symmetry of a Quadratic Function

We do not need to complete the square to obtain the vertex. In almost every case, it is easier to obtain the vertex of a quadratic function f by remembering that its x-coordinate is $h = -\frac{b}{2a}$. The y-coordinate k can then be found by evaluating f at $-\frac{b}{2a}$. That is, $k = f\left(-\frac{b}{2a}\right)$.

Properties of the Graph of a Quadratic Function

$$f(x) = ax^2 + bx + c \qquad a \neq 0$$

Vertex $= \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$ Axis of symmetry: the vertical line $x = -\frac{b}{2a}$ **(2)**

Parabola opens up if $a > 0$; the vertex is a minimum point.
Parabola opens down if $a < 0$; the vertex is a maximum point.

EXAMPLE 2 Locating the Vertex without Graphing

Without graphing, locate the vertex and axis of symmetry of the parabola defined by $f(x) = -3x^2 + 6x + 1$. Does it open up or down?

Solution For this quadratic function, $a = -3$, $b = 6$, and $c = 1$. The x-coordinate of the vertex is

$$h = -\frac{b}{2a} = -\frac{6}{2(-3)} = 1$$

The y-coordinate of the vertex is

$$k = f\left(-\frac{b}{2a}\right) = f(1) = -3(1)^2 + 6(1) + 1 = 4$$

The vertex is located at the point $(1, 4)$. The axis of symmetry is the line $x = 1$. Because $a = -3 < 0$, the parabola opens down. ■

3 Graph a Quadratic Function Using Its Vertex, Axis, and Intercepts

The location of the vertex and intercepts, along with knowledge of whether the graph opens up or down, usually provides enough information to graph $f(x) = ax^2 + bx + c$, $a \neq 0$.

The y-intercept is the value of f at $x = 0$; that is, the y-intercept is $f(0) = c$. The x-intercepts, if there are any, are found by solving the quadratic equation

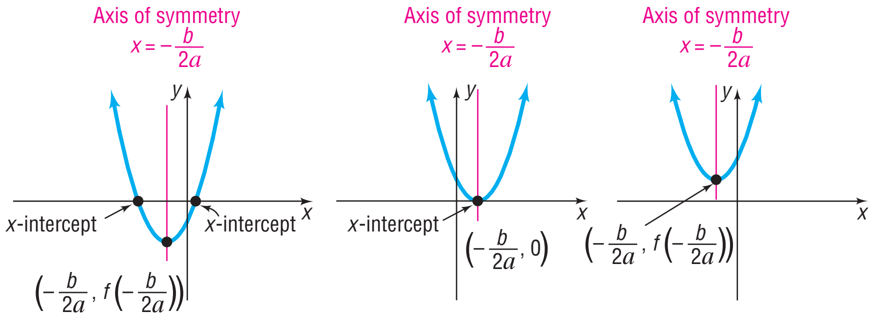
$$ax^2 + bx + c = 0$$

This equation has two, one, or no real solutions, depending on whether the discriminant $b^2 - 4ac$ is positive, 0, or negative. Depending on the value of the discriminant, the graph of f has x-intercepts, as follows:

The *x*-Intercepts of a Quadratic Function

1. If the discriminant $b^2 - 4ac > 0$, the graph of $f(x) = ax^2 + bx + c$ has two distinct x-intercepts so it crosses the x-axis in two places.
2. If the discriminant $b^2 - 4ac = 0$, the graph of $f(x) = ax^2 + bx + c$ has one x-intercept so it touches the x-axis at its vertex.
3. If the discriminant $b^2 - 4ac < 0$, the graph of $f(x) = ax^2 + bx + c$ has no x-intercepts so it does not cross or touch the x-axis.

Figure 18 illustrates these possibilities for parabolas that open up.



(a) $b^2 - 4ac > 0$
Two x-intercepts

(b) $b^2 - 4ac = 0$
One x-intercept

(c) $b^2 - 4ac < 0$
No x-intercepts

Figure 18
$f(x) = ax^2 + bx + c, a > 0$

EXAMPLE 3

How to Graph a Quadratic Function by Hand Using Its Properties

Graph $f(x) = -3x^2 + 6x + 1$ using its properties. Determine the domain and the range of f. Determine where f is increasing and where it is decreasing.

Step-by-Step Solution

Step 1: Determine whether the graph of *f* opens up or down.

In Example 2, it was determined that the graph of $f(x) = -3x^2 + 6x + 1$ opens down because $a = -3 < 0$.

Step 2: Determine the vertex and axis of symmetry of the graph of *f*.

In Example 2, the vertex was found to be at the point whose coordinates are $(1, 4)$. The axis of symmetry is the line $x = 1$.

Step 3: Determine the intercepts of the graph of *f*.

The y-intercept is found by letting $x = 0$. The y-intercept is $f(0) = 1$. The x-intercepts are found by solving the equation $f(x) = 0$.

$$f(x) = 0$$
$$-3x^2 + 6x + 1 = 0 \qquad a = -3, b = 6, c = 1$$

The discriminant $b^2 - 4ac = (6)^2 - 4(-3)(1) = 36 + 12 = 48 > 0$, so the equation has two real solutions and the graph has two x-intercepts. Use the quadratic formula to find that

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-6 + \sqrt{48}}{2(-3)} = \frac{-6 + 4\sqrt{3}}{-6} \approx -0.15$$

and

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-6 - \sqrt{48}}{2(-3)} = \frac{-6 - 4\sqrt{3}}{-6} \approx 2.15$$

The x-intercepts are approximately -0.15 and 2.15.

Step 4: Use the information in Steps 1 through 3 to graph *f*.

The graph is illustrated in Figure 19. Note how the *y*-intercept and the axis of symmetry, $x = 1$, are used to obtain the additional point $(2, 1)$ on the graph.

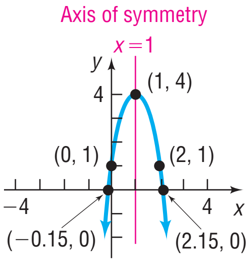


Figure 19 $f(x) = -3x^2 + 6x + 1$

The domain of *f* is the set of all real numbers. Based on the graph, the range of *f* is the interval $(-\infty, 4]$. The function *f* is increasing on the interval $(-\infty, 1]$ and decreasing on the interval $[1, \infty)$.

Graph the function in Example 3 by completing the square and using transformations. Which method do you prefer?

Now Work PROBLEM 33

If the graph of a quadratic function has only one *x*-intercept or no *x*-intercepts, it is usually necessary to plot an additional point to obtain the graph.

EXAMPLE 4

Graphing a Quadratic Function Using Its Vertex, Axis, and Intercepts

(a) Graph $f(x) = x^2 - 6x + 9$ by determining whether the graph opens up or down and by finding its vertex, axis of symmetry, *y*-intercept, and *x*-intercepts, if any.

(b) Determine the domain and the range of *f*.

(c) Determine where *f* is increasing and where it is decreasing.

Solution

(a) **STEP 1:** For $f(x) = x^2 - 6x + 9$, note that $a = 1$, $b = -6$, and $c = 9$. Because $a = 1 > 0$, the parabola opens up.

STEP 2: The *x*-coordinate of the vertex is

$$h = -\frac{b}{2a} = -\frac{-6}{2(1)} = 3$$

The *y*-coordinate of the vertex is

$$k = f(3) = (3)^2 - 6(3) + 9 = 0$$

The vertex is at $(3, 0)$. The axis of symmetry is the line $x = 3$.

STEP 3: The *y*-intercept is $f(0) = 9$. Since the vertex $(3, 0)$ lies on the *x*-axis, the graph touches the *x*-axis at the *x*-intercept.

STEP 4: By using the axis of symmetry and the *y*-intercept at $(0, 9)$, we can locate the additional point $(6, 9)$ on the graph. See Figure 20.

(b) The domain of *f* is the set of all real numbers. Based on the graph, the range of *f* is the interval $[0, \infty)$.

(c) The function *f* is decreasing on the interval $(-\infty, 3]$ and increasing on the interval $[3, \infty)$.

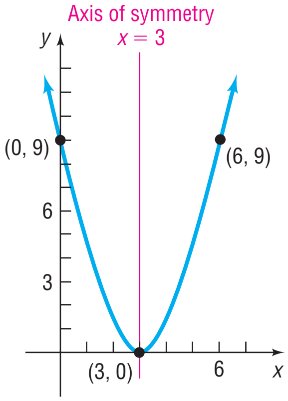


Figure 20 $f(x) = x^2 - 6x + 9$

Now Work PROBLEM 39

EXAMPLE 5 Graphing a Quadratic Function Using Its Vertex, Axis, and Intercepts

(a) Graph $f(x) = 2x^2 + x + 1$ by determining whether the graph opens up or down and by finding its vertex, axis of symmetry, y-intercept, and x-intercepts, if any.

(b) Determine the domain and the range of f.

(c) Determine where f is increasing and where it is decreasing.

Solution

(a) **Step 1:** For $f(x) = 2x^2 + x + 1$, we have $a = 2$, $b = 1$, and $c = 1$. Because $a = 2 > 0$, the parabola opens up.

Step 2: The x-coordinate of the vertex is

$$h = -\frac{b}{2a} = -\frac{1}{4}$$

The y-coordinate of the vertex is

$$k = f\left(-\frac{1}{4}\right) = 2\left(\frac{1}{16}\right) + \left(-\frac{1}{4}\right) + 1 = \frac{7}{8}$$

The vertex is at $\left(-\frac{1}{4}, \frac{7}{8}\right)$. The axis of symmetry is the line $x = -\frac{1}{4}$.

Step 3: The y-intercept is $f(0) = 1$. The x-intercept(s), if any, obey the equation $2x^2 + x + 1 = 0$. The discriminant $b^2 - 4ac = (1)^2 - 4(2)(1) = -7 < 0$. This equation has no real solutions, which means the graph has no x-intercepts.

Step 4: Use the point (0, 1) and the axis of symmetry $x = -\frac{1}{4}$ to locate the additional point $\left(-\frac{1}{2}, 1\right)$ on the graph. See Figure 21.

(b) The domain of f is the set of all real numbers. Based on the graph, the range of f is the interval $\left[\frac{7}{8}, \infty\right)$.

(c) The function f is decreasing on the interval $\left(-\infty, -\frac{1}{4}\right]$ and is increasing on the interval $\left[-\frac{1}{4}, \infty\right)$. ■

Note: In Example 5, since the vertex is above the x-axis and the parabola opens up, we can conclude that the graph of the quadratic function will have no x-intercepts. ■

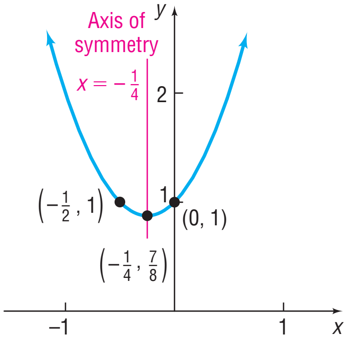


Figure 21 $f(x) = 2x^2 + x + 1$

Now Work PROBLEM 43

4 Find a Quadratic Function Given Its Vertex and One Other Point

If the vertex (h, k) and one additional point on the graph of a quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$, are known, then

$$f(x) = a(x - h)^2 + k \tag{3}$$

can be used to obtain the quadratic function.

EXAMPLE 6 Finding the Quadratic Function Given Its Vertex and One Other Point

Determine the quadratic function whose vertex is $(1, -5)$ and whose y-intercept is -3. The graph of the parabola is shown in Figure 22.

Solution The vertex is $(1, -5)$, so $h = 1$ and $k = -5$. Substitute these values into equation (3).

$$f(x) = a(x - h)^2 + k \quad \textbf{Equation (3)}$$
$$f(x) = a(x - 1)^2 - 5 \quad \boldsymbol{h = 1, k = -5}$$

To determine the value of a, use the fact that $f(0) = -3$ (the y-intercept).

$$f(x) = a(x - 1)^2 - 5$$
$$-3 = a(0 - 1)^2 - 5 \quad \boldsymbol{x = 0, y = f(0) = -3}$$
$$-3 = a - 5$$
$$a = 2$$

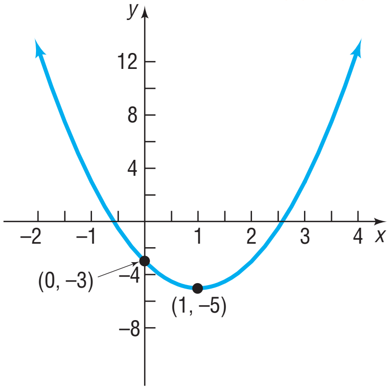


Figure 22 $f(x) = 2x^2 - 4x - 3$

The quadratic function whose graph is shown in Figure 22 is

$$f(x) = a(x - h)^2 + k = 2(x - 1)^2 - 5 = 2x^2 - 4x - 3$$

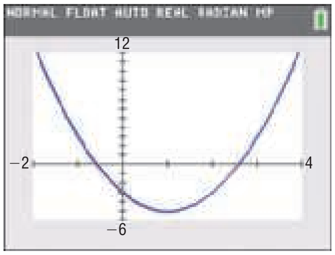


Figure 23 $Y_1 = 2x^2 - 4x - 3$

✓**Check:** Figure 23 shows the graph of $Y_1 = 2x^2 - 4x - 3$ using a TI-84 Plus C. ■

Now Work PROBLEM 49

5 Find the Maximum or Minimum Value of a Quadratic Function

The graph of a quadratic function

$$f(x) = ax^2 + bx + c \qquad a \neq 0$$

is a parabola with vertex at $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$. This vertex is the highest point on the graph if $a < 0$ and the lowest point on the graph if $a > 0$. If the vertex is the highest point $(a < 0)$, then $f\left(-\frac{b}{2a}\right)$ is the **maximum value** of f. If the vertex is the lowest point $(a > 0)$, then $f\left(-\frac{b}{2a}\right)$ is the **minimum value** of f.

EXAMPLE 7 Finding the Maximum or Minimum Value of a Quadratic Function

Determine whether the quadratic function

$$f(x) = x^2 - 4x - 5$$

has a maximum or a minimum value. Then find the maximum or minimum value.

Solution Compare $f(x) = x^2 - 4x - 5$ to $f(x) = ax^2 + bx + c$. Then $a = 1, b = -4$, and $c = -5$. Because $a > 0$, the graph of f opens up, which means the vertex is a minimum point. The minimum value occurs at

$$x = -\frac{b}{2a} \underset{\substack{\uparrow \\ \boldsymbol{a = 1, b = -4}}}{=} -\frac{-4}{2(1)} = \frac{4}{2} = 2$$

The minimum value is

$$f\left(-\frac{b}{2a}\right) = f(2) = 2^2 - 4(2) - 5 = 4 - 8 - 5 = -9$$

✓**Check:** Graph $Y_1 = f(x) = x^2 - 4x - 5$. Use MINIMUM to verify the vertex is $(2, -9)$. ■

Now Work PROBLEM 57

SUMMARY

Steps for Graphing a Quadratic Function $f(x) = ax^2 + bx + c, a \neq 0$

Option 1

STEP 1: Complete the square in x to write the quadratic function in the form $f(x) = a(x - h)^2 + k$.

STEP 2: Graph the function in stages using transformations.

Option 2

STEP 1: Determine whether the parabola opens up $(a > 0)$ or down $(a < 0)$.

STEP 2: Determine the vertex $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

STEP 3: Determine the axis of symmetry, $x = -\frac{b}{2a}$.

STEP 4: Determine the y-intercept, $f(0)$, and the x-intercepts, if any.

(a) If $b^2 - 4ac > 0$, the graph of the quadratic function has two x-intercepts, which are found by solving the equation $ax^2 + bx + c = 0$.

(b) If $b^2 - 4ac = 0$, the vertex is the x-intercept.

(c) If $b^2 - 4ac < 0$, there are no x-intercepts.

STEP 5: Determine an additional point by using the y-intercept and the axis of symmetry.

STEP 6: Plot the points and draw the graph.

4.3 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. List the intercepts of the equation $y = x^2 - 9$. (pp. 165–166)

2. Find the real solutions of the equation $2x^2 + 7x - 4 = 0$. (pp. 110–117)

3. To complete the square of $x^2 - 5x$, you add the number ________. (p. 57)

4. To graph $y = (x - 4)^2$, you shift the graph of $y = x^2$ to the ______ a distance of _______ units. (pp. 256–258)

Concepts and Vocabulary

5. The graph of a quadratic function is called a(n) ________.

6. The vertical line passing through the vertex of a parabola is called the ______________________.

7. The x-coordinate of the vertex of $f(x) = ax^2 + bx + c$, $a \neq 0$, is ________.

8. ***True or False*** The graph of $f(x) = 2x^2 + 3x - 4$ opens up.

9. ***True or False*** The y-coordinate of the vertex of $f(x) = -x^2 + 4x + 5$ is $f(2)$.

10. ***True or False*** If the discriminant $b^2 - 4ac = 0$, the graph of $f(x) = ax^2 + bx + c, a \neq 0$, will touch the x-axis at its vertex.

11. If $b^2 - 4ac > 0$, which of the following conclusions can be made about the graph of $f(x) = ax^2 + bx + c$, $a \neq 0$?
(a) The graph has two distinct x-intercepts.
(b) The graph has no x-intercepts.
(c) The graph has three distinct x-intercepts.
(d) The graph has one x-intercept.

12. If the graph of $f(x) = ax^2 + bx + c$, $a \neq 0$, has a maximum value at its vertex, which of the following conditions must be true?

(a) $-\frac{b}{2a} > 0$ (b) $-\frac{b}{2a} < 0$ (c) $a > 0$ (d) $a < 0$

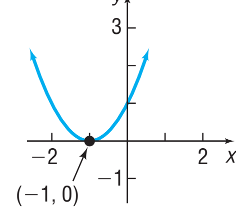
Skill Building

In Problems 13–20, match each graph to one the following functions.

13. $f(x) = x^2 - 1$ **14.** $f(x) = -x^2 - 1$ **15.** $f(x) = x^2 - 2x + 1$ **16.** $f(x) = x^2 + 2x + 1$

17. $f(x) = x^2 - 2x + 2$ **18.** $f(x) = x^2 + 2x$ **19.** $f(x) = x^2 - 2x$ **20.** $f(x) = x^2 + 2x + 2$

A



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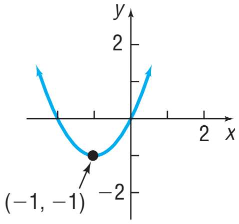

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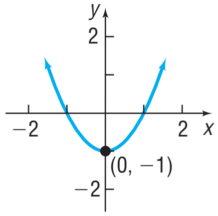

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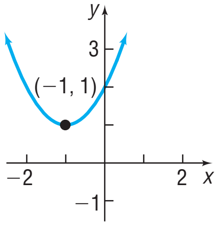

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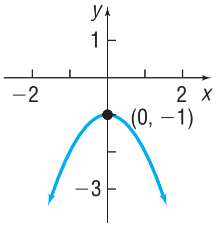

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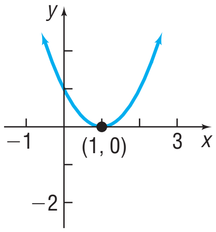

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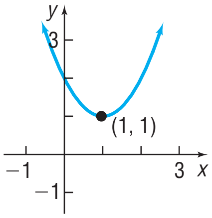

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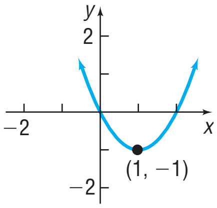


In Problems 21–32, graph the function f by starting with the graph of $y = x^2$ *and using transformations (shifting, compressing, stretching, and/or reflection). Verify your results using a graphing utility.*
[**Hint:** If necessary, write f in the form $f(x) = a(x - h)^2 + k$.]

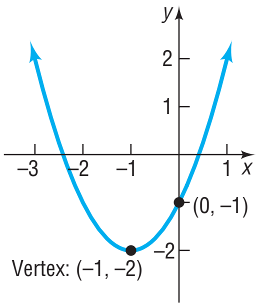
21. $f(x) = \frac{1}{4}x^2$
22. $f(x) = 2x^2 + 4$
23. $f(x) = (x + 2)^2 - 2$
24. $f(x) = (x - 3)^2 - 10$
25. $f(x) = x^2 + 4x + 2$
26. $f(x) = x^2 - 6x - 1$
27. $f(x) = 2x^2 - 4x + 1$
28. $f(x) = 3x^2 + 6x$
29. $f(x) = -x^2 - 2x$
30. $f(x) = -2x^2 + 6x + 2$
31. $f(x) = \frac{1}{2}x^2 + x - 1$
32. $f(x) = \frac{2}{3}x^2 + \frac{4}{3}x - 1$

In Problems 33–48, (a) graph each quadratic function by determining whether its graph opens up or down and by finding its vertex, axis of symmetry, y-intercept, and x-intercepts, if any. (b) Determine the domain and the range of the function. (c) Determine where the function is increasing and where it is decreasing. Verify your results using a graphing utility.

33. $f(x) = x^2 + 2x$
34. $f(x) = x^2 - 4x$
35. $f(x) = -x^2 - 6x$
36. $f(x) = -x^2 + 4x$
37. $f(x) = x^2 + 2x - 8$
38. $f(x) = x^2 - 2x - 3$
39. $f(x) = x^2 + 2x + 1$
40. $f(x) = x^2 + 6x + 9$
41. $f(x) = 2x^2 - x + 2$
42. $f(x) = 4x^2 - 2x + 1$
43. $f(x) = -2x^2 + 2x - 3$
44. $f(x) = -3x^2 + 3x - 2$
45. $f(x) = 3x^2 + 6x + 2$
46. $f(x) = 2x^2 + 5x + 3$
47. $f(x) = -4x^2 - 6x + 2$
48. $f(x) = 3x^2 - 8x + 2$

In Problems 49–54, determine the quadratic function whose graph is given.

49.



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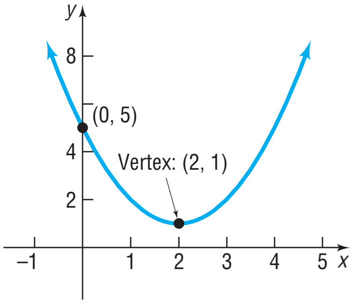

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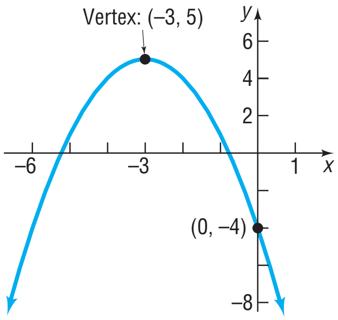

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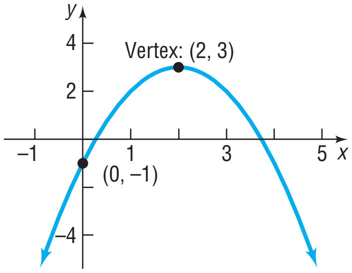

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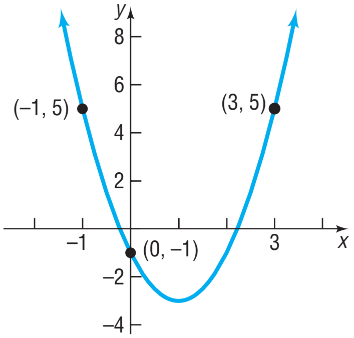

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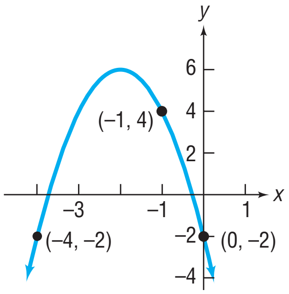

In Problems 55–62, determine, without graphing, whether the given quadratic function has a maximum value or a minimum value and then find the value.

55. $f(x) = 2x^2 + 12x$
56. $f(x) = -2x^2 + 12x$
57. $f(x) = 2x^2 + 12x - 3$
58. $f(x) = 4x^2 - 8x + 3$
59. $f(x) = -x^2 + 10x - 4$
60. $f(x) = -2x^2 + 8x + 3$
61. $f(x) = -3x^2 + 12x + 1$
62. $f(x) = 4x^2 - 4x$

Mixed Practice

In Problems 63–74, (a) graph each function, (b) determine the domain and the range of the function, and (c) determine where the function is increasing and where it is decreasing.

63. $f(x) = x^2 - 2x - 15$

64. $g(x) = x^2 - 2x - 8$

65. $F(x) = 2x - 5$

66. $f(x) = \frac{3}{2}x - 2$

67. $g(x) = -2(x-3)^2 + 2$

68. $h(x) = -3(x+1)^2 + 4$

69. $f(x) = 2x^2 + x + 1$

70. $G(x) = 3x^2 + 2x + 5$

71. $h(x) = -\frac{2}{5}x + 4$

72. $f(x) = -3x + 2$

73. $H(x) = -4x^2 - 4x - 1$

74. $F(x) = -4x^2 + 20x - 25$

Applications and Extensions

75. The graph of the function $f(x) = ax^2 + bx + c$ has vertex at $(0, 2)$ and passes through the point $(1, 8)$. Find a, b, and c.

76. The graph of the function $f(x) = ax^2 + bx + c$ has vertex at $(1, 4)$ and passes through the point $(-1, -8)$. Find a, b, and c.

In Problems 77–82, for the given functions f and g,
(a) Graph f and g on the same Cartesian plane.
(b) Solve $f(x) = g(x)$.
(c) Use the result of part (b) to label the points of intersection of the graphs of f and g.
(d) Shade the region for which $f(x) > g(x)$, that is, the region below f and above g.

77. $f(x) = 2x - 1; \quad g(x) = x^2 - 4$

78. $f(x) = -2x - 1; \quad g(x) = x^2 - 9$

79. $f(x) = -x^2 + 4; \quad g(x) = -2x + 1$

80. $f(x) = -x^2 + 9; \quad g(x) = 2x + 1$

81. $f(x) = -x^2 + 5x; \quad g(x) = x^2 + 3x - 4$

82. $f(x) = -x^2 + 7x - 6; \quad g(x) = x^2 + x - 6$

Answer Problems 83 and 84 using the following: A quadratic function of the form $f(x) = ax^2 + bx + c$ with $b^2 - 4ac > 0$ may also be written in the form $f(x) = a(x - r_1)(x - r_2)$, where r_1 and r_2 are the x-intercepts of the graph of the quadratic function.

83. (a) Find a quadratic function whose x-intercepts are -3 and 1 with $a = 1; a = 2; a = -2; a = 5$.
(b) How does the value of a affect the intercepts?
(c) How does the value of a affect the axis of symmetry?
(d) How does the value of a affect the vertex?
(e) Compare the x-coordinate of the vertex with the midpoint of the x-intercepts. What might you conclude?

84. (a) Find a quadratic function whose x-intercepts are -5 and 3 with $a = 1; a = 2; a = -2; a = 5$.
(b) How does the value of a affect the intercepts?
(c) How does the value of a affect the axis of symmetry?
(d) How does the value of a affect the vertex?
(e) Compare the x-coordinate of the vertex with the midpoint of the x-intercepts. What might you conclude?

---

85. Suppose that $f(x) = x^2 + 4x - 21$.
(a) What is the vertex of f?
(b) What are the x-intercepts of the graph of f?
(c) Solve $f(x) = -21$ for x. What points are on the graph of f?
(d) Use the information obtained in parts (a)–(c) to graph $f(x) = x^2 + 4x - 21$.

86. Suppose that $f(x) = x^2 + 2x - 8$.
(a) What is the vertex of f?
(b) What are the x-intercepts of the graph of f?
(c) Solve $f(x) = -8$ for x. What points are on the graph of f?
(d) Use the information obtained in parts (a)–(c) to graph $f(x) = x^2 + 2x - 8$.

87. Find the point on the line $y = x$ that is closest to the point $(3, 1)$.

[**Hint:** Express the distance d from the point to the line as a function of x, and then find the minimum value of $[d(x)]^2$.]

88. Find the point on the line $y = x + 1$ that is closest to the point $(4, 1)$.

89. Maximizing Revenue Suppose that the manufacturer of a gas clothes dryer has found that, when the unit price is p dollars, the revenue R (in dollars) is

$$R(p) = -4p^2 + 4000p$$

What unit price should be established for the dryer to maximize revenue? What is the maximum revenue?

90. Maximizing Revenue The John Deere company has found that the revenue, in dollars, from sales of riding mowers is a function of the unit price p, in dollars, that it charges. If the revenue R is

$$R(p) = -\frac{1}{2}p^2 + 1900p$$

what unit price p should be charged to maximize revenue? What is the maximum revenue?

91. Minimizing Marginal Cost The **marginal cost** of a product can be thought of as the cost of producing one additional unit of output. For example, if the marginal cost of producing the 50th product is \$6.20, it cost \$6.20 to increase production from 49 to 50 units of output. Suppose the marginal cost C

(in dollars) to produce x thousand digital music players is given by the function

$$C(x) = x^2 - 140x + 7400$$

(a) How many players should be produced to minimize the marginal cost?
(b) What is the minimum marginal cost?

92. Minimizing Marginal Cost (See Problem 91.) The marginal cost C (in dollars) of manufacturing x cell phones (in thousands) is given by

$$C(x) = 5x^2 - 200x + 4000$$

(a) How many cell phones should be manufactured to minimize the marginal cost?
(b) What is the minimum marginal cost?

93. Business The monthly revenue R achieved by selling x wristwatches is figured to be $R(x) = 75x - 0.2x^2$. The monthly cost C of selling x wristwatches is $C(x) = 32x + 1750$.
(a) How many wristwatches must the firm sell to maximize revenue? What is the maximum revenue?
(b) Profit is given as $P(x) = R(x) - C(x)$. What is the profit function?
(c) How many wristwatches must the firm sell to maximize profit? What is the maximum profit?
(d) Provide a reasonable explanation as to why the answers found in parts (a) and (c) differ. Explain why a quadratic function is a reasonable model for revenue.

94. Business The daily revenue R achieved by selling x boxes of candy is figured to be $R(x) = 9.5x - 0.04x^2$. The daily cost C of selling x boxes of candy is $C(x) = 1.25x + 250$.
(a) How many boxes of candy must the firm sell to maximize revenue? What is the maximum revenue?
(b) Profit is given as $P(x) = R(x) - C(x)$. What is the profit function?
(c) How many boxes of candy must the firm sell to maximize profit? What is the maximum profit?
(d) Provide a reasonable explanation as to why the answers found in parts (a) and (c) differ. Explain why a quadratic function is a reasonable model for revenue.

95. Stopping Distance An accepted relationship between stopping distance, d (in feet), and the speed of a car, v (in mph), is $d = 1.1v + 0.06v^2$ on dry, level concrete.
(a) How many feet will it take a car traveling 45 mph to stop on dry, level concrete?
(b) If an accident occurs 200 feet ahead of you, what is the maximum speed you can be traveling to avoid being involved?
(c) What might the term $1.1v$ represent?

96. Birthrate for Unmarried Women In the United States, the birthrate B for unmarried women (births per 1000 unmarried women) whose age is a is modeled by the function $B(a) = -0.30a^2 + 16.26a - 158.90$.
(a) What is the age of unmarried women with the highest birthrate?
(b) What is the highest birthrate of unmarried women?
(c) Evaluate and interpret $B(40)$.

Source: *National Vital Statistics System, 2013*

97. Let $f(x) = ax^2 + bx + c$, where $a, b,$ and c are odd integers. If x is an integer, show that $f(x)$ must be an odd integer.
[**Hint:** x is either an even integer or an odd integer.]

Explaining Concepts: Discussion and Writing

98. Make up a quadratic function that opens down and has only one x-intercept. Compare yours with others in the class. What are the similarities? What are the differences?

99. On one set of coordinate axes, graph the family of parabolas $f(x) = x^2 + 2x + c$ for $c = -3$, $c = 0$, and $c = 1$. Describe the characteristics of a member of this family.

100. On one set of coordinate axes, graph the family of parabolas $f(x) = x^2 + bx + 1$ for $b = -4$, $b = 0$, and $b = 4$. Describe the general characteristics of this family.

101. State the circumstances that cause the graph of a quadratic function $f(x) = ax^2 + bx + c$ to have no x-intercepts.

102. Why does the graph of a quadratic function open up if $a > 0$ and down if $a < 0$?

103. Can a quadratic function have a range of $(-\infty, \infty)$? Justify your answer.

104. What are the possibilities for the number of times the graphs of two different quadratic functions intersect?

Retain Your Knowledge

Problems 105–108 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

105. Determine whether $x^2 + 4y^2 = 16$ is symmetric respect to the x-axis, the y-axis, and/or the origin.

106. Solve the inequality $27 - x \geq 5x + 3$. Write the solution in both set notation and interval notation.

107. Find the center and radius of the circle $x^2 + y^2 - 10x + 4y + 20 = 0$.

108. Write the function whose graph is the graph of $y = \sqrt{x}$, but reflected about the y-axis.

'Are You Prepared?' Answers

1. $(0, -9), (-3, 0), (3, 0)$ **2.** $\left\{-4, \frac{1}{2}\right\}$ **3.** $\frac{25}{4}$ **4.** right; 4

4.4 Build Quadratic Models from Verbal Descriptions and from Data

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Problem Solving (Section 1.6, pp. 137–143)
- Linear Models: Building Linear Functions from Data (Section 4.2, pp. 291–294)

Now Work the **'Are You Prepared?'** problems on page 315.

OBJECTIVES 1 Build Quadratic Models from Verbal Descriptions (p. 310)
2 Build Quadratic Models from Data (p. 314)

In this section we will first discuss models in the form of a quadratic function when a verbal description of the problem is given. We end the section by fitting a quadratic function to data, which is another form of modeling.

When a mathematical model is in the form of a quadratic function, the properties of the graph of the function can provide important information about the model. In particular, we can use the quadratic function to determine the maximum or minimum value of the function. The fact that the graph of a quadratic function has a maximum or minimum value enables us to answer questions involving **optimization**—that is, finding the maximum or minimum values in models.

1 Build Quadratic Models from Verbal Descriptions

In economics, revenue R, in dollars, is defined as the amount of money received from the sale of an item and is equal to the unit selling price p, in dollars, of the item times the number x of units actually sold. That is,

$$R = xp$$

The Law of Demand states that p and x are related: As one increases, the other decreases. The equation that relates p and x is called the **demand equation**. When the demand equation is linear, the revenue model is a quadratic function.

EXAMPLE 1 Maximizing Revenue

The marketing department at Texas Instruments has found that when certain calculators are sold at a price of p dollars per unit, the number x of calculators sold is given by the demand equation

$$x = 21{,}000 - 150p$$

(a) Find a model that expresses the revenue R as a function of the price p.
(b) What is the domain of R?
(c) What unit price should be used to maximize revenue?
(d) If this price is charged, what is the maximum revenue?
(e) How many units are sold at this price?
(f) Graph R.
(g) What price should Texas Instruments charge to collect at least \$675,000 in revenue?

Solution (a) The revenue R is $R = xp$, where $x = 21{,}000 - 150p$.

$$R = xp = (21{,}000 - 150p)p = -150p^2 + 21{,}000p$$

(b) Because x represents the number of calculators sold, we have $x \geq 0$, so $21{,}000 - 150p \geq 0$. Solving this linear inequality gives $p \leq 140$. In addition, Texas Instruments will charge only a positive price for the calculator, so $p > 0$. Combining these inequalities gives the domain of R, which is $\{p \mid 0 < p \leq 140\}$.

(c) The function R is a quadratic function with $a = -150$, $b = 21{,}000$, and $c = 0$. Because $a < 0$, the vertex is the highest point on the parabola. The revenue R is a maximum when the price p is

$$p = -\frac{b}{2a} \underset{\mathbf{a\,=\,-150,\ b\,=\,21{,}000}}{\uparrow}= -\frac{21{,}000}{2(-150)} = \$70.00$$

(d) The maximum revenue R is

$$R(70) = -150(70)^2 + 21{,}000(70) = \$735{,}000$$

(e) The number of calculators sold is given by the demand equation $x = 21{,}000 - 150p$. At a price of $p = \$70$,

$$x = 21{,}000 - 150(70) = 10{,}500$$

calculators are sold.

(f) To graph R, plot the intercept $(140, 0)$ and the vertex $(70, 735\,000)$. See Figure 24 for the graph.

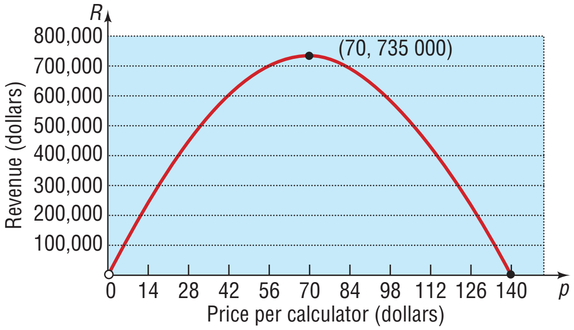

Figure 24

(g) Graph $R = 675{,}000$ and $R(p) = -150p^2 + 21{,}000p$ on the same Cartesian plane. See Figure 25. We find where the graphs intersect by solving

$$\begin{aligned} 675{,}000 &= -150p^2 + 21{,}000p \\ 150p^2 - 21{,}000p + 675{,}000 &= 0 && \textbf{Add } 150p^2 - 21{,}000p \textbf{ to both sides.} \\ p^2 - 140p + 4500 &= 0 && \textbf{Divide both sides by 150.} \\ (p - 50)(p - 90) &= 0 && \textbf{Factor.} \\ p = 50 \text{ or } p &= 90 && \textbf{Use the Zero-Product Property.} \end{aligned}$$

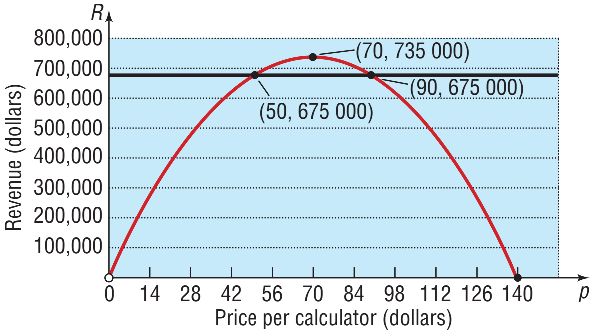

Figure 25

The graphs intersect at $(50, 675\,000)$ and $(90, 675\,000)$. Based on the graph in Figure 25, Texas Instruments should charge between \$50 and \$90 to earn at least \$675,000 in revenue. ■

Now Work PROBLEM 3

EXAMPLE 2 Maximizing the Area Enclosed by a Fence

A farmer has 2000 yards of fence to enclose a rectangular field. What are the dimensions of the rectangle that encloses the most area?

Solution Figure 26 illustrates the situation. The available fence represents the perimeter of the rectangle. If x is the length and w is the width, then

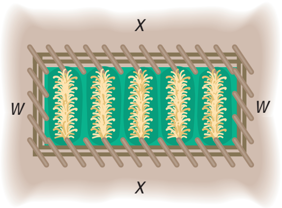

Figure 26

$$2x + 2w = 2000 \tag{1}$$

The area A of the rectangle is

$$A = xw$$

To express A in terms of a single variable, solve equation (1) for w and substitute the result in $A = xw$. Then A involves only the variable x. [You could also solve equation (1) for x and express A in terms of w alone. Try it!]

$$\begin{aligned} 2x + 2w &= 2000 \\ 2w &= 2000 - 2x \\ w &= \frac{2000 - 2x}{2} = 1000 - x \end{aligned}$$

Then the area A is

$$A = xw = x(1000 - x) = -x^2 + 1000x$$

Now, A is a quadratic function of x.

$$A(x) = -x^2 + 1000x \quad \mathbf{a = -1, b = 1000, c = 0}$$

Figure 27 shows the graph of $A(x) = -x^2 + 1000x$. Because $a < 0$, the vertex is a maximum point on the graph of A. The maximum value occurs at

$$x = -\frac{b}{2a} = -\frac{1000}{2(-1)} = 500$$

The maximum value of A is

$$A\left(-\frac{b}{2a}\right) = A(500) = -500^2 + 1000(500) = -250{,}000 + 500{,}000 = 250{,}000$$

The largest rectangle that can be enclosed by 2000 yards of fence has an area of 250,000 square yards. Its dimensions are 500 yards by 500 yards. ■

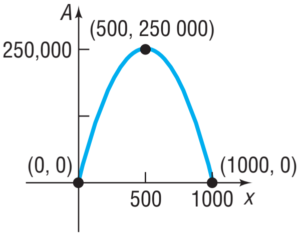

Figure 27 $A(x) = -x^2 + 1000x$

Now Work PROBLEM 7

EXAMPLE 3 Analyzing the Motion of a Projectile

A projectile is fired from a cliff 500 feet above the water at an inclination of 45° to the horizontal, with a muzzle velocity of 400 feet per second. From physics, the height h of the projectile above the water can be modeled by

$$h(x) = \frac{-32x^2}{(400)^2} + x + 500$$

where x is the horizontal distance of the projectile from the base of the cliff. See Figure 28.

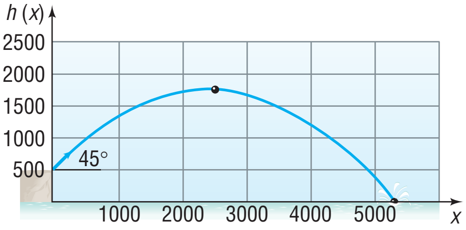

Figure 28

(a) Find the maximum height of the projectile.
(b) How far from the base of the cliff will the projectile strike the water?

Solution (a) The height of the projectile is given by a quadratic function.

$$h(x) = \frac{-32x^2}{(400)^2} + x + 500 = \frac{-1}{5000}x^2 + x + 500$$

We are looking for the maximum value of h. Because $a < 0$, the maximum value occurs at the vertex, whose x-coordinate is

$$x = -\frac{b}{2a} = -\frac{1}{2\left(-\frac{1}{5000}\right)} = \frac{5000}{2} = 2500$$

The maximum height of the projectile is

$$h(2500) = \frac{-1}{5000}(2500)^2 + 2500 + 500 = -1250 + 2500 + 500 = 1750 \text{ ft}$$

(b) The projectile will strike the water when the height is zero. To find the distance x traveled, solve the equation

$$h(x) = \frac{-1}{5000}x^2 + x + 500 = 0$$

The discriminant of this quadratic equation is

$$b^2 - 4ac = 1^2 - 4\left(\frac{-1}{5000}\right)(500) = 1.4$$

Then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1.4}}{2\left(-\frac{1}{5000}\right)} \approx \begin{cases} -458 \\ 5458 \end{cases}$$

Discard the negative solution. The projectile will strike the water at a distance of about 5458 feet from the base of the cliff. ■

Seeing the Concept

Graph

$$h(x) = \frac{-1}{5000}x^2 + x + 500$$
$$0 \le x \le 5500$$

Use MAXIMUM to find the maximum height of the projectile, and use ROOT or ZERO to find the distance from the base of the cliff to where it strikes the water. Compare your results with those obtained in Example 3. ■

Now Work PROBLEM 11

EXAMPLE 4 The Golden Gate Bridge

The Golden Gate Bridge, a suspension bridge, spans the entrance to San Francisco Bay. Its 746-foot-tall towers are 4200 feet apart. The bridge is suspended from two huge cables more than 3 feet in diameter; the 90-foot-wide roadway is 220 feet above the water. The cables are parabolic in shape* and touch the road surface at the center of the bridge. Find the height of the cable above the road at a distance of 1000 feet from the center.

Solution See Figure 29 on the next page. Begin by choosing the placement of the coordinate axes so that the x-axis coincides with the road surface and the origin coincides with the center of the bridge. As a result, the twin towers will be vertical (height $746 - 220 = 526$ feet above the road) and located 2100 feet from the center. Also, the cable, which has the shape of a parabola, will extend from the towers, open up, and have its vertex at $(0, 0)$. This choice of placement of the axes enables the equation of the parabola to have the form $y = ax^2$, $a > 0$. Note that the points $(-2100, 526)$ and $(2100, 526)$ are on the graph.

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*A cable suspended from two towers is in the shape of a **catenary**, but when a horizontal roadway is suspended from the cable, the cable takes the shape of a parabola.

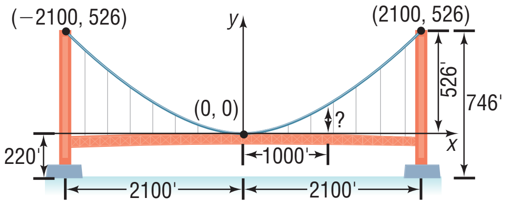


Figure 29

Use these facts to find the value of a in $y = ax^2$.

$$y = ax^2$$

$$526 = a(2100)^2 \quad x = 2100, y = 526$$

$$a = \frac{526}{(2100)^2}$$

The equation of the parabola is

$$y = \frac{526}{(2100)^2}x^2$$

When $x = 1000$, the height of the cable is

$$y = \frac{526}{(2100)^2}(1000)^2 \approx 119.3 \text{ feet}$$

The cable is 119.3 feet above the road at a distance of 1000 feet from the center of the bridge. ■

Now Work PROBLEM 13

2 Build Quadratic Models from Data

In Section 4.2, we found the line of best fit for data that appeared to be linearly related. It was noted that data may also follow a nonlinear relation. Figures 30(a) and (b) show scatter diagrams of data that follow a quadratic relation.

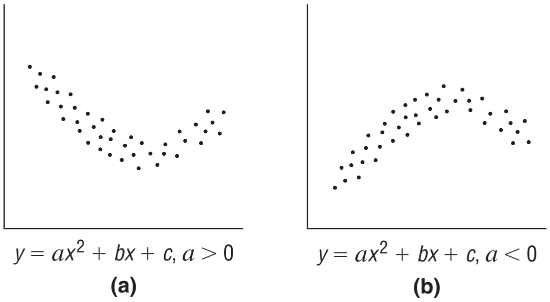


Figure 30

EXAMPLE 5 Fitting a Quadratic Function to Data

The data in Table 8 represent the percentage D of the population that is divorced for various ages x.

(a) Draw a scatter diagram of the data treating age as the independent variable. Comment on the type of relation that may exist between age and percentage of the population divorced.

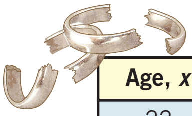
(b) Use a graphing utility to find the quadratic function of best fit that models the relation between age and percentage of the population divorced.

(c) Use the model found in part (b) to approximate the age at which the percentage of the population divorced is greatest.

(d) Use the model found in part (b) to approximate the highest percentage of the population that is divorced.

(e) Use a graphing utility to draw the quadratic function of best fit on the scatter diagram.

Table 8



Age, x	Percentage Divorced, D
22	0.9
27	3.6
32	7.4
37	10.4
42	12.7
50	15.7
60	16.2
70	13.1
80	6.5

Source: United States Statistical Abstract, 2012

Solution

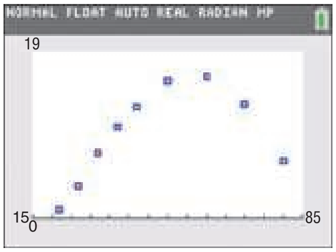
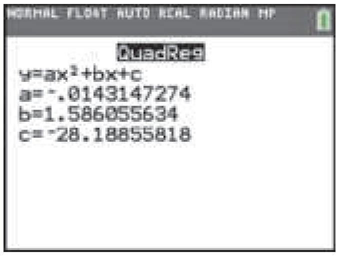


Figure 31



```
NORMAL FLOAT AUTO REAL RADIAN MP
        QuadReg
 y=ax²+bx+c
 a=⁻.0143147274
 b=1.586055634
 c=⁻28.18855818
```

Figure 32

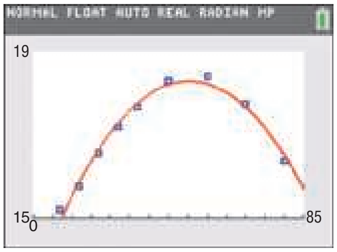


Figure 33

(a) Figure 31 shows the scatter diagram, from which it appears the data follow a quadratic relation, with $a < 0$.

(b) Execute the QUADratic REGression program to obtain the results shown in Figure 32. The output shows the equation $y = ax^2 + bx + c$. The quadratic function of best fit that models the relation between age and percentage divorced is

$$D(x) = -0.0143x^2 + 1.5861x - 28.1886 \quad \textbf{The model}$$

where x represents age and D represents the percentage divorced.

(c) Based on the quadratic function of best fit, the age with the greatest percentage divorced is

$$-\frac{b}{2a} = -\frac{1.5861}{2(-0.0143)} \approx 55 \text{ years}$$

(d) Evaluate the function $D(x)$ at $x = 55$.

$$D(55) = -0.0143(55)^2 + 1.5861(55) - 28.1886 \approx 15.8 \text{ percent}$$

According to the model, 55-year-olds have the highest percentage divorced at 15.8 percent.

(e) Figure 33 shows the graph of the quadratic function found in part (b) drawn on the scatter diagram. ■

Look again at Figure 32. Notice that the output given by the graphing calculator does not include r, the correlation coefficient. Recall that the correlation coefficient is a measure of the strength of a linear relation that exists between two variables. The graphing calculator does not provide an indication of how well the function fits the data in terms of r, since a quadratic function cannot be expressed as a linear function.

Now Work PROBLEM 25

4.4 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Translate the following sentence into a mathematical equation: The total revenue R from selling x hot dogs is \$3 times the number of hot dogs sold. (p. 138)

2. Use a graphing utility to find the line of best fit for the following data: (pp. 293–294)

x	3	5	5	6	7	8
y	10	13	12	15	16	19

Applications and Extensions

3. **Maximizing Revenue** The price p (in dollars) and the quantity x sold of a certain product obey the demand equation

$$p = -\frac{1}{6}x + 100$$

(a) Find a model that expresses the revenue R as a function of x. (Remember, $R = xp$.)
(b) What is the domain of R?
(c) What is the revenue if 200 units are sold?
(d) What quantity x maximizes revenue? What is the maximum revenue?
(e) What price should the company charge to maximize revenue?

4. **Maximizing Revenue** The price p (in dollars) and the quantity x sold of a certain product obey the demand equation

$$p = -\frac{1}{3}x + 100$$

(a) Find a model that expresses the revenue R as a function of x.
(b) What is the domain of R?
(c) What is the revenue if 100 units are sold?
(d) What quantity x maximizes revenue? What is the maximum revenue?
(e) What price should the company charge to maximize revenue?

5. **Maximizing Revenue** The price p (in dollars) and the quantity x sold of a certain product obey the demand equation

$$x = -5p + 100 \qquad 0 < p \le 20$$

(a) Express the revenue R as a function of x.
(b) What is the revenue if 15 units are sold?
(c) What quantity x maximizes revenue? What is the maximum revenue?
(d) What price should the company charge to maximize revenue?
(e) What price should the company charge to earn at least \$480 in revenue?

6. **Maximizing Revenue** The price p (in dollars) and the quantity x sold of a certain product obey the demand equation

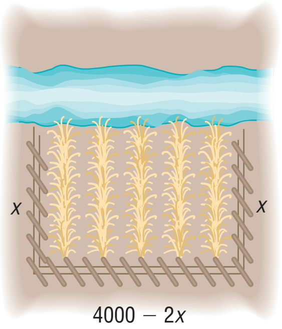
$$x = -20p + 500 \qquad 0 < p \le 25$$

(a) Express the revenue R as a function of x.
(b) What is the revenue if 20 units are sold?
(c) What quantity x maximizes revenue? What is the maximum revenue?
(d) What price should the company charge to maximize revenue?
(e) What price should the company charge to earn at least \$3000 in revenue?

7. **Enclosing a Rectangular Field** David has 400 yards of fencing and wishes to enclose a rectangular area.
(a) Express the area A of the rectangle as a function of the width w of the rectangle.
(b) For what value of w is the area largest?
(c) What is the maximum area?

8. **Enclosing a Rectangular Field** Beth has 3000 feet of fencing available to enclose a rectangular field.
(a) Express the area A of the rectangle as a function of x, where x is the length of the rectangle.
(b) For what value of x is the area largest?
(c) What is the maximum area?

9. **Enclosing the Most Area with a Fence** A farmer with 4000 meters of fencing wants to enclose a rectangular plot that borders on a river. If the farmer does not fence the side along the river, what is the largest area that can be enclosed? (See the figure.)



10. **Enclosing the Most Area with a Fence** A farmer with 2000 meters of fencing wants to enclose a rectangular plot that borders on a straight highway. If the farmer does not fence the side along the highway, what is the largest area that can be enclosed?

11. **Analyzing the Motion of a Projectile** A projectile is fired from a cliff 200 feet above the water at an inclination of 45° to the horizontal, with a muzzle velocity of 50 feet per second. The height h of the projectile above the water is modeled by

$$h(x) = \frac{-32x^2}{(50)^2} + x + 200$$

where x is the horizontal distance of the projectile from the face of the cliff.
(a) At what horizontal distance from the face of the cliff is the height of the projectile a maximum?
(b) Find the maximum height of the projectile.
(c) At what horizontal distance from the face of the cliff will the projectile strike the water?
(d) Using a graphing utility, graph the function h, $0 \le x \le 200$.
(e) Use a graphing utility to verify the solutions found in parts (b) and (c).
(f) When the height of the projectile is 100 feet above the water, how far is it from the cliff?

12. **Analyzing the Motion of a Projectile** A projectile is fired at an inclination of 45° to the horizontal, with a muzzle velocity of 100 feet per second. The height h of the projectile is modeled by

$$h(x) = \frac{-32x^2}{(100)^2} + x$$

where x is the horizontal distance of the projectile from the firing point.
(a) At what horizontal distance from the firing point is the height of the projectile a maximum?
(b) Find the maximum height of the projectile.
(c) At what horizontal distance from the firing point will the projectile strike the ground?
(d) Using a graphing utility, graph the function h, $0 \le x \le 350$.

(e) Use a graphing utility to verify the results obtained in parts (b) and (c).

(f) When the height of the projectile is 50 feet above the ground, how far has it traveled horizontally?

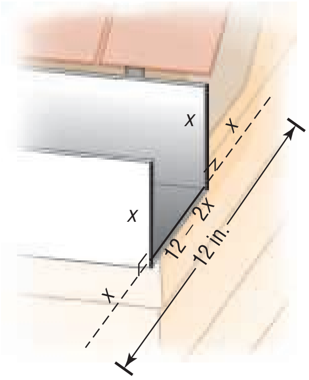
13. Suspension Bridge A suspension bridge with weight uniformly distributed along its length has twin towers that extend 75 meters above the road surface and are 400 meters apart. The cables are parabolic in shape and are suspended from the tops of the towers. The cables touch the road surface at the center of the bridge. Find the height of the cables at a point 100 meters from the center. (Assume that the road is level.)

14. Architecture A parabolic arch has a span of 120 feet and a maximum height of 25 feet. Choose suitable rectangular coordinate axes and find the equation of the parabola. Then calculate the height of the arch at points 10 feet, 20 feet, and 40 feet from the center.

15. Constructing Rain Gutters A rain gutter is to be made of aluminum sheets that are 12 inches wide by turning up the edges 90°. See the illustration.

(a) What depth will provide maximum cross-sectional area and hence allow the most water to flow?

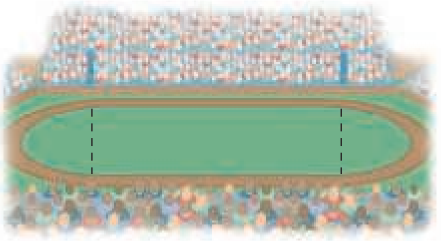
(b) What depths will allow at least 16 square inches of water to flow?



16. Norman Windows A **Norman window** has the shape of a rectangle surmounted by a semicircle of diameter equal to the width of the rectangle. See the picture. If the perimeter of the window is 20 feet, what dimensions will admit the most light (maximize the area)?

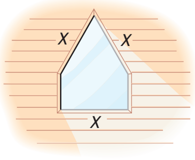
[**Hint:** Circumference of a circle $= 2\pi r$; area of a circle $= \pi r^2$, where r is the radius of the circle.]

17. Constructing a Stadium A track-and-field playing area is in the shape of a rectangle with semicircles at each end. See the figure (top, right). The inside perimeter of the track is to be 1500 meters. What should the dimensions of the rectangle be so that the area of the rectangle is a maximum?



18. Architecture A special window has the shape of a rectangle surmounted by an equilateral triangle. See the figure. If the perimeter of the window is 16 feet, what dimensions will admit the most light?

[**Hint:** Area of an equilateral triangle $= \left(\frac{\sqrt{3}}{4}\right)x^2$, where x is the length of a side of the triangle.]



19. Chemical Reactions A self-catalytic chemical reaction results in the formation of a compound that causes the formation ratio to increase. If the reaction rate V is modeled by

$$V(x) = kx(a - x), \qquad 0 \le x \le a$$

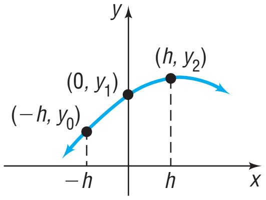
where k is a positive constant, a is the initial amount of the compound, and x is the variable amount of the compound, for what value of x is the reaction rate a maximum?

20. Calculus: Simpson's Rule The figure shows the graph of $y = ax^2 + bx + c$. Suppose that the points $(-h, y_0)$, $(0, y_1)$, and (h, y_2) are on the graph. It can be shown that the area enclosed by the parabola, the x-axis, and the lines $x = -h$ and $x = h$ is

$$\text{Area} = \frac{h}{3}(2ah^2 + 6c)$$

Show that this area may also be given by

$$\text{Area} = \frac{h}{3}(y_0 + 4y_1 + y_2)$$



21. Use the result obtained in Problem 20 to find the area enclosed by $f(x) = -5x^2 + 8$, the x-axis, and the lines $x = -1$ and $x = 1$.

22. Use the result obtained in Problem 20 to find the area enclosed by $f(x) = 2x^2 + 8$, the x-axis, and the lines $x = -2$ and $x = 2$.

23. Use the result obtained in Problem 20 to find the area enclosed by $f(x) = x^2 + 3x + 5$, the x-axis, and the lines $x = -4$ and $x = 4$.

24. Use the result obtained in Problem 20 to find the area enclosed by $f(x) = -x^2 + x + 4$, the x-axis, and the lines $x = -1$ and $x = 1$.

25. Life Cycle Hypothesis An individual's income varies with his or her age. The following table shows the median income I of males of different age groups within the United States for 2012. For each age group, let the class midpoint represent the independent variable, x. For the class "65 years and older," we will assume that the class midpoint is 69.5.

Age	Class Midpoint, x	Median Income, I
15–24 years	19.5	\$10,869
25–34 years	29.5	\$34,113
35–44 years	39.5	\$45,225
45–54 years	49.5	\$46,466
55–64 years	59.5	\$42,176
65 years and older	69.5	\$27,612

Source: U.S. Census Bureau

(a) Use a graphing utility to draw a scatter diagram of the data. Comment on the type of relation that may exist between the two variables.

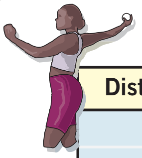
(b) Use a graphing utility to find the quadratic function of best fit that models the relation between age and median income.

(c) Use the function found in part (b) to determine the age at which an individual can expect to earn the most income.

(d) Use the function found in part (b) to predict the peak income earned.

(e) With a graphing utility, graph the quadratic function of best fit on the scatter diagram.

26. Height of a Ball A shot-putter throws a ball at an inclination of 45° to the horizontal. The following data represent the height of the ball h, in feet, at the instant that it has traveled x feet horizontally.



Distance, x	Height, h
20	25
40	40
60	55
80	65
100	71
120	77
140	77
160	75
180	71
200	64

(a) Use a graphing utility to draw a scatter diagram of the data. Comment on the type of relation that may exist between the two variables.

(b) Use a graphing utility to find the quadratic function of best fit that models the relation between distance and height.

(c) Use the function found in part (b) to determine how far the ball will travel before it reaches its maximum height.

(d) Use the function found in part (b) to find the maximum height of the ball.

(e) With a graphing utility, graph the quadratic function of best fit on the scatter diagram.

Mixed Practice

27. Which Model? The following data represent the square footage and rents (dollars per month) for apartments in the La Jolla area of San Diego, California.

Square Footage, x	Rent per Month, R
520	\$1525
621	\$1750
718	\$1785
753	\$1850
850	\$1900
968	\$2130
1020	\$2180

Source: *apartments.com*, 2014

(a) Using a graphing utility, draw a scatter diagram of the data treating square footage as the independent variable. What type of relation appears to exist between square footage and rent?

(b) Based on your response to part (a), find either a linear or a quadratic model that describes the relation between square footage and rent.

(c) Use your model to predict the rent for an apartment in San Diego that is 875 square feet.

28. Which Model? An engineer collects the following data showing the speed s of a Toyota Camry and its average miles per gallon, M.

Speed, s	Miles per Gallon, M
30	18
35	20
40	23
40	25
45	25
50	28
55	30
60	29
65	26
65	25
70	25

(a) Using a graphing utility, draw a scatter diagram of the data, treating speed as the independent variable. What type of relation appears to exist between speed and miles per gallon?

(b) Based on your response to part (a), find either a linear model or a quadratic model that describes the relation between speed and miles per gallon.

(c) Use your model to predict the miles per gallon for a Camry that is traveling 63 miles per hour.

29. **Which Model?** The following data represent the birth rate (births per 1000 population) for women whose age is *a*, in 2012.

Age, *a*	Birth Rate, *B*
16	14.1
19	51.4
22	83.1
27	106.5
32	97.3
37	48.3
42	10.4

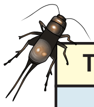
Source: National Vital Statistics System, 2013

(a) Using a graphing utility, draw a scatter diagram of the data, treating age as the independent variable. What type of relation appears to exist between age and birth rate?

(b) Based on your response to part (a), find either a linear or a quadratic model that describes the relation between age and birth rate.

(c) Use your model to predict the birth rate for 35-year-old women.

30. **Which Model?** A cricket makes a chirping noise by sliding its wings together rapidly. Perhaps you have noticed that the rapidity of chirps seems to increase with the temperature. The following data list the temperature (in degrees Fahrenheit) and the number of chirps per second for the striped ground cricket.



Temperature (°F), *x*	Chirps per Second, *C*
88.6	20.0
93.3	19.8
80.6	17.1
69.7	14.7
69.4	15.4
79.6	15.0
80.6	16.0
76.3	14.4
75.2	15.5

Source: Pierce, George W. *The Songs of Insects*. Cambridge, MA Harvard University Press, 1949, pp. 12 – 21

(a) Using a graphing utility, draw a scatter diagram of the data, treating temperature as the independent variable. What type of relation appears to exist between temperature and chirps per second?

(b) Based on your response to part (a), find either a linear or a quadratic model that best describes the relation between temperature and chirps per second.

(c) Use your model to predict the chirps per second if the temperature is 80°F.

Explaining Concepts: Discussion and Writing

31. Refer to Example 1 in this section. Notice that if the price charged for the calculators is \$0 or \$140, then the revenue is \$0. It is easy to explain why revenue would be \$0 if the price charged were \$0, but how can revenue be \$0 if the price charged is \$140?

Retain Your Knowledge

Problems 32–35 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

32. Express as a complex number: $\sqrt{-225}$

33. Find the distance between the points $P_1 = (4, -7)$ and $P_2 = (-1, 5)$.

34. Find the equation of the circle with center $(-6, 0)$ and radius $r = \sqrt{7}$.

35. Solve: $5x^2 + 8x - 3 = 0$

'Are You Prepared?' Answers

1. $R = 3x$ **2.** $y = 1.7826x + 4.0652$

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Solve Inequalities (Section 1.7, pp. 150–154)
- Use Interval Notation (Section 1.7, pp. 147–148)

Now Work the **'Are You Prepared?'** problems on page 322.

OBJECTIVE 1 Solve Inequalities Involving a Quadratic Function (p. 320)

1 Solve Inequalities Involving a Quadratic Function

In this section we solve inequalities that involve quadratic functions. We will accomplish this by using their graphs. For example, to solve the inequality

$$ax^2 + bx + c > 0 \qquad a \neq 0$$

graph the function $f(x) = ax^2 + bx + c$ and, from the graph, determine where it is above the x-axis—that is, where $f(x) > 0$. To solve the inequality $ax^2 + bx + c < 0, a \neq 0$, graph the function $f(x) = ax^2 + bx + c$ and determine where the graph is below the x-axis. If the inequality is not strict, include the x-intercepts in the solution.

EXAMPLE 1 **Solving an Inequality**

Solve the inequality $x^2 - 4x - 12 \leq 0$ and graph the solution set.

By Hand Solution

Graph the function $f(x) = x^2 - 4x - 12$.

y-intercept: $f(0) = -12$ **Evaluate f at 0.**

x-intercepts (if any): $x^2 - 4x - 12 = 0$ **Solve $f(x) = 0$.**

$(x - 6)(x + 2) = 0$ **Factor.**

$x - 6 = 0 \text{ or } x + 2 = 0$ **Apply the Zero Product Property.**

$x = 6 \text{ or } x = -2$

The y-intercept is -12; the x-intercepts are -2 and 6.

The vertex is at $x = -\dfrac{b}{2a} = -\dfrac{-4}{2(1)} = 2$. Since $f(2) = -16$, the vertex is $(2, -16)$. See Figure 34 for the graph.

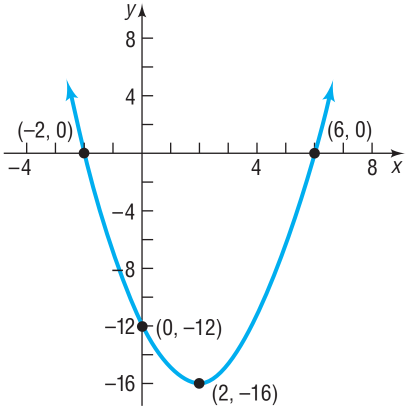


Figure 34 $f(x) = x^2 - 4x - 12$

Because we are solving $f(x) \leq 0$, we must find the x-values for which the graph is below the x-axis, which is between -2 and 6. Since the original inequality is not strict, include the x-intercepts. The solution set is $\{x \mid -2 \leq x \leq 6\}$ or, using interval notation, $[-2, 6]$.

Graphing Utility Solution

Graph $Y_1 = x^2 - 4x - 12$. See Figure 35. Use the ZERO command to find that the x-intercepts of Y_1 are -2 and 6. Because we are solving $f(x) \leq 0$, we must find the x-values for which the graph is below the x-axis, which is between -2 and 6. Since the inequality is not strict, the solution set is $\{x \mid -2 \leq x \leq 6\}$ or, using interval notation, $[-2, 6]$.

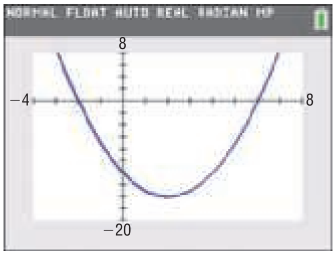


Figure 35 $Y_1 = x^2 - 4x - 12$

See Figure 36 for the graph of the solution set.

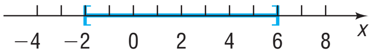


Figure 36

■

Now Work PROBLEM 9

EXAMPLE 2 **Solving an Inequality**

Solve the inequality $2x^2 < x + 10$ and graph the solution set.

Solution ***Option 1*** Rearrange the inequality so that 0 is on the right side.

$$2x^2 < x + 10$$

$$2x^2 - x - 10 < 0 \qquad \textbf{Subtract } x + 10 \textbf{ from both sides.}$$

This inequality is equivalent to the original inequality.

Next graph the function $f(x) = 2x^2 - x - 10$ to find where $f(x) < 0$.

$$\begin{aligned} &y\text{-intercept:} & f(0) &= -10 & &\textbf{Evaluate } f \textbf{ at 0.}\\ &x\text{-intercepts (if any):} & 2x^2 - x - 10 &= 0 & &\textbf{Solve } f(x) = 0.\\ & & (2x-5)(x+2) &= 0 & &\textbf{Factor.}\\ & & 2x - 5 = 0 \quad &\text{or} \quad x + 2 = 0 & &\textbf{Apply the Zero-Product Property.}\\ & & x = \frac{5}{2} \quad &\text{or} \quad x = -2 & & \end{aligned}$$

The y-intercept is -10; the x-intercepts are -2 and $\frac{5}{2}$.

The vertex is at $x = -\frac{b}{2a} = -\frac{-1}{2(2)} = \frac{1}{4}$. Since $f\left(\frac{1}{4}\right) = -10.125$, the vertex is $\left(\frac{1}{4}, -10.125\right)$. See Figure 37 for the graph.

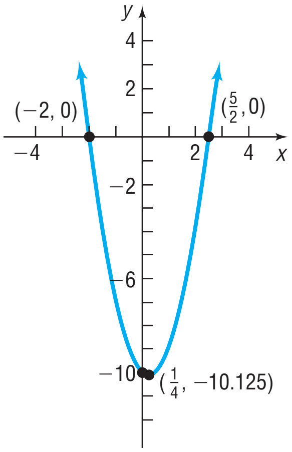


Figure 37 $f(x) = 2x^2 - x - 10$

Because we are solving $f(x) < 0$, we must find where the graph is below the x-axis. This occurs between $x = -2$ and $x = \frac{5}{2}$. Since the inequality is strict, the solution set is $\left\{x \,\middle|\, -2 < x < \frac{5}{2}\right\}$ or, using interval notation, $\left(-2, \frac{5}{2}\right)$.

✓**Check:** Verify the result of Option 1 by graphing $Y_1 = 2x^2 - x - 10$ and using the ZERO feature.

Option 2 If $f(x) = 2x^2$ and $g(x) = x + 10$, then the inequality to be solved is $f(x) < g(x)$. Graph the functions $f(x) = 2x^2$ and $g(x) = x + 10$. See Figure 38. The graphs intersect where $f(x) = g(x)$. Then

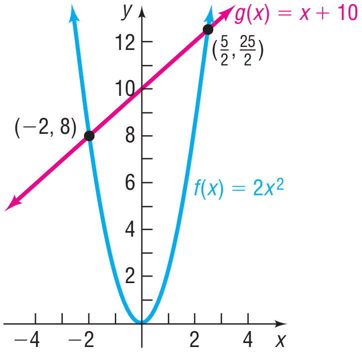


Figure 38

$$\begin{aligned} 2x^2 &= x + 10 & &\boldsymbol{f(x) = g(x)}\\ 2x^2 - x - 10 &= 0 & &\\ (2x-5)(x+2) &= 0 & &\textbf{Factor.}\\ 2x - 5 = 0 \quad \text{or} \quad x + 2 &= 0 & &\textbf{Apply the Zero-Product Property}\\ x = \frac{5}{2} \quad \text{or} \quad x &= -2 & & \end{aligned}$$

The graphs intersect at the points $(-2, 8)$ and $\left(\frac{5}{2}, \frac{25}{2}\right)$. To solve $f(x) < g(x)$, find where the graph of f is below that of g. This happens between the points of intersection. Since the inequality is strict, the solution set is $\left\{x \,\middle|\, -2 < x < \frac{5}{2}\right\}$ or, using interval notation, $\left(-2, \frac{5}{2}\right)$.

✓**Check:** Verify the result of Option 2 by graphing $Y_1 = 2x^2$ and $Y_2 = x + 10$ on the same screen. Use INTERSECT to find the x-coordinates of the points of intersection. Determine where Y_1 is below Y_2 to solve $Y_1 < Y_2$. ■

See Figure 39 for the graph of the solution set.

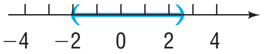


Figure 39

Now Work PROBLEMS 5 AND 13

EXAMPLE 3 Solving an Inequality

Solve the inequality $x^2 + x + 1 > 0$ and graph the solution set.

Solution Graph the function $f(x) = x^2 + x + 1$. The y-intercept is 1; there are no x-intercepts (Do you see why? Check the discriminant). The vertex is at $x = -\frac{b}{2a} = -\frac{1}{2}$. Since $f\left(-\frac{1}{2}\right) = \frac{3}{4}$, the vertex is at $\left(-\frac{1}{2}, \frac{3}{4}\right)$. The points $(1, 3)$ and $(-1, 1)$ are also on the graph. See Figure 40.

Because we are solving $f(x) > 0$, we must find where the graph is above the x-axis. The graph of f lies above the x-axis for all x. The solution set is the set of all real numbers, or $(-\infty, \infty)$. See Figure 41.

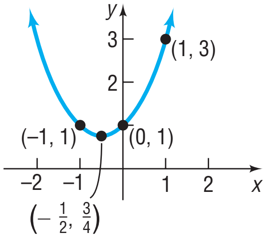


Figure 40 $f(x) = x^2 + x + 1$

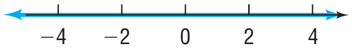


Figure 41

■

Now Work PROBLEM 17

4.5 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

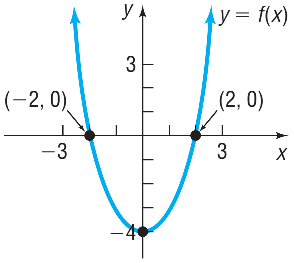
1. Solve the inequality $-3x - 2 < 7$. (pp. 150–151)

2. Write $(-2, 7]$ using inequality notation. (pp. 147–148)

Skill Building

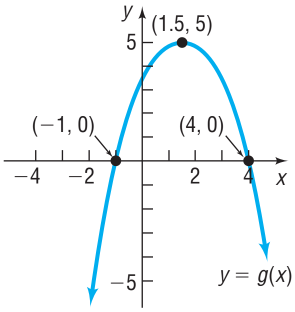
In Problems 3–6, use the figure to solve each inequality.

3.



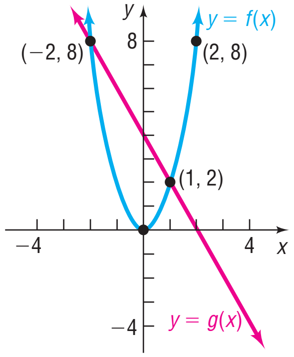
(a) $f(x) > 0$
(b) $f(x) \le 0$

4.



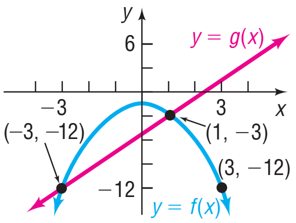
(a) $g(x) < 0$
(b) $g(x) \ge 0$

5.



(a) $g(x) \ge f(x)$
(b) $f(x) > g(x)$

6.



(a) $f(x) < g(x)$
(b) $f(x) \ge g(x)$

In Problems 7–22, solve each inequality.

7. $x^2 - 3x - 10 < 0$
8. $x^2 + 3x - 10 > 0$
9. $x^2 - 4x > 0$
10. $x^2 + 8x > 0$
11. $x^2 - 9 < 0$
12. $x^2 - 1 < 0$
13. $x^2 + x > 12$
14. $x^2 + 7x < -12$
15. $2x^2 < 5x + 3$
16. $6x^2 < 6 + 5x$
17. $x^2 - x + 1 \le 0$
18. $x^2 + 2x + 4 > 0$
19. $4x^2 + 9 < 6x$
20. $25x^2 + 16 < 40x$
21. $6(x^2 - 1) > 5x$
22. $2(2x^2 - 3x) > -9$

Mixed Practice

23. What is the domain of the function $f(x) = \sqrt{x^2 - 16}$?

24. What is the domain of the function $f(x) = \sqrt{x - 3x^2}$?

In Problems 25–32, use the given functions f and g.

(a) Solve $f(x) = 0$.
(b) Solve $g(x) = 0$.
(c) Solve $f(x) = g(x)$.
(d) Solve $f(x) > 0$.
(e) Solve $g(x) \le 0$.
(f) Solve $f(x) > g(x)$.
(g) Solve $f(x) \ge 1$.

25. $f(x) = x^2 - 1$
$g(x) = 3x + 3$

26. $f(x) = -x^2 + 3$
$g(x) = -3x + 3$

27. $f(x) = -x^2 + 1$
$g(x) = 4x + 1$

28. $f(x) = -x^2 + 4$
$g(x) = -x - 2$

29. $f(x) = x^2 - 4$
$g(x) = -x^2 + 4$

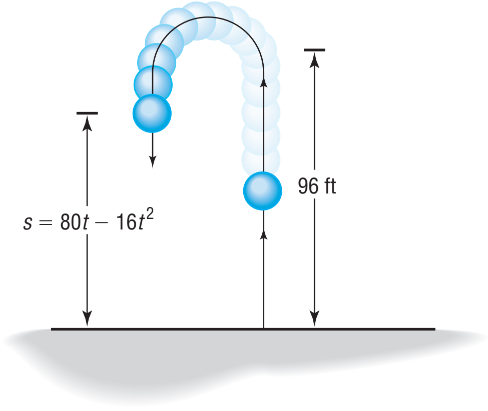
30. $f(x) = x^2 - 2x + 1$
$g(x) = -x^2 + 1$

31. $f(x) = x^2 - x - 2$
$g(x) = x^2 + x - 2$

32. $f(x) = -x^2 - x + 1$
$g(x) = -x^2 + x + 6$

Applications and Extensions

33. Physics A ball is thrown vertically upward with an initial velocity of 80 feet per second. The distance s (in feet) of the ball from the ground after t seconds is $s(t) = 80t - 16t^2$.



(a) At what time t will the ball strike the ground?
(b) For what time t is the ball more than 96 feet above the ground?

34. Physics A ball is thrown vertically upward with an initial velocity of 96 feet per second. The distance s (in feet) of the ball from the ground after t seconds is $s(t) = 96t - 16t^2$.
(a) At what time t will the ball strike the ground?
(b) For what time t is the ball more than 128 feet above the ground?

35. Revenue Suppose that the manufacturer of a gas clothes dryer has found that when the unit price is p dollars, the revenue R (in dollars) is

$$R(p) = -4p^2 + 4000p$$

(a) At what prices p is revenue zero?
(b) For what range of prices will revenue exceed \$800,000?

36. Revenue The John Deere company has found that the revenue from sales of heavy-duty tractors is a function of the unit price p, in dollars, that it charges. If the revenue R, in dollars, is

$$R(p) = -\frac{1}{2}p^2 + 1900p$$

(a) At what prices p is revenue zero?
(b) For what range of prices will revenue exceed \$1,200,000?

37. Artillery A projectile fired from the point $(0, 0)$ at an angle to the positive x-axis has a trajectory given by

$$y = cx - (1 + c^2)\left(\frac{g}{2}\right)\left(\frac{x}{v}\right)^2$$

where
$x =$ horizontal distance in meters
$y =$ height in meters
$v =$ initial muzzle velocity in meters per second (m/sec)
$g =$ acceleration due to gravity $= 9.81$ meters per second squared (m/sec^2)
$c > 0$ is a constant determined by the angle of elevation.

A howitzer fires an artillery round with a muzzle velocity of 897 m/sec.
(a) If the round must clear a hill 200 meters high at a distance of 2000 meters in front of the howitzer, what c values are permitted in the trajectory equation?
(b) If the goal in part (a) is to hit a target on the ground 75 kilometers away, is it possible to do so? If so, for what values of c? If not, what is the maximum distance the round will travel?

Source: *www.answers.com*

38. Runaway Car Using Hooke's Law, we can show that the *work* done in compressing a spring a distance of x feet from its at-rest position is $W = \frac{1}{2}kx^2$, where k is a stiffness constant depending on the spring. It can also be shown that the work done by a body in motion before it comes to rest is given by $\widetilde{W} = \frac{w}{2g}v^2$, where $w =$ weight of the object (lb), $g =$ acceleration due to gravity (32.2 ft/sec^2), and $v =$ object's velocity (in ft/sec). A parking garage has a spring shock absorber at the end of a ramp to stop runaway cars. The spring has a stiffness constant $k = 9450$ lb/ft and must be able to stop a 4000-lb car traveling at 25 mph. What is the least compression required of the spring? Express your answer using feet to the nearest tenth.
[**Hint:** Solve $W > \widetilde{W}, x \geq 0$].

Source: *www.sciforums.com*

Explaining Concepts: Discussion and Writing

39. Show that the inequality $(x - 4)^2 \leq 0$ has exactly one solution.

40. Show that the inequality $(x - 2)^2 > 0$ has one real number that is not a solution.

41. Explain why the inequality $x^2 + x + 1 > 0$ has all real numbers as the solution set.

42. Explain why the inequality $x^2 - x + 1 < 0$ has the empty set as the solution set.

43. Explain the circumstances under which the x-intercepts of the graph of a quadratic function are included in the solution set of a quadratic inequality.

Retain Your Knowledge

Problems 44–47 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

44. Determine the domain of $f(x) = \sqrt{10 - 2x}$.

45. Consider the linear function $f(x) = \frac{2}{3}x - 6$.

(a) Find the intercepts of the graph of f.
(b) Graph f.

46. Determine algebraically whether $f(x) = \frac{-x}{x^2 + 9}$ is even, odd, or neither.

47. Multiply $(6 - 5i)(4 - i)$. Write the answer in the form $a + bi$.

'Are You Prepared?' Answers

1. $\{x \mid x > -3\}$ or $(-3, \infty)$ **2.** $-2 < x \le 7$

Chapter Review

Things to Know

Linear function (p. 281)

$f(x) = mx + b$

Average rate of change $= m$

The graph is a line with slope m and y-intercept b.

Quadratic function (pp. 298–302)

$f(x) = ax^2 + bx + c, a \neq 0$

The graph is a parabola that opens up if $a > 0$ and opens down if $a < 0$.

Vertex: $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$

Axis of symmetry: $x = -\frac{b}{2a}$

y-intercept: $f(0) = c$
x-intercept(s): If any, found by finding the real solutions of the equation $ax^2 + bx + c = 0$

Objectives

Section	You should be able to ...	Examples	Review Exercises
4.1	1 Graph linear functions (p. 281)	1	1(a)–3(a), 1(c)–3(c)
	2 Use average rate of change to identify linear functions (p. 281)	2	1(b)–3(b), 4, 5
	3 Determine whether a linear function is increasing, decreasing, or constant (p. 284)	3	1(d)–3(d)
	4 Build linear models from verbal descriptions (p. 285)	4, 5	22
4.2	1 Draw and interpret scatter diagrams (p. 291)	1	28(a), 29(a)
	2 Distinguish between linear and nonlinear relations (p. 292)	2	28(b), 29(a)
	3 Use a graphing utility to find the line of best fit (p. 293)	4	28(c)
4.3	1 Graph a quadratic function using transformations (p. 299)	1	6–8
	2 Identify the vertex and axis of symmetry of a quadratic function (p. 301)	2	9–14
	3 Graph a quadratic function using its vertex, axis, and intercepts (p. 301)	3–5	9–14
	4 Find a quadratic function given its vertex and one other point (p. 304)	6	20, 21
	5 Find the maximum or minimum value of a quadratic function (p. 305)	7	15–17, 23–26
4.4	1 Build quadratic models from verbal descriptions (p. 310)	1–4	23–27
	2 Build quadratic models from data (p. 314)	5	29
4.5	1 Solve inequalities involving a quadratic function (p. 320)	1–3	18, 19

Review Exercises

In Problems 1–3:
(a) Determine the slope and y-intercept of each linear function.
(b) Find the average rate of change of each function.
(c) Graph each function. Label the intercepts.
(d) Determine whether the function is increasing, decreasing, or constant.

1. $f(x) = 2x - 5$

2. $F(x) = -\frac{1}{3}x + 1$

3. $G(x) = 4$

In Problems 4 and 5, determine whether the function is linear or nonlinear. If the function is linear, find the equation of the line.

4.

x	$y = f(x)$
−1	−2
0	3
1	8
2	13
3	18

5.

x	$y = g(x)$
−1	−3
0	4
1	7
2	6
3	1

In Problems 6–8, graph each quadratic function using transformations (shifting, compressing, stretching, and/or reflecting).

6. $f(x) = (x - 2)^2 + 2$

7. $f(x) = -(x - 4)^2$

8. $f(x) = 2(x + 1)^2 + 4$

In Problems 9–14, (a) graph each quadratic function by determining whether its graph opens up or down and by finding its vertex, axis of symmetry, y-intercept, and x-intercepts, if any. (b) Determine the domain and the range of the function. (c) Determine where the function is increasing and where it is decreasing.

9. $f(x) = x^2 - 4x + 6$

10. $f(x) = -\frac{1}{2}x^2 + 2$

11. $f(x) = -4x^2 + 4x$

12. $f(x) = 9x^2 + 6x + 1$

13. $f(x) = -x^2 + x + \frac{1}{2}$

14. $f(x) = 3x^2 + 4x - 1$

In Problems 15–17, determine whether the given quadratic function has a maximum value or a minimum value, and then find the value.

15. $f(x) = 3x^2 - 6x + 4$

16. $f(x) = -x^2 + 8x - 4$

17. $f(x) = -2x^2 + 4$

In Problems 18–19, solve each quadratic inequality.

18. $x^2 + 6x - 16 < 0$

19. $3x^2 \geq 14x + 5$

In Problems 20 and 21, find the quadratic function for which:

20. Vertex is $(-1, 2)$; contains the point $(1, 6)$

21. Contains the points $(0, 5)$, $(1, 2)$, and $(3, 2)$

---

22. Sales Commissions Bill has just been offered a sales position for a computer company. His salary would be $25,000 per year plus 1% of his total annual sales.

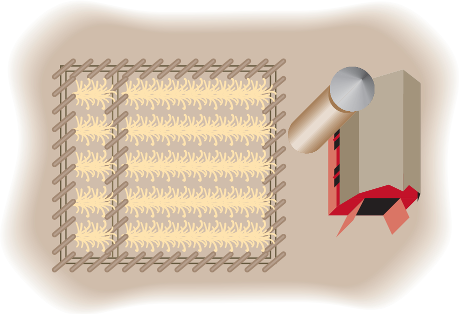
(a) Find a linear function that relates Bill's annual salary, S, to his total annual sales, x.
(b) If Bill's total annual sales were $1,000,000, what would be Bill's salary?
(c) What would Bill have to sell to earn $100,000?
(d) Determine the sales required of Bill for his salary to exceed $150,000.

23. Demand Equation The price p (in dollars) and the quantity x sold of a certain product obey the demand equation

$$p = -\frac{1}{10}x + 150 \quad 0 \leq x \leq 1500$$

(a) Express the revenue R as a function of x.
(b) What is the revenue if 100 units are sold?
(c) What quantity x maximizes revenue? What is the maximum revenue?
(d) What price should the company charge to maximize revenue?

24. Enclosing the Most Area with a Fence A farmer with 10,000 meters of fencing wants to enclose a rectangular field and then divide it into two plots with a fence parallel to one of the sides. See the figure. What is the largest area that can be enclosed?



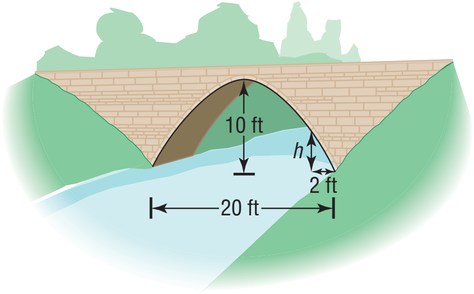
25. **Minimizing Marginal Cost** Callaway Golf Company has determined that the marginal cost C of manufacturing x Big Bertha golf clubs may be expressed by the quadratic function

$$C(x) = 4.9x^2 - 617.4x + 19{,}600$$

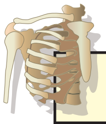
(a) How many clubs should be manufactured to minimize the marginal cost?
(b) At this level of production, what is the marginal cost?

26. **Maximizing Area** A rectangle has one vertex on the line $y = 10 - x$, $x > 0$, another at the origin, one on the positive x-axis, and one on the positive y-axis. Express the area A of the rectangle as a function of x. Find the largest area A that can be enclosed by the rectangle.

27. **Parabolic Arch Bridge** A horizontal bridge is in the shape of a parabolic arch. Given the information shown in the figure, what is the height h of the arch 2 feet from shore?



28. **Bone Length** Research performed at NASA, led by Dr. Emily R. Morey-Holton, measured the lengths of the right humerus and right tibia in 11 rats that were sent to space on Spacelab Life Sciences 2. The following data were collected.



Right Humerus (mm), x	Right Tibia (mm), y
24.80	36.05
24.59	35.57
24.59	35.57
24.29	34.58
23.81	34.20
24.87	34.73
25.90	37.38
26.11	37.96
26.63	37.46
26.31	37.75
26.84	38.50

Source: NASA Life Sciences Data Archive

(a) Draw a scatter diagram of the data treating length of the right humerus as the independent variable.
(b) Based on the scatter diagram, do you think that there is a linear relation between the length of the right humerus and the length of the right tibia?
(c) Use a graphing utility to find the line of best fit relating length of the right humerus and length of the right tibia.
(d) Predict the length of the right tibia on a rat whose right humerus is 26.5 millimeters (mm).

29. **Advertising** A small manufacturing firm collected the following data on advertising expenditures A (in thousands of dollars) and total revenue R (in thousands of dollars).

Advertising	Total Revenue
20	\$6101
22	\$6222
25	\$6350
25	\$6378
27	\$6453
28	\$6423
29	\$6360
31	\$6231

(a) Draw a scatter diagram of the data. Comment on the type of relation that may exist between the two variables.
(b) The quadratic function of best fit to these data is

$$R(A) = -7.76A^2 + 411.88A + 942.72$$

Use this function to determine the optimal level of advertising.
(c) Use the function to predict the total revenue when the optimal level of advertising is spent.
(d) Use a graphing utility to verify that the function given in part (b) is the quadratic function of best fit.
(e) Use a graphing utility to draw a scatter diagram of the data and then graph the quadratic function of best fit on the scatter diagram.

Chapter Test

CHAPTER **Test Prep** VIDEOS

The Chapter Test Prep Videos are step-by-step solutions available in MyMathLab®, or on this text's YouTube Channel. Flip back to the Resources for Success page for a link to this text's YouTube channel.

1. For the linear function $f(x) = -4x + 3$,
  (a) Find the slope and y-intercept.
  (b) Determine whether f is increasing, decreasing, or constant.
  (c) Graph f.

2. Determine whether the given function is linear or nonlinear. If it is linear, determine the equation of the line.

x	y
−2	12
−1	7
0	2
1	−3
2	−8

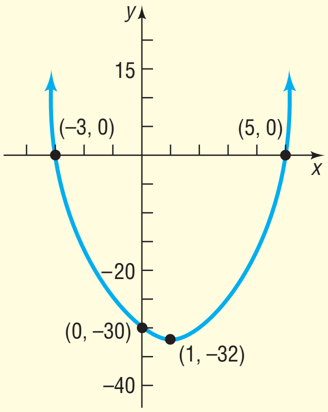
3. Graph $f(x) = (x - 3)^2 - 2$ using transformations.

In Problems 4 and 5,
  (a) Determine whether the graph opens up or down.
  (b) Determine the vertex of the graph of the quadratic function.
  (c) Determine the axis of symmetry of the graph of the quadratic function.
  (d) Determine the intercepts of the graph of the quadratic function.
  (e) Use the information in parts (a)–(d) to graph the quadratic function.
  (f) Based on the graph, determine the domain and the range of the quadratic function.
  (g) Based on the graph, determine where the function is increasing and where it is decreasing.

4. $f(x) = 3x^2 - 12x + 4$

5. $g(x) = -2x^2 + 4x - 5$

6. Determine the quadratic function for the given graph.



7. Determine whether $f(x) = -2x^2 + 12x + 3$ has a maximum or minimum value. Then find the maximum or minimum value.

8. Solve $x^2 - 10x + 24 \geq 0$.

9. The weekly rental cost of a 20-foot recreational vehicle is \$129.50 plus \$0.15 per mile.
  (a) Find a linear function that expresses the cost C as a function of miles driven m.
  (b) What is the rental cost if 860 miles are driven?
  (c) How many miles were driven if the rental cost is \$213.80?

10. The price p (in dollars) and the quantity x sold of a certain product obey the demand equation $p = -\frac{1}{10}x + 1000$.
  (a) Find a model that expresses the revenue R as a function of x.
  (b) What is the revenue if 400 units are sold?
  (c) What quantity x maximizes revenue? What is the maximum revenue?
  (d) What price should the company charge to maximize revenue?

11. Consider these two data sets:

Set A

x	−2	−1	0	0	1	2	2
y	5	2	1	−3	−8	−12	−10

Set B

x	−2	−1	0	0	1	2	2
y	10	4	2	3	5	10	12

One data set follows a linear pattern and one data set follows a quadratic relation.
  (a) Draw a scatter diagram of each data set. Determine which is linear and which is quadratic. For the linear data, indicate whether the relation shows a positive or negative slope. For the quadratic relation, indicate whether the quadratic function of best fit will open up or down.
  (b) For the linear data set, find the line of best fit.
  (c) For the quadratic data set, find the quadratic function of best fit.

Cumulative Review

1. Find the distance between the points $P = (-1, 3)$ and $Q = (4, -2)$. Find the midpoint of the line segment from P to Q.

2. Which of the following points are on the graph of $y = x^3 - 3x + 1$?
(a) $(-2, -1)$ (b) $(2, 3)$ (c) $(3, 1)$

3. Solve the inequality $5x + 3 \geq 0$ and graph the solution set.

4. Find the equation of the line containing the points $(-1, 4)$ and $(2, -2)$. Express your answer in slope–intercept form and graph the line.

5. Find the equation of the line perpendicular to the line $y = 2x + 1$ and containing the point (3, 5). Express your answer in slope–intercept form and graph the line.

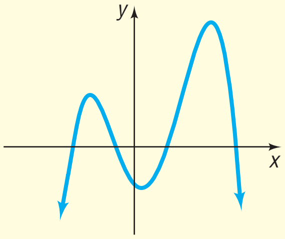
6. Graph the equation $x^2 + y^2 - 4x + 8y - 5 = 0$.

7. Does the following relation represent a function? $\{(-3, 8), (1, 3), (2, 5), (3, 8)\}$.

8. For the function f defined by $f(x) = x^2 - 4x + 1$, find:
(a) $f(2)$
(b) $f(x) + f(2)$
(c) $f(-x)$
(d) $-f(x)$
(e) $f(x + 2)$
(f) $\dfrac{f(x + h) - f(x)}{h}$ $h \neq 0$

9. Find the domain of $h(z) = \dfrac{3z - 1}{6z - 7}$.

10. Is the following graph the graph of a function?



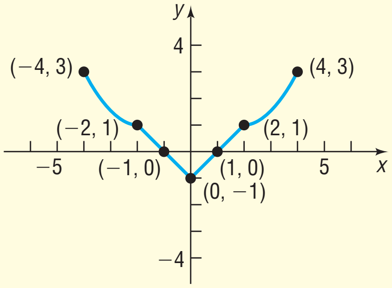
11. Consider the function $f(x) = \dfrac{x}{x + 4}$.
(a) Is the point $\left(1, \dfrac{1}{4}\right)$ on the graph of f?
(b) If $x = -2$, what is $f(x)$? What point is on the graph of f?
(c) If $f(x) = 2$, what is x? What point is on the graph of f?

12. Is the function $f(x) = \dfrac{x^2}{2x + 1}$ even, odd, or neither?

13. Approximate the local maximum values and local minimum values of $f(x) = x^3 - 5x + 1$ on $[-4, 4]$. Determine where the function is increasing and where it is decreasing.

14. If $f(x) = 3x + 5$ and $g(x) = 2x + 1$,
(a) Solve $f(x) = g(x)$.
(b) Solve $f(x) > g(x)$.

15. For the graph of the function f,



(a) Find the domain and the range of f.
(b) Find the intercepts.
(c) Is the graph of f symmetric with respect to the x-axis, the y-axis, or the origin?
(d) Find $f(2)$.
(e) For what value(s) of x is $f(x) = 3$?
(f) Solve $f(x) < 0$.
(g) Graph $y = f(x) + 2$.
(h) Graph $y = f(-x)$.
(i) Graph $y = 2f(x)$.
(j) Is f even, odd, or neither?
(k) Find the interval(s) on which f is increasing.

Chapter Projects

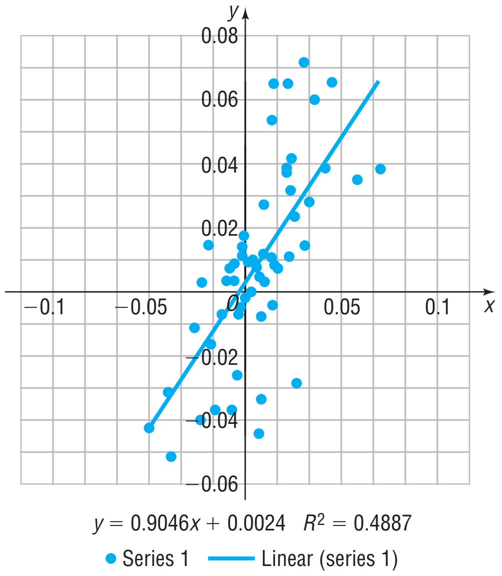
I. The Beta of a Stock You want to invest in the stock market but are not sure which stock to purchase. Information is the key to making an informed investment decision. One piece of information that many stock analysts use is the beta of the stock. Go to Wikipedia (*http://en.wikipedia.org/wiki/Beta_%28finance%29*) and research what beta measures and what it represents.

1. **Approximating the beta of a stock.** Choose a well-known company such as Google or Coca-Cola. Go to a website such as Yahoo! Finance (*http://finance.yahoo.com/*) and find the weekly closing price of the company's stock for the past year. Then find the closing price of the Standard & Poor's 500 (S&P500) for the same time period.

   To get the historical prices in Yahoo! Finance, select Historical Prices from the left menu. Choose the appropriate time period. Select Weekly and Get Prices. Finally, select Download to Spreadsheet. Repeat this for the S&P500, and copy the data into the same spreadsheet. Finally, rearrange the data in chronological order. Be sure to expand the selection to sort all the data. Now, using the adjusted close price, compute the percentage change in price for each week, using the formula % change $= \dfrac{P_1 - P_o}{P_o}$. For example, if week 1 price is in cell D1 and week 2 price is in cell D2, then % change $= \dfrac{D2 - D1}{D1}$. Repeat this for the S&P500 data.

2. **Using Excel to draw a scatter diagram.** Treat the percentage change in the S&P500 as the independent variable and the percentage change in the stock you chose as the dependent variable. The easiest way to draw a scatter diagram in Excel is to place the two columns of data next to each other (for example, have the percentage change in the S&P500 in column F and the percentage change in the stock you chose in column G). Then highlight the data and select the Scatter Diagram icon under Insert. Comment on the type of relation that appears to exist between the two variables.

3. **Finding beta.** To find beta requires that we find the line of best fit using least-squares regression. The easiest approach is to click inside the scatter diagram. Select the Chart Elements icon (+). Check the box for Trendline, select the arrow to the right, and choose More Options. Select Linear and check the box for Display Equation on chart. The line of best fit appears on the scatter diagram. See below.



The line of best fit for this data is $y = 0.9046x + 0.0024$. You may click on Chart Title or either axis title and insert the appropriate names. The beta is the slope of the line of best fit, 0.9046. We interpret this by saying, "If the S&P500 increases by 1%, then this stock will increase by 0.9%, on average." Find the beta of your stock and provide an interpretation. NOTE: Another way to use Excel to find the line of best fit requires using the Data Analysis Tool Pack under add-ins.

The following projects are available on the Instructor's Resource Center (IRC):

II. Cannons A battery commander uses the weight of a missile, its initial velocity, and the position of its gun to determine where the missile will travel.

III. First and Second Differences Finite differences provide a numerical method that is used to estimate the graph of an unknown function.

IV. CBL Experiment Computer simulation is used to study the physical properties of a bouncing ball.

5 Polynomial and Rational Functions

Day Length

Day length is the length of time each day from the moment the upper limb of the sun's disk appears above the horizon during sunrise to the moment when the upper limb disappears below the horizon during sunset. The length of a day depends on the day of the year as well as on the latitude of the location. Latitude gives the location of a point on Earth north or south of the equator. In the Internet Project at the end of this chapter, we use information from the chapter to investigate the relation between day length and latitude for a specific day of the year.

—See the Internet-based Chapter Project I—

Outline

••• A Look Back

In Chapter 3, we began our discussion of functions. We defined domain, range, and independent and dependent variables, found the value of a function, and graphed functions. We continued our study of functions by listing the properties that a function might have, such as being even or odd, and created a library of functions, naming key functions and listing their properties, including their graphs.

In Chapter 4, we discussed linear functions and quadratic functions, which belong to the class of *polynomial functions*.

A Look Ahead •••

In this chapter, we look at two general classes of functions, polynomial functions and rational functions, and examine their properties. Polynomial functions are arguably the simplest expressions in algebra. For this reason, they are often used to approximate other, more complicated functions. Rational functions are ratios of polynomial functions.

5.1 Polynomial Functions and Models

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Polynomials (Chapter R, Section R.4, pp. 41–42)
- Using a Graphing Utility to Approximate Local Maxima and Local Minima (Section 3.3, pp. 237–238)
- Intercepts of a Function (Section 3.2, pp. 223–225)
- Graphing Techniques: Transformations (Section 3.5, pp. 256–264)
- Intercepts (Section 2.1, pp. 165–166)

Now Work the **'Are You Prepared?'** problems on page 346.

OBJECTIVES
1 Identify Polynomial Functions and Their Degree (p. 331)
2 Graph Polynomial Functions Using Transformations (p. 334)
3 Identify the Real Zeros of a Polynomial Function and Their Multiplicity (p. 335)
4 Analyze the Graph of a Polynomial Function (p. 342)
5 Build Cubic Models from Data (p. 345)

1 Identify Polynomial Functions and Their Degree

In Chapter 4, we studied the linear function $f(x) = mx + b$, which can be written as

$$f(x) = a_1 x + a_0$$

and the quadratic function $f(x) = ax^2 + bx + c, a \neq 0$, which can be written as

$$f(x) = a_2 x^2 + a_1 x + a_0 \qquad a_2 \neq 0$$

Each of these functions is an example of a *polynomial function*.

DEFINITION

A **polynomial function** in one variable is a function of the form

$$\boxed{f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \qquad \textbf{(1)}}$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants, called the **coefficients** of the polynomial, $n \geq 0$ is an integer, and x is the variable. If $a_n \neq 0$, it is called the **leading coefficient**, and n is the **degree** of the polynomial.

The domain of a polynomial function is the set of all real numbers.

In Words
A polynomial function is a sum of monomials.

The monomials that make up a polynomial are called its **terms**. If $a_n \neq 0$, $a_n x^n$ is called the **leading term**; a_0 is called the **constant term**. If all of the coefficients are 0, the polynomial is called the **zero polynomial**, which has no degree.

Polynomials are usually written in **standard form**, beginning with the nonzero term of highest degree and continuing with terms in descending order according to degree. If a power of x is missing, it is because its coefficient is zero.

Polynomial functions are among the simplest expressions in algebra. They are easy to evaluate: only addition and repeated multiplication are required. Because of this, they are often used to approximate other, more complicated functions. In this section, we investigate properties of this important class of functions.

EXAMPLE 1 **Identifying Polynomial Functions**

Determine which of the following are polynomial functions. For those that are, state the degree; for those that are not, tell why not. Write each polynomial in standard form, and then identify the leading term and the constant term.

(a) $p(x) = 5x^3 - \frac{1}{4}x^2 - 9$ (b) $f(x) = x + 2 - 3x^4$ (c) $g(x) = \sqrt{x}$

(d) $h(x) = \dfrac{x^2 - 2}{x^3 - 1}$ (e) $G(x) = 8$ (f) $H(x) = -2x^3(x - 1)^2$

Solution (a) p is a polynomial function of degree 3, and it is already in standard form. The leading term is $5x^3$, and the constant term is -9.

(b) f is a polynomial function of degree 4. Its standard form is $f(x) = -3x^4 + x + 2$. The leading term is $-3x^4$, and the constant term is 2.

(c) g is not a polynomial function because $g(x) = \sqrt{x} = x^{\frac{1}{2}}$, so the variable x is raised to the $\frac{1}{2}$ power, which is not a nonnegative integer.

(d) h is not a polynomial function. It is the ratio of two distinct polynomials, and the polynomial in the denominator is of positive degree.

(e) G is a nonzero constant polynomial function so it is of degree 0. The polynomial is in standard form. The leading term and constant term are both 8.

(f) $H(x) = -2x^3(x-1)^2 = -2x^3(x^2 - 2x + 1) = -2x^5 + 4x^4 - 2x^3$. So, H is a polynomial function of degree 5. Because $H(x) = -2x^5 + 4x^4 - 2x^3$, the leading term is $-2x^5$. Since no constant term is shown, the constant term is 0. Do you see a way to find the degree of H without multiplying it out? ■

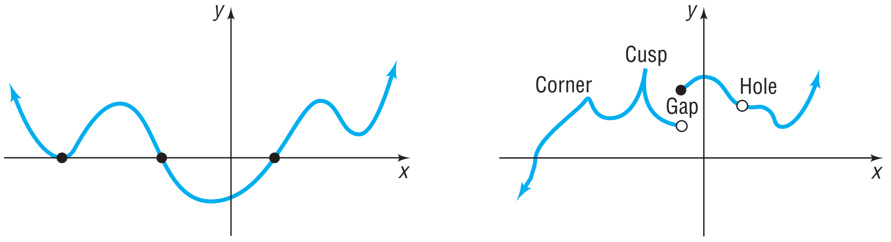
Now Work PROBLEMS 17 AND 21

We have already discussed in detail polynomial functions of degrees 0, 1, and 2. See Table 1 for a summary of the properties of the graphs of these polynomial functions.

Table 1

Degree	Form	Name	Graph
No degree	$f(x) = 0$	Zero function	The x-axis
0	$f(x) = a_0, \quad a_0 \neq 0$	Constant function	Horizontal line with y-intercept a_0
1	$f(x) = a_1x + a_0, \quad a_1 \neq 0$	Linear function	Nonvertical, nonhorizontal line with slope a_1 and y-intercept a_0
2	$f(x) = a_2x^2 + a_1x + a_0, \quad a_2 \neq 0$	Quadratic function	Parabola: graph opens up if $a_2 > 0$; graph opens down if $a_2 < 0$

One objective of this section is to analyze the graph of a polynomial function. If you take a course in calculus, you will learn that the graph of every polynomial function is both smooth and continuous. By **smooth**, we mean that the graph contains no sharp corners or cusps; by **continuous**, we mean that the graph has no gaps or holes and can be drawn without lifting your pencil from the paper. See Figures 1(a) and 1(b).



(a) Graph of a polynomial function: smooth, continuous

(b) Cannot be the graph of a polynomial function

Figure 1

Power Functions

We begin the analysis of the graph of a polynomial function by discussing *power functions,* a special kind of polynomial function.

DEFINITION

A **power function of degree n** is a monomial function of the form

$$f(x) = ax^n \qquad (2)$$

where a is a real number, $a \neq 0$, and $n > 0$ is an integer. ■

In Words

A power function is defined by a single monomial.

Examples of power functions are

$f(x) = 3x$	$f(x) = -5x^2$	$f(x) = 8x^3$	$f(x) = -5x^4$
degree 1	**degree 2**	**degree 3**	**degree 4**

The graph of a power function of degree 1, $f(x) = ax$, is a straight line, with slope a, that passes through the origin. The graph of a power function of degree 2, $f(x) = ax^2$, is a parabola, with vertex at the origin, that opens up if $a > 0$ and opens down if $a < 0$.

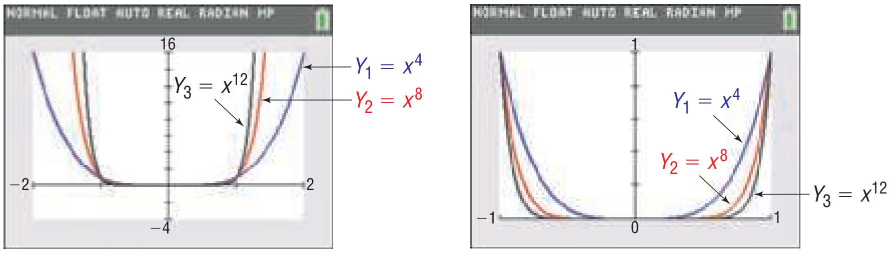
If we know how to graph a power function of the form $f(x) = x^n$, a compression or stretch and, perhaps, a reflection about the x-axis will enable us to obtain the graph of $g(x) = ax^n$. Consequently, we shall concentrate on graphing power functions of the form $f(x) = x^n$.

We begin with power functions of even degree of the form $f(x) = x^n$, $n \geq 2$ and n even.

Exploration

Using your graphing utility and the viewing window $-2 \leq x \leq 2$, $-4 \leq y \leq 16$, graph the function $Y_1 = f(x) = x^4$. On the same screen, graph $Y_2 = g(x) = x^8$. Now, also on the same screen, graph $Y_3 = h(x) = x^{12}$. What do you notice about the graphs as the magnitude of the exponent increases? Repeat this procedure for the viewing window $-1 \leq x \leq 1$, $0 \leq y \leq 1$. What do you notice?

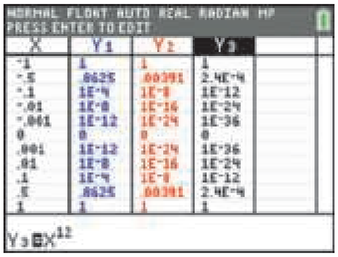
Result See Figures 2(a) and 2(b).



(a) (b)

Figure 2 $Y_1 = x^4$; $Y_2 = x^8$; $Y_3 = x^{12}$

Table 2



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X	Y1	Y2	Y3
-1	1	1	1
-.5	.0625	.00391	2.4E-4
-.1	1E-4	1E-8	1E-12
-.01	1E-8	1E-16	1E-24
-.001	1E-12	1E-24	1E-36
0	0	0	0
.001	1E-12	1E-24	1E-36
.01	1E-8	1E-16	1E-24
.1	1E-4	1E-8	1E-12
.5	.0625	.00391	2.4E-4
1	1	1	1

Y3=X^12

The domain of $f(x) = x^n$, $n \geq 2$ and n even, is the set of all real numbers, and the range is the set of nonnegative real numbers. Such a power function is an even function (do you see why?), so its graph is symmetric with respect to the y-axis. Its graph always contains the origin $(0, 0)$ and the points $(-1, 1)$ and $(1, 1)$.

If $n = 2$, the graph is the parabola $y = x^2$ that opens up, with vertex at the origin. For large n, it appears that the graph coincides with the x-axis near the origin, but it does not; the graph actually touches the x-axis only at the origin. See Table 2, where $Y_1 = x^4$, $Y_2 = x^8$, and $Y_3 = x^{12}$. For x close to 0, the values of y are positive and close to 0. Also, for large n, it may appear that for $x < -1$ or for $x > 1$ the graph is vertical, but it is not; it is only increasing very rapidly. That is, as the values of x approach negative infinity $(-\infty)$, the values of $f(x)$ approach ∞. In calculus, we denote this as $\lim_{x \to -\infty} f(x) = \infty$. Using this notation, we also see $\lim_{x \to \infty} f(x) = \infty$. These limits describe the end behavior of the graph of f, as we will see later in this section. If you TRACE along one of the graphs, these distinctions will be clear. ■

Note: Don't forget how graphing calculators express scientific notation. In Table 2, 1E −8 means 1×10^{-8}. ■

To summarize:

Properties of Power Functions, $f(x) = x^n$, n Is an Even Integer

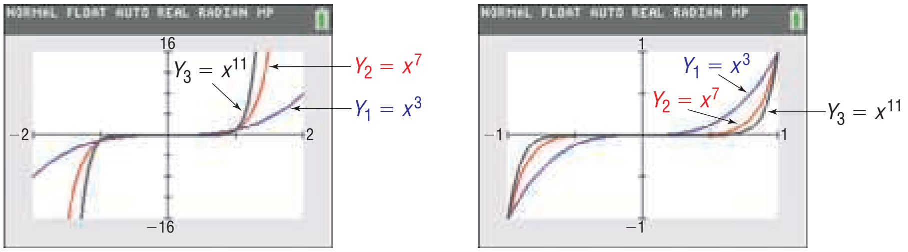
1. f is an even function, so its graph is symmetric with respect to the y-axis.
2. The domain is the set of all real numbers. The range is the set of nonnegative real numbers.
3. The graph always contains the points $(-1, 1)$, $(0, 0)$, and $(1, 1)$.
4. As the exponent n increases in magnitude, the graph is steeper when $x < -1$ or $x > 1$; but for x near the origin, the graph tends to flatten out and lie closer to the x-axis.

Now we consider power functions of odd degree of the form $f(x) = x^n$, n odd.

Exploration

Using your graphing utility and the viewing window $-2 \leq x \leq 2$, $-16 \leq y \leq 16$, graph the function $Y_1 = f(x) = x^3$. On the same screen, graph $Y_2 = g(x) = x^7$ and $Y_3 = h(x) = x^{11}$. What do you notice about the graphs as the magnitude of the exponent increases? Repeat this procedure for the viewing window $-1 \leq x \leq 1$, $-1 \leq y \leq 1$. What do you notice?

Result The graphs on your screen should look like Figures 3(a) and 3(b).



(a) **(b)**

Figure 3 $Y_1 = x^3$; $Y_2 = x^7$; $Y_3 = x^{11}$

The domain and the range of $f(x) = x^n$, $n \geq 3$ and n odd, are the set of real numbers. Such a power function is an odd function (do you see why?), so its graph is symmetric with respect to the origin. Its graph always contains the origin $(0, 0)$ and the points $(-1, -1)$ and $(1, 1)$.

It appears that the graph coincides with the x-axis near the origin, but it does not; the graph actually crosses the x-axis only at the origin. Also, it appears that as x increases the graph is vertical, but it is not; it is increasing very rapidly. That is, $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$. These limits describe the end behavior of the graph of f. TRACE along the graphs to verify these distinctions. ■

To summarize:

Properties of Power Functions, $f(x) = x^n$, n Is an Odd Integer

1. f is an odd function, so its graph is symmetric with respect to the origin.
2. The domain and the range are the set of all real numbers.
3. The graph always contains the points $(-1, -1)$, $(0, 0)$, and $(1, 1)$.
4. As the exponent n increases in magnitude, the graph is steeper when $x < -1$ or $x > 1$, but for x near the origin, the graph tends to flatten out and lie closer to the x-axis.

2 Graph Polynomial Functions Using Transformations

The methods of shifting, compression, stretching, and reflection studied in Section 3.5, when used with the facts just presented, enable us to graph polynomial functions that are transformations of power functions.

EXAMPLE 2 Graphing a Polynomial Function Using Transformations

Graph: $f(x) = 1 - x^5$

Solution It is helpful to rewrite f as $f(x) = -x^5 + 1$. Figure 4 shows the required stages.

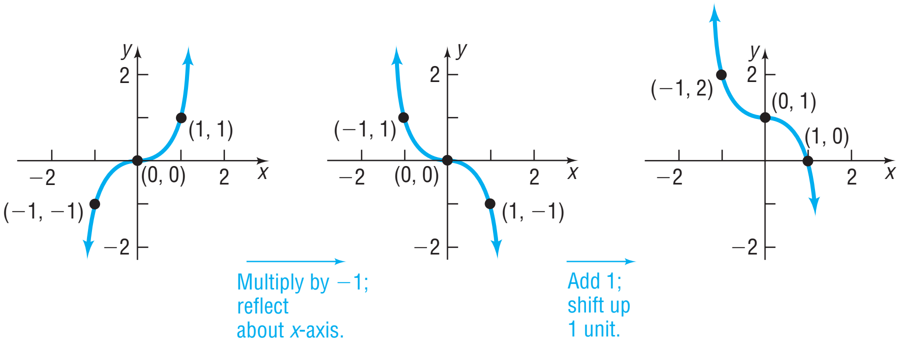


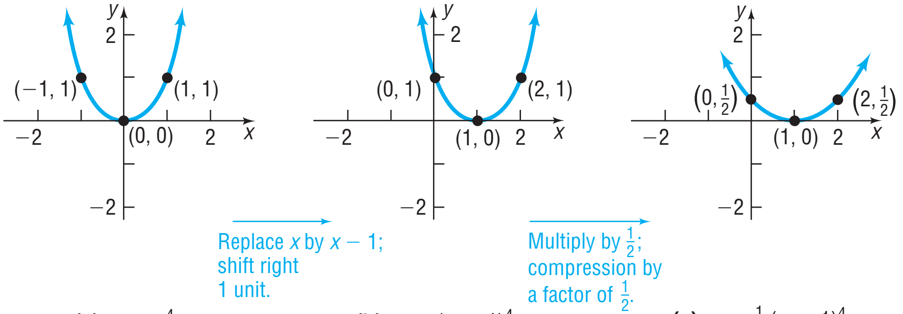
Figure 4 **(a)** $y = x^5$ **(b)** $y = -x^5$ **(c)** $y = -x^5 + 1 = 1 - x^5$

✓**Check:** Verify the graph of f by graphing $Y_1 = 1 - x^5$ on a graphing utility. ■

EXAMPLE 3 Graphing a Polynomial Function Using Transformations

Graph: $f(x) = \frac{1}{2}(x - 1)^4$

Solution Figure 5 shows the required stages.



(a) $y = x^4$ **(b)** $y = (x - 1)^4$ **(c)** $y = \frac{1}{2}(x - 1)^4$

Figure 5

✓**Check:** Verify the graph of f by graphing $Y_1 = \frac{1}{2}(x - 1)^4$ on a graphing utility.

Now Work PROBLEMS 29 AND 35

3 Identify the Real Zeros of a Polynomial Function and Their Multiplicity

Figure 6 shows the graph of a polynomial function with four x-intercepts. Notice that at the x-intercepts, the graph must either cross the x-axis or touch the x-axis. Consequently, between consecutive x-intercepts the graph is either above the x-axis or below the x-axis.

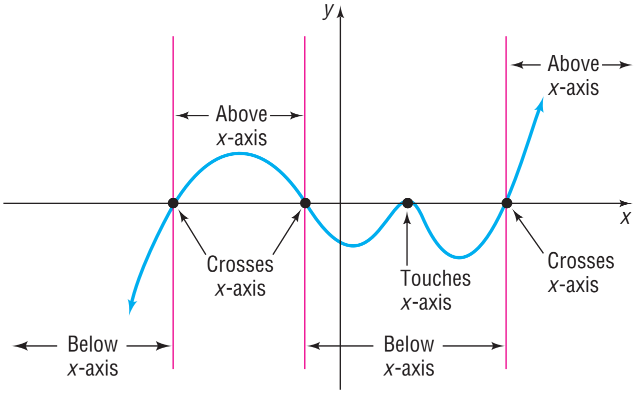


Figure 6 Graph of a polynomial function

If a polynomial function f is factored completely, it is easy to locate the x-intercepts of the graph by solving the equation $f(x) = 0$ using the Zero-Product Property. For example, if $f(x) = (x - 1)^2(x + 3)$ then the solutions of the equation

$$f(x) = (x - 1)^2(x + 3) = 0$$

are identified as 1 and -3. That is, $f(1) = 0$ and $f(-3) = 0$.

DEFINITION

If f is a function and r is a real number for which $f(r) = 0$, then r is called a **real zero** of f.

As a consequence of this definition, the following statements are equivalent.

- r is a real zero of a polynomial function f.
- r is an x-intercept of the graph of f.
- $x - r$ is a factor of f.
- r is a real solution to the equation $f(x) = 0$.

So the real zeros of a polynomial function are the x-intercepts of its graph, and they are found by solving the equation $f(x) = 0$.

EXAMPLE 4 Finding a Polynomial Function from Its Zeros

(a) Find a polynomial of degree 3 whose zeros are -3, 2, and 5.
(b) Use a graphing utility to graph the polynomial found in part (a) to verify your result.

Solution (a) If r is a real zero of a polynomial function f, then $x - r$ is a factor of f. This means that $x - (-3) = x + 3$, $x - 2$, and $x - 5$ are factors of f. As a result, any polynomial function of the form

$$f(x) = a(x + 3)(x - 2)(x - 5)$$

where a is a nonzero real number, qualifies. The value of a causes a stretch, compression, or reflection, but it does not affect the x-intercepts of the graph. Do you know why?

(b) We choose to graph f with $a = 1$. Then

$$f(x) = (x + 3)(x - 2)(x - 5) = x^3 - 4x^2 - 11x + 30$$

Figure 7 shows the graph of f. Notice that the x-intercepts are -3, 2, and 5. ■

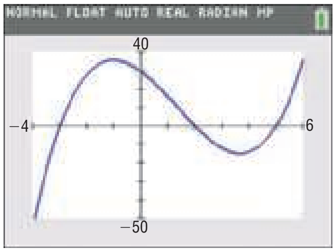


Figure 7 $f(x) = x^3 - 4x^2 - 11x + 30$

Seeing the Concept

Graph the function found in Example 4 for $a = 2$ and $a = -1$. Does the value of a affect the zeros of f? How does the value of a affect the graph of f? ■

Now Work PROBLEM 43

If the same factor $x - r$ occurs more than once, r is called a **repeated**, or **multiple**, **zero of f**. More precisely, we have the following definition.

DEFINITION If $(x - r)^m$ is a factor of a polynomial f and $(x - r)^{m+1}$ is not a factor of f, then r is called a **zero of multiplicity m of f**.* ■

EXAMPLE 5 Identifying Zeros and Their Multiplicities

For the polynomial function

$$f(x) = 5x^2(x + 2)\left(x - \frac{1}{2}\right)^4$$

0 is a zero of multiplicity 2 because the exponent on the factor $x = x - 0$ is 2.

-2 is a zero of multiplicity 1 because the exponent on the factor $x + 2$ is 1.

$\frac{1}{2}$ is a zero of multiplicity 4 because the exponent on the factor $x - \frac{1}{2}$ is 4. ■

In Words
The multiplicity of a zero is the number of times its corresponding factor occurs.

Now Work PROBLEM 57(a)

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*Some texts use the terms **multiple root** and **root of multiplicity *m***.

In Example 5, notice that if you add the multiplicities $(1 + 2 + 4 = 7)$, you obtain the degree of the polynomial function.

Suppose that it is possible to factor completely a polynomial function and, as a result, locate all the x-intercepts of its graph (the real zeros of the function). The following example illustrates the role that the multiplicity of an x-intercept plays.

EXAMPLE 6 Investigating the Role of Multiplicity

For the polynomial function $f(x) = (x + 1)^2(x - 2)$:

(a) Find the x- and y-intercepts of the graph of f.
(b) Using a graphing utility, graph the polynomial function.
(c) For each x-intercept, determine whether it is of odd or even multiplicity.

Solution (a) The y-intercept is $f(0) = (0 + 1)^2\,(0 - 2) = -2$. The x-intercepts satisfy the equation

$$f(x) = (x + 1)^2(x - 2) = 0$$

from which we find that

$$\begin{array}{rcr} (x + 1)^2 = 0 & \text{or} & x - 2 = 0 \\ x = -1 & \text{or} & x = 2 \end{array}$$

The x-intercepts are -1 and 2.

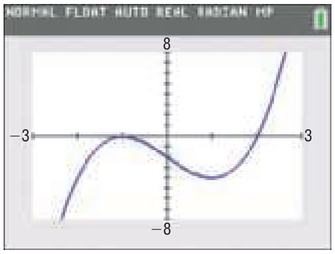
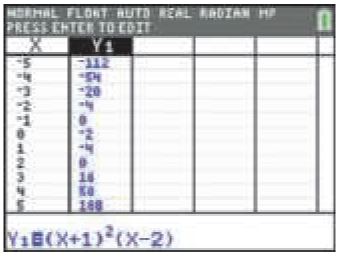


Figure 8 $Y_1 = (x + 1)^2(x - 2)$

(b) See Figure 8 for the graph of f.

(c) We can see from the factored form of f that -1 is a zero or root of multiplicity 2, and 2 is a zero or root of multiplicity 1; so -1 is of even multiplicity and 2 is of odd multiplicity. ■

Table 3



NORMAL FLOAT AUTO REAL RADIAN MP
PRESS ENTER TO EDIT

X	Y₁
-5	-112
-4	-54
-3	-20
-2	-4
-1	0
0	-2
1	-4
2	0
3	16
4	50
5	108

Y₁=(X+1)²(X−2)

We can use a TABLE to further analyze the graph. See Table 3. The sign of $f(x)$ is the same on each side of $x = -1$, and the graph of f just *touches* the x-axis at $x = -1$ (a zero of *even* multiplicity). The sign of $f(x)$ changes from one side of $x = 2$ to the other, and the graph of f *crosses* the x-axis at $x = 2$ (a zero of *odd* multiplicity). These observations suggest the following result:

If *r* Is a Zero of Even Multiplicity

Numerically: The sign of $f(x)$ does not change from one side to the other side of r.

Graphically: The graph of f **touches** the x-axis at r.

If *r* Is a Zero of Odd Multiplicity

Numerically: The sign of $f(x)$ changes from one side to the other side of r.

Graphically: The graph of f **crosses** the x-axis at r.

Now Work PROBLEM 57(b)

Turning Points

Points on the graph where the graph changes from an increasing function to a decreasing function, or vice versa, are called **turning points**.* Each turning point yields a local maximum or a local minimum (see Section 3.3).

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*Graphing utilities can be used to approximate turning points. Calculus is needed to find the exact location of turning points for most polynomial functions.

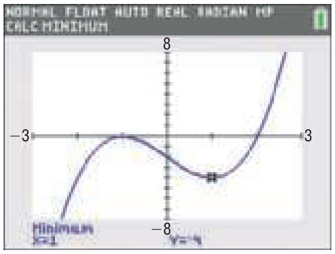


Figure 9 Local minimum of $f(x) = (x + 1)^2(x - 2)$

Look at Figure 9. The graph of $f(x) = (x + 1)^2(x - 2) = x^3 - 3x - 2$ has a turning point at $(-1, 0)$. After utilizing MINIMUM, we find that the graph also has a turning point at $(1, -4)$.

Exploration

Graph $Y_1 = x^3$, $Y_2 = x^3 - x$, and $Y_3 = x^3 + 3x^2 + 4$. How many turning points do you see? Graph $Y_1 = x^4$, $Y_2 = x^4 - \frac{4}{3}x^3$, and $Y_3 = x^4 - 2x^2$. How many turning points do you see? How does the number of turning points compare to the degree? ■

The following theorem from calculus supplies the answer to the question posed in the Exploration.

THEOREM

Turning Points

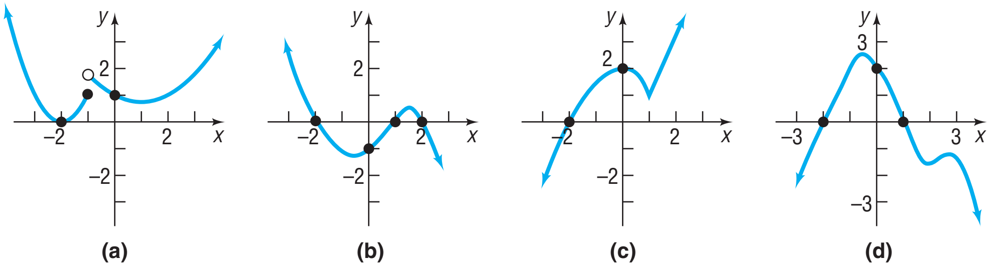
If f is a polynomial function of degree n, then f has at most $n - 1$ turning points.

If the graph of a polynomial function f has $n - 1$ turning points, then the degree of f is at least n. ■

Based on the first part of the theorem, a polynomial function of degree 5 will have at most $5 - 1 = 4$ turning points. Based on the second part of the theorem, if a polynomial function has 3 turning points, then its degree must be at least 4.

EXAMPLE 7 Identifying the Graph of a Polynomial Function

Which of the graphs in Figure 10 could be the graph of a polynomial function? For those that could, list the real zeros and state the least degree the polynomial function can have. For those that could not, say why not.



(a) (b) (c) (d)

Figure 10

Solution

(a) The graph in Figure 10(a) cannot be the graph of a polynomial function because of the gap that occurs at $x = -1$. Remember, the graph of a polynomial function is continuous—no gaps or holes. (See Figure 1.)

(b) The graph in Figure 10(b) could be the graph of a polynomial function because the graph is smooth and continuous. It has three real zeros, at -2, 1, and 2. Since the graph has two turning points, the degree of the polynomial function must be at least 3.

(c) The graph in Figure 10(c) cannot be the graph of a polynomial function because of the cusp at $x = 1$. Remember, the graph of a polynomial function is smooth.

(d) The graph in Figure 10(d) could be the graph of a polynomial function. It has two real zeros, at -2 and 1. Since the graph has three turning points, the degree of the polynomial function is at least 4. ■

Now Work PROBLEMS 57(c) AND 69

End Behavior

One last remark about Figure 8. For very large values of x, either positive or negative, the graph of $f(x) = (x+1)^2(x-2)$ looks like the graph of $y = x^3$. To see why, write f in the form

$$f(x) = (x+1)^2(x-2) = x^3 - 3x - 2 = x^3\left(1 - \frac{3}{x^2} - \frac{2}{x^3}\right)$$

For large values of x, either positive or negative, the terms $\frac{3}{x^2}$ and $\frac{2}{x^3}$ are close to 0. Do you see why? Evaluate $\frac{3}{x^2}$ and $\frac{2}{x^3}$ for $x = 10, 100, 1000$ and for $x = -10, -100, -1000$. What happens to the value of each expression for large values of $|x|$? So, for large values of $|x|$,

$$f(x) = x^3 - 3x - 2 = x^3\left(1 - \frac{3}{x^2} - \frac{2}{x^3}\right) \approx x^3$$

The behavior of the graph of a function for large values of x, either positive or negative, is referred to as its **end behavior**.

THEOREM

End Behavior

For large values of x, either positive or negative, the graph of the polynomial function

$$\boxed{f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \quad a_n \neq 0}$$

resembles the graph of the power function

$$\boxed{y = a_n x^n}$$

In Words

The end behavior of a polynomial function resembles that of its leading term.

For example, if $f(x) = -2x^3 + 5x^2 + x - 4$, then the graph of f will behave like the graph of $y = -2x^3$ for very large values of x, either positive or negative. We can see that the graphs of f and $y = -2x^3$ "behave" the same by considering Table 4 and Figure 11.

Table 4

x	$f(x)$	$y = -2x^3$
10	−1494	−2000
100	−1,949,904	−2,000,000
500	−248,749,504	−250,000,000
1000	−1,994,999,004	−2,000,000,000

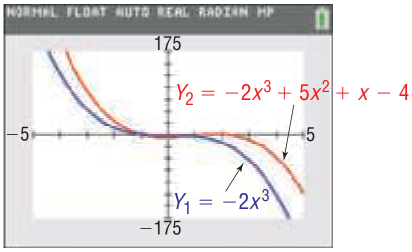


Figure 11

Notice that, as x becomes a larger and larger positive number, the values of f become larger and larger negative numbers. When this happens, we say that f is **unbounded in the negative direction**. Rather than using words to describe the behavior of the graph of the function, we explain its behavior using notation. We can symbolize "the value of f becomes a larger and larger negative number as x becomes a larger and larger positive number" by writing $f(x) \to -\infty$ as $x \to \infty$ (read "the values of f approach negative infinity as x approaches infinity"). In calculus, **limits** are used to convey these ideas. There we use the symbolism $\lim_{x\to\infty} f(x) = -\infty$, read "the limit of $f(x)$ as x approaches infinity equals negative infinity," to mean that $f(x) \to -\infty$ as $x \to \infty$.

When the value of a limit equals infinity (or negative infinity), we mean that the values of the function are unbounded in the positive (or negative) direction and call the limit an **infinite limit**. When we discuss limits as x becomes unbounded in the negative direction or unbounded in the positive direction, we are discussing **limits at infinity**.

Note: Infinity (∞) and negative infinity ($-\infty$) are not numbers. Rather, they are symbols that represent unboundedness. ■

Look back at Figures 2 and 3. Based on the preceding theorem and the previous discussion on power functions, the end behavior of a polynomial function can be of only four types. See Figure 12.

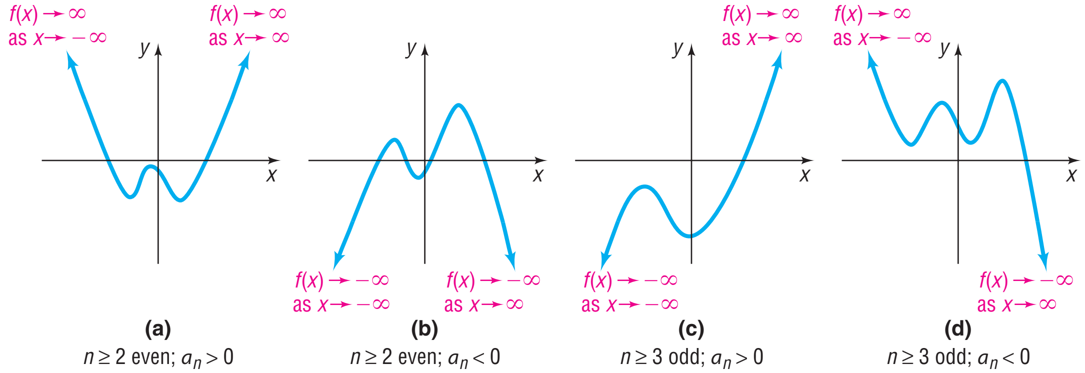


Figure 12 End behavior of $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

For example, if $f(x) = -2x^4 + x^3 + 4x^2 - 7x + 1$, the graph of f will resemble the graph of the power function $y = -2x^4$ for large $|x|$. The graph of f will behave like Figure 12(b) for large $|x|$.

Now Work PROBLEM 57(d)

EXAMPLE 8 Identifying the Graph of a Polynomial Function

Which of the graphs in Figure 13 could be the graph of

$$f(x) = x^4 + ax^3 + bx^2 - 5x - 6$$

where $a > 0, b > 0$?

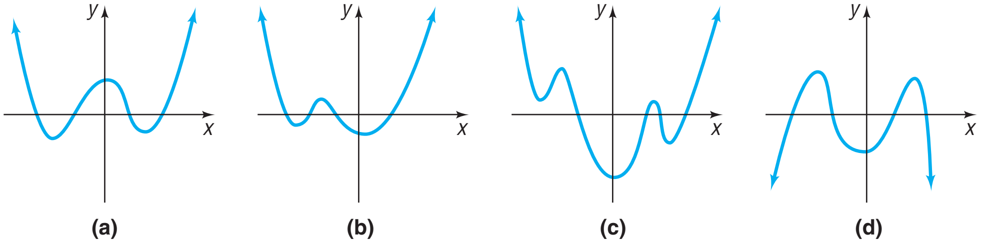


Figure 13

Solution The y-intercept of f is $f(0) = -6$. We can eliminate the graph in Figure 13(a), whose y-intercept is positive.

We are not able to solve $f(x) = 0$ to find the x-intercepts of f, so we move on to investigate the turning points of each graph. Since f is of degree 4, the graph of f has at most 3 turning points. Eliminate the graph in Figure 13(c) because that graph has 5 turning points.

Now look at end behavior. For large values of x, the graph of f will behave like the graph of $y = x^4$. This eliminates the graph in Figure 13(d), whose end behavior is like the graph of $y = -x^4$.

Only the graph in Figure 13(b) could be the graph of

$$f(x) = x^4 + ax^3 + bx^2 - 5x - 6$$

where $a > 0, b > 0$. ■

EXAMPLE 9 Writing a Polynomial Function from Its Graph

Write a polynomial function whose graph is shown in Figure 14 (use the smallest degree possible).

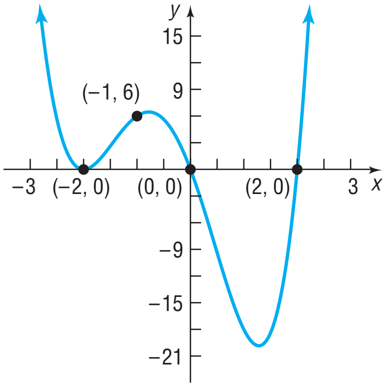


Figure 14

Solution The x-intercepts are -2, 0, and 2. Therefore, the polynomial must have the factors $(x + 2)$, x, and $(x - 2)$, respectively. There are three turning points, so the degree of the polynomial must be at least 4. The graph touches the x-axis at $x = -2$, so -2 must have an even multiplicity. The graph crosses the x-axis at $x = 0$ and $x = 2$, so 0 and 2 must have odd multiplicities. Using the smallest degree possible (1 for odd multiplicity and 2 for even multiplicity), we can write

$$f(x) = ax(x + 2)^2(x - 2)$$

All that remains is to find the leading coefficient, a. From Figure 14, the point $(-1, 6)$ must lie on the graph.

$$6 = a(-1)(-1 + 2)^2(-1 - 2) \quad f(-1) = 6$$
$$6 = 3a$$
$$2 = a$$

The polynomial function $f(x) = 2x(x + 2)^2(x - 2)$ would have the graph in Figure 14.

✓ **Check:** Graph $Y_1 = 2x(x + 2)^2(x - 2)$ using a graphing utility to verify this result. ■

Now Work PROBLEMS 73 AND 77

SUMMARY

Graph of a Polynomial Function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \quad a_n \neq 0$

Degree of the polynomial function f: n
y-intercept: $f(0) = a_0$.
Graph is smooth and continuous.
Maximum number of turning points: $n - 1$
At a zero of even multiplicity: The graph of f touches the x-axis.
At a zero of odd multiplicity: The graph of f crosses the x-axis.
Between zeros, the graph of f is either above or below the x-axis.
End behavior: For large $|x|$, the graph of f behaves like the graph of $y = a_n x^n$.

4 Analyze the Graph of a Polynomial Function

EXAMPLE 10

How to Analyze the Graph of a Polynomial Function

Analyze the graph of the polynomial function $f(x) = (2x + 1)(x - 3)^2$.

Step-by-Step Solution

Step 1: Determine the end behavior of the graph of the function.

$$\begin{aligned} f(x) &= (2x + 1)(x - 3)^2 \\ &= (2x + 1)(x^2 - 6x + 9) && \textbf{Square the binomial difference.} \\ &= 2x^3 - 12x^2 + 18x + x^2 - 6x + 9 && \textbf{Multiply.} \\ &= 2x^3 - 11x^2 + 12x + 9 && \textbf{Combine like terms.} \end{aligned}$$

The polynomial function f is of degree 3. The graph of f behaves like $y = 2x^3$ for large values of $|x|$.

Step 2: Find the x- and y-intercepts of the graph of the function.

The y-intercept is $f(0) = 9$. To find the x-intercepts, solve $f(x) = 0$.

$$f(x) = 0$$

$$(2x + 1)(x - 3)^2 = 0$$

$$2x + 1 = 0 \quad \text{or} \quad (x - 3)^2 = 0$$

$$x = -\frac{1}{2} \quad \text{or} \quad x - 3 = 0$$

$$x = 3$$

The x-intercepts are $-\frac{1}{2}$ and 3.

Step 3: Determine the zeros of the function and their multiplicity. Use this information to determine whether the graph crosses or touches the x-axis at each x-intercept.

The zeros of f are $-\frac{1}{2}$ and 3. The zero $-\frac{1}{2}$ is a zero of multiplicity 1, so the graph of f crosses the x-axis at $x = -\frac{1}{2}$. The zero 3 is a zero of multiplicity 2, so the graph of f touches the x-axis at $x = 3$.

Step 4: Use a graphing utility to graph the function.

See Figure 15 for the graph of f.

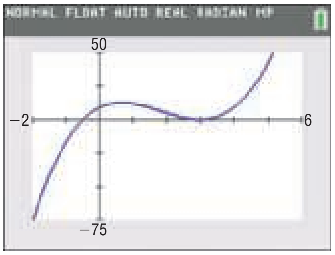


Figure 15 $Y_1 = (2x + 1)(x - 3)^2$

Step 5: Approximate the turning points of the graph.

From the graph of f shown in Figure 15, we see that f has two turning points. Using MAXIMUM, one turning point is at $(0.67, 12.70)$, rounded to two decimal places. Using MINIMUM, the other turning point is at $(3, 0)$.

Step 6: Use the information in Steps 1 to 5 to draw a complete graph of the function by hand.

Figure 16 shows a graph of *f* using the information in Steps 1 through 5.

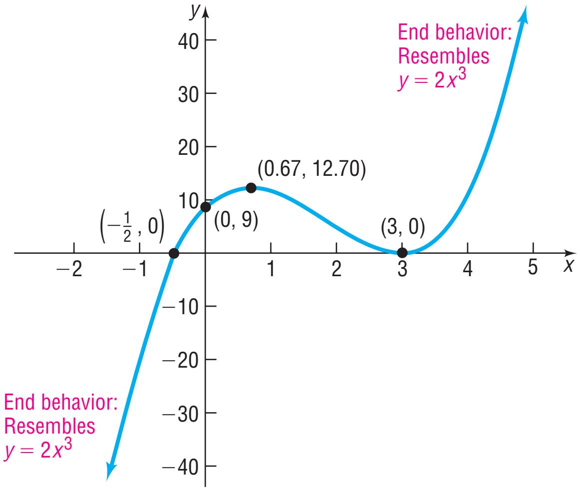


Figure 16 $f(x) = (2x + 1)(x - 3)^2$

Step 7: Find the domain and the range of the function.

The domain and the range of *f* is the set of all real numbers.

Step 8: Use the graph to determine where the function is increasing and where it is decreasing.

Based on the graph, *f* is increasing on the intervals $(-\infty, 0.67]$ and $[3, \infty)$. Also, *f* is decreasing on the interval $[0.67, 3]$. ■

SUMMARY

Analyzing the Graph of a Polynomial Function

STEP 1: Determine the end behavior of the graph of the function.

STEP 2: Find the *x*- and *y*-intercepts of the graph of the function.

STEP 3: Determine the zeros of the function and their multiplicity. Use this information to determine whether the graph crosses or touches the *x*-axis at each *x*-intercept.

STEP 4: Use a graphing utility to graph the function.

STEP 5: Approximate the turning points of the graph.

STEP 6: Use the information in Steps 1 through 5 to draw a complete graph of the function by hand.

STEP 7: Find the domain and the range of the function.

STEP 8: Use the graph to determine where the function is increasing and where it is decreasing.

Now Work PROBLEM 81

For polynomial functions that have noninteger coefficients and for polynomials that are not easily factored, we utilize the graphing utility early in the analysis. This is because the amount of information that can be obtained from algebraic analysis is limited.

EXAMPLE 11

How to Use a Graphing Utility to Analyze the Graph of a Polynomial Function

Analyze the graph of the polynomial function

$$f(x) = x^3 + 2.48x^2 - 4.3155x + 2.484406$$

Step-by-Step Solution

Step 1: Determine the end behavior of the graph of the function.

The polynomial function f is of degree 3. The graph of f behaves like $y = x^3$ for large values of $|x|$.

Step 2: Graph the function using a graphing utility.

See Figure 17 for the graph of f.

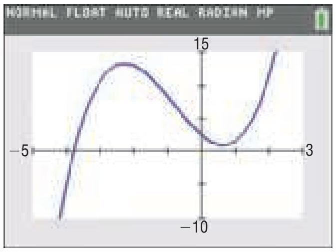


Figure 17 $Y_1 = x^3 + 2.48x^2 - 4.3155x + 2.484406$

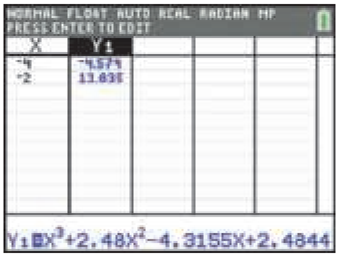
Step 3: Use a graphing utility to approximate the x- and y-intercepts of the graph.

The y-intercept is $f(0) = 2.484406$. In Example 10 the polynomial function was factored, so it was easy to find the x-intercepts algebraically. However, it is not readily apparent how to factor f in this example. Therefore, use a graphing utility's ZERO (or ROOT or SOLVE) feature to find the lone x-intercept, -3.79, rounded to two decimal places.

Step 4: Use a graphing utility to create a TABLE to find points on the graph around each x-intercept.

Table 5 shows values of x around the x-intercept. The points $(-4, -4.57)$ and $(-2, 13.04)$ are on the graph.

Table 5



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X	Y1
-4	-4.574
-2	13.035

Y1=X³+2.48X²−4.3155X+2.4844

Step 5: Approximate the turning points of the graph.

From the graph of f shown in Figure 17, we see that f has two turning points. Using MAXIMUM, one turning point is at $(-2.28, 13.36)$, rounded to two decimal places. Using MINIMUM, the other turning point is at $(0.63, 1)$, rounded to two decimal places.

Step 6: Use the information in Steps 1 through 5 to draw a complete graph of the function by hand.

Figure 18 shows a graph of f using the information in Steps 1 to 5.

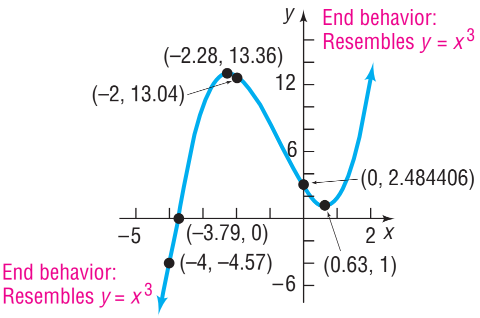


Figure 18 $f(x) = x^3 + 2.48x^2 - 4.3155x + 2.484406$

Step 7: Find the domain and the range of the function.

The domain and the range of f are the set of all real numbers.

Step 8: Use the graph to determine where the function is increasing and where it is decreasing.

Based on the graph, f is increasing on the intervals $(-\infty, -2.28]$ and $[0.63, \infty)$. Also, f is decreasing on the interval $[-2.28, 0.63]$. ■

SUMMARY

Using a Graphing Utility to Analyze the Graph of a Polynomial Function

STEP 1: Determine the end behavior of the graph of the function.

STEP 2: Graph the function using a graphing utility.

STEP 3: Use a graphing utility to approximate the x- and y-intercepts of the graph.

STEP 4: Use a graphing utility to create a TABLE to find points on the graph around each x-intercept.

STEP 5: Approximate the turning points of the graph.

STEP 6: Use the information in Steps 1 through 5 to draw a complete graph of the function by hand.

STEP 7: Find the domain and the range of the function.

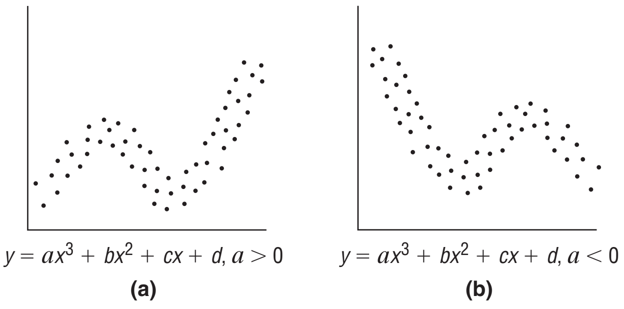
STEP 8: Use the graph to determine where the function is increasing and where it is decreasing.

Now Work PROBLEM 99

5 Build Cubic Models from Data

In Section 4.2 we found the line of best fit from data, and in Section 4.4 we found the quadratic function of best fit. It is also possible to find polynomial functions of best fit. However, most statisticians do not recommend finding polynomial functions of best fit of degree higher than 3.

Data that follow a cubic relation should look like Figure 19(a) or 19(b).



$y = ax^3 + bx^2 + cx + d, a > 0$
(a)

$y = ax^3 + bx^2 + cx + d, a < 0$
(b)

Figure 19 Cubic relation

EXAMPLE 12 A Cubic Function of Best Fit

The data in Table 6 on the next page represent the weekly cost C (in thousands of dollars) of printing x thousand textbooks.

(a) Draw a scatter diagram of the data using x as the independent variable and C as the dependent variable. Comment on the type of relation that may exist between the two variables x and C.

(b) Using a graphing utility, find the cubic function of best fit $C = C(x)$ that models the relation between number of texts and cost.

(c) Graph the cubic function of best fit on your scatter diagram.

(d) Use the function found in part (b) to predict the cost of printing 22 thousand texts per week.

Table 6

Number of Textbooks, x (thousands)	Cost, C ($1000s)
0	100
5	128.1
10	144
13	153.5
17	161.2
18	162.6
20	166.3
23	178.9
25	190.2
27	221.8

Solution

(a) Figure 20 shows the scatter diagram. A cubic relation may exist between the two variables.

(b) Upon executing the CUBIC REGression program, we obtain the results shown in Figure 21. The output that the utility provides shows the equation $y = ax^3 + bx^2 + cx + d$. The cubic function of best fit to the data is $C(x) = 0.0155x^3 - 0.5951x^2 + 9.1502x + 98.4327$.

(c) Figure 22 shows the graph of the cubic function of best fit on the scatter diagram. The function fits the data reasonably well.

(d) Evaluate the function $C(x)$ at $x = 22$.

$$C(22) = 0.0155(22)^3 - 0.5951(22)^2 + 9.1502(22) + 98.4327 \approx 176.8$$

The model predicts that the cost of printing 22 thousand textbooks in a week will be 176.8 thousand dollars—that is, $176,800. ■

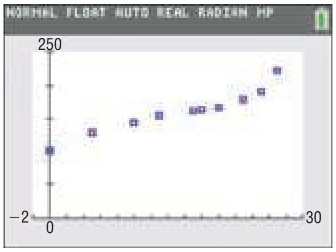
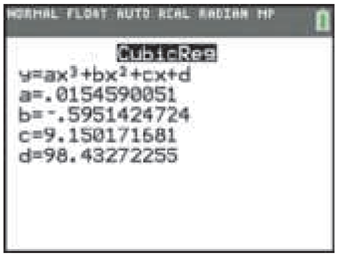


Figure 20



```
CubicReg
y=ax³+bx²+cx+d
a=.0154590051
b=-.5951424724
c=9.150171681
d=98.43272255
```

Figure 21

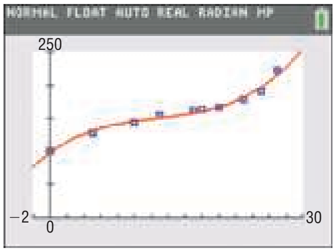


Figure 22

In Example 12, notice that the cubic function of best fit suggests that as the number of textbooks printed increases, cost also increases. That is, $\lim_{x\to\infty} C(x) = \infty$.

5.1 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. The intercepts of the equation $9x^2 + 4y = 36$ are _____. (pp. 165–166)
2. Is the expression $4x^3 - 3.6x^2 - \sqrt{2}$ a polynomial? If so, what is its degree? (pp. 41–42)
3. To graph $y = x^2 - 4$, you would shift the graph of $y = x^2$ _______ a distance of ___ units. (pp. 256–258)
4. Use a graphing utility to approximate (rounded to two decimal places) the local maximum value and local minimum value of $f(x) = x^3 - 2x^2 - 4x + 5$, for $-3 \le x \le 3$. (pp. 237–238)
5. ***True or False*** The x-intercepts of the graph of a function $y = f(x)$ are the real solutions of the equation $f(x) = 0$. (pp. 223–225)
6. If $g(5) = 0$, what point is on the graph of g? What is the corresponding x-intercept of the graph of g? (pp. 223–225)

Concepts and Vocabulary

7. The graph of every polynomial function is both _________ and _____________.
8. If r is a real zero of even multiplicity of a polynomial function f, then the graph of f _________ (crosses/touches) the x-axis at r.
9. The graphs of power functions of the form $f(x) = x^n$, where n is an even integer, always contain the points _________, ________, and ________.
10. If r is a solution to the equation $f(x) = 0$, name three additional statements that can be made about f and r, assuming f is a polynomial function.
11. The points at which a graph changes direction (from increasing to decreasing or decreasing to increasing) are called ______________.

12. For the function $f(x) = 3x^4$, $\lim_{x\to-\infty} f(x) =$ _____ and $\lim_{x\to\infty} f(x) =$ _____.

13. If $f(x) = -2x^5 + x^3 - 5x^2 + 7$, then $\lim_{x\to-\infty} f(x) =$ _____ and $\lim_{x\to\infty} f(x) =$ _______.

14. Explain what the notation $\lim_{x\to\infty} f(x) = -\infty$ means.

15. The _________ of a zero is the number of times its corresponding factor occurs.
(a) degree (b) multiplicity (c) turning point (d) limit

16. The graph of $y = 5x^6 - 3x^4 + 2x - 9$ has at most how many turning points?
(a) -9 (b) 14 (c) 6 (d) 5

Skill Building

In Problems 17–28, determine which functions are polynomial functions. For those that are, state the degree. For those that are not, tell why not. Write each polynomial in standard form. Then identify the leading term and the constant term.

17. $f(x) = 4x + x^3$

18. $f(x) = 5x^2 + 4x^4$

19. $g(x) = \dfrac{1 - x^2}{2}$

20. $h(x) = 3 - \dfrac{1}{2}x$

21. $f(x) = 1 - \dfrac{1}{x}$

22. $f(x) = x(x - 1)$

23. $g(x) = x^{3/2} - x^2 + 2$

24. $h(x) = \sqrt{x}(\sqrt{x} - 1)$

25. $F(x) = 5x^4 - \pi x^3 + \dfrac{1}{2}$

26. $F(x) = \dfrac{x^2 - 5}{x^3}$

27. $G(x) = 2(x - 1)^2(x^2 + 1)$

28. $G(x) = -3x^2(x + 2)^3$

In Problems 29–42, use transformations of the graph of $y = x^4$ or $y = x^5$ to graph each function.

29. $f(x) = (x + 1)^4$

30. $f(x) = (x - 2)^5$

31. $f(x) = x^5 - 3$

32. $f(x) = x^4 + 2$

33. $f(x) = \dfrac{1}{2}x^4$

34. $f(x) = 3x^5$

35. $f(x) = -x^5$

36. $f(x) = -x^4$

37. $f(x) = (x - 1)^5 + 2$

38. $f(x) = (x + 2)^4 - 3$

39. $f(x) = 2(x + 1)^4 + 1$

40. $f(x) = \dfrac{1}{2}(x - 1)^5 - 2$

41. $f(x) = 4 - (x - 2)^5$

42. $f(x) = 3 - (x + 2)^4$

In Problems 43–50, form a polynomial function whose real zeros and degree are given. Answers will vary depending on the choice of a leading coefficient.

43. Zeros: $-1, 1, 3$; degree 3

44. Zeros: $-2, 2, 3$; degree 3

45. Zeros: $-3, 0, 4$; degree 3

46. Zeros: $-4, 0, 2$; degree 3

47. Zeros: $-4, -1, 2, 3$; degree 4

48. Zeros: $-3, -1, 2, 5$; degree 4

49. Zeros: -1, multiplicity 1; 3, multiplicity 2; degree 3

50. Zeros: -2, multiplicity 2; 4, multiplicity 1; degree 3

In Problems 51–56, find the polynomial function with the given zeros whose graph passes through the given point.

51. Zeros: $-3, 1, 4$
Point: $(6, 180)$

52. Zeros: $-2, 0, 2$
Point: $(-4, 16)$

53. Zeros: $-1, 0, 2, 4$
Point: $\left(\dfrac{1}{2}, 63\right)$

54. Zeros: $-5, -1, 2, 6$
Point: $\left(\dfrac{5}{2}, 15\right)$

55. Zeros: -1 (multiplicity 2), 1 (multiplicity 2)
Point: $(-2, 45)$

56. Zeros: 0 (multiplicity 1), -1, 3 (multiplicity 2)
Point: $(1, -48)$

In Problems 57–68, for each polynomial function:
(a) List each real zero and its multiplicity.
(b) Determine whether the graph crosses or touches the x-axis at each x-intercept.
(c) Determine the maximum number of turning points on the graph.
(d) Determine the end behavior; that is, find the power function that the graph of f resembles for large values of $|x|$.

57. $f(x) = 3(x - 7)(x + 3)^2$

58. $f(x) = 4(x + 4)(x + 3)^3$

59. $f(x) = 4(x^2 + 1)(x - 2)^3$

60. $f(x) = 2(x - 3)(x^2 + 4)^3$

61. $f(x) = -2\left(x + \dfrac{1}{2}\right)^2(x + 4)^3$

62. $f(x) = \left(x - \dfrac{1}{3}\right)^2(x - 1)^3$

63. $f(x) = (x - 5)^3(x + 4)^2$

64. $f(x) = (x + \sqrt{3})^2(x - 2)^4$

65. $f(x) = 3(x^2 + 8)(x^2 + 9)^2$

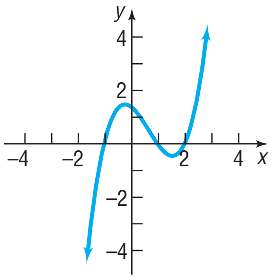
66. $f(x) = -2(x^2 + 3)^3$

67. $f(x) = -2x^2(x^2 - 2)$

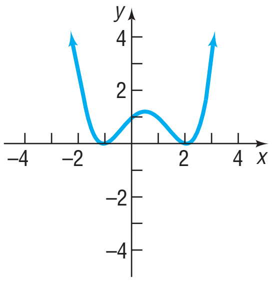
68. $f(x) = 4x(x^2 - 3)$

In Problems 69–72, identify which of the graphs could be the graph of a polynomial function. For those that could, list the real zeros and state the least degree the polynomial can have. For those that could not, say why not.

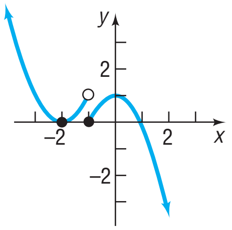
69.



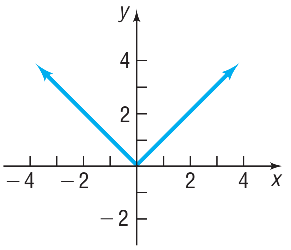
70.



71.

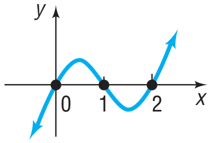


72.



In Problems 73–76, construct a polynomial function that might have the given graph. (More than one answer may be possible.)

73.



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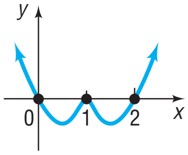

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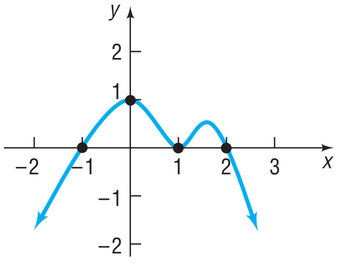

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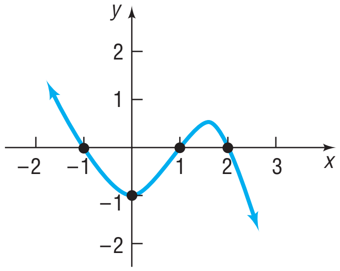
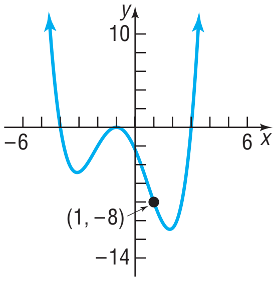


In Problems 77–80, write a polynomial function whose graph is shown (use the smallest degree possible).

77.



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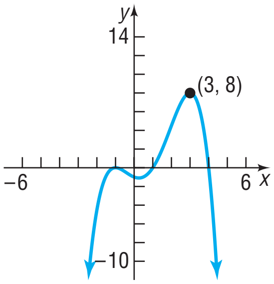

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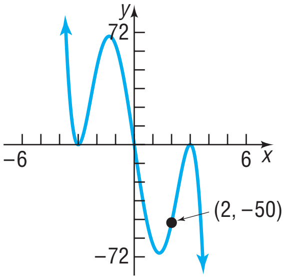
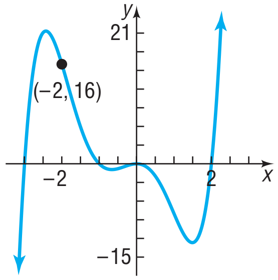


80.



In Problems 81–98, analyze each polynomial function by following Steps 1 through 8 on page 343.

81. $f(x) = x^2(x - 3)$

82. $f(x) = x(x + 2)^2$

83. $f(x) = (x + 4)^2(1 - x)$

84. $f(x) = (x - 1)(x + 3)^2$

85. $f(x) = -2(x + 2)(x - 2)^3$

86. $f(x) = -\frac{1}{2}(x + 4)(x - 1)^3$

87. $f(x) = (x + 1)(x - 2)(x + 4)$

88. $f(x) = (x - 1)(x + 4)(x - 3)$

89. $f(x) = x^2(x - 2)(x + 2)$

90. $f(x) = x^2(x - 3)(x + 4)$

91. $f(x) = (x + 1)^2(x - 2)^2$

92. $f(x) = (x - 4)^2(x + 2)^2$

93. $f(x) = x^2(x + 3)(x + 1)$

94. $f(x) = x^2(x - 3)(x - 1)$

95. $f(x) = 5x(x^2 - 4)(x + 3)$

96. $f(x) = (x - 2)^2(x + 2)(x + 4)$

97. $f(x) = x^2(x - 2)(x^2 + 3)$

98. $f(x) = x^2(x^2 + 1)(x + 4)$

In Problems 99–106, analyze each polynomial function f by following Steps 1 through 8 on page 345.

99. $f(x) = x^3 + 0.2x^2 - 1.5876x - 0.31752$

100. $f(x) = x^3 - 0.8x^2 - 4.6656x + 3.73248$

101. $f(x) = x^3 + 2.56x^2 - 3.31x + 0.89$

102. $f(x) = x^3 - 2.91x^2 - 7.668x - 3.8151$

103. $f(x) = x^4 - 2.5x^2 + 0.5625$

104. $f(x) = x^4 - 18.5x^2 + 50.2619$

105. $f(x) = 2x^4 - \pi x^3 + \sqrt{5}x - 4$

106. $f(x) = -1.2x^4 + 0.5x^2 - \sqrt{3}x + 2$

Mixed Practice

In Problems 107–114, analyze each polynomial function by following Steps 1 through 8 on page 343.

[**Hint:** You will need to first factor the polynomial].

107. $f(x) = 4x - x^3$

108. $f(x) = x - x^3$

109. $f(x) = x^3 + x^2 - 12x$

110. $f(x) = x^3 + 2x^2 - 8x$

111. $f(x) = 2x^4 + 12x^3 - 8x^2 - 48x$

112. $f(x) = 4x^3 + 10x^2 - 4x - 10$

113. $f(x) = -x^5 - x^4 + x^3 + x^2$

114. $f(x) = -x^5 + 5x^4 + 4x^3 - 20x^2$

In Problems 115–118, construct a polynomial function f with the given characteristics.

115. Zeros: $-3, 1, 4$; degree 3; y-intercept: 36

116. Zeros: $-4, -1, 2$; degree 3; y-intercept: 16

117. Zeros: -5(multiplicity 2); 2 (multiplicity 1); 4 (multiplicity 1); degree 4; contains the point $(3, 128)$

118. Zeros: -4 (multiplicity 1); 0 (multiplicity 3); 2 (multiplicity 1); degree 5; contains the point $(-2, 64)$

119. $G(x) = (x + 3)^2(x - 2)$
(a) Identify the x-intercepts of the graph of G.
(b) What are the x-intercepts of the graph of $y = G(x + 3)$?

120. $h(x) = (x + 2)(x - 4)^3$
(a) Identify the x-intercepts of the graph of h.
(b) What are the x-intercepts of the graph of $y = h(x - 2)$?

Applications and Extensions

121. Hurricanes In 2012, Hurricane Sandy struck the East Coast of the United States, killing 147 people and causing an estimated \$75 billion in damage. With a gale diameter of about 1000 miles, it was the largest ever to form over the Atlantic Basin. The accompanying data represent the number of major hurricane strikes in the Atlantic Basin (category 3, 4, or 5) each decade from 1921 to 2010.

Decade, x	Major Hurricanes Striking Atlantic Basin, H
1921–1930, 1	17
1931–1940, 2	16
1941–1950, 3	29
1951–1960, 4	33
1961–1970, 5	27
1971–1980, 6	16
1981–1990, 7	16
1991–2000, 8	27
2001–2010, 9	33

Source: *National Oceanic & Atmospheric Administration*

(a) Draw a scatter diagram of the data. Comment on the type of relation that may exist between the two variables.
(b) Use a graphing utility to find the cubic function of best fit that models the relation between decade and number of major hurricanes.
(c) Use the model found in part (b) to predict the number of major hurricanes that struck the Atlantic Basin between 1961 and 1970.
(d) With a graphing utility, draw a scatter diagram of the data and then graph the cubic function of best fit on the scatter diagram.
(e) Concern has risen about the increase in the number and intensity of hurricanes, but some scientists believe this is just a natural fluctuation that could last another decade or two. Use your model to predict the number of major hurricanes that will strike the Atlantic Basin between 2011 and 2020. Is your result reasonable? How does this result suggest using end behavior of models to make long-term predictions is dangerous?

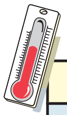
122. Poverty Rates The following data represent the percentage of people in the United States living below the poverty level.

Year, t	Percent below Poverty Level, p	Year, t	Percent below Poverty Level, p
1990, 1	13.5	2001, 12	11.7
1991, 2	14.2	2002, 13	12.1
1992, 3	14.8	2003, 14	12.5
1993, 4	15.1	2004, 15	12.7
1994, 5	14.5	2005, 16	12.6
1995, 6	13.8	2006, 17	12.3
1996, 7	13.7	2007, 18	12.5
1997, 8	13.3	2008, 19	13.2
1998, 9	12.7	2009, 20	14.3
1999, 10	11.9	2010, 21	15.1
2000, 11	11.3	2011, 22	15.0

Source: *U.S. Census Bureau*

(a) With a graphing utility, draw a scatter diagram of the data. Comment on the type of relation that appears to exist between the two variables.
(b) Decide on a function of best fit to these data (linear, quadratic, or cubic), and use this function to predict the percentage of people in the United States who were living below the poverty level in 2013 $(t = 24)$. Compare your prediction to the actual value of 14.5.
(c) Draw the function of best fit on the scatter diagram drawn in part (a).

123. Temperature The following data represent the temperature T (°Fahrenheit) in Kansas City, Missouri, x hours after midnight on March 15, 2015.



Hours after Midnight, x	Temperature (°F), T
3	43.0
6	39.0
9	44.1
12	62.1
15	71.1
18	71.6
21	60.1
24	59.0

Source: *The Weather Underground*

(a) Draw a scatter diagram of the data. Comment on the type of relation that may exist between the two variables.
(b) Find the average rate of change in temperature from 9 AM to 12 noon.
(c) What is the average rate of change in temperature from 3 PM to 6 PM?
(d) Decide on a function of best fit to these data (linear, quadratic, or cubic) and use this function to predict the temperature at 5 PM.
(e) With a graphing utility, draw a scatter diagram of the data and then graph the function of best fit on the scatter diagram.
(f) Interpret the y-intercept.

124. Future Value of Money Suppose that you make deposits of \$500 at the beginning of every year into an Individual Retirement Account (IRA) earning interest r (expressed as a decimal). At the beginning of the first year, the value of the account will be \$500; at the beginning of the second year, the value of the account, will be

$$\underbrace{\$500 + \$500r}_{\textbf{Value of 1st deposit}} + \underbrace{\$500}_{\textbf{Value of 2nd deposit}} = \$500(1+r) + \$500 = 500r + 1000$$

(a) Verify that the value of the account at the beginning of the third year is $T(r) = 500r^2 + 1500r + 1500$.
(b) The account value at the beginning of the fourth year is $F(r) = 500r^3 + 2000r^2 + 3000r + 2000$. If the annual rate of interest is $5\% \;= 0.05$, what will be the value of the account at the beginning of the fourth year?

125. A Geometric Series In calculus, you will learn that certain functions can be approximated by polynomial functions. We will explore one such function now.

(a) Using a graphing utility, create a table of values with

$$Y_1 = f(x) = \frac{1}{1-x} \quad \text{and} \quad Y_2 = g_2(x) = 1 + x + x^2 + x^3$$

for $-1 < x < 1$ with $\Delta\text{Tbl} = 0.1$.

(b) Using a graphing utility, create a table of values with

$$Y_1 = f(x) = \frac{1}{1-x} \quad \text{and}$$

$$Y_2 = g_3(x) = 1 + x + x^2 + x^3 + x^4$$

for $-1 < x < 1$ with $\Delta\text{Tbl} = 0.1$.

(c) Using a graphing utility, create a table of values with

$$Y_1 = f(x) = \frac{1}{1-x} \quad \text{and}$$

$$Y_2 = g_4(x) = 1 + x + x^2 + x^3 + x^4 + x^5$$

for $-1 < x < 1$ with $\Delta\text{Tbl} = 0.1$.

(d) What do you notice about the values of the function as more terms are added to the polynomial? Are there some values of x for which the approximations are better?

126. If $f(x) = x^3$, graph $f(2x)$.

Explaining Concepts: Discussion and Writing

127. Write a few paragraphs that provide a general strategy for graphing a polynomial function. Be sure to mention the following: degree, intercepts, end behavior, and turning points.

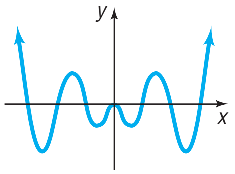
128. Make up a polynomial that has the following characteristics: crosses the x-axis at -1 and 4, touches the x-axis at 0 and 2, and is above the x-axis between 0 and 2. Give your polynomial to a fellow classmate and ask for a written critique.

129. Make up two polynomials, not of the same degree, with the following characteristics: crosses the x-axis at -2, touches the x-axis at 1, and is above the x-axis between -2 and 1. Give your polynomials to a fellow classmate and ask for a written critique.

130. The graph of a polynomial function is always smooth and continuous. Name a function studied earlier that is smooth but not continuous. Name one that is continuous but not smooth.

131. Which of the following statements are true regarding the graph of the cubic polynomial $f(x) = x^3 + bx^2 + cx + d$? (Give reasons for your conclusions.)
(a) It intersects the y-axis in one and only one point.
(b) It intersects the x-axis in at most three points.
(c) It intersects the x-axis at least once.
(d) For $|x|$ very large, it behaves like the graph of $y = x^3$.
(e) It is symmetric with respect to the origin.
(f) It passes through the origin.

132. The illustration shows the graph of a polynomial function.



(a) Is the degree of the polynomial even or odd?
(b) Is the leading coefficient positive or negative?
(c) Is the function even, odd, or neither?
(d) Why is x^2 necessarily a factor of the polynomial?
(e) What is the minimum degree of the polynomial?
(f) Formulate five different polynomials whose graphs could look like the one shown. Compare yours to those of other students. What similarities do you see? What differences?

133. Design a polynomial function with the following characteristics: degree 6; four distinct real zeros, one of multiplicity 3; y-intercept 3; behaves like $y = -5x^6$ for large values of $|x|$. Is this polynomial unique? Compare your polynomial with those of other students. What terms will be the same as everyone else's? Add some more characteristics, such as symmetry or naming the real zeros. How does this modify the polynomial?

134. Can the graph of a polynomial function have no y-intercept? Can it have no x-intercepts? Explain.

Retain Your Knowledge

Problems 135–138 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

135. Find the equation of the line that contains the point $(2, -3)$ and is perpendicular to the line $5x - 2y = 6$.

136. Find the domain of the function $h(x) = \frac{x-3}{x+5}$.

137. Use the quadratic formula to find the zeros of the function $f(x) = 4x^2 + 8x - 3$.

138. Solve: $|5x - 3| = 7$.

'Are You Prepared?' Answers

1. $(-2, 0), (2, 0), (0, 9)$ **2.** Yes; 3 **3.** Down; 4 **4.** Local maximum value 6.48 at $x = -0.67$; local minimum value -3 at $x = 2$

5. True **6.** $(5, 0); 5$

5.2 The Real Zeros of a Polynomial Function

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Evaluating Functions (Section 3.1 pp. 210–213)
- Factoring Polynomials (Chapter R, Section R.5, pp. 50–56)
- Synthetic Division (Chapter R, Section R.6, pp. 59–62)
- Polynomial Division (Chapter R, Section R.4, pp. 45–48)
- Solve a Quadratic Equation (Section 1.3, pp. 110–116)

Now Work the **'Are You Prepared?'** problems on page 363.

OBJECTIVES
1 Use the Remainder and Factor Theorems (p. 352)
2 Use Descartes' Rule of Signs to Determine the Number of Positive and the Number of Negative Real Zeros of a Polynomial Function (p. 354)
3 Use the Rational Zeros Theorem to List the Potential Rational Zeros of a Polynomial Function (p. 355)
4 Find the Real Zeros of a Polynomial Function (p. 356)
5 Solve Polynomial Equations (p. 359)
6 Use the Theorem for Bounds on Zeros (p. 359)
7 Use the Intermediate Value Theorem (p. 362)

In Section 5.1, we were able to identify the real zeros of a polynomial function because either the polynomial function was in factored form or it could be easily factored. But how do we find the real zeros of a polynomial function if it is not factored or cannot be easily factored?

Recall that if r is a real zero of a polynomial function f then $f(r) = 0$, r is an x-intercept of the graph of f, $x - r$ is a factor of f, and r is a solution of the equation $f(x) = 0$. For example, if $x - 4$ is a factor of f, then 4 is a real zero of f and 4 is a solution to the equation $f(x) = 0$. For polynomial functions, we have seen the importance of the real zeros for graphing. In most cases, however, the real zeros of a polynomial function are difficult to find using algebraic methods. No nice formulas like the quadratic formula are available to help us find zeros for polynomial functions of degree 3 or higher. Formulas do exist for solving any third- or fourth-degree polynomial equation, but they are somewhat complicated. No general formulas exist for polynomial equations of degree 5 or higher. Refer to the Historical Feature at the end of this section for more information.

1 Use the Remainder and Factor Theorems

When one polynomial (the dividend) is divided by another (the divisor), a quotient polynomial and a remainder are obtained, the remainder being either the zero polynomial or a polynomial whose degree is less than the degree of the divisor. To check, verify that

$$(\text{Quotient})(\text{Divisor}) + \text{Remainder} = \text{Dividend}$$

This checking routine is the basis for a famous theorem called the **division algorithm*** **for polynomials**, which we now state without proof.

THEOREM

Division Algorithm for Polynomials

If $f(x)$ and $g(x)$ denote polynomial functions and if $g(x)$ is a polynomial function whose degree is greater than zero, then there are unique polynomial functions $q(x)$ and $r(x)$ such that

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)} \quad \text{or} \quad \underset{\substack{\uparrow \\ \textbf{dividend}}}{f(x)} = \underset{\substack{\uparrow \\ \textbf{quotient}}}{q(x)}\,\underset{\substack{\uparrow \\ \textbf{divisor}}}{g(x)} + \underset{\substack{\uparrow \\ \textbf{remainder}}}{r(x)} \tag{1}$$

where $r(x)$ is either the zero polynomial or a polynomial function of degree less than that of $g(x)$.

In equation (1), $f(x)$ is the **dividend**, $g(x)$ is the **divisor**, $q(x)$ is the **quotient**, and $r(x)$ is the **remainder**.

If the divisor $g(x)$ is a first-degree polynomial function of the form

$$g(x) = x - c \qquad c \text{ a real number}$$

then the remainder $r(x)$ is either the zero polynomial or a polynomial function of degree 0. As a result, for such divisors, the remainder is some number, say R, and

$$f(x) = (x - c)q(x) + R \tag{2}$$

This equation is an identity in x and is true for all real numbers x. Suppose that $x = c$. Then equation (2) becomes

$$f(c) = (c - c)q(c) + R$$
$$f(c) = R$$

Substitute $f(c)$ for R in equation (2) to obtain

$$f(x) = (x - c)q(x) + f(c) \tag{3}$$

which proves the **Remainder Theorem**.

REMAINDER THEOREM

Let f be a polynomial function. If $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

---

*A systematic process in which certain steps are repeated a finite number of times is called an **algorithm**. For example, long division is an algorithm.

EXAMPLE 1 Using the Remainder Theorem

Find the remainder if $f(x) = x^3 - 4x^2 - 5$ is divided by

(a) $x - 3$ (b) $x + 2$

Solution (a) We could use long division or synthetic division, but it is easier to use the Remainder Theorem, which says that the remainder is $f(3)$.

$$f(3) = (3)^3 - 4(3)^2 - 5 = 27 - 36 - 5 = -14$$

The remainder is -14.

(b) To find the remainder when $f(x)$ is divided by $x + 2 = x - (-2)$, evaluate $f(-2)$.

$$f(-2) = (-2)^3 - 4(-2)^2 - 5 = -8 - 16 - 5 = -29$$

The remainder is -29. ■

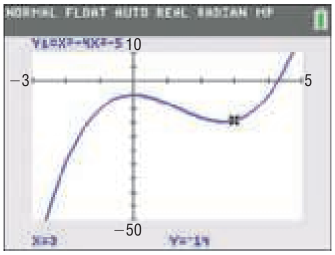


Figure 23 Value of a function

Compare the method used in Example 1(a) with the method used in Example 1 of Chapter R, Section R.6. Which method do you prefer? Give reasons.

COMMENT A graphing utility provides another way to find the value of a function using the eVALUEate feature. Consult your manual for details. See Figure 23 for the result of Example 1(a). ■

An important and useful consequence of the Remainder Theorem is the **Factor Theorem**.

FACTOR THEOREM

Let f be a polynomial function. Then $x - c$ is a factor of $f(x)$ if and only if $f(c) = 0$. ■

The Factor Theorem actually consists of two separate statements:

1. If $f(c) = 0$, then $x - c$ is a factor of $f(x)$.
2. If $x - c$ is a factor of $f(x)$, then $f(c) = 0$.

The proof requires two parts.

Proof

1. Suppose that $f(c) = 0$. Then, by equation (3), we have

$$f(x) = (x - c)q(x)$$

for some polynomial function $q(x)$. That is, $x - c$ is a factor of $f(x)$.

2. Suppose that $x - c$ is a factor of $f(x)$. Then there is a polynomial function q such that

$$f(x) = (x - c)q(x)$$

Replacing x by c, we find that

$$f(c) = (c - c)q(c) = 0 \cdot q(c) = 0$$

This completes the proof. ■

One use of the Factor Theorem is to determine whether a polynomial has a particular factor.

EXAMPLE 2 Using the Factor Theorem

Use the Factor Theorem to determine whether the function

$$f(x) = 2x^3 - x^2 + 2x - 3$$

has the factor

(a) $x - 1$ (b) $x + 2$

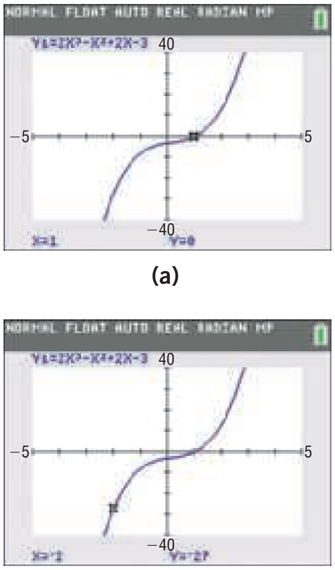
Solution The Factor Theorem states that if $f(c) = 0$ then $x - c$ is a factor.

(a) Because $x - 1$ is of the form $x - c$ with $c = 1$, find the value of $f(1)$. We choose to use substitution.

$$f(1) = 2(1)^3 - (1)^2 + 2(1) - 3 = 2 - 1 + 2 - 3 = 0$$

See also Figure 24(a). By the Factor Theorem, $x - 1$ is a factor of $f(x)$.

(b) To test the factor $x + 2$, first write it in the form $x - c$. Since $x + 2 = x - (-2)$, find the value of $f(-2)$. See Figure 24(b). Because $f(-2) = -27 \neq 0$, conclude from the Factor Theorem that $x - (-2) = x + 2$ is not a factor of $f(x)$. ■



(a)

(b)

Figure 24

Now Work PROBLEM 11

In Example 2(a), $x - 1$ was found to be a factor of f. To write f in factored form, use long division or synthetic division. Using synthetic division,

$$\begin{array}{r|rrrr} 1 & 2 & -1 & 2 & -3 \\ & & 2 & 1 & 3 \\ \hline & 2 & 1 & 3 & 0 \end{array}$$

The quotient is $q(x) = 2x^2 + x + 3$ with a remainder of 0, as expected. Write f in factored form as

$$f(x) = 2x^3 - x^2 + 2x - 3 = (x - 1)(2x^2 + x + 3)$$

WARNING Remember that in order for synthetic division to be used, the divisor must be of the form $x - c$. ■

The next theorem concerns the number of real zeros that a polynomial function may have. In counting the zeros of a polynomial function, count each zero as many times as its multiplicity.

THEOREM

Number of Real Zeros

A polynomial function cannot have more real zeros than its degree. ■

Proof The proof is based on the Factor Theorem. If r is a real zero of a polynomial function f, then $f(r) = 0$ and, hence, $x - r$ is a factor of $f(x)$. Each real zero corresponds to a factor of degree 1. Because f cannot have more first-degree factors than its degree, the result follows. ■

2 Use Descartes' Rule of Signs to Determine the Number of Positive and the Number of Negative Real Zeros of a Polynomial Function

Descartes' Rule of Signs provides information about the number and location of the real zeros of a polynomial function written in standard form (omitting terms with a 0 coefficient). It utilizes the number of variations in the sign of the coefficients of $f(x)$ and $f(-x)$.

For example, the following polynomial function has two variations in the signs of the coefficients.

$$f(x) = \underbrace{-3x^7 + 4x^4}_{-\text{ to }+} + \underbrace{3x^2 - 2x}_{+\text{ to }-} - 1$$

Replacing x by $-x$ gives

$$\begin{aligned} f(-x) &= -3(-x)^7 + 4(-x)^4 + 3(-x)^2 - 2(-x) - 1 \\ &= 3x^7 + 4x^4 + \underbrace{3x^2 + 2x}_{+\text{ to }-} - 1 \end{aligned}$$

which has one variation in sign.

THEOREM

Descartes' Rule of Signs

Let f denote a polynomial function written in standard form.

- The number of positive real zeros of f either equals the number of variations in the sign of the nonzero coefficients of $f(x)$ or else equals that number less an even integer.
- The number of negative real zeros of f either equals the number of variations in the sign of the nonzero coefficients of $f(-x)$ or else equals that number less an even integer.

We shall not prove Descartes' Rule of Signs. Let's see how it is used.

EXAMPLE 3

Using the Number of Real Zeros Theorem and Descartes' Rule of Signs

Discuss the real zeros of $f(x) = 3x^7 - 4x^4 + 3x^3 + 2x^2 - x - 3$.

Solution Because the polynomial is of degree 7, by the Number of Real Zeros Theorem there are at most seven real zeros. Since there are three variations in the sign of the nonzero coefficients of $f(x)$, by Descartes' Rule of Signs we expect either three positive real zeros or one positive real zero. To continue, look at $f(-x)$.

$$f(-x) = -3x^7 - 4x^4 - 3x^3 + 2x^2 + x - 3$$

There are two variations in sign, so we expect either two negative real zeros or no negative real zeros. Equivalently, we now know that the graph of f has either three positive x-intercepts or one positive x-intercept and two negative x intercepts or no negative x-intercepts. ■

Now Work PROBLEM 21

3 Use the Rational Zeros Theorem to List the Potential Rational Zeros of a Polynomial Function

The next result, called the **Rational Zeros Theorem**, provides information about the rational zeros of a polynomial function *with integer coefficients*.

THEOREM

Rational Zeros Theorem

Let f be a polynomial function of degree 1 or higher of the form

$$\boxed{f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \quad a_n \neq 0, \quad a_0 \neq 0}$$

where each coefficient is an integer. If $\frac{p}{q}$, in lowest terms, is a rational zero of f, then p must be a factor of a_0 and q must be a factor of a_n.

EXAMPLE 4

Listing Potential Rational Zeros

List the potential rational zeros of

$$f(x) = 2x^3 + 11x^2 - 7x - 6$$

Solution Because f has integer coefficients, we may use the Rational Zeros Theorem. First, list all the integers p that are factors of the constant term $a_0 = -6$ and all the integers q that are factors of the leading coefficient $a_3 = 2$.

p: $\pm 1, \pm 2, \pm 3, \pm 6$ **Factors of -6**

q: $\pm 1, \pm 2$ **Factors of 2**

Now form all possible ratios $\frac{p}{q}$.

$$\frac{p}{q}: \pm 1, \pm 2, \pm 3, \pm 6, \pm\frac{1}{2}, \pm\frac{3}{2}$$

If f has a rational zero, it will be found in this list, which contains 12 possibilities. ■

Now Work PROBLEM 33

In Words

For the polynomial function $f(x) = 2x^3 + 11x^2 - 7x - 6$, we know 5 is not a zero, because 5 is not in the list of potential rational zeros. However, -1 may or may not be a zero.

Be sure that you understand what the Rational Zeros Theorem says: For a polynomial function with integer coefficients, *if* there is a rational zero, it is one of those listed. It may be the case that the function does not have any rational zeros.

The Rational Zeros Theorem provides a list of potential rational zeros of a function f. If we graph f, we can get a better sense of the location of the x-intercepts and test to see if they are rational. We can also use the potential rational zeros to select our initial viewing window to graph f and then adjust the window based on the results. The graphs shown throughout the text will be those obtained after setting the final viewing window.

4 Find the Real Zeros of a Polynomial Function

EXAMPLE 5 How to Find the Real Zeros of a Polynomial Function

Find the real zeros of the polynomial function $f(x) = 2x^3 + 11x^2 - 7x - 6$. Write f in factored form.

Step-by-Step Solution

Step 1: Determine the maximum number of zeros. Also determine the number of positive and negative real zeros.

Since f is a polynomial function of degree 3, there are at most three real zeros. From Descartes' Rule of Signs, there is one positive real zero. Also, since $f(-x) = -2x^3 + 11x^2 + 7x - 6$, there are two negative real zeros or no negative real zeros.

Step 2: If the polynomial function has integer coefficients, use the Rational Zeros Theorem to identify those rational numbers that potentially can be zeros.

List the potential rational zeros obtained in Example 4:

$$\pm 1, \pm 2, \pm 3, \pm 6, \pm\frac{1}{2}, \pm\frac{3}{2}$$

Step 3: Using a graphing utility, graph the polynomial function.

Figure 25 shows the graph of f. We see that f has three zeros: one near -6, one between -1 and 0, and one near 1.

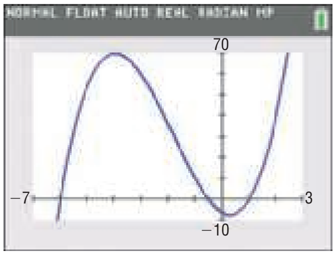


Figure 25 $Y_1 = 2x^3 + 11x^2 - 7x - 6$

Step 4: Use the Factor Theorem to determine if the potential rational zero is a zero. If it is, use synthetic division or long division to factor the polynomial function. Repeat Step 4 until all the zeros of the polynomial function have been identified and the polynomial function is completely factored.

From our list of potential rational zeros, we test -6 to determine if it is a zero of f. Because

$$\begin{aligned} f(-6) &= 2(-6)^3 + 11(-6)^2 - 7(-6) - 6 \\ &= 2(-216) + 11(36) + 42 - 6 \\ &= -432 + 396 + 36 \\ &= 0 \end{aligned}$$

we know that -6 is a zero and $x - (-6) = x + 6$ is a factor of f. Use long division or synthetic division to factor f. (We will not show the division here, but you are encouraged to verify the results shown.) After dividing f by $x + 6$, the quotient is $2x^2 - x - 1$, so

$$\begin{aligned} f(x) &= 2x^3 + 11x^2 - 7x - 6 \\ &= (x + 6)(2x^2 - x - 1) \end{aligned}$$

Now any solution of the equation $2x^2 - x - 1 = 0$ will be a zero of f. Because of this, the equation $2x^2 - x - 1 = 0$ is called a **depressed equation** of f. Because any solution to the equation $2x^2 - x - 1 = 0$ is a zero of f, work with the depressed equation to find the remaining zeros of f.

The depressed equation $2x^2 - x - 1 = 0$ is a quadratic equation with discriminant $b^2 - 4ac = (-1)^2 - 4(2)(-1) = 9 > 0$. The equation has two real solutions, which can be found by factoring.

$$\begin{gathered} 2x^2 - x - 1 = (2x + 1)(x - 1) = 0 \\ 2x + 1 = 0 \quad \text{or} \quad x - 1 = 0 \\ x = -\frac{1}{2} \quad \text{or} \quad x = 1 \end{gathered}$$

The zeros of f are -6, $-\frac{1}{2}$, and 1.

Factor f completely as follows:

$$f(x) = 2x^3 + 11x^2 - 7x - 6 = (x + 6)(2x^2 - x - 1) = (x + 6)(2x + 1)(x - 1)$$

Notice that the three zeros of f are in the list of potential rational zeros in Step 2, and confirm what was expected from Descartes' Rule of Signs. ■

SUMMARY

Steps for Finding the Real Zeros of a Polynomial Function

STEP 1: Use the degree of the polynomial function to determine the maximum number of zeros. Use Descartes' Rule of Signs to determine the number of positive and negative real zeros.

STEP 2: If the polynomial function has integer coefficients, use the Rational Zeros Theorem to identify those rational numbers that potentially can be zeros.

STEP 3: Graph the polynomial function using a graphing utility to find the best choice of potential rational zeros to test.

STEP 4: Use the Factor Theorem to determine if the potential rational zero is a zero. If it is, use synthetic division or long division to factor the polynomial function. Each time that a zero (and thus a factor) is found, repeat Step 4 on the depressed equation. In attempting to find the zeros, remember to use (if possible) the factoring techniques that you already know (special products, factoring by grouping, and so on).

EXAMPLE 6 Finding the Real Zeros of a Polynomial Function

Find the real zeros of $f(x) = x^6 + 4x^5 - 16x^3 - 37x^2 - 84x - 84$. Write f in factored form.

Solution

STEP 1: There are at most six real zeros. There is one positive real zero and there are five, three, or one negative real zeros.

STEP 2: To obtain the list of potential rational zeros, write the factors p of $a_0 = -84$ and the factors q of the leading coefficient $a_6 = 1$.

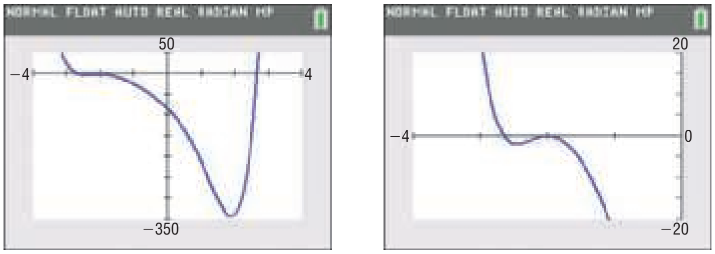
$$p: \quad \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 7, \pm 12, \pm 14, \pm 21, \pm 28, \pm 42, \pm 84$$

$$q: \quad \pm 1$$

The potential rational zeros consist of all possible quotients $\frac{p}{q}$:

$$\frac{p}{q}: \quad \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 7, \pm 12, \pm 14, \pm 21, \pm 28, \pm 42, \pm 84$$

STEP 3: Figure 26 shows the graph of f. The graph has the characteristics expected of the given polynomial function of degree 6: no more than five turning points, y-intercept -84, and it behaves like $y = x^6$ for large $|x|$.



(a) **(b)**

Figure 26 $Y_1 = x^6 + 4x^5 - 16x^3 - 37x^2 - 84x - 84$

STEP 4: From Figure 26(b), -2 appears to be a zero. Additionally, -2 is a potential rational zero. Evaluate $f(-2)$ and find that $f(-2) = 0$. By the Factor Theorem, $x + 2$ is a factor of f. We use synthetic division to factor f.

$$\begin{array}{r|rrrrrrr} -2 & 1 & 4 & 0 & -16 & -37 & -84 & -84 \\ & & -2 & -4 & 8 & 16 & 42 & 84 \\ \hline & 1 & 2 & -4 & -8 & -21 & -42 & 0 \end{array}$$

Factor f as

$$f(x) = (x + 2)(x^5 + 2x^4 - 4x^3 - 8x^2 - 21x - 42)$$

Now work with the first depressed equation:

$$q_1(x) = x^5 + 2x^4 - 4x^3 - 8x^2 - 21x - 42 = 0$$

Repeat Step 4: In looking back at Figure 26(b), it appears that -2 might be a zero of even multiplicity. Check the potential rational zero -2 again using synthetic division.

$$\begin{array}{r|rrrrrr} -2 & 1 & 2 & -4 & -8 & -21 & -42 \\ & & -2 & 0 & 8 & 0 & 42 \\ \hline & 1 & 0 & -4 & 0 & -21 & 0 \end{array}$$

Since $q_1(-2) = 0$, then $x + 2$ is a factor and

$$f(x) = (x + 2)(x + 2)(x^4 - 4x^2 - 21)$$

Repeat Step 4: The depressed equation $q_2(x) = x^4 - 4x^2 - 21 = 0$ can be factored.

$$x^4 - 4x^2 - 21 = (x^2 - 7)(x^2 + 3) = 0$$

$$x^2 - 7 = 0 \quad \text{or} \quad x^2 + 3 = 0$$

$$x^2 = 7$$

$$x = \pm\sqrt{7}$$

Since $x^2 + 3 = 0$ has no real solutions, the real zeros of f are $-\sqrt{7}$, $\sqrt{7}$, and -2, with -2 being a zero of multiplicity 2. The factored form of f is

$$\begin{aligned} f(x) &= x^6 + 4x^5 - 16x^3 - 37x^2 - 84x - 84 \\ &= (x + 2)^2(x + \sqrt{7})(x - \sqrt{7})(x^2 + 3) \end{aligned}$$

Note that one positive real zero and three negative real zeros agrees with our result from Step 1. ■

Now Work PROBLEM 51

5 Solve Polynomial Equations

EXAMPLE 7 **Solving a Polynomial Equation**

Solve the equation: $x^6 + 4x^5 - 16x^3 - 37x^2 - 84x - 84 = 0$

Solution The solutions of this equation are the zeros of the polynomial function

$$f(x) = x^6 + 4x^5 - 16x^3 - 37x^2 - 84x - 84$$

Using the result of Example 6, the real zeros are $-2, -\sqrt{7}$, and $\sqrt{7}$. These are the real solutions of the equation $x^6 + 4x^5 - 16x^3 - 37x^2 - 84x - 84 = 0$. ■

Now Work PROBLEM 75

In Example 6, the quadratic factor $x^2 + 3$ that appears in the factored form of $f(x)$ is called *irreducible*, because the polynomial $x^2 + 3$ cannot be factored over the real numbers. In general, a quadratic factor $ax^2 + bx + c$ is **irreducible** if it cannot be factored over the real numbers, that is, if it is prime over the real numbers.

Refer again to Examples 5 and 6. The polynomial function of Example 5 has three real zeros, and its factored form contains three linear factors. The polynomial function of Example 6 has three distinct real zeros, and its factored form contains three distinct linear factors and one irreducible quadratic factor.

THEOREM

Every polynomial function (with real coefficients) can be uniquely factored into a product of linear factors and/or irreducible quadratic factors. ■

We prove this result in Section 5.3, and, in fact, shall draw several additional conclusions about the zeros of a polynomial function. One conclusion is worth noting now. If a polynomial function (with real coefficients) is of odd degree, then it must contain at least one linear factor. (Do you see why? Consider the end behavior of polynomial functions of odd degree.) This means that it must have at least one real zero.

COROLLARY

A polynomial function (with real coefficients) of odd degree has at least one real zero. ■

6 Use the Theorem for Bounds on Zeros

One challenge in using a graphing utility is to set the viewing window so that a complete graph is obtained. The next theorem is a tool that can be used to find bounds on the zeros. This will assure that the function does not have any zeros outside these bounds. Then using these bounds to set Xmin and Xmax assures that all the x-intercepts appear in the viewing window.

A number M is an **upper bound** to the zeros of a polynomial f if no zero of f is greater than M. The number m is a **lower bound** if no zero of f is less than m. Accordingly, if m is a lower bound and M is an upper bound to the zeros of a polynomial function f, then

$$m \le \text{any zero of } f \le M$$

COMMENT Knowing the values of a lower bound m and an upper bound M may enable you to eliminate some potential rational zeros—that is, any potential zeros outside of the interval $[m, M]$. ■

THEOREM

Bounds on Zeros

Let f denote a polynomial function whose leading coefficient is positive.

- If $M > 0$ is a real number and if the third row in the process of synthetic division of f by $x - M$ contains only numbers that are positive or zero, then M is an upper bound to the zeros of f.
- If $m < 0$ is a real number and if the third row in the process of synthetic division of f by $x - m$ contains numbers that alternate positive (or 0) and negative (or 0), then m is a lower bound to the zeros of f. ■

Note: When finding a lower bound, remember that a 0 can be treated as either positive or negative, but not both. For example, 3, 0, 5 would be considered to alternate sign, whereas 3, 0, −5 would not. ■

Proof (Outline) We give only an outline of the proof of the first part of the theorem. Suppose that M is a positive real number, and the third row in the process of synthetic division of the polynomial f by $x - M$ contains only numbers that are positive or 0. Then there are a quotient q and a remainder R such that

$$f(x) = (x - M)q(x) + R$$

where the coefficients of $q(x)$ are positive or 0 and the remainder $R \geq 0$. Then, for any $x > M$, we must have $x - M > 0$, $q(x) > 0$, and $R \geq 0$, so that $f(x) > 0$. That is, there is no zero of f larger than M. The proof of the second part follows similar reasoning. ■

In finding bounds, it is preferable to find the smallest upper bound and largest lower bound. This will require repeated synthetic division until a desired pattern is observed. For simplicity, we will consider only potential rational zeros that are integers. If a bound is not found using these values, continue checking positive and/or negative integers until you find both an upper and a lower bound.

EXAMPLE 8 Finding Upper and Lower Bounds of Zeros

For the polynomial function $f(x) = 2x^3 + 11x^2 - 7x - 6$, use the Bounds on Zeros Theorem to find integer upper and lower bounds to the zeros of f.

Solution From Example 4, the potential rational zeros of f are $\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$.

To find an upper bound, start with the smallest positive integer that is a potential rational zero, which is 1. Continue checking 2, 3, and 6 (and then subsequent positive integers), if necessary, until an upper bound is found. To find a lower bound, start with the largest negative integer that is a potential rational zero, which is −1. Continue checking −2, −3, and −6 (and then subsequent negative integers), if necessary, until a lower bound is found. Table 7 summarizes the results of doing repeated synthetic divisions by showing only the third row of each division. For example, the first row of the table shows the result of dividing $f(x)$ by $x - 1$.

$$\begin{array}{r|rrrr} 1 & 2 & 11 & -7 & -6 \\ & & 2 & 13 & 6 \\ \hline & 2 & 13 & 6 & 0 \end{array}$$

Table 7 Synthetic Division Summary

r	Coefficients of $q(x)$			Remainder
1	2	13	6	0
−1	2	9	−16	10
−2	2	7	−21	36
−3	2	5	−22	60
−6	2	−1	−1	0
−7	2	−3	14	−104

Upper bound → 1 — All nonnegative

Lower bound → −7 — Alternating Signs

Note: Keep track of any zeros that are found when looking for bounds. ■

For $r = 1$, the third row of synthetic division contains only numbers that are positive or 0, so we know there are no zeros greater than 1. Since the third row of synthetic division for $r = -7$ results in alternating positive (or 0) and negative (or 0) values, we know that −7 is a lower bound. There are no zeros less than −7. Notice that in looking for bounds, two zeros were discovered. These zeros are 1 and −6. ■

EXAMPLE 9 Obtaining Graphs Using Bounds on Zeros

Obtain a graph for the polynomial function.

$$f(x) = 2x^3 + 11x^2 - 7x - 6$$

Solution Based on Example 8, every zero lies between -7 and 1. Using Xmin $= -7$ and Xmax $= 1$, we graph $Y_1 = f(x) = 2x^3 + 11x^2 - 7x - 6$. Figure 27 shows the graph obtained using ZOOM-FIT.

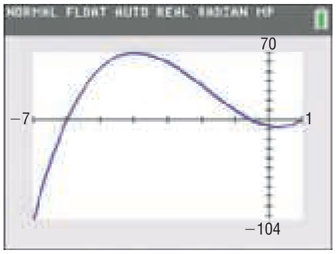


Figure 27

■

Now Work PROBLEM 45

The next example shows how to proceed when some of the coefficients of the polynomial are not integers.

EXAMPLE 10 Finding the Real Zeros of a Polynomial Function

Find all the real zeros of the polynomial function

$$f(x) = x^5 - 1.8x^4 - 17.78x^3 + 31.61x^2 + 37.9x - 8.7$$

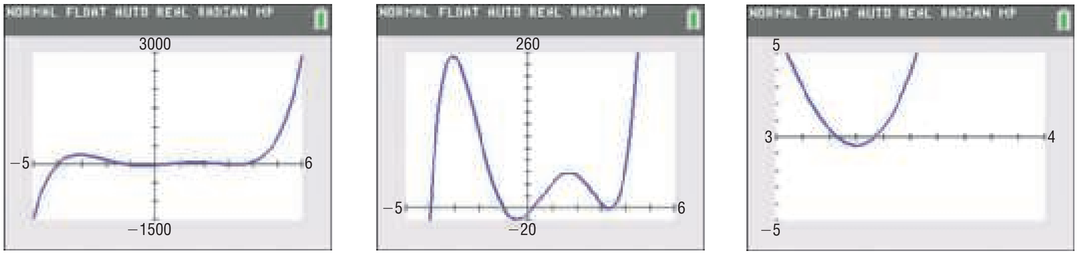
Round answers to two decimal places.

Solution **STEP 1:** There are at most five real zeros. There are three positive zeros or one positive zero, and there are two negative zeros or no negative zeros.

STEP 2: Since there are noninteger coefficients, the Rational Zeros Theorem does not apply.

STEP 3: Determine the bounds on the zeros of f. Using synthetic division with successive integers, beginning with ± 1, a lower bound is -5 and an upper bound is 6.

Every real zero of f lies between -5 and 6. Figure 28(a) shows the graph of f with Xmin $= -5$ and Xmax $= 6$. Figure 28(b) shows a graph of f after adjusting the viewing window to improve the graph.



(a) (b) (c)

Figure 28

STEP 4: From Figure 28(b), we see that f appears to have four x-intercepts: one near -4, one near -1, one between 0 and 1, and one near 3. The x-intercept near 3 might be a zero of even multiplicity since the graph seems to touch the x-axis at that point.

Using ZERO (or ROOT), the zero between 0 and 1 is found to be 0.20 (rounded to two decimal places), and the zeros -4 and -1 are confirmed. Zooming in shows that there are in fact two distinct zeros near 3. See Figure 28(c). The two remaining zeros are 3.23 and 3.37, each of which is rounded to two decimal places. ■

Now Work PROBLEM 69

7 Use the Intermediate Value Theorem

The Intermediate Value Theorem requires that the function be *continuous*. Although calculus is needed to explain the meaning precisely, we have already said that, very basically, a function f is continuous when its graph can be drawn without lifting pencil from paper, that is, when the graph contains no holes or jumps or gaps. Every polynomial function is continuous.

INTERMEDIATE VALUE THEOREM

Let f denote a continuous function. If $a < b$ and if $f(a)$ and $f(b)$ are of opposite sign, then f has at least one zero between a and b.

Although the proof of this result requires advanced methods in calculus, it is easy to "see" why the result is true. Look at Figure 29.

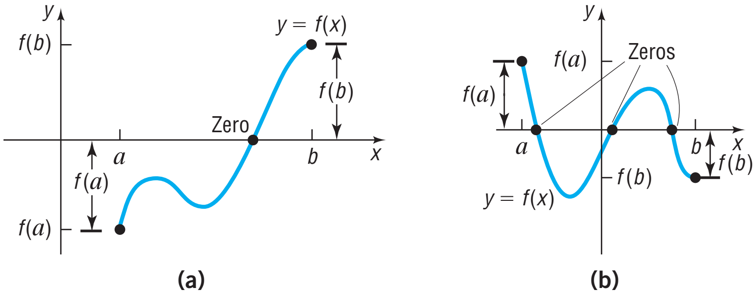


Figure 29 If $f(a)$ and $f(b)$ are of opposite sign and if f is continuous, there is at least one zero between a and b.

The Intermediate Value Theorem together with the TABLE feature of a graphing utility provides a basis for finding zeros.

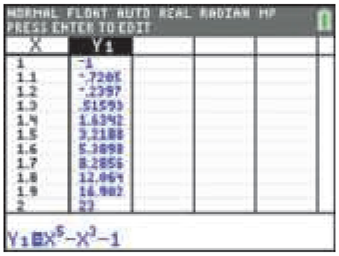
EXAMPLE 11 Using the Intermediate Value Theorem and a Graphing Utility to Locate Zeros

Find the positive zero of $f(x) = x^5 - x^3 - 1$ correct to two decimal places.

Solution Because $f(1) = -1 < 0$ and $f(2) = 23 > 0$, we know from the Intermediate Value Theorem that the zero lies between 1 and 2. Divide the interval $[1, 2]$ into 10 equal subintervals, and use the TABLE feature of a graphing utility to evaluate f at the endpoints of the intervals. See Table 8.

We can conclude that the zero is between 1.2 and 1.3 since $f(1.2) < 0$ and $f(1.3) > 0$. Now divide the interval $[1.2, 1.3]$ into 10 equal subintervals and evaluate f at each endpoint. See Table 9. The zero lies between 1.23 and 1.24, and so, correct to two decimal places, the zero is 1.23.

Table 8

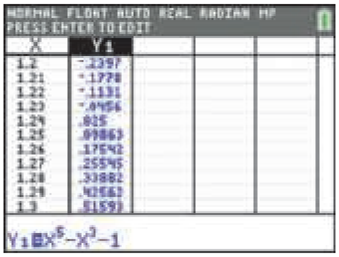


NORMAL FLOAT AUTO REAL RADIAN MP
PRESS ENTER TO EDIT

X	Y_1
1	-1
1.1	-.72049
1.2	-.23968
1.3	.51593
1.4	1.6342
1.5	3.2188
1.6	5.3898
1.7	8.2856
1.8	12.064
1.9	16.902
2	23

$Y_1 = X^5 - X^3 - 1$

Table 9



NORMAL FLOAT AUTO REAL RADIAN MP
PRESS ENTER TO EDIT

X	Y_1
1.2	-.23968
1.21	-.17782
1.22	-.11314
1.23	-.04556
1.24	.025
1.25	.09863
1.26	.17542
1.27	.25545
1.28	.33882
1.29	.42563
1.3	.51593

$Y_1 = X^5 - X^3 - 1$

Now Work PROBLEM 87

There are many other numerical techniques for approximating the zeros of a polynomial function. The one outlined in Example 11 (a variation of the *bisection method*) has the advantages that it will always work, it can be programmed rather easily on a computer, and each time it is used another decimal place of accuracy is achieved. See Problem 111 for the bisection method, which places the zero in a succession of intervals, with each new interval being half the length of the preceding one.

Historical Feature

Formulas for the solution of third- and fourth-degree polynomial equations exist, and, while not very practical, they do have an interesting history.

In the 1500s in Italy, mathematical contests were a popular pastime, and people who possessed methods for solving problems kept them secret. (Solutions that were published were already common knowledge.) Niccolo of Brescia (1499–1557), commonly referred to as Tartaglia ("the stammerer"), had the secret for solving cubic (third-degree) equations, which gave him a decided advantage in the contests. Girolamo Cardano (1501–1576) found out that Tartaglia had the secret, and, being interested in cubics, he requested it from Tartaglia. The reluctant Tartaglia hesitated for some time, but finally, swearing Cardano to secrecy with midnight oaths by candlelight, told him the secret. Cardano then published the solution in his book *Ars Magna* (1545), giving Tartaglia the credit but rather compromising the secrecy. Tartaglia exploded into bitter recriminations, and each wrote pamphlets that reflected on the other's mathematics, moral character, and ancestry.

The quartic (fourth-degree) equation was solved by Cardano's student Lodovico Ferrari, and this solution also was included, with credit and this time with permission, in the *Ars Magna*.

Attempts were made to solve the fifth-degree equation in similar ways, all of which failed. In the early 1800s, P. Ruffini, Niels Abel, and Evariste Galois all found ways to show that it is not possible to solve fifth-degree equations by formula, but the proofs required the introduction of new methods. Galois's methods eventually developed into a large part of modern algebra.

Historical Problems

Problems 1–8 develop the Tartaglia–Cardano solution of the cubic equation, and show why it is not altogether practical.

1. Show that the general cubic equation $y^3 + by^2 + cy + d = 0$ can be transformed into an equation of the form $x^3 + px + q = 0$ by using the substitution $y = x - \frac{b}{3}$.
2. In the equation $x^3 + px + q = 0$, replace x by $H + K$. Let $3HK = -p$, and show that $H^3 + K^3 = -q$.
3. Based on Problem 2, we have the two equations

$$3HK = -p \quad \text{and} \quad H^3 + K^3 = -q$$

Solve for K in $3HK = -p$ and substitute into $H^3 + K^3 = -q$. Then show that

$$H = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

[**Hint:** Look for an equation that is quadratic in form.]

4. Use the solution for H from Problem 3 and the equation $H^3 + K^3 = -q$ to show that

$$K = \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

5. Use the results from Problems 2 to 4 to show that the solution of $x^3 + px + q = 0$ is

$$x = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

6. Use the result of Problem 5 to solve the equation $x^3 - 6x - 9 = 0$.
7. Use a calculator and the result of Problem 5 to solve the equation $x^3 + 3x - 14 = 0$.
8. Use the methods of this section to solve the equation $x^3 + 3x - 14 = 0$.

5.2 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Find $f(-1)$ if $f(x) = 2x^2 - x$. (pp. 210–212)
2. Factor the expression $6x^2 + x - 2$. (pp. 55–56)
3. Find the quotient and remainder if $3x^4 - 5x^3 + 7x - 4$ is divided by $x - 3$. (pp. 45–48 or 59–62)
4. Find the zeros of $f(x) = x^2 + x - 3$. (pp. 113–115)

Concepts and Vocabulary

5. If $f(x) = q(x)g(x) + r(x)$, the function $r(x)$ is called the __________.

   (a) remainder (b) dividend (c) quotient (d) divisor
6. When a polynomial function f is divided by $x - c$, the remainder is ______.
7. Given $f(x) = 3x^4 - 2x^3 + 7x - 2$, how many sign changes are there in the coefficients of $f(-x)$?

   (a) 0 (b) 1 (c) 2 (d) 3
8. ***True or False*** Every polynomial function of degree 3 with real coefficients has exactly three real zeros.
9. If f is a polynomial function and $x - 4$ is a factor of f, then $f(4) =$ ____.
10. ***True or False*** If f is a polynomial function of degree 4 and if $f(2) = 5$, then

$$\frac{f(x)}{x-2} = p(x) + \frac{5}{x-2}$$

where $p(x)$ is a polynomial function of degree 3.

Skill Building

In Problems 11–20, use the Remainder Theorem to find the remainder when $f(x)$ is divided by $x - c$. Then use the Factor Theorem to determine whether $x - c$ is a factor of $f(x)$.

11. $f(x) = 4x^3 - 3x^2 - 8x + 4; x - 2$

12. $f(x) = -4x^3 + 5x^2 + 8; x + 3$

13. $f(x) = 3x^4 - 6x^3 - 5x + 10; x - 2$

14. $f(x) = 4x^4 - 15x^2 - 4; x - 2$

15. $f(x) = 3x^6 + 82x^3 + 27; x + 3$

16. $f(x) = 2x^6 - 18x^4 + x^2 - 9; x + 3$

17. $f(x) = 4x^6 - 64x^4 + x^2 - 15; x + 4$

18. $f(x) = x^6 - 16x^4 + x^2 - 16; x + 4$

19. $f(x) = 2x^4 - x^3 + 2x - 1; x - \frac{1}{2}$

20. $f(x) = 3x^4 + x^3 - 3x + 1; x + \frac{1}{3}$

In Problems 21–32, use Descartes' Rule of Signs to determine how many positive and how many negative zeros each polynomial function may have. Do not attempt to find the zeros.

21. $f(x) = -4x^7 + x^3 - x^2 + 2$

22. $f(x) = 5x^4 + 2x^2 - 6x - 5$

23. $f(x) = 2x^6 - 3x^2 - x + 1$

24. $f(x) = -3x^5 + 4x^4 + 2$

25. $f(x) = 3x^3 - 2x^2 + x + 2$

26. $f(x) = -x^3 - x^2 + x + 1$

27. $f(x) = -x^4 + x^2 - 1$

28. $f(x) = x^4 + 5x^3 - 2$

29. $f(x) = x^5 + x^4 + x^2 + x + 1$

30. $f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$

31. $f(x) = x^6 - 1$

32. $f(x) = x^6 + 1$

In Problems 33–44, determine the maximum number of real zeros that each polynomial function may have. Then list the potential rational zeros of each polynomial function. Do not attempt to find the zeros.

33. $f(x) = 3x^4 - 3x^3 + x^2 - x + 1$

34. $f(x) = x^5 - x^4 + 2x^2 + 3$

35. $f(x) = x^5 - 6x^2 + 9x - 3$

36. $f(x) = 2x^5 - x^4 - x^2 + 1$

37. $f(x) = -4x^3 - x^2 + x + 2$

38. $f(x) = 6x^4 - x^2 + 2$

39. $f(x) = 6x^4 - x^2 + 9$

40. $f(x) = -4x^3 + x^2 + x + 6$

41. $f(x) = 2x^5 - x^3 + 2x^2 + 12$

42. $f(x) = 3x^5 - x^2 + 2x + 18$

43. $f(x) = 6x^4 + 2x^3 - x^2 + 20$

44. $f(x) = -6x^3 - x^2 + x + 10$

In Problems 45–50, find the bounds to the zeros of each polynomial function. Use the bounds to obtain a complete graph of f.

45. $f(x) = 2x^3 + x^2 - 1$

46. $f(x) = 3x^3 - 2x^2 + x + 4$

47. $f(x) = x^3 - 5x^2 - 11x + 11$

48. $f(x) = 2x^3 - x^2 - 11x - 6$

49. $f(x) = x^4 + 3x^3 - 5x^2 + 9$

50. $f(x) = 4x^4 - 12x^3 + 27x^2 - 54x + 81$

In Problems 51–68, find the real zeros of f. Use the real zeros to factor f.

51. $f(x) = x^3 + 2x^2 - 5x - 6$

52. $f(x) = x^3 + 8x^2 + 11x - 20$

53. $f(x) = 2x^3 - 13x^2 + 24x - 9$

54. $f(x) = 2x^3 - 5x^2 - 4x + 12$

55. $f(x) = 3x^3 + 4x^2 + 4x + 1$

56. $f(x) = 3x^3 - 7x^2 + 12x - 28$

57. $f(x) = x^3 - 10x^2 + 28x - 16$

58. $f(x) = x^3 + 6x^2 + 6x - 4$

59. $f(x) = x^4 + x^3 - 3x^2 - x + 2$

60. $f(x) = x^4 - x^3 - 6x^2 + 4x + 8$

61. $f(x) = 21x^4 + 22x^3 - 99x^2 - 72x + 28$

62. $f(x) = 54x^4 - 57x^3 - 323x^2 + 20x + 12$

63. $f(x) = x^3 - 8x^2 + 17x - 6$

64. $f(x) = 2x^4 + 11x^3 - 5x^2 - 43x + 35$

65. $f(x) = 4x^4 + 7x^2 - 2$

66. $f(x) = 4x^4 + 15x^2 - 4$

67. $f(x) = 4x^5 - 8x^4 - x + 2$

68. $f(x) = 4x^5 + 12x^4 - x - 3$

In Problems 69–74, find the real zeros of f. If necessary, round to two decimal places.

69. $f(x) = x^3 + 3.2x^2 - 16.83x - 5.31$

70. $f(x) = x^3 + 3.2x^2 - 7.25x - 6.3$

71. $f(x) = x^4 - 1.4x^3 - 33.71x^2 + 23.94x + 292.41$

72. $f(x) = x^4 + 1.2x^3 - 7.46x^2 - 4.692x + 15.2881$

73. $f(x) = x^3 + 19.5x^2 - 1021x + 1000.5$

74. $f(x) = x^3 + 42.2x^2 - 664.8x + 1490.4$

In Problems 75–84, find the real solutions of each equation.

75. $x^4 - x^3 + 2x^2 - 4x - 8 = 0$

76. $2x^3 + 3x^2 + 2x + 3 = 0$

77. $3x^3 + 4x^2 - 7x + 2 = 0$

78. $2x^3 - 3x^2 - 3x - 5 = 0$

79. $3x^3 - x^2 - 15x + 5 = 0$

80. $2x^3 - 11x^2 + 10x + 8 = 0$

81. $x^4 + 4x^3 + 2x^2 - x + 6 = 0$

82. $x^4 - 2x^3 + 10x^2 - 18x + 9 = 0$

83. $x^3 - \frac{2}{3}x^2 + \frac{8}{3}x + 1 = 0$

84. $x^3 - \frac{2}{3}x^2 + 3x - 2 = 0$

In Problems 85–90, use the Intermediate Value Theorem to show that each function has a zero in the given interval. Approximate the zero correct to two decimal places.

85. $f(x) = 8x^4 - 2x^2 + 5x - 1;\quad [0, 1]$

86. $f(x) = x^4 + 8x^3 - x^2 + 2;\quad [-1, 0]$

87. $f(x) = 2x^3 + 6x^2 - 8x + 2;\quad [-5, -4]$

88. $f(x) = 3x^3 - 10x + 9;\quad [-3, -2]$

89. $f(x) = x^5 - x^4 + 7x^3 - 7x^2 - 18x + 18;\quad [1.4, 1.5]$

90. $f(x) = x^5 - 3x^4 - 2x^3 + 6x^2 + x + 2;\quad [1.7, 1.8]$

Mixed Practice

In Problems 91–98, analyze each polynomial function using Steps 1 through 8 on page 343 in Section 5.1.

91. $f(x) = x^3 + 2x^2 - 5x - 6$
[**Hint:** See Problem 51.]

92. $f(x) = x^3 + 8x^2 + 11x - 20$
[**Hint:** See Problem 52.]

93. $f(x) = x^4 + x^3 - 3x^2 - x + 2$
[**Hint:** See Problem 59.]

94. $f(x) = x^4 - x^3 - 6x^2 + 4x + 8$
[**Hint:** See Problem 60.]

95. $f(x) = 4x^5 - 8x^4 - x + 2$
[**Hint:** See Problem 67.]

96. $f(x) = 4x^5 + 12x^4 - x - 3$
[**Hint:** See Problem 68.]

97. $f(x) = 6x^4 - 37x^3 + 58x^2 + 3x - 18$

98. $f(x) = 20x^4 + 73x^3 + 46x^2 - 52x - 24$

Applications and Extensions

99. Find k such that $f(x) = x^3 - kx^2 + kx + 2$ has the factor $x - 2$.

100. Find k such that $f(x) = x^4 - kx^3 + kx^2 + 1$ has the factor $x + 2$.

101. What is the remainder when $f(x) = 2x^{20} - 8x^{10} + x - 2$ is divided by $x - 1$?

102. What is the remainder when $f(x) = -3x^{17} + x^9 - x^5 + 2x$ is divided by $x + 1$?

103. Use the Factor Theorem to prove that $x - c$ is a factor of $x^n - c^n$ for any positive integer n.

104. Use the Factor Theorem to prove that $x + c$ is a factor of $x^n + c^n$ if $n \geq 1$ is an odd integer.

105. One solution of the equation $x^3 - 8x^2 + 16x - 3 = 0$ is 3. Find the sum of the remaining solutions.

106. One solution of the equation $x^3 + 5x^2 + 5x - 2 = 0$ is -2. Find the sum of the remaining solutions.

107. Geometry What is the length of the edge of a cube if, after a slice 1-inch thick is cut from one side, the volume remaining is 294 cubic inches?

108. Geometry What is the length of the edge of a cube if its volume could be doubled by an increase of 6 centimeters in one edge, an increase of 12 centimeters in a second edge, and a decrease of 4 centimeters in the third edge?

109. Let $f(x)$ be a polynomial function whose coefficients are integers. Suppose that r is a real zero of f and that the leading coefficient of f is 1. Use the Rational Zeros Theorem to show that r is either an integer or an irrational number.

110. Prove the Rational Zeros Theorem.

[**Hint:** Let $\frac{p}{q}$, where p and q have no common factors except 1 and -1, be a zero of the polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

whose coefficients are all integers. Show that

$$a_n p^n + a_{n-1}p^{n-1}q + \cdots + a_1 pq^{n-1} + a_0 q^n = 0$$

Now, because p is a factor of the first n terms of this equation, p must also be a factor of the term $a_0 q^n$. Since p is not a factor of q (why?), p must be a factor of a_0. Similarly, q must be a factor of a_n.]

111. Bisection Method for Approximating Zeros of a Function *f* We begin with two consecutive integers, a and $a + 1$, such that $f(a)$ and $f(a + 1)$ are of opposite sign. Evaluate f at the midpoint m_1 of a and $a + 1$. If $f(m_1) = 0$, then m_1 is the zero of f, and we are finished. Otherwise, $f(m_1)$ is of opposite sign to either $f(a)$ or $f(a + 1)$. Suppose that it is

$f(a)$ and $f(m_1)$ that are of opposite sign. Now evaluate f at the midpoint m_2 of a and m_1. Repeat this process until the desired degree of accuracy is obtained. Note that each iteration places the zero in an interval whose length is half that of the previous interval. Use the bisection method to approximate the zero of $f(x) = 8x^4 - 2x^2 + 5x - 1$ in the interval $[0, 1]$ correct to three decimal places. Verify your result using a graphing utility.

[**Hint:** The process ends when both endpoints agree to the desired number of decimal places.]

Discussion and Writing

112. Is $\frac{1}{3}$ a zero of $f(x) = 2x^3 + 3x^2 - 6x + 7$? Explain.

113. Is $\frac{1}{3}$ a zero of $f(x) = 4x^3 - 5x^2 - 3x + 1$? Explain.

114. Is $\frac{3}{5}$ a zero of $f(x) = 2x^6 - 5x^4 + x^3 - x + 1$? Explain.

115. Is $\frac{2}{3}$ a zero of $f(x) = x^7 + 6x^5 - x^4 + x + 2$? Explain.

Retain Your Knowledge

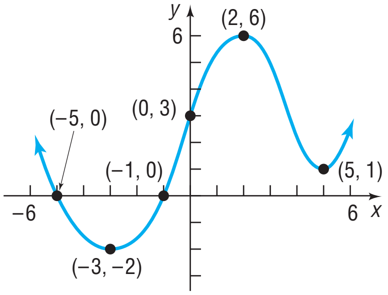
Problems 116–119 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

116. Solve $2x - 5y = 3$ for y.

117. Express the inequality $3 \leq x < 8$ using interval notation.

118. Find the intercepts of the graph of the equation $3x + y^2 = 12$.

119. Use the figure to determine the interval(s) on which the function is increasing.



'Are You Prepared?' Answers

1. 3

2. $(3x + 2)(2x - 1)$

3. Quotient: $3x^3 + 4x^2 + 12x + 43$; remainder: 125

4. $\left\{\frac{-1 - \sqrt{13}}{2}, \frac{-1 + \sqrt{13}}{2}\right\}$

5.3 Complex Zeros; Fundamental Theorem of Algebra

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Complex Numbers (Section 1.4, pp. 121–125)
- Complex Solutions of a Quadratic Equation (Section 1.4, pp. 125–127)

Now Work the **'Are You Prepared?'** problems on page 370.

OBJECTIVES
1 Use the Conjugate Pairs Theorem (p. 367)
2 Find a Polynomial Function with Specified Zeros (p. 368)
3 Find the Complex Zeros of a Polynomial Function (p. 369)

In Section 1.3, we found the real solutions of a quadratic equation. That is, we found the real zeros of a polynomial function of degree 2. Then, in Section 1.4 we found the complex solutions of a quadratic equation. That is, we found the complex zeros of a polynomial function of degree 2.

In Section 5.2, we found the real zeros of polynomial functions of degree 3 or higher. In this section we will find the *complex zeros* of polynomial functions of degree 3 or higher.

DEFINITION

A variable in the complex number system is referred to as a **complex variable**. A **complex polynomial function** f of degree n is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad \textbf{(1)}$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are complex numbers, $a_n \neq 0$, n is a nonnegative integer, and x is a complex variable. As before, a_n is called the **leading coefficient** of f. A complex number r is called a **complex zero** of f if $f(r) = 0$.

In most of our work the coefficients in (1) will be real numbers.

We have learned that some quadratic equations have no real solutions, but that in the complex number system every quadratic equation has a solution, either real or complex. The next result, proved by Karl Friedrich Gauss (1777–1855) when he was 22 years old,* extends this idea to polynomial equations of degree 3 or higher. In fact, this result is so important and useful that it has become known as the **Fundamental Theorem of Algebra**.

FUNDAMENTAL THEOREM OF ALGEBRA

Every complex polynomial function $f(x)$ of degree $n \geq 1$ has at least one complex zero.

We shall not prove this result, as the proof is beyond the scope of this book. However, using the Fundamental Theorem of Algebra and the Factor Theorem, we can prove the following result:

THEOREM

Every complex polynomial function $f(x)$ of degree $n \geq 1$ can be factored into n linear factors (not necessarily distinct) of the form

$$f(x) = a_n(x - r_1)(x - r_2) \cdot \cdots \cdot (x - r_n) \qquad \textbf{(2)}$$

where $a_n, r_1, r_2, \ldots, r_n$ are complex numbers. That is, every complex polynomial function of degree $n \geq 1$ has exactly n complex zeros, some of which may repeat.

Proof Let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

By the Fundamental Theorem of Algebra, f has at least one zero, say r_1. Then, by the Factor Theorem, $x - r_1$ is a factor, and

$$f(x) = (x - r_1)q_1(x)$$

where $q_1(x)$ is a complex polynomial function of degree $n - 1$ whose leading coefficient is a_n. Repeating this argument n times, we arrive at

$$f(x) = (x - r_1)(x - r_2) \cdot \cdots \cdot (x - r_n)q_n(x)$$

where $q_n(x)$ is a complex polynomial function of degree $n - n = 0$ whose leading coefficient is a_n. That is, $q_n(x) = a_n x^0 = a_n$, and so

$$f(x) = a_n(x - r_1)(x - r_2) \cdot \cdots \cdot (x - r_n)$$

We conclude that every complex polynomial function $f(x)$ of degree $n \geq 1$ has exactly n (not necessarily distinct) zeros. ■

1 Use the Conjugate Pairs Theorem

The Fundamental Theorem of Algebra can be used to obtain valuable information about the complex zeros of polynomial functions whose coefficients are real numbers.

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*In all, Gauss gave four different proofs of this theorem, the first one in 1799 being the subject of his doctoral dissertation.

CONJUGATE PAIRS THEOREM

Let $f(x)$ be a polynomial function whose coefficients are real numbers. If $r = a + bi$ is a zero of f, the complex conjugate $\overline{r} = a - bi$ is also a zero of f.

In other words, for polynomial functions whose coefficients are real numbers, the complex zeros occur in conjugate pairs. This result should not be all that surprising since the complex zeros of a quadratic function occurred in conjugate pairs.

Proof Let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers and $a_n \neq 0$. If $r = a + bi$ is a zero of f, then $f(r) = f(a + bi) = 0$, so

$$a_n r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0$$

Take the conjugate of both sides to get

$$\overline{a_n r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0} = \overline{0}$$

$$\overline{a_n r^n} + \overline{a_{n-1}r^{n-1}} + \cdots + \overline{a_1 r} + \overline{a_0} = \overline{0}$$

The conjugate of a sum equals the sum of the conjugates (see Section 1.4).

$$\overline{a_n}(\overline{r})^n + \overline{a_{n-1}}(\overline{r})^{n-1} + \cdots + \overline{a_1}\,\overline{r} + \overline{a_0} = \overline{0}$$

The conjugate of a product equals the product of the conjugates.

$$a_n(\overline{r})^n + a_{n-1}(\overline{r})^{n-1} + \cdots + a_1\overline{r} + a_0 = 0$$

The conjugate of a real number equals the real number.

This last equation states that $f(\overline{r}) = 0$; that is, $\overline{r} = a - bi$ is a zero of f. ■

The importance of this result should be clear. Once we know that, say, $3 + 4i$ is a zero of a polynomial function with real coefficients, then we know that $3 - 4i$ is also a zero. This result has an important corollary.

COROLLARY

A polynomial function f of odd degree with real coefficients has at least one real zero.

Proof Because complex zeros occur as conjugate pairs in a polynomial function with real coefficients, there will always be an even number of zeros that are not real numbers. Consequently, since f is of odd degree, one of its zeros has to be a real number. ■

For example, the polynomial function $f(x) = x^5 - 3x^4 + 4x^3 - 5$ has at least one zero that is a real number, since f is of degree 5 (odd) and has real coefficients.

EXAMPLE 1 Using the Conjugate Pairs Theorem

A polynomial function f of degree 5 whose coefficients are real numbers has the zeros $1, 5i$, and $1 + i$. Find the remaining two zeros.

Solution Since f has coefficients that are real numbers, complex zeros appear as conjugate pairs. It follows that $-5i$, the conjugate of $5i$, and $1 - i$, the conjugate of $1 + i$, are the two remaining zeros. ■

Now Work PROBLEM 7

2 Find a Polynomial Function with Specified Zeros

EXAMPLE 2 Finding a Polynomial Function Whose Zeros Are Given

(a) Find a polynomial function f of degree 4 whose coefficients are real numbers and that has the zeros $1, 1$, and $-4 + i$.

(b) Graph the function found in part (a) to verify your result.

Solution (a) Since $-4 + i$ is a zero, by the Conjugate Pairs Theorem, $-4 - i$ must also be a zero of f. Because of the Factor Theorem, if $f(c) = 0$, then $x - c$ is a factor of $f(x)$. So f can now be written as

$$f(x) = a(x-1)(x-1)[x-(-4+i)][x-(-4-i)]$$

where a is any real number. Letting $a = 1$, we obtain

$$\begin{aligned} f(x) &= (x-1)(x-1)[x-(-4+i)][x-(-4-i)] \\ &= (x^2-2x+1)[(x+4)-i][(x+4)+i] \\ &= (x^2-2x+1)((x+4)^2-i^2) \\ &= (x^2-2x+1)(x^2+8x+16-(-1)) \\ &= (x^2-2x+1)(x^2+8x+17) \\ &= x^4+8x^3+17x^2-2x^3-16x^2-34x+x^2+8x+17 \\ &= x^4+6x^3+2x^2-26x+17 \end{aligned}$$

(b) A quick analysis of the polynomial function f tells us what to expect:

At most three turning points
For large $|x|$, the graph behaves like $y = x^4$.
A repeated real zero at 1 so the graph touches the x-axis at 1
The only x-intercept is at 1. The y-intercept is 17.

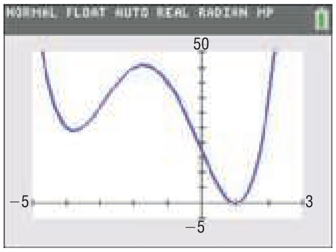


Figure 30
$f(x) = x^4 + 6x^3 + 2x^2 - 26x + 17$

Figure 30 shows the complete graph. (Do you see why? The graph has exactly three turning points and the degree of the polynomial function is 4.) ■

Now Work PROBLEM 17

Now we can prove the theorem we conjectured earlier in Section 5.2.

THEOREM

Every polynomial function with real coefficients can be uniquely factored over the real numbers into a product of linear factors and/or irreducible quadratic factors. ■

Exploration

Graph the function found in Example 2 for $a = 2$ and $a = -1$. Does the value of a affect the zeros of f? How does the value of a affect the graph of f? ■

Proof Every complex polynomial function f of degree n has exactly n zeros and can be factored into a product of n linear factors. If its coefficients are real, then those zeros that are complex numbers always occur as conjugate pairs. As a result, if $r = a + bi$ is a complex zero, then so is $\bar{r} = a - bi$. Consequently, when the linear factors $x - r$ and $x - \bar{r}$ of f are multiplied, we have

$$(x-r)(x-\bar{r}) = x^2 - (r+\bar{r})x + r\bar{r} = x^2 - 2ax + a^2 + b^2$$

This second-degree polynomial has real coefficients and is irreducible (over the real numbers). Thus, the factors of f are either linear or irreducible quadratic factors. ■

3 Find the Complex Zeros of a Polynomial Function

The steps for finding the complex zeros of a polynomial function are the same as those for finding the real zeros.

EXAMPLE 3 Finding the Complex Zeros of a Polynomial Function

Find the complex zeros of the polynomial function

$$f(x) = 3x^4 + 5x^3 + 25x^2 + 45x - 18$$

Write f in factored form.

Solution **STEP 1:** The degree of f is 4. So f has four complex zeros. From Descartes' Rule of Signs, there is one positive real zero. Also, since $f(-x) = 3x^4 - 5x^3 + 25x^2 - 45x - 18$, there are three negative real zeros or one negative real zero.

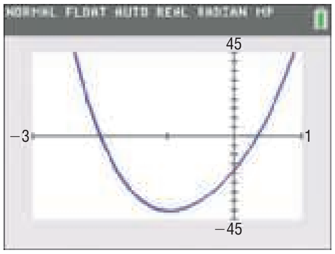


Figure 31
$f(x) = 3x^4 + 5x^3 + 25x^2 + 45x - 18$

STEP 2: The Rational Zeros Theorem provides information about the potential rational zeros of polynomial functions with integer coefficients. For this polynomial function (which has integer coefficients), the potential rational zeros are

$$\pm\frac{1}{3}, \pm\frac{2}{3}, \pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$$

STEP 3: Figure 31 shows the graph of f. The graph has the characteristics expected of this polynomial function of degree 4: It behaves like $y = 3x^4$ for large $|x|$ and has y-intercept -18. There are x-intercepts near -2 and between 0 and 1.

STEP 4: Because $f(-2) = 0$, we know that -2 is a zero of f and $x - (-2) = x + 2$ is a factor of f. Use long division or synthetic division to factor f. Using synthetic division,

$$\begin{array}{r|rrrrr} -2 & 3 & 5 & 25 & 45 & -18 \\ & & -6 & 2 & -54 & 18 \\ \hline & 3 & -1 & 27 & -9 & 0 \end{array}$$

So $f(x) = (x + 2)(3x^3 - x^2 + 27x - 9)$. The depressed equation is

$$q_1(x) = 3x^3 - x^2 + 27x - 9 = 0$$

Repeat Step 4: The depressed equation $3x^3 - x^2 + 27x - 9 = 0$ can be factored by grouping.

$$3x^3 - x^2 + 27x - 9 = 0$$

$$x^2(3x - 1) + 9(3x - 1) = 0 \qquad \text{Factor } x^2 \text{ from } 3x^3 - x^2 \text{ and 9 from } 27x - 9.$$

$$(x^2 + 9)(3x - 1) = 0 \qquad \text{Factor out the common factor } 3x - 1.$$

$$x^2 + 9 = 0 \quad \text{or} \quad 3x - 1 = 0 \qquad \text{Apply the Zero-Product Property.}$$

$$x^2 = -9 \quad \text{or} \quad 3x = 1$$

$$x = -3i, \quad x = 3i \quad \text{or} \quad x = \frac{1}{3}$$

The four complex zeros of f are $-3i$, $3i$, -2, and $\frac{1}{3}$.

The factored form of f is

$$\begin{aligned} f(x) &= 3x^4 + 5x^3 + 25x^2 + 45x - 18 \\ &= (x + 3i)(x - 3i)(x + 2)(3x - 1) \\ &= 3(x + 3i)(x - 3i)(x + 2)\left(x - \frac{1}{3}\right) \end{aligned}$$

■

Now Work PROBLEM 33

5.3 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Find the sum and the product of the complex numbers $3 - 2i$ and $-3 + 5i$. (pp. 122–123)

2. In the complex number system, find the complex solutions of the equation $x^2 + 2x + 2 = 0$. (pp. 125–127)

Concepts and Vocabulary

3. Every polynomial function of odd degree with real coefficients has at least _____ real zero(s).

4. If $3 + 4i$ is a zero of a polynomial function of degree 5 with real coefficients, then so is ________.

5. ***True or False*** A polynomial function of degree n with real coefficients has exactly n complex zeros. At most n of them are real zeros.

6. ***True or False*** A polynomial function of degree 4 with real coefficients could have $-3, 2 + i, 2 - i$, and $-3 + 5i$ as its zeros.

Skill Building

In Problems 7–16, information is given about a polynomial function $f(x)$ whose coefficients are real numbers. Find the remaining zeros of f.

7. Degree 3; zeros: $3, 4 - i$

8. Degree 3; zeros: $4, 3 + i$

9. Degree 4; zeros: $i, 1 + i$

10. Degree 4; zeros: $1, 2, 2 + i$

11. Degree 5; zeros: $1, i, 2i$

12. Degree 5; zeros: $0, 1, 2, i$

13. Degree 4; zeros: $i, 2, -2$

14. Degree 4; zeros: $2 - i, -i$

15. Degree 6; zeros: $2, 2 + i, -3 - i, 0$

16. Degree 6; zeros: $i, 3 - 2i, -2 + i$

In Problems 17–22, form a polynomial function f(x) with real coefficients having the given degree and zeros. Answers will vary depending on the choice of the leading coefficient. Use a graphing utility to graph the function and verify the result.

17. Degree 4; zeros: $3 + 2i$; 4, multiplicity 2

18. Degree 4; zeros: i; $1 + 2i$

19. Degree 5; zeros: 2; $-i$; $1 + i$

20. Degree 6; zeros: i; $4 - i$; $2 + i$

21. Degree 4; zeros: 3, multiplicity 2; $-i$

22. Degree 5; zeros: 1, multiplicity 3; $1 + i$

In Problems 23–30, use the given zero to find the remaining zeros of each function.

23. $f(x) = x^3 - 4x^2 + 4x - 16$; zero: $2i$

24. $g(x) = x^3 + 3x^2 + 25x + 75$; zero: $-5i$

25. $f(x) = 2x^4 + 5x^3 + 5x^2 + 20x - 12$; zero: $-2i$

26. $h(x) = 3x^4 + 5x^3 + 25x^2 + 45x - 18$; zero: $3i$

27. $h(x) = x^4 - 9x^3 + 21x^2 + 21x - 130$; zero: $3 - 2i$

28. $f(x) = x^4 - 7x^3 + 14x^2 - 38x - 60$; zero: $1 + 3i$

29. $h(x) = 3x^5 + 2x^4 + 15x^3 + 10x^2 - 528x - 352$; zero: $-4i$

30. $g(x) = 2x^5 - 3x^4 - 5x^3 - 15x^2 - 207x + 108$; zero: $3i$

In Problems 31–40, find the complex zeros of each polynomial function. Write f in factored form.

31. $f(x) = x^3 - 1$

32. $f(x) = x^4 - 1$

33. $f(x) = x^3 - 8x^2 + 25x - 26$

34. $f(x) = x^3 + 13x^2 + 57x + 85$

35. $f(x) = x^4 + 5x^2 + 4$

36. $f(x) = x^4 + 13x^2 + 36$

37. $f(x) = x^4 + 2x^3 + 22x^2 + 50x - 75$

38. $f(x) = x^4 + 3x^3 - 19x^2 + 27x - 252$

39. $f(x) = 3x^4 - x^3 - 9x^2 + 159x - 52$

40. $f(x) = 2x^4 + x^3 - 35x^2 - 113x + 65$

Mixed Practice

41. Given $f(x) = 2x^3 - 14x^2 + bx - 3$ with $f(2) = 0$, $g(x) = x^3 + cx^2 - 8x + 30$, with the zero $x = 3 - i$, and b and c real numbers, find $(f \cdot g)(1)$.†

42. Let f be the polynomial function of degree 4 with real coefficients, leading coefficient 1, and zeros $x = 3 + i, 2, -2$. Let g be the polynomial function of degree 4 with intercept $(0, -4)$ and zeros $x = i, 2i$. Find $(f + g)(1)$.†

43. The complex zeros of $f(x) = x^4 + 1$ For the function $f(x) = x^4 + 1$:

(a) Factor f into the product of two irreducible quadratics. (***Hint:*** Complete the square by adding and subtracting $2x^2$.)
(b) Find the zeros of f by finding the zeros of each irreducible quadratic.

† Courtesy of the Joliet Junior College Mathematics Department

Discussion and Writing

In Problems 44 and 45, explain why the facts given are contradictory.

44. f is a polynomial function of degree 3 whose coefficients are real numbers; its zeros are $2, i$, and $3 + i$.

45. f is a polynomial function of degree 3 whose coefficients are real numbers; its zeros are $4 + i, 4 - i$, and $2 + i$.

46. f is a polynomial function of degree 4 whose coefficients are real numbers; two of its zeros are -3 and $4 - i$. Explain why one of the remaining zeros must be a real number. Write down one of the missing zeros.

47. f is a polynomial function of degree 4 whose coefficients are real numbers; three of its zeros are $2, 1 + 2i$, and $1 - 2i$. Explain why the remaining zero must be a real number.

48. For the polynomial function $f(x) = x^2 + 2ix - 10$:

(a) Verify that $3 - i$ is a zero of f.

(b) Verify that $3 + i$ is not a zero of f.

(c) Explain why these results do not contradict the Conjugate Pairs Theorem.

Retain Your Knowledge

Problems 49–52 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

49. Draw a scatter diagram for the given data.

x	−1	1	2	5	8	10
y	−4	0	3	1	5	7

50. Solve: $\sqrt{3 - x} = 5$

51. Multiply: $(2x - 5)(3x^2 + x - 4)$

52. Find the area and circumference of a circle with a diameter of 6 feet.

'Are You Prepared?' Answers

1. Sum: $3i$; product: $1 + 21i$

2. $-1 - i, \quad -1 + i$

5.4 Properties of Rational Functions

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Rational Expressions (Chapter R, Section R.7, pp. 63–71)
- Polynomial Division (Chapter R, Section R.4, pp. 45–48)
- Graph of $f(x) = \frac{1}{x}$ (Section 2.1, Example 6, pp. 169–170)
- Graphing Techniques: Transformations (Section 3.5, pp. 256–264)

Now Work the **'Are You Prepared?'** problems on page 379.

OBJECTIVES
1 Find the Domain of a Rational Function (p. 372)
2 Find the Vertical Asymptotes of a Rational Function (p. 376)
3 Find the Horizontal or Oblique Asymptote of a Rational Function (p. 377)

Ratios of integers are called *rational numbers*. Similarly, ratios of polynomial functions are called *rational functions*. Examples of rational functions are

$$R(x) = \frac{x^2 - 4}{x^2 + x + 1} \qquad F(x) = \frac{x^3}{x^2 - 4} \qquad G(x) = \frac{3x^2}{x^4 - 1}$$

DEFINITION

A **rational function** is a function of the form

$$R(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomial functions and q is not the zero polynomial. The domain of a rational function is the set of all real numbers except those for which the denominator q is 0. ■

1 Find the Domain of a Rational Function

EXAMPLE 1 Finding the Domain of a Rational Function

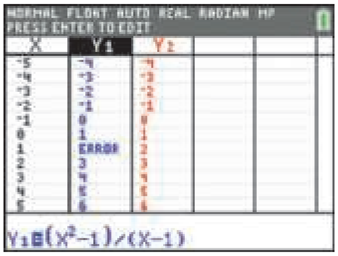
(a) The domain of $R(x) = \frac{2x^2 - 4}{x + 5}$ is the set of all real numbers x except -5; that is, the domain is $\{x \mid x \neq -5\}$.

(b) The domain of $R(x) = \frac{1}{x^2 - 4}$ is the set of all real numbers x except -2 and 2; that is, the domain is $\{x \mid x \neq -2, x \neq 2\}$.

(c) The domain of $R(x) = \frac{x^3}{x^2 + 1}$ is the set of all real numbers.

(d) The domain of $R(x) = \frac{x^2 - 1}{x - 1}$ is the set of all real numbers x except 1; that is, the domain is $\{x \mid x \neq 1\}$. ■

Table 10



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X	Y₁	Y₂
-5	-4	-4
-4	-3	-3
-3	-2	-2
-2	-1	-1
-1	0	0
0	1	1
1	ERROR	2
2	3	3
3	4	4
4	5	5
5	6	6

Y₁■(X²−1)/(X−1)

Although $\frac{x^2 - 1}{x - 1}$ simplifies to $x + 1$, it is important to observe that the functions

$$R(x) = \frac{x^2 - 1}{x - 1} \quad \text{and} \quad f(x) = x + 1$$

are not equal, since the domain of R is $\{x \mid x \neq 1\}$ and the domain of f is the set of all real numbers. Notice in Table 10 there is an error message for $Y_1 = R(x) = \frac{x^2 - 1}{x - 1}$, but there is no error message for $Y_2 = f(x) = x + 1$.

Now Work PROBLEM 17

If $R(x) = \frac{p(x)}{q(x)}$ is a rational function and if p and q have no common factors, then the rational function R is said to be in **lowest terms**. For a rational function $R(x) = \frac{p(x)}{q(x)}$ in lowest terms, the real zeros, if any, of the numerator in the domain of R are the x-intercepts of the graph of R and so play a major role in the graph of R. The real zeros of the denominator of R [that is, the numbers x, if any, for which $q(x) = 0$], although not in the domain of R, also play a major role in the graph of R.

We have already discussed the properties of the rational function $y = \frac{1}{x}$. (Refer to Example 6, page 169.) The next rational function that we take up is $H(x) = \frac{1}{x^2}$.

EXAMPLE 2 Graphing $y = \frac{1}{x^2}$

Analyze the graph of $H(x) = \frac{1}{x^2}$.

Solution The domain of $H(x) = \frac{1}{x^2}$ consists of all real numbers x except 0. The graph has no y-intercept, since x can never equal 0. The graph has no x-intercept because the equation $H(x) = 0$ has no solution. Therefore, the graph of H will not cross either coordinate axis.

Because

$$H(-x) = \frac{1}{(-x)^2} = \frac{1}{x^2} = H(x)$$

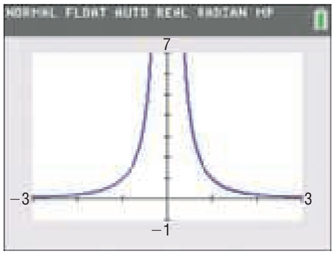
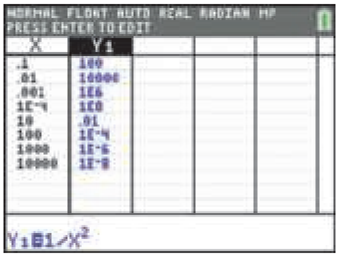


Figure 32 $Y_1 = \frac{1}{x^2}$

H is an even function, so its graph is symmetric with respect to the y-axis.

See Figure 32. Notice that the graph confirms the conclusions just reached. But what happens to the graph as the values of x get closer and closer to 0? We use a TABLE to answer the question. See Table 11 on the following page. The first four rows show that as the values of x approach (get closer to) 0, the values of $H(x)$ become larger and larger positive numbers. When this happens, we say that $H(x)$ is **unbounded in the positive direction**. We symbolize this by writing $H(x) \to \infty$ [read as "$H(x)$ **approaches infinity**"]. In calculus, we use limit notation, $\lim_{x \to 0} H(x) = \infty$, read as "the limit of $H(x)$ as x approaches 0 is infinity," to mean that $H(x) \to \infty$ as $x \to 0$.

Table 11



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X	Y1
.1	100
.01	10000
.001	1E6
1E-4	1E8
10	.01
100	1E-4
1000	1E-6
10000	1E-8

$Y_1 = 1/X^2$

Look at the last four rows of Table 11. As $x \to \infty$, the values of $H(x)$ approach 0 (the end behavior of the graph). This is expressed in calculus by writing $\lim_{x\to\infty} H(x) = 0$. Remember, on the calculator 1E−4 means 1×10^{-4} or 0.0001.

Figure 33 shows the graph of $H(x) = \dfrac{1}{x^2}$ drawn by hand. Notice the use of red dashed lines to convey the ideas discussed.

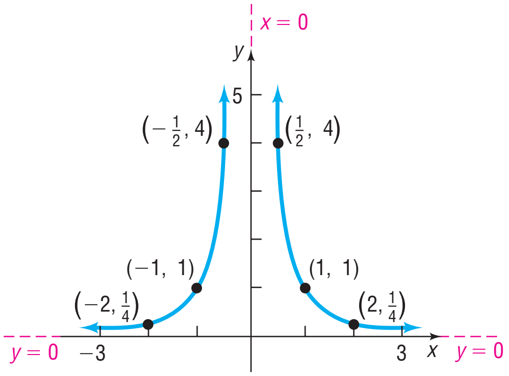


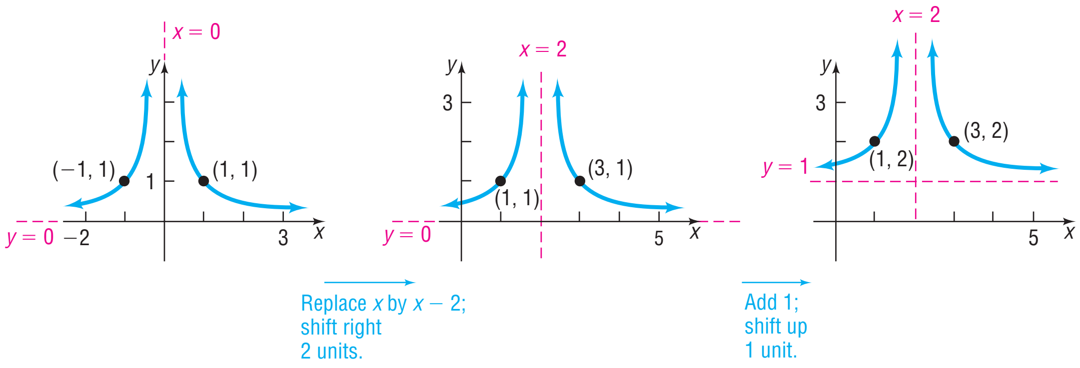
Figure 33 $H(x) = \dfrac{1}{x^2}$

■

EXAMPLE 3 Using Transformations to Graph a Rational Function

Graph the rational function: $R(x) = \dfrac{1}{(x-2)^2} + 1$

Solution The domain of R is the set of all real numbers except $x = 2$. To graph R, start with the graph of $y = \dfrac{1}{x^2}$. See Figure 34 for the stages.



(a) $y = \dfrac{1}{x^2}$ **(b)** $y = \dfrac{1}{(x-2)^2}$ **(c)** $y = \dfrac{1}{(x-2)^2} + 1$

Figure 34

✓**Check:** Graph $Y_1 = \dfrac{1}{(x-2)^2} + 1$ using a graphing utility to verify the graph obtained in Figure 34(c). ■

Now Work PROBLEM 35

Asymptotes

Notice that the y-axis in Figure 34(a) is transformed into the vertical line $x = 2$ in Figure 34(c), and the x-axis in Figure 34(a) is transformed into the horizontal line $y = 1$ in Figure 34(c). The **Exploration** that follows will help us analyze the role of these lines.

Exploration

(a) Using a graphing utility and the TABLE feature, evaluate the function $H(x) = \dfrac{1}{(x-2)^2} + 1$ at $x = 10$, 100, 1000, and 10,000. What happens to the values of H as x becomes unbounded in the positive direction, expressed as $\lim_{x\to\infty} H(x)$?

(b) Evaluate H at $x = -10$, -100, -1000, and $-10{,}000$. What happens to the values of H as x becomes unbounded in the negative direction, expressed as $\lim_{x\to-\infty} H(x)$?

(c) Evaluate H at $x = 1.5, 1.9, 1.99, 1.999$, and 1.9999. What happens to the values of H as x approaches 2, $x < 2$, expressed as $\lim_{x\to2^-} H(x)$?

(d) Evaluate H at $x = 2.5, 2.1, 2.01, 2.001$, and 2.0001. What happens to the values of H as x approaches 2, $x > 2$, expressed as $\lim_{x\to2^+} H(x)$?

Result

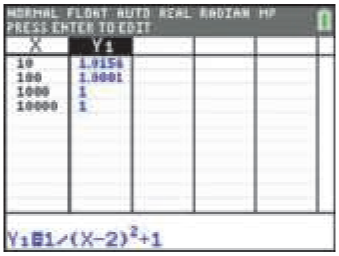
(a) Table 12 shows the values of $Y_1 = H(x)$ as x approaches ∞. Notice that the values of H are approaching 1, so $\lim_{x\to\infty} H(x) = 1$.

(b) Table 13 shows the values of $Y_1 = H(x)$ as x approaches $-\infty$. Again the values of H are approaching 1, so $\lim_{x\to-\infty} H(x) = 1$.

(c) From Table 14 we see that, as x approaches 2, $x < 2$, the values of H are increasing without bound, so $\lim_{x\to2^-} H(x) = \infty$.

(d) Finally, Table 15 reveals that, as x approaches 2, $x > 2$, the values of H are increasing without bound, so $\lim_{x\to2^+} H(x) = \infty$.

Table 12

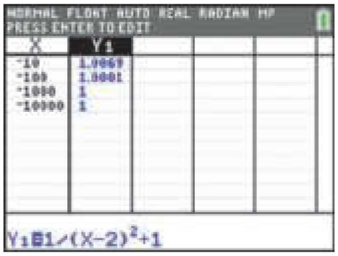


NORMAL FLOAT AUTO REAL RADIAN MP
PRESS ENTER TO EDIT

X	Y₁
10	1.0156
100	1.0001
1000	1
10000	1

$Y_1 = 1/(X-2)^2+1$

Table 13



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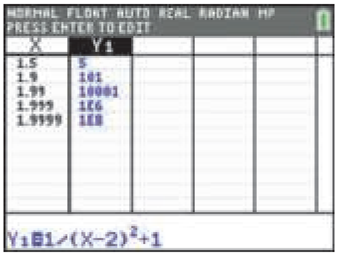
X	Y₁
⁻10	1.0069
⁻100	1.0001
⁻1000	1
⁻10000	1

$Y_1 = 1/(X-2)^2+1$

Table 14

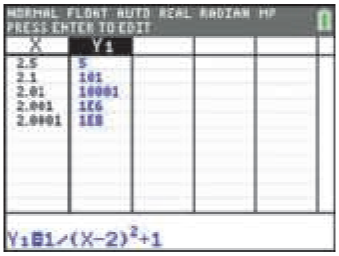


NORMAL FLOAT AUTO REAL RADIAN MP
PRESS ENTER TO EDIT

X	Y₁
1.5	5
1.9	101
1.99	10001
1.999	1E6
1.9999	1E8

$Y_1 = 1/(X-2)^2+1$

Table 15



NORMAL FLOAT AUTO REAL RADIAN MP
PRESS ENTER TO EDIT

X	Y₁
2.5	5
2.1	101
2.01	10001
2.001	1E6
2.0001	1E8

$Y_1 = 1/(X-2)^2+1$

■

The results of the Exploration reveal an important property of rational functions. The vertical line $x = 2$ and the horizontal line $y = 1$ in Figure 34(c) are called *asymptotes* of the graph of H.

DEFINITION

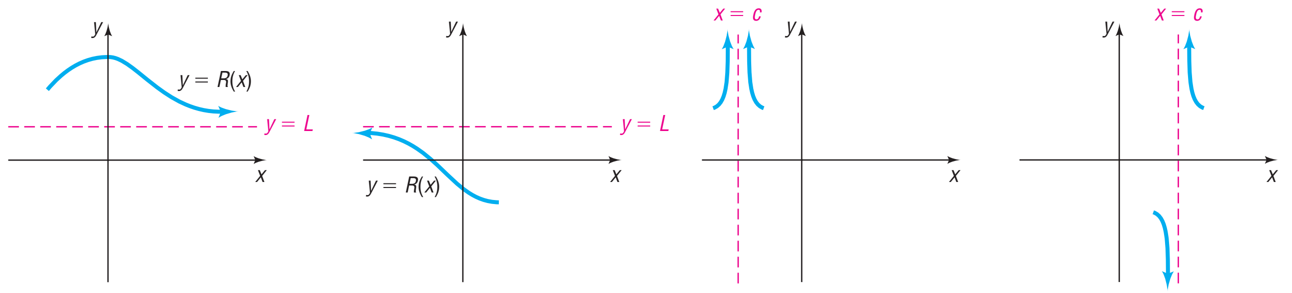
Let R denote a function:

If, as $x \to -\infty$ or as $x \to \infty$, the values of $R(x)$ approach some fixed number L, then the line $y = L$ is a **horizontal asymptote** of the graph of R. [Refer to Figures 35(a) and (b).]

If, as x approaches some number c, the values $|R(x)| \to \infty$, then the line $x = c$ is a **vertical asymptote** of the graph of R. The graph of R never intersects a vertical asymptote. [Refer to Figures 35(c) and (d).]

■

A horizontal asymptote, when it occurs, describes the end behavior of the graph as $x \to \infty$ or as $x \to -\infty$. **The graph of a function may intersect a horizontal asymptote.**



(a) End behavior: As $x \to \infty$, the values of $R(x)$ approach L [expressed as $\lim_{x\to\infty} R(x) = L$]. That is, the points on the graph of R are getting closer to the line $y = L$; $y = L$ is a horizontal asymptote.

(b) End behavior: As $x \to -\infty$, the values of $R(x)$ approach L [expressed as $\lim_{x\to-\infty} R(x) = L$]. That is, the points on the graph of R are getting closer to the line $y = L$; $y = L$ is a horizontal asymptote.

(c) As x approaches c, the values of $R(x) \to \infty$ [for $x < c$, this is expressed as $\lim_{x\to c^-} R(x) = \infty$; for $x > c$, this is expressed as $\lim_{x\to c^+} R(x) = \infty$]. That is, the points on the graph of R are getting closer to the line $x = c$; $x = c$ is a vertical asymptote.

(d) As x approaches c, the values of $|R(x)| \to \infty$ [for $x < c$, this is expressed as $\lim_{x\to c^-} R(x) = -\infty$; for $x > c$, this is expressed as $\lim_{x\to c^+} R(x) = \infty$]. That is, the points on the graph of R are getting closer to the line $x = c$; $x = c$ is a vertical asymptote.

Figure 35

A vertical asymptote, when it occurs, describes the behavior of the graph when x is close to some number c. **The graph of a function will never intersect a vertical asymptote**.

There is a third possibility. If, as $x \to -\infty$ or as $x \to \infty$, the value of a rational function $R(x)$ approaches a linear expression $ax + b$, $a \neq 0$, then the line $y = ax + b$, $a \neq 0$, is an **oblique (or slant) asymptote** of R. Figure 36 shows an oblique asymptote. An oblique asymptote, when it occurs, describes the end behavior of the graph. **The graph of a function may intersect an oblique asymptote**.

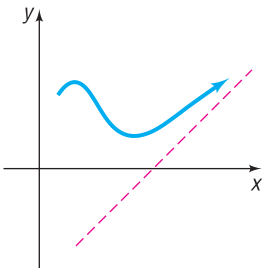


Figure 36 Oblique asymptote

Now Work PROBLEM 27

2 Find the Vertical Asymptotes of a Rational Function

The vertical asymptotes of a rational function $R(x) = \dfrac{p(x)}{q(x)}$, in lowest terms, are located at the real zeros of the denominator, $q(x)$. Suppose that r is a real zero of q, so $x - r$ is a factor of q. As x approaches r, symbolized as $x \to r$, the values of $x - r$ approach 0, causing the ratio to become unbounded; that is, $|R(x)| \to \infty$. Based on the definition, we conclude that the line $x = r$ is a vertical asymptote.

THEOREM

Locating Vertical Asymptotes

A rational function $R(x) = \dfrac{p(x)}{q(x)}$, *in lowest terms*, will have a vertical asymptote $x = r$ if r is a real zero of the denominator q. That is, if $x - r$ is a factor of the denominator q of a rational function $R(x) = \dfrac{p(x)}{q(x)}$, in lowest terms, R will have the vertical asymptote $x = r$.

WARNING If a rational function is not in lowest terms, an application of this theorem may result in an incorrect listing of vertical asymptotes.

EXAMPLE 4 **Finding Vertical Asymptotes**

Find the vertical asymptotes, if any, of the graph of each rational function.

(a) $F(x) = \dfrac{x + 3}{x - 1}$ (b) $R(x) = \dfrac{x}{x^2 - 4}$

(c) $H(x) = \dfrac{x^2}{x^2 + 1}$ (d) $G(x) = \dfrac{x^2 - 9}{x^2 + 4x - 21}$

Solution

(a) F is in lowest terms, and the only zero of the denominator is 1. The line $x = 1$ is the vertical asymptote of the graph of F.

(b) R is in lowest terms, and the zeros of the denominator $x^2 - 4$ are -2 and 2. The lines $x = -2$ and $x = 2$ are the vertical asymptotes of the graph of R.

(c) H is in lowest terms, and the denominator has no real zeros, because the equation $x^2 + 1$ has no real solutions. The graph of H has no vertical asymptotes.

(d) Factor $G(x)$ to determine whether it is in lowest terms.

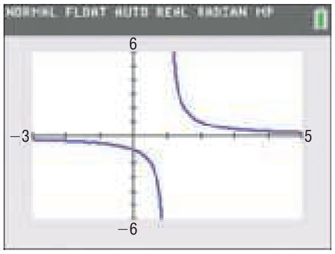
$$G(x) = \frac{x^2 - 9}{x^2 + 4x - 21} = \frac{(x + 3)(x - 3)}{(x + 7)(x - 3)} = \frac{x + 3}{x + 7} \qquad x \neq 3$$

The only zero of the denominator of $G(x)$ in lowest terms is -7. The line $x = -7$ is the only vertical asymptote of the graph of G.

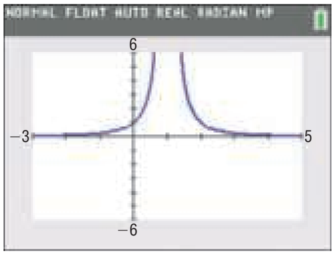
WARNING In Example 4(b), the vertical asymptotes are $x = -2$ and $x = 2$. Do not say that the vertical asymptotes are -2 and 2.

As Example 4 points out, rational functions can have no vertical asymptotes, one vertical asymptote, or more than one vertical asymptote.

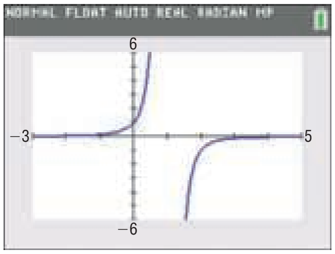
For polynomial functions, we saw how the multiplicities of zeros could be used to determine the behavior of the graph around each x-intercept. In a similar way, the multiplicities of the zeros in the denominator of a rational function, in lowest terms, can be used to determine the behavior of the graph around each vertical asymptote. Consider the following Exploration.



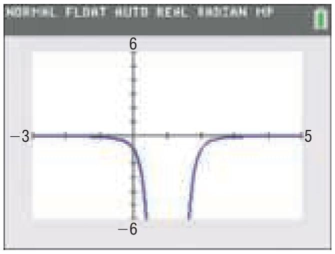
(a) $R_1(x)$; odd multiplicity



(b) $R_2(x)$; even multiplicity



(c) $R_3(x)$; odd multiplicity



(d) $R_4(x)$; even multiplicity

Figure 37

Exploration

Graph each of the following rational functions using a graphing utility and describe the behavior of the graph near the vertical asymptote $x = 1$.

$$R_1(x) = \frac{1}{x-1} \quad R_2(x) = \frac{1}{(x-1)^2} \quad R_3(x) = -\frac{1}{(x-1)^3} \quad R_4(x) = -\frac{1}{(x-1)^4}$$

Result For R_1, as x approaches 1 (odd multiplicity) from the left side of the vertical asymptote, the values of $R_1(x)$ approach negative infinity. That is, $\lim_{x \to 1^-} R_1(x) = -\infty$. As x approaches 1 from the right side of the vertical asymptote, the values of $R_1(x)$ approach infinity. That is, $\lim_{x \to 1^+} R_1(x) = \infty$. See Figure 37(a). Below, we summarize the behavior near $x = 1$ for the remaining rational functions:

- For R_2: $\lim_{x \to 1^-} R_2(x) = \infty$ and $\lim_{x \to 1^+} R_2(x) = \infty$ [See Figure 37(b)]
- For R_3: $\lim_{x \to 1^-} R_3(x) = \infty$ and $\lim_{x \to 1^+} R_3(x) = -\infty$ [See Figure 37(c)]
- For R_4: $\lim_{x \to 1^-} R_4(x) = -\infty$ and $\lim_{x \to 1^+} R_4(x) = -\infty$ [See Figure 37(d)] ■

The results of the Exploration lead to the following conclusion.

Multiplicity and Vertical Asymptotes

- If the multiplicity of any zero of the denominator of a rational function in lowest terms that gives rise to a vertical asymptote is odd, the graph approaches ∞ on one side of the vertical asymptote and approaches $-\infty$ on the other side.
- If the multiplicity of any zero of the denominator of a rational function in lowest terms that gives rise to a vertical asymptote is even, the graph approaches either ∞ or $-\infty$ on both sides of the vertical asymptote.

Now Work PROBLEMS 45, 47, AND 49 (FIND THE VERTICAL ASYMPTOTES, IF ANY)

3 Find the Horizontal or Oblique Asymptote of a Rational Function

To find horizontal or oblique asymptotes, we need to know how the value of the function behaves as $x \to -\infty$ or as $x \to \infty$. That is, we need to determine the end behavior of the function. This can be done by examining the degrees of the numerator and denominator, and the respective power functions that each resembles. For example, consider the rational function

$$R(x) = \frac{3x-2}{5x^2-7x+1}$$

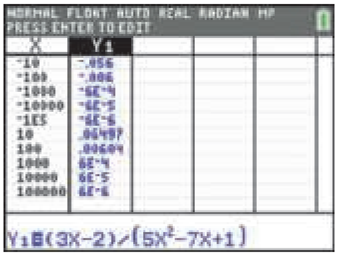
The degree of the numerator, 1, is less than the degree of the denominator, 2. When $|x|$ is very large, the numerator of R can be approximated by the power function $y = 3x$, and the denominator can be approximated by the power function $y = 5x^2$. This means

$$R(x) = \frac{3x-2}{5x^2-7x+1} \underset{\substack{\uparrow \\ \text{For } |x| \text{ very large}}}{\approx} \frac{3x}{5x^2} = \frac{3}{5x} \underset{\substack{\uparrow \\ \text{As } x \to -\infty \text{ or } x \to \infty}}{\to} 0$$

which shows that the line $y = 0$ is a horizontal asymptote. We verify this in Table 16.

This result is true for all rational functions that are **proper** (that is, the degree of the numerator is less than the degree of the denominator). If a rational function is **improper** (that is, if the degree of the numerator is greater than or equal to the degree of the denominator), there could be a horizontal asymptote, an oblique asymptote, or neither. The summary on the next page details how to find horizontal or oblique asymptotes.

Table 16



X	Y₁
-10	-.056
-100	-.006
-1000	-6E-4
-10000	-6E-5
-1E5	-6E-6
10	.06497
100	.00604
1000	6E-4
10000	6E-5
100000	6E-6

Y₁=(3X−2)/(5X²−7X+1)

Finding a Horizontal or Oblique Asymptote of a Rational Function

Consider the rational function

$$R(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

in which the degree of the numerator is n and the degree of the denominator is m.

1. If $n < m$ (the degree of the numerator is less than the degree of the denominator), the line $y = 0$ is a horizontal asymptote.
2. If $n = m$ (the degree of the numerator equals the degree of the denominator), the line $y = \frac{a_n}{b_m}$ is a horizontal asymptote. (That is, the horizontal asymptote equals the ratio of the leading coefficients.)
3. If $n = m + 1$ (the degree of the numerator is one more than the degree of the denominator), the line $y = ax + b$ is an oblique asymptote, which is the quotient found using long division. However, if R in lowest terms is a linear function, then we agree that R has no horizontal or oblique asymptote.
4. If $n \geq m + 2$ (the degree of the numerator is two or more greater than the degree of the denominator), there are no horizontal or oblique asymptotes. The end behavior of the graph will resemble the power function $y = \frac{a_n}{b_m}x^{n-m}$.

Note: A rational function will never have both a horizontal asymptote and an oblique asymptote. A rational function may have neither a horizontal nor an oblique asymptote. ■

We illustrate each of the possibilities in Examples 5 through 8.

EXAMPLE 5

Finding a Horizontal Asymptote

Find the horizontal asymptote, if one exists, of the graph of

$$R(x) = \frac{4x^3 - 5x + 2}{7x^5 + 2x^4 - 3x}$$

Solution Since the degree of the numerator, 3, is less than the degree of the denominator, 5, the rational function R is proper. The line $y = 0$ is a horizontal asymptote of the graph of R. ■

EXAMPLE 6

Finding a Horizontal or Oblique Asymptote

Find the horizontal or oblique asymptote, if one exists, of the graph of

$$H(x) = \frac{3x^4 - x^2}{x^3 - x^2 + 1}$$

Solution Since the degree of the numerator, 4, is exactly one greater than the degree of the denominator, 3, the rational function H has an oblique asymptote. Find the asymptote by using long division.

$$\begin{array}{r@{}l}
 & 3x + 3 \\
x^3 - x^2 + 1 & \overline{)3x^4 \qquad\qquad - x^2} \\
 & \underline{3x^4 - 3x^3 \qquad + 3x} \\
 & \qquad 3x^3 - x^2 - 3x \\
 & \qquad \underline{3x^3 - 3x^2 \qquad + 3} \\
 & \qquad\qquad 2x^2 - 3x - 3
\end{array}$$

As a result,

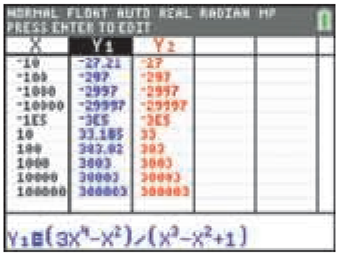
$$H(x) = \frac{3x^4 - x^2}{x^3 - x^2 + 1} = 3x + 3 + \frac{2x^2 - 3x - 3}{x^3 - x^2 + 1}$$

As $x \to -\infty$ or as $x \to \infty$,

$$\frac{2x^2 - 3x - 3}{x^3 - x^2 + 1} \approx \frac{2x^2}{x^3} = \frac{2}{x} \to 0$$

So, as $x \to -\infty$ or as $x \to \infty$, we have $H(x) \to 3x + 3$. See Table 17 with $Y_1 = H(x)$ and $Y_2 = 3x + 3$. As $x \to \infty$ or $x \to -\infty$, the difference in values

Table 17



X	Y1	Y2
-10	-27.21	-27
-100	-297	-297
-1000	-2997	-2997
-10000	-29997	-29997
-1E5	-3E5	-3E5
10	33.185	33
100	303.02	303
1000	3003	3003
10000	30003	30003
100000	300003	300003

Y1=(3X^4-X^2)/(X^3-X^2+1)

between Y_1 and Y_2 becomes indistinguishable. Put another way, as $x \rightarrow \pm\infty$, the graph of H will behave like the graph of $y = 3x + 3$. The graph of the rational function H has an oblique asymptote $y = 3x + 3$. ■

EXAMPLE 7 Finding a Horizontal or Oblique Asymptote

Find the horizontal or oblique asymptote, if one exists, of the graph of

$$R(x) = \frac{8x^2 - x + 2}{4x^2 - 1}$$

Solution Since the degree of the numerator, 2, equals the degree of the denominator, 2, the rational function R has a horizontal asymptote equal to the ratio of the leading coefficients.

$$y = \frac{a_n}{b_m} = \frac{8}{4} = 2$$

To see why the horizontal asymptote equals the ratio of the leading coefficients, investigate the behavior of R as $x \rightarrow -\infty$ or as $x \rightarrow \infty$. When $|x|$ is very large, the numerator of R can be approximated by the power function $y = 8x^2$, and the denominator can be approximated by the power function $y = 4x^2$. This means that as $x \rightarrow -\infty$ or as $x \rightarrow \infty$,

$$R(x) = \frac{8x^2 - x + 2}{4x^2 - 1} \approx \frac{8x^2}{4x^2} = \frac{8}{4} = 2$$

The graph of the rational function R has a horizontal asymptote $y = 2$. The graph of R will behave like $y = 2$ as $x \rightarrow \pm\infty$.

✓ **Check:** Verify the results by creating a TABLE with $Y_1 = R(x)$ and $Y_2 = 2$. ■

EXAMPLE 8 Finding a Horizontal or Oblique Asymptote

Find the horizontal or oblique asymptote, if one exists, of the graph of

$$G(x) = \frac{2x^5 - x^3 + 2}{x^3 - 1}$$

Solution Since the degree of the numerator, 5, is greater than the degree of the denominator, 3, by more than one, the rational function G has no horizontal or oblique asymptote. The end behavior of the graph will resemble the power function $y = 2x^{5-3} = 2x^2$.

To see why this is the case, investigate the behavior of G as $x \rightarrow -\infty$ or as $x \rightarrow \infty$. When $|x|$ is very large, the numerator of G can be approximated by the power function $y = 2x^5$, and the denominator can be approximated by the power function $y = x^3$. This means as $x \rightarrow -\infty$ or as $x \rightarrow \infty$,

$$G(x) = \frac{2x^5 - x^3 + 2}{x^3 - 1} \approx \frac{2x^5}{x^3} = 2x^{5-3} = 2x^2$$

Since this is not linear, the graph of G has no horizontal or oblique asymptote. The graph of G will behave like $y = 2x^2$ as $x \rightarrow \pm\infty$. ■

Now Work PROBLEMS 45, 47, AND 49 (FIND THE HORIZONTAL OR OBLIQUE ASYMPTOTE, IF ONE EXISTS.)

5.4 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. ***True or False*** The quotient of two polynomial expressions is a rational expression. (p. 63)
2. What are the quotient and remainder when $3x^4 - x^2$ is divided by $x^3 - x^2 + 1$. (pp. 45–48)
3. Graph $y = \frac{1}{x}$. (pp. 169–170)
4. Graph $y = 2(x + 1)^2 - 3$ using transformations. (pp. 256–264)

Concepts and Vocabulary

5. ***True or False*** The domain of every rational function is the set of all real numbers.

6. If, as $x \to -\infty$ or as $x \to \infty$, the values of $R(x)$ approach some fixed number L, then the line $y = L$ is a ________ ________ of the graph of R.

7. If, as x approaches some number c, the values of $|R(x)| \to \infty$, then the line $x = c$ is a ________ ________ of the graph of R.

8. For a rational function R, if the degree of the numerator is less than the degree of the denominator, then R is ________.

9. ***True or False*** The graph of a rational function may intersect a horizontal asymptote.

10. ***True or False*** The graph of a rational function may intersect a vertical asymptote.

11. If a rational function is proper, then ________ is a horizontal asymptote.

12. ***True or False*** If the degree of the numerator of a rational function equals the degree of the denominator, then the ratio of the leading coefficients gives rise to the horizontal asymptote.

13. If $R(x) = \dfrac{p(x)}{q(x)}$ is a rational function and if p and q have no common factors, then R is ________.
(a) improper (b) proper
(c) undefined (d) in lowest terms

14. Which type of asymptote, when it occurs, describes the behavior of a graph when x is close to some number?
(a) vertical (b) horizontal
(c) oblique (d) all of these

Skill Building

In Problems 15–26, find the domain of each rational function.

15. $R(x) = \dfrac{4x}{x - 3}$

16. $R(x) = \dfrac{5x^2}{3 + x}$

17. $H(x) = \dfrac{-4x^2}{(x - 2)(x + 4)}$

18. $G(x) = \dfrac{6}{(x + 3)(4 - x)}$

19. $F(x) = \dfrac{3x(x - 1)}{2x^2 - 5x - 3}$

20. $Q(x) = \dfrac{-x(1 - x)}{3x^2 + 5x - 2}$

21. $R(x) = \dfrac{x}{x^3 - 8}$

22. $R(x) = \dfrac{x}{x^4 - 1}$

23. $H(x) = \dfrac{3x^2 + x}{x^2 + 4}$

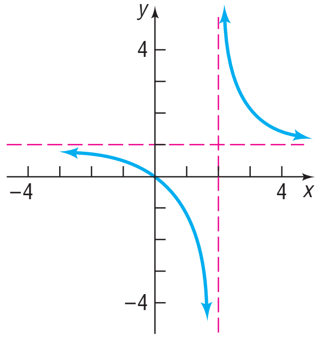
24. $G(x) = \dfrac{x - 3}{x^4 + 1}$

25. $R(x) = \dfrac{3(x^2 - x - 6)}{4(x^2 - 9)}$

26. $F(x) = \dfrac{-2(x^2 - 4)}{3(x^2 + 4x + 4)}$

In Problems 27–32, use the graph shown to find
(a) The domain and range of each function *(b) The intercepts, if any* *(c) Horizontal asymptotes, if any*
(d) Vertical asymptotes, if any *(e) Oblique asymptotes, if any*

27.



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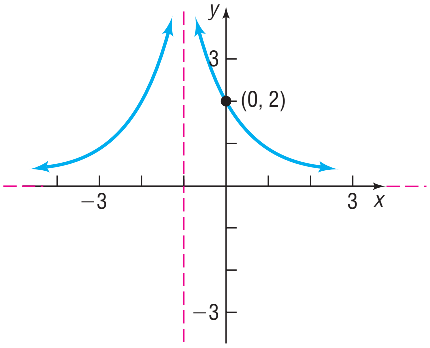

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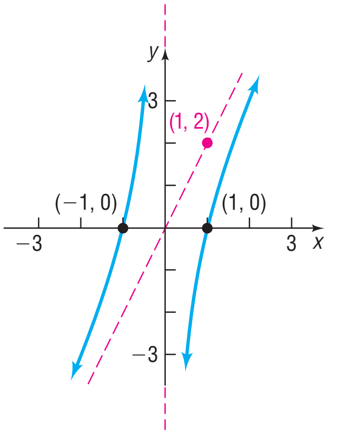

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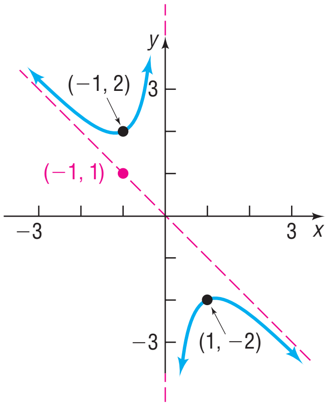

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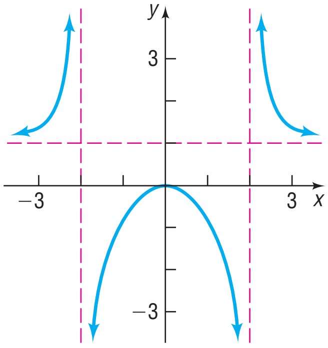

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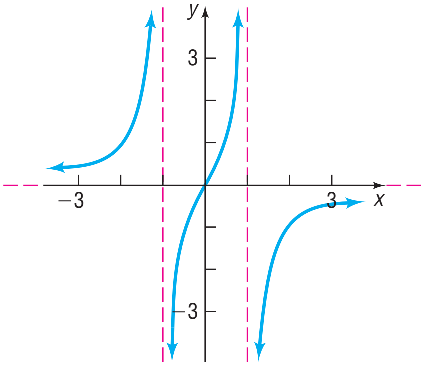


In Problems 33–44, (a) graph the rational function using transformations, (b) use the final graph to find the domain and range, and (c) use the final graph to list any vertical, horizontal, or oblique asymptotes.

33. $F(x) = 2 + \dfrac{1}{x}$

34. $Q(x) = 3 + \dfrac{1}{x^2}$

35. $R(x) = \dfrac{1}{(x-1)^2}$

36. $R(x) = \dfrac{3}{x}$

37. $H(x) = \dfrac{-2}{x+1}$

38. $G(x) = \dfrac{2}{(x+2)^2}$

39. $R(x) = \dfrac{-1}{x^2+4x+4}$

40. $R(x) = \dfrac{1}{x-1} + 1$

41. $G(x) = 1 + \dfrac{2}{(x-3)^2}$

42. $F(x) = 2 - \dfrac{1}{x+1}$

43. $R(x) = \dfrac{x^2-4}{x^2}$

44. $R(x) = \dfrac{x-4}{x}$

In Problems 45–56, find the vertical, horizontal, and oblique asymptotes, if any, of each rational function.

45. $R(x) = \dfrac{3x}{x+4}$

46. $R(x) = \dfrac{3x+5}{x-6}$

47. $H(x) = \dfrac{x^3-8}{x^2-5x+6}$

48. $G(x) = \dfrac{x^3+1}{x^2-5x-14}$

49. $T(x) = \dfrac{x^3}{x^4-1}$

50. $P(x) = \dfrac{4x^2}{x^3-1}$

51. $Q(x) = \dfrac{2x^2-5x-12}{3x^2-11x-4}$

52. $F(x) = \dfrac{x^2+6x+5}{2x^2+7x+5}$

53. $R(x) = \dfrac{6x^2+7x-5}{3x+5}$

54. $R(x) = \dfrac{8x^2+26x-7}{4x-1}$

55. $G(x) = \dfrac{x^4-1}{x^2-x}$

56. $F(x) = \dfrac{x^4-16}{x^2-2x}$

Applications and Extensions

57. Gravity In physics, it is established that the acceleration due to gravity, g (in meters/sec^2), at a height h meters above sea level is given by

$$g(h) = \frac{3.99 \times 10^{14}}{(6.374 \times 10^6 + h)^2}$$

where 6.374×10^6 is the radius of Earth in meters.

(a) What is the acceleration due to gravity at sea level?

(b) The Willis Tower in Chicago, Illinois, is 443 meters tall. What is the acceleration due to gravity at the top of the Willis Tower?

(c) The peak of Mount Everest is 8848 meters above sea level. What is the acceleration due to gravity on the peak of Mount Everest?

(d) Find the end behavior of g. That is, find $\lim_{h\to\infty} g(h)$. What does the result suggest?

(e) Solve $g(h) = 0$. How do you interpret your answer?

58. Population Model A rare species of insect was discovered in the Amazon Rain Forest. To protect the species, environmentalists declared the insect endangered and transplanted the insect into a protected area. The population P of the insect t months after being transplanted is

$$P(t) = \frac{50(1 + 0.5t)}{2 + 0.01t}$$

(a) How many insects were discovered? In other words, what was the population when $t = 0$?

(b) What will the population be after 5 years?

(c) Determine the end behavior of P. What is the largest population that the protected area can sustain?

59. Resistance in Parallel Circuits From Ohm's Law for circuits, it follows that the total resistance R_{tot} of two components hooked in parallel is given by the equation

$$R_{\text{tot}} = \frac{R_1 R_2}{R_1 + R_2}$$

where R_1 and R_2 are the individual resistances.

(a) Let $R_1 = 10$ ohms, and graph R_{tot} as a function of R_2.

(b) Find and interpret any asymptotes of the graph obtained in part (a).

(c) If $R_2 = 2\sqrt{R_1}$, what value of R_1 will yield an R_{tot} of 17 ohms?

60. Newton's Method In calculus you will learn that if

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

is a polynomial function, then the *derivative* of $p(x)$ is

$$p'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2 x + a_1$$

Newton's Method is an efficient method for approximating the x-intercepts (or real zeros) of a function, such as $p(x)$. The following steps outline Newton's Method.

STEP 1: Select an initial value x_0 that is somewhat close to the x-intercept being sought.

STEP 2: Find values for x using the relation

$$x_{n+1} = x_n - \frac{p(x_n)}{p'(x_n)} \qquad n = 1, 2, \ldots$$

until you get two consecutive values x_n and x_{n+1} that agree to whatever decimal place accuracy you desire.

STEP 3: The approximate zero will be x_{n+1}.

Consider the polynomial $p(x) = x^3 - 7x - 40$.

(a) Evaluate $p(5)$ and $p(-3)$.

(b) What might we conclude about a zero of p? Explain.

(c) Use Newton's Method to approximate an x-intercept, r, $-3 < r < 5$, of $p(x)$ to four decimal places.

(d) Use a graphing utility to graph $p(x)$ and verify your answer in part (c).

(e) Using a graphing utility, evaluate $p(r)$ to verify your result.

61. Exploration The standard form of the rational function $R(x) = \dfrac{mx + b}{cx + d}$, where $c \neq 0$, is $R(x) = a\left(\dfrac{1}{x - h}\right) + k$. To write a rational function in standard form requires long division.

(a) Write the rational function $R(x) = \dfrac{2x + 3}{x - 1}$ in standard form by writing R in the form

$$\text{Quotient} + \frac{\text{remainder}}{\text{divisor}}$$

(b) Graph R using transformations.

(c) Determine the vertical asymptote and the horizontal asymptote of R.

62. Exploration Repeat Problem 61 for the rational function $R(x) = \dfrac{-6x + 16}{2x - 7}$.

Explaining Concepts: Discussion and Writing

63. If the graph of a rational function R has the vertical asymptote $x = 4$, the factor $x - 4$ must be present in the denominator of R. Explain why.

64. If the graph of a rational function R has the horizontal asymptote $y = 2$, the degree of the numerator of R equals the degree of the denominator of R. Explain why.

65. The graph of a rational function cannot have both a horizontal and an oblique asymptote? Explain why.

66. Make up a rational function that has $y = 2x + 1$ as an oblique asymptote. Explain the methodology that you used.

Retain Your Knowledge

Problems 67–70 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

67. Find the equation of a vertical line passing through the point $(5, -3)$.

68. Solve: $\dfrac{2}{5}(3x - 7) + 1 = \dfrac{x}{4} - 2$

69. Determine whether the graph of the equation $2x^3 - xy^2 = 4$ is symmetric with respect to the x-axis, the y-axis, the origin, or none of these.

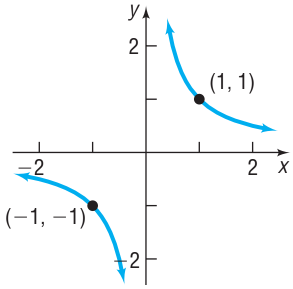
70. What are the points of intersection of the graphs of the functions $f(x) = -3x + 2$ and $g(x) = x^2 - 2x - 4$?

‘Are You Prepared?’ Answers

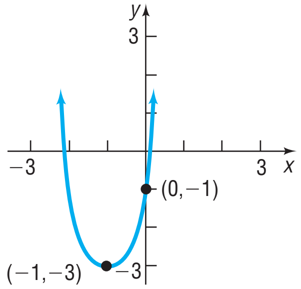
1. True

2. Quotient: $3x + 3$; remainder: $2x^2 - 3x - 3$

3.



4.



5.5 The Graph of a Rational Function

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Finding Intercepts (Section 2.1, pp. 165–166)

Now Work the **‘Are You Prepared?’** problem on page 390.

OBJECTIVES 1 Analyze the Graph of a Rational Function (p. 382)
2 Solve Applied Problems Involving Rational Functions (p. 389)

1 Analyze the Graph of a Rational Function

Graphing utilities make the task of graphing rational functions less time consuming. However, the results of algebraic analysis must be taken into account before drawing conclusions based on the graph provided by the utility. In the next example we illustrate how to use the information collected in the last section in conjunction with the graphing utility to analyze the graph of a rational function $R(x) = \dfrac{p(x)}{q(x)}$.

EXAMPLE 1 How to Analyze the Graph of a Rational Function

Analyze the graph of the rational function: $R(x) = \dfrac{x-1}{x^2-4}$

Step-by-Step Solution

Step 1: Factor the numerator and denominator of *R*. Find the domain of the rational function.

$$R(x) = \frac{x-1}{x^2-4} = \frac{x-1}{(x+2)(x-2)}$$

The domain of R is $\{x \mid x \neq -2, x \neq 2\}$.

Step 2: Write *R* in lowest terms.

Because there are no common factors between the numerator and denominator, R is in lowest terms.

Step 3: Locate the intercepts of the graph. Use multiplicity to determine the bahavior of the graph of *R* at each *x*-intercept.

Since 0 is in the domain of R, the y-intercept is $R(0) = \frac{1}{4}$. The x-intercepts are found by determining the real zeros of the numerator of R written in lowest terms. By solving $x - 1 = 0$, the only real zero of the numerator is 1. So the only x-intercept of the graph of R is 1. The multiplicity of 1 is odd, so the graph will cross the x-axis at $x = 1$.

Step 4: Locate the vertical asymptotes. Determine the behavior of the graph on either side of each vertical asymptote.

To locate the vertical asymptotes, find the zeros of the denominator with the rational function in lowest terms. With R written in lowest terms, we find that the graph of R has two vertical asymptotes: the lines $x = -2$ and $x = 2$. The multiplicities of the zeros that give rise to the vertical asymptotes are both odd. Therefore, the graph will approach ∞ on one side of each vertical asymptote, and it will approach $-\infty$ on the other side.

Step 5: Locate the horizontal or oblique asymptote. Determine points, if any, at which the graph of *R* intersects this asymptote.

Because the degree of the numerator is less than the degree of the denominator, R is proper and the line $y = 0$ (the x-axis) is a horizontal asymptote of the graph. To determine if the graph of R intersects the horizontal asymptote, solve the equation $R(x) = 0$:

$$\frac{x-1}{x^2-4} = 0$$
$$x - 1 = 0$$
$$x = 1$$

The only solution is $x = 1$, so the graph of R intersects the horizontal asymptote at $(1, 0)$.

Step 6: Graph *R* using a graphing utility.

The analysis in Steps 1 through 5 helps us to determine an appropriate viewing window to obtain a complete graph. Figure 38 shows the graph of $R(x) = \dfrac{x-1}{x^2-4}$. You should confirm that all the algebraic conclusions that we came to in Steps 1 through 5 are part of the graph. For example, the graph has a horizontal asymptote at $y = 0$ and vertical asymptotes at $x = -2$ and $x = 2$. The y-intercept is $\frac{1}{4}$ and the x-intercept is 1.

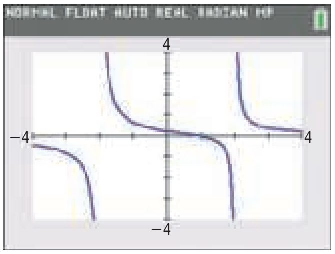


Figure 38 $Y_1 = (x-1)/(x^2-4)$

Step 7: Use the results obtained in Steps 1 through 6 to graph R by hand.

Figure 39 shows the graph of R.

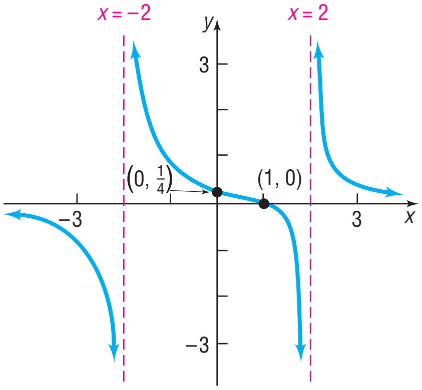


Figure 39 $R(x) = \dfrac{x-1}{x^2-4}$

■

SUMMARY

Analyzing the Graph of a Rational Function

STEP 1: Factor the numerator and denominator of R. Find the domain of the rational function.

STEP 2: Write R in lowest terms.

STEP 3: Locate the intercepts of the graph. The y-intercept, if there is one, is $R(0)$. The x-intercepts, if any, of $R(x) = \dfrac{p(x)}{q(x)}$ in lowest terms satisfy the equation $p(x) = 0$. Use multiplicity to determine the behavior of the graph of R at each x-intercept.

STEP 4: Locate the vertical asymptotes. The vertical asymptotes, if any, of $R(x) = \dfrac{p(x)}{q(x)}$ in lowest terms are found by identifying the real zeros of $q(x)$. Each zero of the denominator gives rise to a vertical asymptote. Use multiplicity to determine the behavior of the graph of R on either side of each vertical asymptote.

STEP 5: Locate the horizontal or oblique asymptote, if one exists, using the procedure given in Section 5.4. Determine points, if any, at which the graph of R intersects this asymptote.

STEP 6: Graph R using a graphing utility.

STEP 7: Use the results obtained in Steps 1 through 6 to graph R by hand.

Now Work PROBLEM 7

EXAMPLE 2 Analyzing the Graph of a Rational Function

Analyze the graph of the rational function: $R(x) = \dfrac{x^2-1}{x}$

Solution

STEP 1: $R(x) = \dfrac{x^2-1}{x} = \dfrac{(x+1)(x-1)}{x}$

The domain of R is $\{x \mid x \neq 0\}$.

STEP 2: R is in lowest terms.

STEP 3: The graph has two x-intercepts, -1 and 1, each with odd multiplicity. The graph will cross the x-axis at each point. There is no y-intercept, since x cannot equal 0.

Step 4: The real zero of the denominator with R in lowest terms is 0, so the graph of $R(x)$ has the line $x = 0$ (the y-axis) as a vertical asymptote. The multiplicity is odd, so the graph will approach ∞ on one side of the asymptote and $-\infty$ on the other.

Step 5: Since the degree of the numerator, 2, is one more than the degree of the denominator, 1, the rational function R is improper and will have an oblique asymptote. To find the oblique asymptote, use long division.

$$\begin{array}{r} x \\ x\overline{)\,x^2-1} \\ \underline{x^2} \\ -1 \end{array}$$

The quotient is x, so the line $y = x$ is an oblique asymptote of the graph. To determine whether the graph of R intersects the asymptote $y = x$, solve the equation $R(x) = x$.

$$\begin{aligned} R(x) = \frac{x^2-1}{x} &= x \\ x^2 - 1 &= x^2 \\ -1 &= 0 \quad \textbf{Impossible} \end{aligned}$$

The equation $\dfrac{x^2-1}{x} = x$ has no solution, so the graph of $R(x)$ does not intersect the line $y = x$.

Note: Because the denominator of the rational function is a monomial, we can also find the oblique asymptote as follows:

$$\frac{x^2-1}{x} = \frac{x^2}{x} - \frac{1}{x} = x - \frac{1}{x}$$

Since $\dfrac{1}{x} \rightarrow 0$ as $x \rightarrow \infty$, $y = x$ is an oblique asymptote. ■

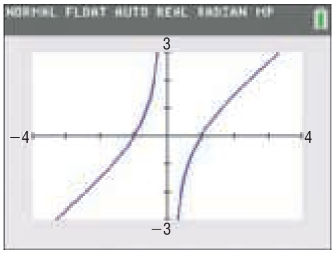


Figure 40 $Y_1 = \dfrac{x^2-1}{x}$

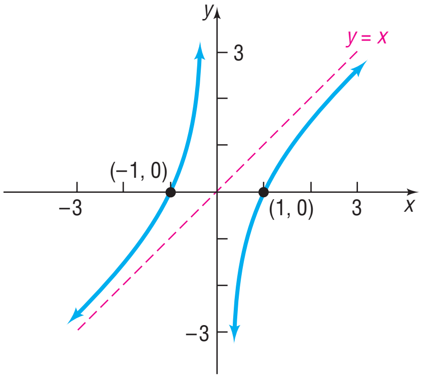


Figure 41 $R(x) = \dfrac{x^2-1}{x}$

Step 6: See Figure 40. We see from the graph that there is no y-intercept and there are two x-intercepts, -1 and 1. We can also see that there is a vertical asymptote at $x = 0$.

Step 7: Using the information gathered in Steps 1 through 6, we obtain the graph of R shown in Figure 41. Notice how the oblique asymptote is used as a guide in graphing the rational function by hand. ■

Now Work PROBLEM 15

EXAMPLE 3 Analyzing the Graph of a Rational Function

Analyze the graph of the rational function: $R(x) = \dfrac{x^4+1}{x^2}$

Solution

Step 1: R is completely factored. The domain of R is $\{x \mid x \neq 0\}$.

Step 2: R is in lowest terms.

Step 3: There is no y-intercept. Since $x^4 + 1 = 0$ has no real solutions, there are no x-intercepts.

Step 4: R is in lowest terms, so $x = 0$ (the y-axis) is a vertical asymptote of R. The multiplicity of 0 is even, so the graph will approach either ∞ or $-\infty$ on both sides of the asymptote.

Step 5: Since the degree of the numerator, 4, is two more than the degree of the denominator, 2, the rational function will not have a horizontal or oblique asymptote. Find the end behavior of R. As $|x| \rightarrow \infty$,

$$R(x) = \frac{x^4+1}{x^2} \approx \frac{x^4}{x^2} = x^2$$

The graph of R will approach the graph of $y = x^2$ as $x \rightarrow -\infty$ and as $x \rightarrow \infty$. The graph of R does not intersect $y = x^2$. Do you know why?

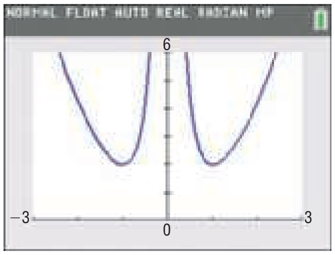


Figure 42 $Y_1 = \dfrac{x^4 + 1}{x^2}$

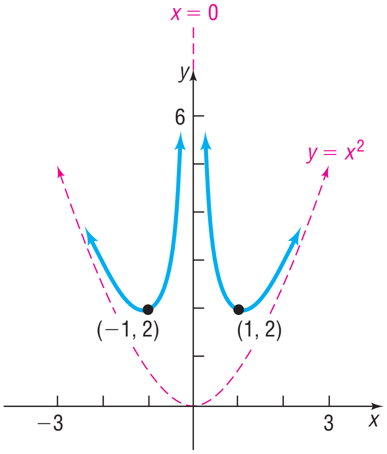


Figure 43

Step 6: See Figure 42. Notice the vertical asymptote at $x = 0$, and note that there are no intercepts.

Step 7: Since the graph of R is above the x-axis, and the multiplicity of the zero that gives rise to the vertical asymptote, $x = 0$, is even, the graph will approach the vertical asymptote $x = 0$ at the top to the left of $x = 0$ and at the top to the right of $x = 0$. See Figure 43. ■

Note: Notice that R in Example 3 is an even function. Do you see the symmetry about the y-axis in the graph of R? ■

EXAMPLE 4 Analyzing the Graph of a Rational Function

Analyze the graph of the rational function: $R(x) = \dfrac{3x^2 - 3x}{x^2 + x - 12}$

Solution

Step 1: Factor R to get

$$R(x) = \frac{3x(x - 1)}{(x + 4)(x - 3)}$$

The domain of R is $\{x \mid x \neq -4, x \neq 3\}$.

Step 2: R is in lowest terms.

Step 3: The graph has two x-intercepts, 0 and 1, each with odd multiplicity. The graph will cross the x-axis at both points. The y-intercept is $R(0) = 0$.

Step 4: The real zeros of the denominator of R with R in lowest terms are -4 and 3. So the graph of R has two vertical asymptotes: $x = -4$ and $x = 3$. The multiplicities of the values that give rise to the asymptotes are both odd, so the graph will approach ∞ on one side of each asymptote and $-\infty$ on the other side.

Step 5: Since the degree of the numerator equals the degree of the denominator, the graph has a horizontal asymptote. To find it, form the quotient of the leading coefficient of the numerator, 3, and the leading coefficient of the denominator, 1. The graph of R has the horizontal asymptote $y = 3$. To find out whether the graph of R intersects the asymptote, solve the equation $R(x) = 3$.

$$R(x) = \frac{3x^2 - 3x}{x^2 + x - 12} = 3$$

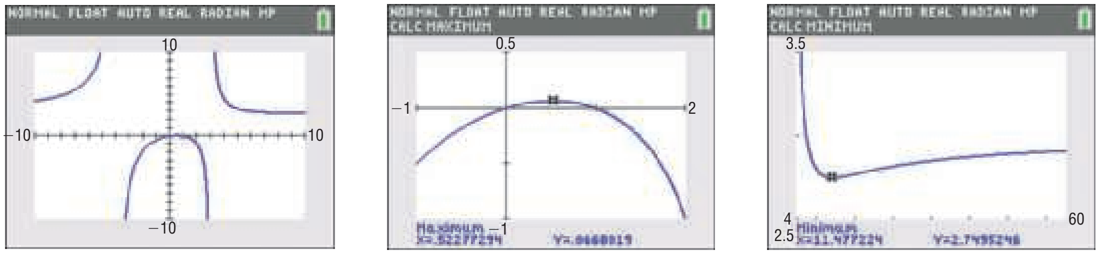
$$3x^2 - 3x = 3x^2 + 3x - 36$$

$$-6x = -36$$

$$x = 6$$

The graph intersects the line $y = 3$ at $x = 6$, and $(6, 3)$ is a point on the graph of R.

Step 6: Figure 44(a) shows the graph of R.



(a) (b) (c)

Figure 44 $Y_1 = \dfrac{3x^2 - 3x}{x^2 + x - 12}$

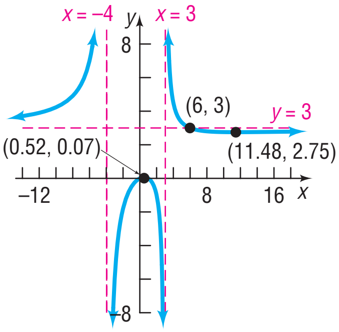


Figure 45 $R(x) = \dfrac{3x^2 - 3x}{x^2 + x - 12}$

Step 7: Figure 44(a) does not clearly show the graph between the two x-intercepts, 0 and 1. Because the zeros in the numerator, 0 and 1, are of odd multiplicity (both are multiplicity 1), we know that the graph of R crosses the x-axis at 0 and 1. Therefore, the graph of R is above the x-axis for $0 < x < 1$. To see this part better, we graph R for $-1 \le x \le 2$ in Figure 44(b). Using MAXIMUM, we approximate the turning point to be $(0.52, 0.07)$, rounded to two decimal places.

Figure 44(a) also does not display the graph of R crossing the horizontal asymptote at $(6, 3)$. To see this part better, we graph R for $4 \le x \le 60$ in Figure 44(c). Using MINIMUM, we approximate the turning point to be $(11.48, 2.75)$, rounded to two decimal places.

Using this information along with the information gathered in Steps 1 through 6, we obtain the graph of R shown in Figure 45. ■

EXAMPLE 5 Analyzing the Graph of a Rational Function with a Hole

Analyze the graph of the rational function: $R(x) = \dfrac{2x^2 - 5x + 2}{x^2 - 4}$

Solution

Step 1: Factor R and obtain

$$R(x) = \frac{(2x - 1)(x - 2)}{(x + 2)(x - 2)}$$

The domain of R is $\{x \mid x \neq -2, x \neq 2\}$.

Step 2: In lowest terms,

$$R(x) = \frac{2x - 1}{x + 2} \qquad x \neq -2, x \neq 2$$

Step 3: The graph has one x-intercept, 0.5, with odd multiplicity. The graph will cross the x-axis at $x = \dfrac{1}{2}$. The y-intercept is $R(0) = -0.5$.

Step 4: Since $x + 2$ is the only factor of the denominator of $R(x)$ *in lowest terms,* the graph has one vertical asymptote, $x = -2$. However, the rational function is undefined at both $x = 2$ and $x = -2$. The multiplicity of -2 is odd, so the graph will approach ∞ on one side of the asymptote and $-\infty$ on the other side.

Step 5: Since the degree of the numerator equals the degree of the denominator, the graph has a horizontal asymptote. To find it, form the quotient of the leading coefficient of the numerator, 2, and the leading coefficient of the denominator, 1. The graph of R has the horizontal asymptote $y = 2$. To find whether the graph of R intersects the asymptote, solve the equation $R(x) = 2$.

$$R(x) = \frac{2x-1}{x+2} = 2$$
$$2x - 1 = 2(x+2)$$
$$2x - 1 = 2x + 4$$
$$-1 = 4 \qquad \textbf{Impossible}$$

The graph does not intersect the line $y = 2$.

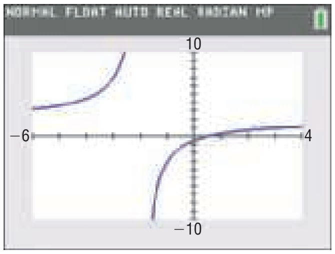
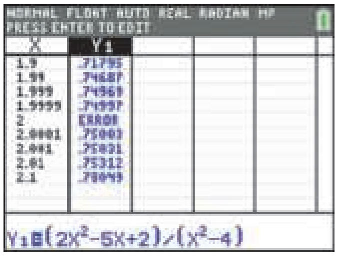


Figure 46 $Y_1 = \dfrac{2x^2 - 5x + 2}{x^2 - 4}$

Step 6: Figure 46 shows the graph of $R(x)$. Notice that the graph has one vertical asymptote at $x = -2$. Also, the function appears to be continuous at $x = 2$.

Step 7: The analysis presented thus far does not explain the behavior of the graph at $x = 2$. We use the TABLE feature of our graphing utility to determine the behavior of the graph of R as x approaches 2. See Table 18. From the table, we conclude that the value of R approaches 0.75 as x approaches 2. This result is further verified by evaluating R in lowest terms at $x = 2$. We conclude that there is a hole in the graph at $(2, 0.75)$. Using the information gathered in Steps 1 through 6, we obtain the graph of R shown in Figure 47.

Table 18



NORMAL FLOAT AUTO REAL RADIAN MP
PRESS ENTER TO EDIT

X	Y₁
1.9	.71795
1.99	.74687
1.999	.74969
1.9999	.74997
2	ERROR
2.0001	.75003
2.001	.75031
2.01	.75312
2.1	.78049

Y₁=(2X²−5X+2)/(X²−4)

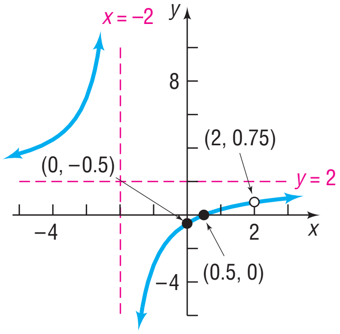


Figure 47 $R(x) = \dfrac{2x^2 - 5x + 2}{x^2 - 4}$ ■

As Example 5 shows, **the values excluded from the domain of a rational function give rise to either vertical asymptotes or holes**.

Now Work PROBLEM 33

EXAMPLE 6 Constructing a Rational Function from Its Graph

Make up a rational function that might have the graph shown in Figure 48.

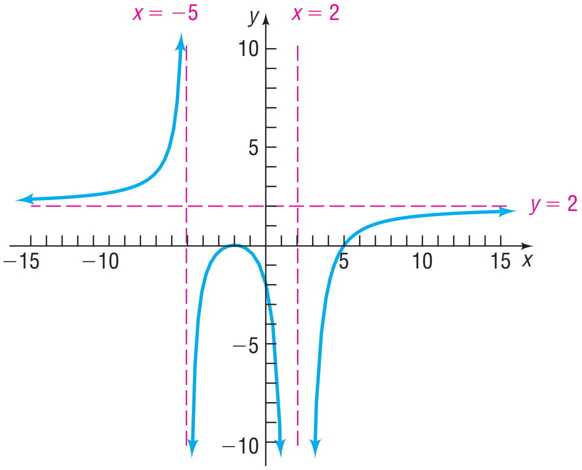


Figure 48

Solution The numerator of a rational function $R(x) = \frac{p(x)}{q(x)}$ in lowest terms determines the x-intercepts of its graph. The graph shown in Figure 48 has x-intercepts -2 (even multiplicity; graph touches the x-axis) and 5 (odd multiplicity; graph crosses the x-axis). So one possibility for the numerator is $p(x) = (x+2)^2(x-5)$.

The denominator of a rational function in lowest terms determines the vertical asymptotes of its graph. The vertical asymptotes of the graph are $x = -5$ and $x = 2$. Since $R(x)$ approaches ∞ from the left of $x = -5$ and $R(x)$ approaches $-\infty$ from the right of $x = -5$, we know that $(x+5)$ is a factor of odd multiplicity in $q(x)$. Also, $R(x)$ approaches $-\infty$ from both sides of $x = 2$, so $(x-2)$ is a factor of even multiplicity in $q(x)$. A possibility for the denominator is $q(x) = (x+5)(x-2)^2$.

So far we have $R(x) = \frac{(x+2)^2(x-5)}{(x+5)(x-2)^2}$. The horizontal asymptote of the graph given in Figure 48 is $y = 2$, so we know that the degree of the numerator must equal the degree of the denominator, and the quotient of leading coefficients must be $\frac{2}{1}$. This leads to $R(x) = \frac{2(x+2)^2(x-5)}{(x+5)(x-2)^2}$. Figure 49 shows the graph of R drawn on a graphing utility. Since Figure 49 looks similar to Figure 48, we have found a rational function R for the graph in Figure 48. ■

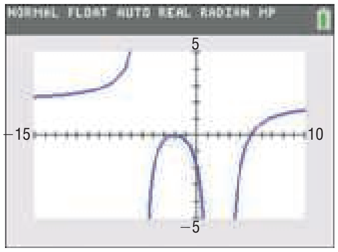


Figure 49 $Y_1 = \frac{2(x+2)^2(x-5)}{(x+5)(x-2)^2}$

Now Work PROBLEM 51

2 Solve Applied Problems Involving Rational Functions

EXAMPLE 7 Finding the Least Cost of a Can

Reynolds Metal Company manufactures aluminum cans in the shape of a cylinder with a capacity of 500 cubic centimeters (cm^3), or $\frac{1}{2}$ liter. The top and bottom of the can are made of a special aluminum alloy that costs 0.05¢ per square centimeter (cm^2). The sides of the can are made of material that costs 0.02¢ per square centimeter.

(a) Express the cost of material for the can as a function of the radius r of the can.
(b) Use a graphing utility to graph the function $C = C(r)$.
(c) What value of r will result in the least cost?
(d) What is this least cost?

Solution (a) Figure 50 illustrates the situation. Notice that the material required to produce a cylindrical can of height h and radius r consists of a rectangle of area $2\pi rh$ and two circles, each of area πr^2. The total cost C (in cents) of manufacturing the can is therefore

$$C = \text{Cost of the top and bottom} + \text{Cost of the side}$$

$$= \underbrace{2(\pi r^2)}_{\substack{\text{Total area}\\ \text{of top and}\\ \text{bottom}}}\ \underbrace{(0.05)}_{\substack{\text{Cost/unit}\\ \text{area}}} + \underbrace{(2\pi rh)}_{\substack{\text{Total}\\ \text{area of}\\ \text{side}}}\ \underbrace{(0.02)}_{\substack{\text{Cost/unit}\\ \text{area}}}$$

$$= 0.10\pi r^2 + 0.04\pi rh$$

But there is the additional restriction that the height h and radius r must be chosen so that the volume V of the can is 500 cm³. Since $V = \pi r^2 h$,

$$500 = \pi r^2 h \quad \text{or} \quad h = \frac{500}{\pi r^2}$$

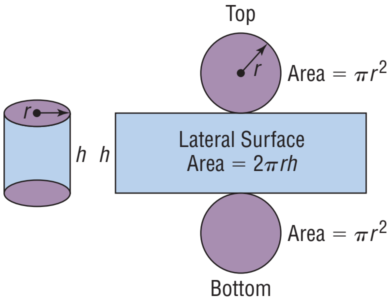


Figure 50 Surface of a cylinder

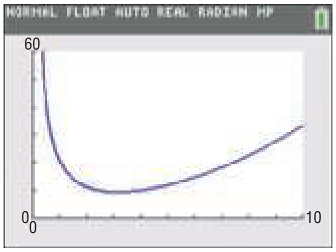


Figure 51 $Y_1 = \dfrac{0.10\pi x^3 + 20}{x}$

Substituting this expression for h, the cost C, in cents, as a function of the radius r, is

$$C(r) = 0.10\pi r^2 + 0.04\pi r \frac{500}{\pi r^2} = 0.10\pi r^2 + \frac{20}{r} = \frac{0.10\pi r^3 + 20}{r}$$

(b) See Figure 51 for the graph of $C = C(r)$.

(c) Using the MINIMUM command, the cost is least for a radius of about 3.17 cm.

(d) The least cost is $C(3.17) \approx 9.47$¢. ■

Now Work PROBLEM 63

5.5 Assess Your Understanding

'Are You Prepared?' *The answer is given at the end of these exercises. If you get the wrong answer, read the pages listed in red.*

1. Find the intercepts of the graph of the equation $y = \dfrac{x^2 - 1}{x^2 - 4}$. (pp. 165–166)

Concepts and Vocabulary

2. ***True or False*** Every rational function has at least one asymptote.

3. Which type of asymptote will never intersect the graph of a rational function?
(a) horizontal (b) oblique (c) vertical (d) all of these

4. Identify the y-intercept of the graph of $R(x) = \dfrac{6(x-1)}{(x+1)(x+2)}$.
(a) -3 (b) -2 (c) -1 (d) 1

5. $R(x) = \dfrac{x(x-2)^2}{x-2}$
(a) Find the domain of R.
(b) Find the x-intercepts of R.

6. ***True or False*** The graph of a rational function sometimes has a hole.

Skill Building

In Problems 7–50, follow Steps 1 through 7 on page 384 to analyze the graph of each function.

7. $R(x) = \dfrac{x+1}{x(x+4)}$
8. $R(x) = \dfrac{x}{(x-1)(x+2)}$
9. $R(x) = \dfrac{3x+3}{2x+4}$
10. $R(x) = \dfrac{2x+4}{x-1}$
11. $R(x) = \dfrac{3}{x^2-4}$
12. $R(x) = \dfrac{6}{x^2-x-6}$
13. $P(x) = \dfrac{x^4+x^2+1}{x^2-1}$
14. $Q(x) = \dfrac{x^4-1}{x^2-4}$
15. $H(x) = \dfrac{x^3-1}{x^2-9}$
16. $G(x) = \dfrac{x^3+1}{x^2+2x}$
17. $R(x) = \dfrac{x^2}{x^2+x-6}$
18. $R(x) = \dfrac{x^2+x-12}{x^2-4}$
19. $G(x) = \dfrac{x}{x^2-4}$
20. $G(x) = \dfrac{3x}{x^2-1}$
21. $R(x) = \dfrac{3}{(x-1)(x^2-4)}$
22. $R(x) = \dfrac{-4}{(x+1)(x^2-9)}$
23. $H(x) = \dfrac{x^2-1}{x^4-16}$
24. $H(x) = \dfrac{x^2+4}{x^4-1}$
25. $F(x) = \dfrac{x^2-3x-4}{x+2}$
26. $F(x) = \dfrac{x^2+3x+2}{x-1}$
27. $R(x) = \dfrac{x^2+x-12}{x-4}$
28. $R(x) = \dfrac{x^2-x-12}{x+5}$
29. $F(x) = \dfrac{x^2+x-12}{x+2}$
30. $G(x) = \dfrac{x^2-x-12}{x+1}$
31. $R(x) = \dfrac{x(x-1)^2}{(x+3)^3}$
32. $R(x) = \dfrac{(x-1)(x+2)(x-3)}{x(x-4)^2}$
33. $R(x) = \dfrac{x^2+x-12}{x^2-x-6}$
34. $R(x) = \dfrac{x^2+3x-10}{x^2+8x+15}$
35. $R(x) = \dfrac{6x^2-7x-3}{2x^2-7x+6}$
36. $R(x) = \dfrac{8x^2+26x+15}{2x^2-x-15}$
37. $R(x) = \dfrac{x^2+5x+6}{x+3}$
38. $R(x) = \dfrac{x^2+x-30}{x+6}$
39. $H(x) = \dfrac{3x-6}{4-x^2}$
40. $H(x) = \dfrac{2-2x}{x^2-1}$
41. $F(x) = \dfrac{x^2-5x+4}{x^2-2x+1}$
42. $F(x) = \dfrac{x^2-2x-15}{x^2+6x+9}$

43. $G(x) = \dfrac{x}{(x+2)^2}$

44. $G(x) = \dfrac{2-x}{(x-1)^2}$

45. $f(x) = x + \dfrac{1}{x}$

46. $f(x) = 2x + \dfrac{9}{x}$

47. $f(x) = x^2 + \dfrac{1}{x}$

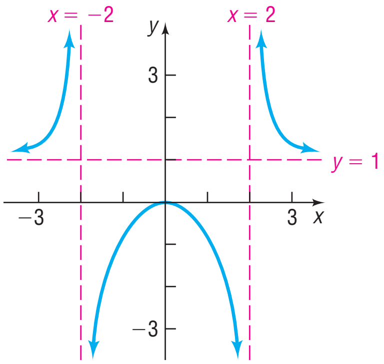
48. $f(x) = 2x^2 + \dfrac{16}{x}$

49. $f(x) = x + \dfrac{1}{x^3}$

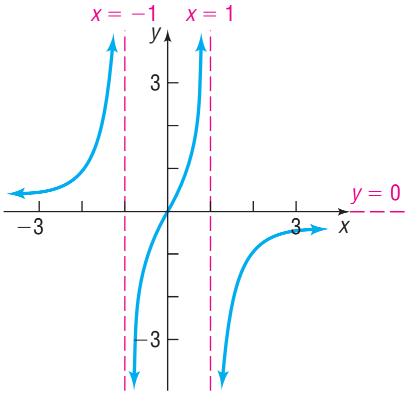
50. $f(x) = 2x + \dfrac{9}{x^3}$

In Problems 51–54, find a rational function that might have the given graph. (More than one answer might be possible.)

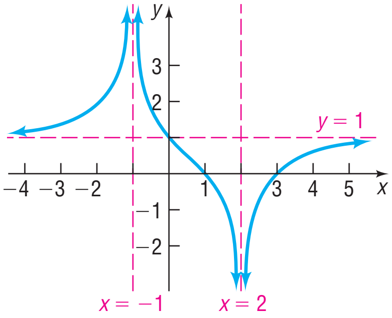
51.



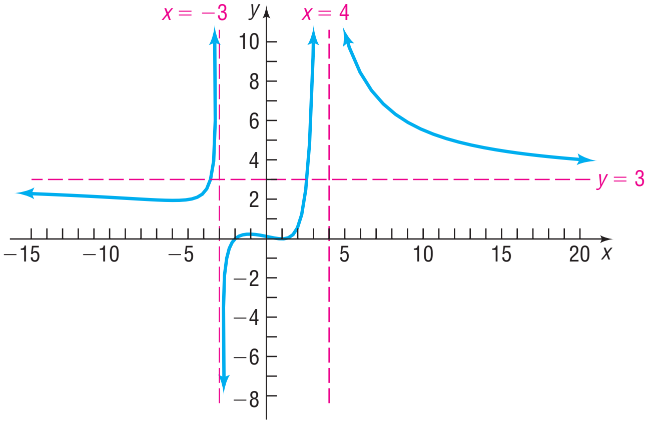
52.



53.



54.



Applications and Extensions

55. Probability Suppose you attend a fundraiser where each person in attendance is given a ball, each with a different number. The balls are numbered 1 through x. Each person in attendance places his or her ball in an urn. After dinner, a ball is chosen at random from the urn. The probability that your ball is selected is $\frac{1}{x}$. Therefore, the probability that your ball is not chosen is $1 - \frac{1}{x}$. Graph $P(x) = 1 - \frac{1}{x}$ using transformations. Comment on the likelihood of your ball *not* being chosen as x increases.

56. Waiting in Line Suppose that two employees at a fast-food restaurant can serve customers at the rate of 6 customers per minute. Further suppose that customers are arriving at the restaurant at the rate of x customers per minute. Then the average waiting time T, in minutes, spent waiting in line and having your order taken and filled is given by the function $T(x) = -\frac{1}{x-6}$, where $0 < x < 6$. Graph this function using transformations.

57. Drug Concentration The concentration C of a certain drug in a patient's bloodstream t hours after injection is given by

$$C(t) = \frac{t}{2t^2 + 1}$$

(a) Find the horizontal asymptote of $C(t)$. What happens to the concentration of the drug as t increases?

(b) Using your graphing utility, graph $C = C(t)$.

(c) Determine the time at which the concentration is highest.

58. Drug Concentration The concentration C of a certain drug in a patient's bloodstream t minutes after injection is given by

$$C(t) = \frac{50t}{t^2 + 25}$$

(a) Find the horizontal asymptote of $C(t)$. What happens to the concentration of the drug as t increases?

(b) Using your graphing utility, graph $C = C(t)$.

(c) Determine the time at which the concentration is highest.

59. Minimum Cost A rectangular area adjacent to a river is to be fenced in; no fence is needed on the river side. The enclosed area is to be 1000 square feet. Fencing for the side parallel to the river is \$5 per linear foot, and fencing for the other two sides is \$8 per linear foot; the four corner posts are \$25 apiece. Let x be the length of one of the sides perpendicular to the river.

(a) Write a function $C(x)$ that describes the cost of the project.
(b) What is the domain of C?
(c) Use a graphing utility to graph $C = C(x)$.
(d) Find the dimensions of the cheapest enclosure.

Source: *www.uncwil.edu/courses/math111hb/PandR/rational/rational.html*

60. Doppler Effect The Doppler effect (named after Christian Doppler) is the change in the pitch (frequency) of the sound from a source (s) as heard by an observer (o) when one or both are in motion. If we assume both the source and the observer are moving in the same direction, the relationship is

$$f' = f_a\left(\frac{v - v_o}{v - v_s}\right)$$

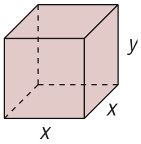
where f' = perceived pitch by the observer
f_a = actual pitch of the source
v = speed of sound in air (assume 772.4 mph)
v_o = speed of the observer
v_s = speed of the source

Suppose that you are traveling down the road at 45 mph and you hear an ambulance (with siren) coming toward you from the rear. The actual pitch of the siren is 600 hertz (Hz).

(a) Write a function $f'(v_s)$ that describes this scenario.
(b) If $f' = 620$ Hz, find the speed of the ambulance.
(c) Use a graphing utility to graph the function.
(d) Verify your answer from part (b).

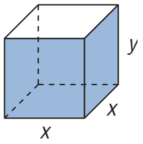
Source: *www.kettering.edu/~drussell/*

61. Minimizing Surface Area United Parcel Service has contracted you to design a closed box with a square base that has a volume of 10,000 cubic inches. See the illustration.



(a) Express the surface area S of the box as a function of x.
(b) Using a graphing utility, graph the function found in part (a).
(c) What is the minimum amount of cardboard that can be used to construct the box?
(d) What are the dimensions of the box that minimize the surface area?
(e) Why might UPS be interested in designing a box that minimizes the surface area?

62. Minimizing Surface Area United Parcel Service has contracted you to design an open box with a square base that has a volume of 5000 cubic inches. See the illustration.



(a) Express the surface area S of the box as a function of x.
(b) Using a graphing utility, graph the function found in part (a).
(c) What is the minimum amount of cardboard that can be used to construct the box?
(d) What are the dimensions of the box that minimize the surface area?
(e) Why might UPS be interested in designing a box that minimizes the surface area?

63. Cost of a Can A can in the shape of a right circular cylinder is required to have a volume of 500 cubic centimeters. The top and bottom are made of material that costs 0.06¢ per square centimeter, while the sides are made of material that costs 0.04¢ per square centimeter.

(a) Express the total cost C of the material as a function of the radius r of the cylinder. (Refer to Figure 50.)
(b) Graph $C = C(r)$. For what value of r is the cost C a minimum?

64. Material Needed to Make a Drum A steel drum in the shape of a right circular cylinder is required to have a volume of 100 cubic feet.

(a) Express the amount A of material required to make the drum as a function of the radius r of the cylinder.
(b) How much material is required if the drum's radius is 3 feet?
(c) How much material is required if the drum's radius is 4 feet?
(d) How much material is required if the drum's radius is 5 feet?
(e) Graph $A = A(r)$. For what value of r is A smallest?

Explaining Concepts: Discussion and Writing

65. Graph each of the following functions:

$$y = \frac{x^2 - 1}{x - 1} \qquad y = \frac{x^3 - 1}{x - 1}$$

$$y = \frac{x^4 - 1}{x - 1} \qquad y = \frac{x^5 - 1}{x - 1}$$

Is $x = 1$ a vertical asymptote? Why not? What is happening for $x = 1$? What do you conjecture about $y = \frac{x^n - 1}{x - 1}$, $n \geq 1$ an integer, for $x = 1$?

66. Graph each of the following functions:

$$y = \frac{x^2}{x-1} \qquad y = \frac{x^4}{x-1} \qquad y = \frac{x^6}{x-1} \qquad y = \frac{x^8}{x-1}$$

What similarities do you see? What differences?

67. Write a few paragraphs that provide a general strategy for graphing a rational function. Be sure to mention the following: proper, improper, intercepts, and asymptotes.

68. Create a rational function that has the following characteristics: crosses the x-axis at 2; touches the x-axis at -1; one vertical asymptote at $x = -5$ and another at $x = 6$; and one horizontal asymptote, $y = 3$. Compare your function to a fellow classmate's. How do they differ? What are their similarities?

69. Create a rational function that has the following characteristics: crosses the x-axis at 3; touches the x-axis at -2; one vertical asymptote, $x = 1$; and one horizontal asymptote, $y = 2$. Give your rational function to a fellow classmate and ask for a written critique of your rational function.

70. Create a rational function with the following characteristics: three real zeros, one of multiplicity 2; y-intercept 1; vertical asymptotes, $x = -2$ and $x = 3$; oblique asymptote, $y = 2x + 1$. Is this rational function unique? Compare your function with those of other students. What will be the same as everyone else's? Add some more characteristics, such as symmetry or naming the real zeros. How does this modify the rational function?

71. Explain the circumstances under which the graph of a rational function will have a hole.

Retain Your Knowledge

Problems 72–75 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

72. Subtract: $(4x^3 - 7x - 1) - (5x^2 - 9x - 3)$

73. Solve: $\dfrac{3x}{3x+1} = \dfrac{x-2}{x+5}$

74. Find the maximum value of $f(x) = -\dfrac{2}{3}x^2 + 6x - 5$.

75. Approximate $\dfrac{\sqrt{5}-3}{\sqrt{7}+2}$. Round your answer to three decimal places.

'Are You Prepared?' Answer

1. $\left(0, \frac{1}{4}\right)$, $(1, 0)$, $(-1, 0)$

5.6 Polynomial and Rational Inequalities

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Solving Linear Inequalities (Section 1.7, pp. 150–151)
- Solving Quadratic Inequalities (Section 4.5, pp. 320–322)

Now Work the **'Are You Prepared?'** problems on page 398.

OBJECTIVES **1** Solve Polynomial Inequalities Algebraically and Graphically (p. 393)
2 Solve Rational Inequalities Algebraically and Graphically (p. 395)

1 Solve Polynomial Inequalities Algebraically and Graphically

In this section we solve inequalities that involve polynomials of degree 3 and higher, along with inequalities that involve rational functions. To help understand the algebraic procedure for solving such inequalities, we use the information obtained in Sections 5.1, 5.2, and 5.5 about the graphs of polynomial and rational functions. The approach follows the same logic used to solve inequalities involving quadratic functions.

EXAMPLE 1 Solving a Polynomial Inequality Using Its Graph

Solve $(x+3)(x-1)^2 > 0$ by graphing $f(x) = (x+3)(x-1)^2$.

By Hand Graphical Solution

Graph $f(x) = (x+3)(x-1)^2$ and determine the intervals of x for which the graph is above the x-axis. Do you see why? These values of x result in $f(x)$ being positive. Using Steps 1 through 8 on page 343, we obtain the graph shown in Figure 52.

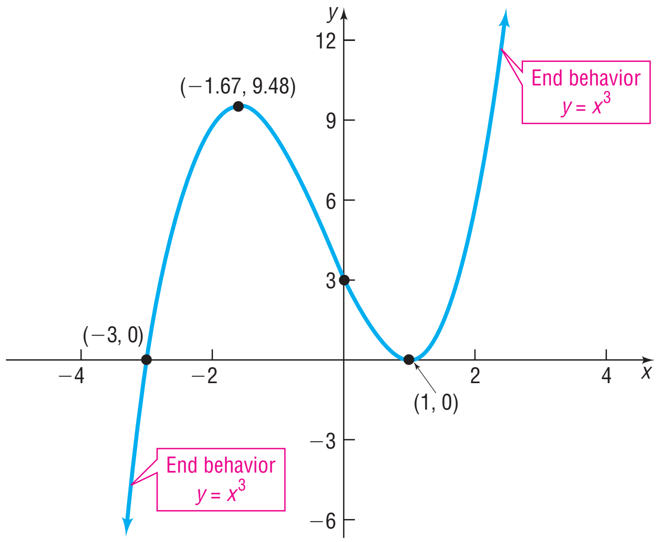


Figure 52 $f(x) = (x+3)(x-1)^2$

From the graph, we can see that $f(x) > 0$ for $-3 < x < 1$ or $x > 1$. The solution set is $\{x \mid -3 < x < 1 \text{ or } x > 1\}$ or, using interval notation, $(-3, 1) \cup (1, \infty)$. ■

Graphing Utility Solution

Graph $Y_1 = (x+3)(x-1)^2$. See Figure 53. Using the ZERO command, find that the x-intercepts of the graph of Y_1 are -3 and 1. The graph of Y_1 is above the x-axis (and therefore f is positive) for $-3 < x < 1$ or $x > 1$. Therefore, the solution set is $\{x \mid -3 < x < 1 \text{ or } x > 1\}$ or, using interval notation, $(-3, 1) \cup (1, \infty)$.

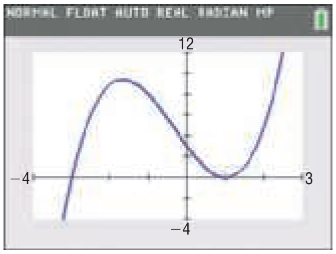


Figure 53 $Y_1 = (x+3)(x-1)^2$ ■

Now Work PROBLEM 9

The results of Example 1 lead to the following approach to solving polynomial inequalities algebraically. Suppose that the polynomial inequality is in one of the forms

$$f(x) < 0 \qquad f(x) > 0 \qquad f(x) \le 0 \qquad f(x) \ge 0$$

Locate the zeros (x-intercepts of the graph) of the polynomial function f. We know that the sign of f can change on either side of an x-intercept, so we use these zeros to divide the real number line into intervals. Then we know that on each interval the graph of f is either above $[f(x) > 0]$ or below $[f(x) < 0]$ the x-axis.

EXAMPLE 2 How to Solve a Polynomial Inequality Algebraically

Solve the inequality $x^4 > x$ algebraically, and graph the solution set.

Step-by-Step Solution

Step 1: Write the inequality so that a polynomial expression f is on the left side and zero is on the right side.

Rearrange the inequality so that 0 is on the right side.

$$x^4 > x$$
$$x^4 - x > 0 \quad \text{Subtract } x \text{ from both sides of the inequality.}$$

This inequality is equivalent to the one we wish to solve.

Step 2: Determine the real zeros (*x*-intercepts of the graph) of *f*.

Find the zeros of $f(x) = x^4 - x$ by solving $x^4 - x = 0$.

$$x^4 - x = 0$$

$$x(x^3 - 1) = 0 \quad \text{Factor out } x.$$

$$x(x-1)(x^2 + x + 1) = 0 \quad \text{Factor the difference of two cubes.}$$

$$x = 0 \quad \text{or} \quad x - 1 = 0 \quad \text{or} \quad x^2 + x + 1 = 0 \quad \text{Set each factor equal to zero and solve.}$$

$$x = 0 \quad \text{or} \quad x = 1$$

The equation $x^2 + x + 1 = 0$ has no real solutions. Do you see why?

Step 3: Use the zeros found in Step 2 to divide the real number line into intervals.

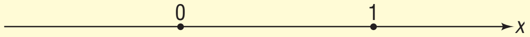
Use the zeros to separate the real number line into three intervals:

$$(-\infty, 0) \qquad (0, 1) \qquad (1, \infty)$$

Step 4: Select a number in each interval, evaluate *f* at the number, and determine whether *f* is positive or negative. If *f* is positive, all values of *f* in the interval are positive. If *f* is negative, all values of *f* in the interval are negative.

Select a test number in each interval found in Step 3 and evaluate $f(x) = x^4 - x$ at each number to determine if $f(x)$ is positive or negative. See Table 19.

Table 19

			
Interval	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
Number chosen	-1	$\frac{1}{2}$	2
Value of *f*	$f(-1) = 2$	$f\left(\frac{1}{2}\right) = -\frac{7}{16}$	$f(2) = 14$
Conclusion	Positive	Negative	Positive

Note: If the inequality is not strict ($\le$ or $\ge$), include the solutions of $f(x) = 0$ in the solution set. ■

Since we want to know where $f(x)$ is positive, conclude that $f(x) > 0$ for all numbers x for which $x < 0$ or $x > 1$. Because the original inequality is strict, numbers x that satisfy the equation $x^4 = x$ are not solutions. The solution set of the inequality $x^4 > x$ is $\{x | x < 0 \text{ or } x > 1\}$ or, using interval notation, $(-\infty, 0) \cup (1, \infty)$.

Figure 54 shows the graph of the solution set.

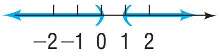


Figure 54

✓**Check:** Graph $Y_1 = x^4$ and $Y_2 = x$ on the same screen. Use INTERSECT to find where $Y_1 = Y_2$. Then determine where $Y_1 > Y_2$. ■

The Role of Multiplicity in Solving Polynomial Inequalities

In Example 2, we used the number -1 and found that f is positive for all $x < 0$. Because the "cut point" of 0 is the result of a zero of odd multiplicity (x is a factor to the first power), we know that the sign of f will change on either side of 0, so for $0 < x < 1$, f will be negative. Similarly, we know that f will be positive for $x > 1$, since the multiplicity of the zero 1 is odd. Therefore, the solution set of $x^4 > x$ is $\{x | x < 0 \text{ or } x > 1\}$ or, using interval notation, $(-\infty, 0) \cup (1, \infty)$.

Now Work PROBLEM 21

2 Solve Rational Inequalities Algebraically and Graphically

Just as we presented a graphical approach to help understand the algebraic procedure for solving inequalities involving polynomials, we present a graphical approach to help understand the algebraic procedure for solving inequalities involving rational expressions.

EXAMPLE 3 Solving a Rational Inequality Using Its Graph

Solve $\dfrac{x-1}{x^2-4} \geq 0$ by graphing $R(x) = \dfrac{x-1}{x^2-4}$.

By Hand Graphical Solution

Graph $R(x) = \dfrac{x-1}{x^2-4}$ and determine the intervals of x such that the graph is above or on the x-axis. Do you see why? These values of x result in $R(x)$ being positive or zero. We graphed $R(x) = \dfrac{x-1}{x^2-4}$ in Example 1 from Section 5.5 (p. 383). We reproduce the graph in Figure 55.

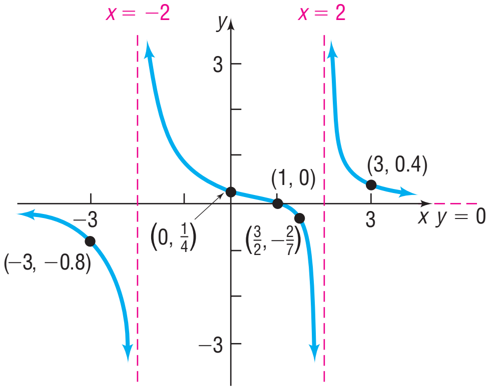


Figure 55 $R(x) = \dfrac{x-1}{x^2-4}$

From the graph, we can see that $R(x) \geq 0$ for $-2 < x \leq 1$ or $x > 2$. The solution set is $\{x \mid -2 < x \leq 1 \text{ or } x > 2\}$ or, using interval notation, $(-2, 1] \cup (2, \infty)$.

Graphing Utility Solution

Graph $Y_1 = \dfrac{x-1}{x^2-4}$. See Figure 56. Using the ZERO command, find that the x-intercept of the graph of Y_1 is 1. The graph of Y_1 is above the x-axis (and, therefore, R is positive) for $-2 < x < 1$ or $x > 2$. Since the inequality is not strict, include 1 in the solution set. Therefore, the solution set is $\{x \mid -2 < x \leq 1 \text{ or } x > 2\}$ or, using interval notation, $(-2, 1] \cup (2, \infty)$. Do you see why we do not include 2?

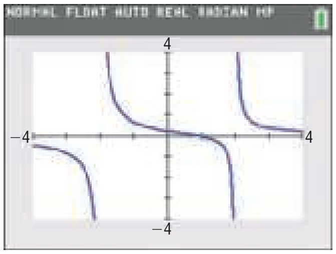


Figure 56 $Y_1 = \dfrac{x-1}{x^2-4}$

Now Work PROBLEMS 15 AND 33

To solve a rational inequality algebraically, we follow the same approach that we used to solve a polynomial inequality algebraically. However, we must also identify the zeros of the denominator of the rational function, because the sign of a rational function may change on either side of a vertical asymptote. Convince yourself of this by looking at Figure 55. Notice the function values are negative for $x < -2$ and are positive for $x > -2$ (but less than 1).

EXAMPLE 4 How to Solve a Rational Inequality Algebraically

Solve the inequality $\dfrac{3x^2+13x+9}{(x+2)^2} \leq 3$ algebraically, and graph the solution set.

Step-by-Step Solution

Step 1: Write the inequality so that a rational expression f is on the left side and zero is on the right side.

Rearrange the inequality so that 0 is on the right side.

$$\frac{3x^2+13x+9}{(x+2)^2} \leq 3$$

$$\frac{3x^2+13x+9}{x^2+4x+4} - 3 \leq 0 \qquad \textbf{Subtract 3 from both sides of the inequality; Expand } (x+2)^2.$$

$$\frac{3x^2+13x+9}{x^2+4x+4} - 3 \cdot \frac{x^2+4x+4}{x^2+4x+4} \leq 0 \qquad \textbf{Multiply 3 by } \frac{x^2+4x+4}{x^2+4x+4}.$$

$$\frac{3x^2+13x+9-3x^2-12x-12}{x^2+4x+4} \leq 0 \qquad \textbf{Write as a single quotient.}$$

$$\frac{x-3}{(x+2)^2} \leq 0 \qquad \textbf{Combine like terms.}$$

Step 2: Determine the real zeros (x-intercepts of the graph) of f and the real numbers for which f is undefined.

The zero of $f(x) = \dfrac{x-3}{(x+2)^2}$ is 3. Also, f is undefined for $x = -2$.

Step 3: Use the zeros and undefined values found in Step 2 to divide the real number line into intervals.

Use the zero and the undefined value to separate the real number line into three intervals:

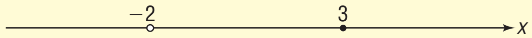
$$(-\infty, -2) \quad (-2, 3) \quad (3, \infty)$$

Step 4: Select a number in each interval, evaluate f at the number, and determine whether $f(x)$ is positive or negative. If $f(x)$ is positive, all values of f in the interval are positive. If $f(x)$ is negative, all values of f in the interval are negative.

Note: If the inequality is not strict ($\leq$ or $\geq$), include the solutions of $f(x) = 0$ in the solution set. ■

Select a test number in each interval from Step 3, and evaluate f at each number to determine whether $f(x)$ is positive or negative. See Table 20.

Table 20

			
Interval	$(-\infty, -2)$	$(-2, 3)$	$(3, \infty)$
Number chosen	-3	0	4
Value of *f*	$f(-3) = -6$	$f(0) = -\frac{3}{4}$	$f(4) = \frac{1}{36}$
Conclusion	Negative	Negative	Positive

Since we want to know where $f(x)$ is negative or zero, we conclude that $f(x) \leq 0$ for all numbers for which $x < -2$ or $-2 < x \leq 3$. Notice that we do not include -2 in the solution because -2 is not in the domain of f. The solution set of the inequality $\dfrac{3x^2 + 13x + 9}{(x+2)^2} \leq 3$ is $\{x \mid x < -2 \text{ or } -2 < x \leq 3\}$ or, using interval notation, $(-\infty, -2) \cup (-2, 3]$. Figure 57 shows the graph of the solution set.

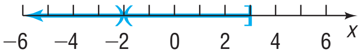


Figure 57

✓**Check:** Graph $Y_1 = \dfrac{3x^2 + 13x + 9}{(x+2)^2}$ and $Y_2 = 3$ on the same screen. Use INTERSECT to find where $Y_1 = Y_2$. Then determine where $Y_1 \leq Y_2$. ■

The Role of Multiplicity in Solving Rational Inequalities

In Example 4, we used the number -3 and found that R is negative for all $x < -2$. Because the "cut point" of -2 is the result of a zero of even multiplicity, we know the sign of R will not change on either side of -2, so for $-2 < x < 3$, R will also be negative. Because the "cut point" of 3 is the result of a zero of odd multiplicity, the sign of R will change on either side of 3, so for $x > 3$, R will be positive. Therefore, the solution set of $\dfrac{3x^2 + 13x + 9}{(x+2)^2} \leq 3$ is $\{x \mid x < -2 \text{ or } -2 < x \leq 3\}$ or, using interval notation, $(-\infty, -2) \cup (-2, 3]$.

Now Work PROBLEM 39

SUMMARY

Steps for Solving Polynomial and Rational Inequalities Algebraically

STEP 1: Write the inequality so that a polynomial or rational expression f is on the left side and zero is on the right side in one of the following forms:

$$f(x) > 0 \qquad f(x) \geq 0 \qquad f(x) < 0 \qquad f(x) \leq 0$$

For rational expressions, be sure that the left side is written as a single quotient, and find the domain of f.

STEP 2: Determine the real numbers at which the expression f equals zero and, if the expression is rational, the real numbers at which the expression f is undefined.

STEP 3: Use the numbers found in Step 2 to separate the real number line into intervals.

STEP 4: Select a number in each interval and evaluate f at the number.

(a) If the value of f is positive, then $f(x) > 0$ for all numbers x in the interval.

(b) If the value of f is negative, then $f(x) < 0$ for all numbers x in the interval.

If the inequality is not strict ($\geq$ or $\leq$), include the solutions of $f(x) = 0$ that are in the domain of f in the solution set. Be careful to exclude values of x where f is undefined.

5.6 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Solve the inequality $3 - 4x > 5$. Graph the solution set. (pp. 150–151)

2. Solve the inequality $x^2 - 5x \le 24$. Graph the solution set. (pp. 320–322)

Concepts and Vocabulary

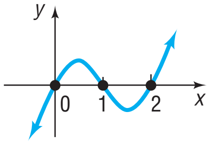
3. Which of the following could be a test number for the interval $-2 < x < 5$?
(a) -3 (b) -2 (c) 4 (d) 7

4. ***True or False*** The graph of $f(x) = \frac{x}{x-3}$ is above the x-axis for $x < 0$ or $x > 3$, so the solution set of the inequality $\frac{x}{x-3} \ge 0$ is $\{x \mid x \le 0 \text{ or } x \ge 3\}$.

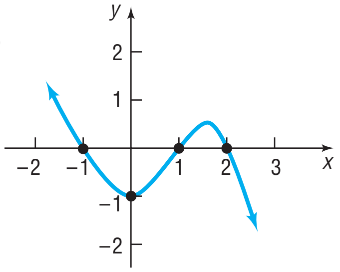
Skill Building

In Problems 5–8, use the graph of the function f to solve the inequality.

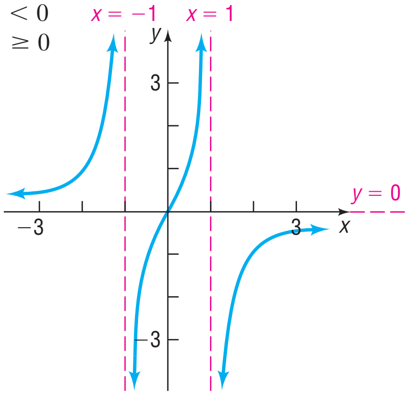
5. (a) $f(x) > 0$
(b) $f(x) \le 0$



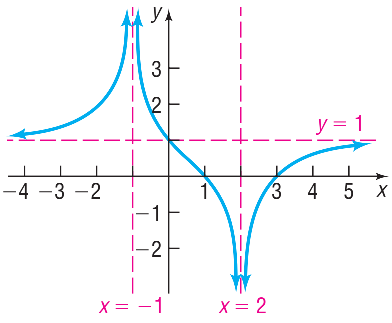
6. (a) $f(x) < 0$
(b) $f(x) \ge 0$



7. (a) $f(x) < 0$
(b) $f(x) \ge 0$



8. (a) $f(x) > 0$
(b) $f(x) \le 0$



In Problems 9–14, solve the inequality by using the graph of the function.

[**Hint:** The graphs were drawn in Problems 81–86 of Section 5.1.]

9. Solve $f(x) < 0$, where $f(x) = x^2(x - 3)$.

10. Solve $f(x) \le 0$, where $f(x) = x(x + 2)^2$.

11. Solve $f(x) \ge 0$, where $f(x) = (x + 4)^2(1 - x)$.

12. Solve $f(x) > 0$, where $f(x) = (x - 1)(x + 3)^2$.

13. Solve $f(x) \le 0$, where $f(x) = -2(x + 2)(x - 2)^3$.

14. Solve $f(x) < 0$, where $f(x) = -\frac{1}{2}(x + 4)(x - 1)^3$.

In Problems 15–18, solve the inequality by using the graph of the function.

[**Hint:** The graphs were drawn in Problems 7–10 of Section 5.5.]

15. Solve $R(x) > 0$, where $R(x) = \frac{x+1}{x(x+4)}$.

16. Solve $R(x) < 0$, where $R(x) = \frac{x}{(x-1)(x+2)}$.

17. Solve $R(x) \le 0$, where $R(x) = \frac{3x+3}{2x+4}$.

18. Solve $R(x) \ge 0$, where $R(x) = \frac{2x+4}{x-1}$.

In Problems 19–48, solve each inequality algebraically.

19. $(x - 5)^2(x + 2) < 0$

20. $(x - 5)(x + 2)^2 > 0$

21. $x^3 - 4x^2 > 0$

22. $x^3 + 8x^2 < 0$

23. $2x^3 > -8x^2$

24. $3x^3 < -15x^2$

25. $(x-1)(x-2)(x-3) \le 0$

26. $(x+1)(x+2)(x+3) \le 0$

27. $x^3 - 2x^2 - 3x > 0$

28. $x^3 + 2x^2 - 3x > 0$

29. $x^4 > x^2$

30. $x^4 < 9x^2$

31. $x^4 > 1$

32. $x^3 > 1$

33. $\dfrac{x+1}{x-1} > 0$

34. $\dfrac{x-3}{x+1} > 0$

35. $\dfrac{(x-1)(x+1)}{x} \le 0$

36. $\dfrac{(x-3)(x+2)}{x-1} \le 0$

37. $\dfrac{(x-2)^2}{x^2-1} \ge 0$

38. $\dfrac{(x+5)^2}{x^2-4} \ge 0$

39. $\dfrac{x+4}{x-2} \le 1$

40. $\dfrac{x+2}{x-4} \ge 1$

41. $\dfrac{3x-5}{x+2} \le 2$

42. $\dfrac{x-4}{2x+4} \ge 1$

43. $\dfrac{1}{x-2} < \dfrac{2}{3x-9}$

44. $\dfrac{5}{x-3} > \dfrac{3}{x+1}$

45. $\dfrac{x^2(3+x)(x+4)}{(x+5)(x-1)} \ge 0$

46. $\dfrac{x(x^2+1)(x-2)}{(x-1)(x+1)} \ge 0$

47. $\dfrac{(3-x)^3(2x+1)}{x^3-1} < 0$

48. $\dfrac{(2-x)^3(3x-2)}{x^3+1} < 0$

Mixed Practice

In Problems 49–60, solve each inequality algebraically.

49. $(x+1)(x-3)(x-5) > 0$

50. $(2x-1)(x+2)(x+5) < 0$

51. $7x - 4 \ge -2x^2$

52. $x^2 + 3x \ge 10$

53. $\dfrac{x+1}{x-3} \le 2$

54. $\dfrac{x-1}{x+2} \ge -2$

55. $3(x^2-2) < 2(x-1)^2 + x^2$

56. $(x-3)(x+2) < x^2 + 3x + 5$

57. $6x - 5 < \dfrac{6}{x}$

58. $x + \dfrac{12}{x} < 7$

59. $x^3 - 9x \le 0$

60. $x^3 - x \ge 0$

In Problems 61 and 62, (a) find the zeros of each function, (b) factor each function over the real numbers, (c) graph each function by hand, and (d) solve $f(x) < 0$.

61. $f(x) = 2x^4 + 11x^3 - 11x^2 - 104x - 48$

62. $f(x) = 4x^5 - 19x^4 + 32x^3 - 31x^2 + 28x - 12$

In Problems 63–66, (a) graph each function by hand, and (b) solve $f(x) \ge 0$.

63. $f(x) = \dfrac{x^2+5x-6}{x^2-4x+4}$

64. $f(x) = \dfrac{2x^2+9x+9}{x^2-4}$

65. $f(x) = \dfrac{x^3+2x^2-11x-12}{x^2-x-6}$

66. $f(x) = \dfrac{x^3-6x^2+9x-4}{x^2+x-20}$

Applications and Extensions

67. For what positive numbers will the cube of a number exceed four times its square?

68. For what positive numbers will the cube of a number be less than the number?

69. What is the domain of the function $f(x) = \sqrt{x^4 - 16}$?

70. What is the domain of the function $f(x) = \sqrt{x^3 - 3x^2}$?

71. What is the domain of the function $f(x) = \sqrt{\dfrac{x-2}{x+4}}$?

72. What is the domain of the function $f(x) = \sqrt{\dfrac{x-1}{x+4}}$?

In Problems 73–76, determine where the graph of f is below the graph of g by solving the inequality $f(x) \le g(x)$. Graph f and g together.

73. $f(x) = x^4 - 1$
$g(x) = -2x^2 + 2$

74. $f(x) = x^4 - 1$
$g(x) = x - 1$

75. $f(x) = x^4 - 4$
$g(x) = 3x^2$

76. $f(x) = x^4$
$g(x) = 2 - x^2$

77. Average Cost Suppose that the daily cost C of manufacturing bicycles is given by $C(x) = 80x + 5000$. Then the average daily cost $\overline{C}$ is given by $\overline{C}(x) = \dfrac{80x + 5000}{x}$. How many bicycles must be produced each day for the average cost to be no more than \$100?

78. Average Cost See Problem 77. Suppose that the government imposes a \$1000-per-day tax on the bicycle manufacturer so that the daily cost C of manufacturing x bicycles is now given by $C(x) = 80x + 6000$. Now the average daily cost $\overline{C}$ is given by $\overline{C}(x) = \dfrac{80x + 6000}{x}$. How many bicycles must be produced each day for the average cost to be no more than \$100?

79. Bungee Jumping Originating on Pentecost Island in the Pacific, the practice of a person jumping from a high place harnessed to a flexible attachment was introduced to Western culture in 1979 by the Oxford University Dangerous Sport Club. One important parameter to know before attempting a bungee jump is the amount the cord will stretch at the bottom of the fall. The stiffness of the cord is related to the amount of stretch by the equation

$$K = \frac{2W(S + L)}{S^2}$$

where W = weight of the jumper (pounds)
K = cord's stiffness (pounds per foot)
L = free length of the cord (feet)
S = stretch (feet)

(a) A 150-pound person plans to jump off a ledge attached to a cord of length 42 feet. If the stiffness of the cord is no less than 16 pounds per foot, how much will the cord stretch?

(b) If safety requirements will not permit the jumper to get any closer than 3 feet to the ground, what is the minimum height required for the ledge in part (a)?

Source: *American Institute of Physics, Physics News Update, No. 150, November 5, 1993.*

80. Gravitational Force According to Newton's Law of Universal Gravitation, the attractive force F between two bodies is given by

$$F = G\frac{m_1 m_2}{r^2}$$

where m_1, m_2 = the masses of the two bodies
r = distance between the two bodies
G = gravitational constant $= 6.6742 \times 10^{-11}$ newtons · meter^2 · kilogram^{-2}

Suppose an object is traveling directly from Earth to the moon. The mass of Earth is 5.9742×10^{24} kilograms, the mass of the moon is 7.349×10^{22} kilograms, and the mean distance from Earth to the moon is 384,400 kilometers. For an object between Earth and the moon, how far from Earth is the force on the object due to the moon greater than the force on the object due to Earth?

Source: *www.solarviews.com;en.wikipedia.org*

81. Field Trip Mrs. West has decided to take her fifth grade class to a play. The manager of the theater agreed to discount the regular \$40 price of the ticket by \$0.20 for each ticket sold. The cost of the bus, \$500, will be split equally among the students. How many students must attend to keep the cost per student at or below \$40?

Explaining Concepts: Discussion and Writing

82. Make up an inequality that has no solution. Make up one that has exactly one solution.

83. The inequality $x^4 + 1 < -5$ has no solution. Explain why.

84. A student attempted to solve the inequality $\frac{x+4}{x-3} \le 0$ by multiplying both sides of the inequality by $x - 3$ to get $x + 4 \le 0$. This led to a solution of $\{x \mid x \le -4\}$. Is the student correct? Explain.

85. Write a rational inequality whose solution set is $\{x \mid -3 < x \le 5\}$.

Retain Your Knowledge

Problems 86–89 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

86. Solve: $9 - 2x \le 4x + 1$

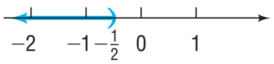
87. Given $f(x) = x^2 + 3x - 2$, find $f(x - 2)$.

88. Factor completely: $6x^4y^4 + 3x^3y^5 - 18x^2y^6$

89. Suppose y varies directly with $\sqrt{x}$. Write a general formula to describe the variation if $y = 2$ when $x = 9$.

'Are You Prepared?' Answers

1. $\left\{x \middle| x < -\frac{1}{2}\right\}$ or $\left(-\infty, -\frac{1}{2}\right)$



2. $\{x \mid -3 \le x \le 8\}$ or $[-3, 8]$

Chapter Review

Things to Know

Power function (pp. 332–334)

$f(x) = x^n, \quad n \ge 2$ even

Domain: all real numbers Range: nonnegative real numbers

Passes through $(-1, 1), (0, 0), (1, 1)$

Even function

Decreasing on $(-\infty, 0]$, increasing on $[0, \infty)$

$f(x) = x^n$, $n \geq 3$ odd	Domain: all real numbers Range: all real numbers Passes through $(-1, -1)$, $(0, 0)$, $(1, 1)$ Odd function Increasing on $(-\infty, \infty)$
Polynomial function (pp. 331–341) $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0, \quad a_n \neq 0$	Domain: all real numbers At most $n - 1$ turning points End behavior: Behaves like $y = a_nx^n$ for large $\lvert x \rvert$
Real zeros of a polynomial function *f* (p. 335)	Real numbers for which $f(x) = 0$; the real zeros of *f* are the *x*-intercepts of the graph of *f*.
Remainder Theorem (p. 352)	If a polynomial function $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.
Factor Theorem (p. 353)	$x - c$ is a factor of a polynomial function $f(x)$ if and only if $f(c) = 0$.
Descartes' Rule of Signs (p. 355)	Let *f* denote a polynomial function written in standard form. • The number of positive real zeros of *f* either equals the number of variations in the sign of the nonzero coefficients of $f(x)$ or else equals that number less an even integer. • The number of negative real zeros of *f* either equals the number of variations in the sign of the nonzero coefficients of $f(-x)$ or else equals that number less an even integer.
Rational Zeros Theorem (p. 355)	Let *f* be a polynomial function of degree 1 or higher of the form $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \quad a_n \neq 0, a_0 \neq 0$ where each coefficient is an integer. If $\frac{p}{q}$, in lowest terms, is a rational zero of f, then p must be a factor of a_0, and q must be a factor of a_n.
Intermediate Value Theorem (p. 362)	Let f be a continuous function. If $a < b$ and $f(a)$ and $f(b)$ are of opposite sign, then there is at least one real zero of f between a and b.
Fundamental Theorem of Algebra (p. 367)	Every complex polynomial function $f(x)$ of degree $n \geq 1$ has at least one complex zero.
Conjugate Pairs Theorem (p. 368)	Let $f(x)$ be a polynomial function whose coefficients are real numbers. If $r = a + bi$ is a zero of f, then its complex conjugate $\bar{r} = a - bi$ is also a zero of *f*.
Rational function (pp. 372–379) $R(x) = \frac{p(x)}{q(x)}$	Domain: $\{x \mid q(x) \neq 0\}$
p, q are polynomial functions and q is not the zero polynomial.	Vertical asymptotes: With $R(x)$ in lowest terms, if $q(r) = 0$ for some real number, then $x = r$ is a vertical asymptote. Horizontal or oblique asymptote: See the summary on pages 377–378.

Objectives

Section		You should be able to . . .	Examples	Review Exercises
5.1	1	Identify polynomial functions and their degree (p. 331)	1	1–4
	2	Graph polynomial functions using transformations (p. 334)	2, 3	5–7
	3	Identify the real zeros of a polynomial function and their multiplicity (p. 335)	4–9	8–11
	4	Analyze the graph of a polynomial function (p. 342)	10, 11	8–11
	5	Build cubic models from data (p. 345)	12	51
5.2	1	Use the Remainder and Factor Theorems (p. 352)	1, 2	12–14
	2	Use Descartes' Rule of Signs to determine the number of positive and the number of negative real zeros of a polynomial function. (p. 354)	3	15, 16
	3	Use the Rational Zeros Theorem to list the potential rational zeros of a polynomial function (p. 355)	4	17–20
	4	Find the real zeros of a polynomial function (p. 356)	5, 6	18–20
	5	Solve polynomial equations (p. 359)	7	21, 22
	6	Use the Theorem for Bounds on Zeros (p. 359)	8, 9, 10	23, 24
	7	Use the Intermediate Value Theorem (p. 362)	11	25–28
5.3	1	Use the Conjugate Pairs Theorem (p. 367)	1	29, 30
	2	Find a polynomial function with specified zeros (p.368)	2	29, 30
	3	Find the complex zeros of a polynomial function (p. 369)	3	31–34

Section		You should be able to . . .	Examples	Review Exercises
5.4	1	Find the domain of a rational function (p. 372)	1–3	35, 36
	2	Find the vertical asymptotes of a rational function (p. 376)	4	35, 36, 44
	3	Find the horizontal or oblique asymptote of a rational function (p. 377)	5–8	35, 36, 44
5.5	1	Analyze the graph of a rational function (p. 382)	1–6	37–42
	2	Solve applied problems involving rational functions (p. 389)	7	50
5.6	1	Solve polynomial inequalities algebraically and graphically (p. 393)	1, 2	43, 45, 46
	2	Solve rational inequalities algebraically and graphically (p. 395)	3, 4	44, 47–49

Review Exercises

In Problems 1–4, determine whether the function is a polynomial function, rational function, or neither. For those that are polynomial functions, state the degree. For those that are not polynomial functions, tell why not.

1. $f(x) = 4x^5 - 3x^2 + 5x - 2$ **2.** $f(x) = \dfrac{3x^5}{2x+1}$ **3.** $f(x) = 3x^2 + 5x^{1/2} - 1$ **4.** $f(x) = 3$

In Problems 5–7, graph each function using transformations (shifting, compressing, stretching, and reflection). Show all the stages.

5. $f(x) = (x+2)^3$ **6.** $f(x) = -(x-1)^4$ **7.** $f(x) = (x-1)^4 + 2$

In Problems 8–11, analyze each polynomial function by following Steps 1 through 8 on page 343.

8. $f(x) = x(x+2)(x+4)$ **9.** $f(x) = (x-2)^2(x+4)$

10. $f(x) = -2x^3 + 4x^2$ **11.** $f(x) = (x-1)^2(x+3)(x+1)$

In Problems 12 and 13, find the remainder R when $f(x)$ is divided by $g(x)$. Is g a factor of f?

12. $f(x) = 8x^3 - 3x^2 + x + 4;\quad g(x) = x - 1$ **13.** $f(x) = x^4 - 2x^3 + 15x - 2;\quad g(x) = x + 2$

14. Find the value of $f(x) = 12x^6 - 8x^4 + 1$ at $x = 4$.

In Problems 15 and 16, use Descartes' Rule of Signs to determine how many positive and how many negative zeros each polynomial function may have. Do not attempt to find the zeros.

15. $f(x) = 12x^8 - x^7 + 8x^4 - 2x^3 + x + 3$ **16.** $f(x) = -6x^5 + x^4 + 5x^3 + x + 1$

17. List all the potential rational zeros of $f(x) = 12x^8 - x^7 + 6x^4 - x^3 + x - 3$.

In Problems 18–20, use the Rational Zeros Theorem to find all the real zeros of each polynomial function. Use the zeros to factor f over the real numbers.

18. $f(x) = x^3 - 3x^2 - 6x + 8$ **19.** $f(x) = 4x^3 + 4x^2 - 7x + 2$ **20.** $f(x) = x^4 - 4x^3 + 9x^2 - 20x + 20$

In Problems 21 and 22, solve each equation in the real number system.

21. $2x^4 + 2x^3 - 11x^2 + x - 6 = 0$ **22.** $2x^4 + 7x^3 + x^2 - 7x - 3 = 0$

In Problems 23 and 24, find bounds to the real zeros of each polynomial function. Obtain a complete graph of f using a graphing utility.

23. $f(x) = x^3 - x^2 - 4x + 2$ **24.** $f(x) = 2x^3 - 7x^2 - 10x + 35$

In Problems 25 and 26, use the Intermediate Value Theorem to show that each polynomial function has a zero in the given interval.

25. $f(x) = 3x^3 - x - 1;\ [0, 1]$ **26.** $f(x) = 8x^4 - 4x^3 - 2x - 1;\ [0, 1]$

In Problems 27 and 28, each polynomial function has exactly one positive zero. Approximate the zero correct to two decimal places.

27. $f(x) = x^3 - x - 2$ **28.** $f(x) = 8x^4 - 4x^3 - 2x - 1$

In Problems 29 and 30, information is given about a complex polynomial function $f(x)$ whose coefficients are real numbers. Find the remaining zeros of f. Then find a polynomial function with real coefficients that has the zeros.

29. Degree 3; zeros: $4 + i, 6$ **30.** Degree 4; zeros: $i, 1 + i$

In Problems 31–34, find the complex zeros of each polynomial function $f(x)$. Write f in factored form.

31. $f(x) = x^3 - 3x^2 - 6x + 8$

32. $f(x) = 4x^3 + 4x^2 - 7x + 2$

33. $f(x) = x^4 - 4x^3 + 9x^2 - 20x + 20$

34. $f(x) = 2x^4 + 2x^3 - 11x^2 + x - 6$

In Problems 35 and 36, find the domain of each rational function. Find any horizontal, vertical, or oblique asymptotes.

35. $R(x) = \dfrac{x + 2}{x^2 - 9}$

36. $R(x) = \dfrac{x^2 + 3x + 2}{(x + 2)^2}$

In Problems 37–42, analyze each rational function following Steps 1–7 given on page 384.

37. $R(x) = \dfrac{2x - 6}{x}$

38. $H(x) = \dfrac{x + 2}{x(x - 2)}$

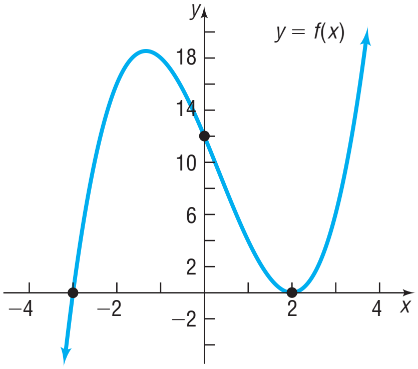
39. $R(x) = \dfrac{x^2 + x - 6}{x^2 - x - 6}$

40. $F(x) = \dfrac{x^3}{x^2 - 4}$

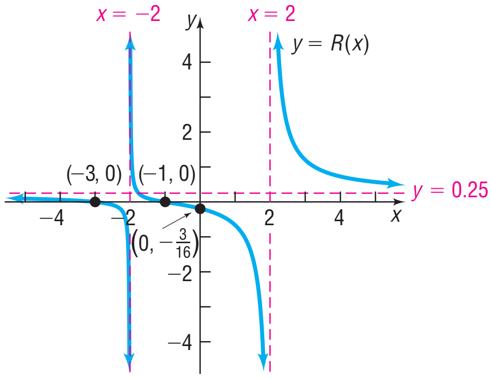
41. $R(x) = \dfrac{2x^4}{(x - 1)^2}$

42. $G(x) = \dfrac{x^2 - 4}{x^2 - x - 2}$

43. Use the graph below of a polynomial function $y = f(x)$ to solve (a) $f(x) = 0$, (b) $f(x) > 0$, (c) $f(x) \le 0$, and (d) determine f.



44. Use the graph below of a rational function $y = R(x)$ to (a) identify the horizontal asymptote of R, (b) identify the vertical asymptotes of R, (c) solve $R(x) < 0$, (d) solve $R(x) \ge 0$, and (e) determine R.



In Problems 45–49, solve each inequality. Graph the solution set.

45. $x^3 + x^2 < 4x + 4$

46. $x^3 + 4x^2 \ge x + 4$

47. $\dfrac{2x - 6}{1 - x} < 2$

48. $\dfrac{(x - 2)(x - 1)}{x - 3} \ge 0$

49. $\dfrac{x^2 - 8x + 12}{x^2 - 16} > 0$

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50. Making a Can A can in the shape of a right circular cylinder is required to have a volume of 250 cubic centimeters.

(a) Express the amount A of material to make the can as a function of the radius r of the cylinder.

(b) How much material is required if the can is of radius 3 centimeters?

(c) How much material is required if the can is of radius 5 centimeters?

(d) Graph $A = A(r)$. For what value of r is A smallest?

51. Housing Prices The data in the table represent the January median new-home prices in the United States for the years shown.

(a) With a graphing utility, draw a scatter diagram of the data. Comment on the type of relation that appears to exist between the two variables.

(b) Decide on the function of best fit to these data (linear, quadratic, or cubic), and use this function to predict the median new-home price in the United States for January 2020 $(t = 9)$.

(c) Draw the function of best fit on the scatter diagram obtained in part (a).

Year, *t*	Median Price, *P* ($1000s)
2004, 1	209.5
2006, 2	244.9
2008, 3	232.4
2010, 4	218.2
2012, 5	221.7
2014, 6	262.7

52. The illustration shows the graph of a polynomial function.

(a) Is the degree of the polynomial even or odd?

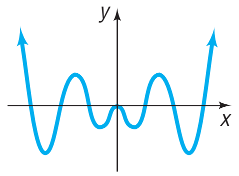
(b) Is the leading coefficient positive or negative?

(c) Is the function even, odd, or neither?

(d) Why is x^2 necessarily a factor of the polynomial?

(e) What is the minimum degree of the polynomial?

(f) Formulate five different polynomial functions whose graphs could look like the one shown. Compare yours to those of other students. What similarities do you see? What differences?



Chapter Test

CHAPTER Test Prep VIDEOS

The Chapter Test Prep Videos are step-by-step solutions available in MyMathLab®, or on this text's YouTube™ Channel. Flip back to the Resources for Success page for a link to this text's YouTube channel.

1. Graph $f(x) = (x-3)^4 - 2$ using transformations.

2. For the polynomial function $g(x) = 2x^3 + 5x^2 - 28x - 15$,

(a) Determine the maximum number of real zeros that the function may have.

(b) List the potential rational zeros.

(c) Determine the real zeros of g. Factor g over the reals.

(d) Find the x- and y-intercepts of the graph of g.

(e) Determine whether the graph crosses or touches the x-axis at each x-intercept.

(f) Find the power function that the graph of g resembles for large values of $|x|$.

(g) Approximate the turning points on the graph of g.

(h) Put all the information together to obtain the graph of g.

3. Find the complex zeros of $f(x) = x^3 - 4x^2 + 25x - 100$.

4. Solve $3x^3 + 2x - 1 = 8x^2 - 4$ in the complex number system.

In Problems 5 and 6, find the domain of each function. Find any horizontal, vertical, or oblique asymptotes.

5. $g(x) = \dfrac{2x^2 - 14x + 24}{x^2 + 6x - 40}$ **6.** $r(x) = \dfrac{x^2 + 2x - 3}{x + 1}$

7. Sketch the graph of the function in Problem 6. Label all intercepts, vertical asymptotes, horizontal asymptotes, and oblique asymptotes.

In Problems 8 and 9, write a function that meets the given conditions.

8. Fourth-degree polynomial function with real coefficients; zeros: $-2, 0, 3 + i$.

9. Rational function; asymptotes: $y = 2, x = 4$; domain: $\{x \mid x \neq 4, x \neq 9\}$.

10. Use the Intermediate Value Theorem to show that the function $f(x) = -2x^2 - 3x + 8$ has at least one real zero on the interval $[0, 4]$.

11. Solve: $\dfrac{x+2}{x-3} < 2$

Cumulative Review

1. Find the distance between the points $P = (1, 3)$ and $Q = (-4, 2)$.

2. Solve the inequality $x^2 \geq x$ and graph the solution set.

3. Solve the inequality $x^2 - 3x < 4$ and graph the solution set.

4. Find a linear function with slope -3 that contains the point $(-1, 4)$. Graph the function.

5. Find the equation of the line parallel to the line $y = 2x + 1$ and containing the point $(3, 5)$. Express your answer in slope–intercept form and graph the line.

6. Graph the equation $y = x^3$.

7. Does the relation $\{(3, 6), (1, 3), (2, 5), (3, 8)\}$ represent a function? Why or why not?

8. Solve the equation $x^3 - 6x^2 + 8x = 0$.

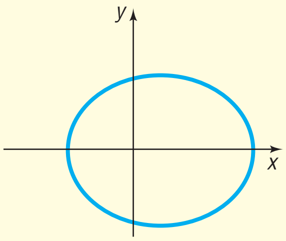
9. Solve the inequality $3x + 2 \leq 5x - 1$ and graph the solution set.

10. Find the center and the radius of the circle described by the equation $x^2 + 4x + y^2 - 2y - 4 = 0$. Graph the circle.

11. For the equation $y = x^3 - 9x$, determine the intercepts and test for symmetry.

12. Find an equation of the line perpendicular to $3x - 2y = 7$ that contains the point $(1, 5)$.

13. Is the following the graph of a function? Why or why not?



14. For the function $f(x) = x^2 + 5x - 2$, find

(a) $f(3)$ (b) $f(-x)$

(c) $-f(x)$ (d) $f(3x)$

(e) $\dfrac{f(x+h) - f(x)}{h}$ $h \neq 0$

15. Answer the following questions regarding the function

$$f(x) = \frac{x+5}{x-1}$$

(a) What is the domain of f?

(b) Is the point $(2, 6)$ on the graph of f?

(c) If $x = 3$, what is $f(x)$? What point is on the graph of f?

(d) If $f(x) = 9$, what is x? What point is on the graph of f?

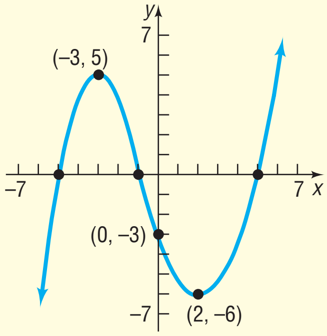
(e) Is f a polynomial or rational function?

16. Graph the function $f(x) = -3x + 7$.

17. Graph $f(x) = 2x^2 - 4x + 1$ by determining whether its graph opens up or down and by finding its vertex, axis of symmetry, y-intercept, and x-intercepts, if any.

18. Find the average rate of change of $f(x) = x^2 + 3x + 1$ from 1 to 2. Use this result to find the equation of the secant line containing $(1, f(1))$ and $(2, f(2))$.

19. In parts (a) to (f), use the following graph.



(a) Determine the intercepts.

(b) Based on the graph, tell whether the graph is symmetric with respect to the x-axis, the y-axis, and/or the origin.

(c) Based on the graph, tell whether the function is even, odd, or neither.

(d) List the intervals on which f is increasing. List the intervals on which f is decreasing.

(e) List the numbers, if any, at which f has a local maximum value. What are these local maximum values?

(f) List the numbers, if any, at which f has a local minimum value. What are these local minimum values?

20. Determine algebraically whether the function

$$f(x) = \frac{5x}{x^2 - 9}$$

is even, odd, or neither.

21. For the function $f(x) = \begin{cases} 2x + 1 & \text{if } -3 < x < 2 \\ -3x + 4 & \text{if } x \geq 2 \end{cases}$

(a) Find the domain of f.

(b) Locate any intercepts.

(c) Graph the function.

(d) Based on the graph, find the range.

22. Graph the function $f(x) = -3(x + 1)^2 + 5$ using transformations.

23. Suppose that $f(x) = x^2 - 5x + 1$ and $g(x) = -4x - 7$.

(a) Find $f + g$ and state its domain.

(b) Find $\frac{f}{g}$ and state its domain.

24. Demand Equation The price p (in dollars) and the quantity x sold of a certain product obey the demand equation

$$p = -\frac{1}{10}x + 150, \qquad 0 \leq x \leq 1500$$

(a) Express the revenue R as a function of x.

(b) What is the revenue if 100 units are sold?

(c) What quantity x maximizes revenue? What is the maximum revenue?

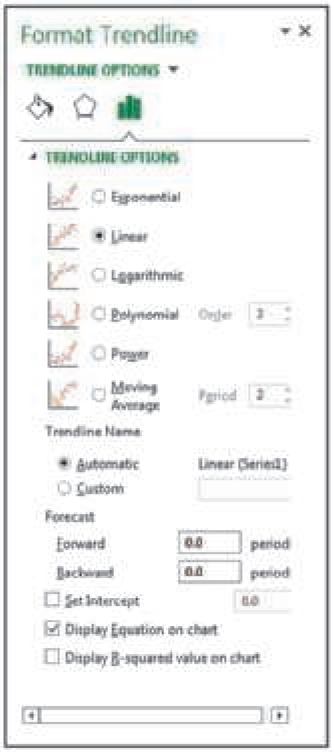
(d) What price should the company charge to maximize revenue?

Chapter Projects

I. Length of Day Go to *http://en.wikipedia.org/wiki/Latitude* and read about latitude through the subhead "Effect of Latitude." Now go to *http://www.orchidculture.com/COD/daylength.html#0N*.

1. For a particular day of the year, record in a table the length of day for the equator (0°N), 5°N, 10°N, . . . , 60°N. Enter the data into an Excel spreadsheet, TI graphing calculator, or some other spreadsheet capable of finding linear, quadratic, and cubic functions of best fit.
2. Draw a scatter diagram of the data with latitude as the independent variable and length of day as the dependent variable using Excel, a TI graphing calculator, or some other spreadsheet. The Chapter 4 project describes how to draw a scatter diagram in Excel.

3. Determine the linear function of best fit. Graph the linear function of best fit on the scatter diagram. To do this in Excel, right click on any data point in the scatter diagram. Now click the Add Chart Element menu, select Trendline, and then select More Trendline Options. Select the Linear radio button and select Display Equation on Chart. See Figure 58. Move the Trendline Options window off to the side and you will see the linear function of best fit displayed on the scatter diagram. Do you think the function accurately describes the relation between latitude and length of day?



○ Exponential
● Linear
○ Logarithmic
○ Polynomial Order 2
○ Power
○ Moving Average Period 2

Trendline Name
● Automatic Linear (Series1)
○ Custom

Forecast
Forward 0.0 period
Backward 0.0 period
☐ Set Intercept 0.0
☑ Display Equation on chart
☐ Display R-squared value on chart

Figure 58

4. Determine the quadratic function of best fit. Graph the quadratic function of best fit on the scatter diagram. To do this in Excel, click on any data point in the scatter diagram. Now click the Add Chart Element menu, select Trendline, and then select More Trendline Options. Select the Polynomial radio button with Order set to 2. Select Display Equation on chart. Move the Trendline Options window off to the side and you will see the quadratic function of best fit displayed on the scatter diagram. Do you think the function accurately describes the relation between latitude and length of day?

5. Determine the cubic function of best fit. Graph the cubic function of best fit on the scatter diagram. To do this in Excel, click on any data point in the scatter diagram. Now click the Add Chart Element menu, select Trendline, and then select More Trendline Options. Select the Polynomial radio button with Order set to 3. Select Display Equation on chart. Move the Trendline Options window off to the side and you will see the cubic function of best fit displayed on the scatter diagram. Do you think the function accurately describes the relation between latitude and length of day?

6. Which of the three models seems to fit the data best? Explain your reasoning.

7. Use your model to predict the hours of daylight on the day you selected for Chicago (41.85 degrees north latitude). Go to the Old Farmer's Almanac or another website to determine the hours of daylight in Chicago for the day you selected. How do the two compare?

Citation: Excel © 2013 Microsoft Corporation. Used with permission from Microsoft.

The following project is available at the Instructor's Resource Center (IRC):

II. **Theory of Equations** The coefficients of a polynomial function can be found if its zeros are known, an advantage of using polynomials in modeling.