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ULTIMATE STABILITY OF NONLINEAR TIME-VARYING SYSTEMS WITH MULTIPLE DELAYS

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Abstract. The ultimate stability of nonlinear time-varying systems with multiple delays and bounded disturbances are investigated in this paper. Based on some comparison techniques via differential inequalities, explicit delay-independent conditions are derived for determining an ultimate bound such that all state trajectories of the system converge exponentially within that bound. The obtained results also guarantee exponential stability of the system when the input disturbance vector is ignored. Numerical simulations are given to illustrate the effectiveness of the obtained results.
Keywords: Ultimate stability, exponential convergence, time-varying systems, bounded disturbances, M-matrix.

1. Introduction

In practical systems, there usually exists an interval of time between stimulation and the system response [1]. This interval of time is often known as the time delay of a system. Since time-delay unavoidably occurs in engineering systems and usually is a source of bad performance, oscillations or instability [2], stability analysis and control of time-delay systems are essential and of great importance for theoretical and practical reasons [3]. This problem has attracted considerable attention from the mathematics and control communities, see, for example, [4-9].

When considering the long-time behavior of a system, the framework of Lyapunov stability theory and its extensions for time-delay systems, the Lyapunov-Krasovskii and Lyapunov-Razumikhin, have been extensively developed [3]. However, realistic systems usually exhibit nonlinear characteristics for which the theoretical definitions in the sense of Lyapunov can be quite restrictive [10]. Namely, the desired state of a system may be mathematically unstable in the sense of Lyapunov, but the response of the system oscillates close enough to this state for its performance to be considered as acceptable.

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Furthermore, in many stabilization problems, especially for systems that may lack an equilibrium point due to the presence of disturbances or constrained states, the aim is to bring those states close to certain sets rather than to a particular state [11-15]. In such situations, the concept of ultimate stability, also known as practical stability is more suitable and meaningful. Ultimate stability with a fixed bound [16] was first proposed in [17], retaken and systematically introduced in [18] to address some potential practical limitations of Lyapunov stability. These stability notions not only provide information on the stability of the system, but also characterize its transient behavior with estimations of the bounds on the system trajectories. During the past decade, considerable research efforts have been devoted to study the practical stability of dynamical systems. To this point, we refer the reader to some recent papers [10,13-15,19-23] and the cited references therein.

Although ultimate stability provides a more relaxed concept of stability, only a few results concerning this problem have been reported especially for nonlinear time-varying systems with multiple delays. Furthermore, when dealing with time-varying systems with delays, the developed methodologies such as Lyapunov-Krasovskii functional method and its variants either lead to matrix Riccati differential equations (RDEs) or indefinite linear matrix inequalities (LMIs). So far, there has been no efficient computational tool available to solve RDEs or indefinite LMIs. In addition, the constructive approaches proposed in the aforementioned works seem not applicable to time-varying systems. Therefore, an alternative and efficient approach to address the problem of ultimate stability of time-varying systems with delays is necessary and motivation for our present research.

In this paper, we consider the problem of ultimate stability for a class of nonlinear time-varying systems with multiple time-varying delays and bounded disturbances. A constructive approach based on some comparison techniques is presented to derive explicit delay-independent conditions for determining an ultimate bound ensuring that all state trajectories of the system converge exponentially within that bound after an initial transient period. The derived conditions also guarantee exponential stability in the sense of Lyapunov when the input disturbance vector is ignored.

2. Preliminaries

Notation: $\underline{n} = \{1, 2, \ldots, n\}$ for a positive integer n. $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote the n-dimensional vector space with the norm $\|x\|_\infty = \max_{i\in\underline{n}} |x_i|$ and the set of $m \times n$ real matrices, respectively. A comparison between vectors will be understood componentwise. Specifically, for $u = (u_i)$ and $v = (v_i)$ in $\mathbb{R}^n$, $u \geq v$ means $u_i \geq v_i$ for all $i \in \underline{n}$ and if $u_i > v_i$ for all $i \in \underline{n}$ then we write $u \gg v$ instead of $u > v$. $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0\}$ and $int(\mathbb{R}^n_+) = \{x \in \mathbb{R}^n : x \gg 0\}$. By denoting $v_{\min} = \min_{i\in\underline{n}} v_i$ then $v_{\min} > 0$ for any vector $v = (v_i) \in int(\mathbb{R}^n_+)$. We also specifically use the notation $\alpha^+ = \max\{\alpha, 0\}$ for a real number α, that means $\alpha^+ = \alpha$ if and only if $\alpha > 0$, otherwise $\alpha^+ = 0$.

Consider a class of nonlinear time-varying systems with multiple time-varying delays of the form

$$\begin{aligned}\dot{x}(t) &= \mathcal{A}(t)x(t) + W_0(t)F(x(t))\\ &\quad + W_1(t)G(x(t-\tau(t))) + d(t),\ t \geq 0,\\ x(t) &= \phi(t), \quad t \in [-\tau_{\max}, 0].\end{aligned} \tag{2.1}$$

System (2.1) can be written explicitly as follows:

$$\begin{aligned}\dot{x}_i(t) &= a_i(t)x_i(t) + \sum_{j=1}^{n} w_{ij}^0(t) f_j(x_j(t))\\ &\quad + \sum_{j=1}^{n} w_{ij}^1(t) g_j(x_j(t-\tau_{ij}(t))) + d_i(t),\ t \geq 0,\\ x_i(t) &= \phi_i(t),\ t \in [-\tau_{\max}, 0],\ i \in \underline{n},\end{aligned} \tag{2.2}$$

where $x(t) = (x_i(t)) \in \mathbb{R}^n$ and $d(t) = (d_i(t)) \in \mathbb{R}^n$ are the state vector and exogenous disturbance vector, respectively, $\mathcal{A}(t) = \mathrm{diag}(a_i(t))$, $W_0(t) = (w_{ij}^0(t))$ and $W_1(t) = (w_{ij}^1(t))$ are time-varying system matrices whose elements are assumed to be continuous on $\mathbb{R}_+$, nonlinear functions $f_j(.), g_j(.) : \mathbb{R} \to \mathbb{R}, j \in \underline{n}$, are continuous, $\tau_{ij}(t)$ are heterogeneous time-varying delays and $\phi(.) \in C([-\tau_{\max}, 0], \mathbb{R}^n)$ is the vector-valued initial function specifying the initial state of the system, $\phi(t) = (\phi_i(t)) \in \mathbb{R}^n$. Let us denote $|\phi_i| = \sup_{-\tau_{\max} \leq t \leq 0} |\phi_i(t)|$ and $\|\phi\|_\infty = \max_{i \in \underline{n}} |\phi_i|$.

Note that the system (2.1) is quite general which includes LTI systems with delays [10], linear time-varying systems with time-varying delays [24] or neural networks [25] as some special cases.

Definition 2.1. *System* (2.1) *is said to be ultimately stable if there exists a bound $\mu > 0$ such that for any $\phi(.) \in C([-\tau_{\max}, 0], \mathbb{R}^n)$, there exists a transient time $T = T(\mu, \phi) \geq 0$ such that $\|x(t, \phi)\|_\infty \leq \mu$ for all $t \geq T$.*

Our aim in this paper is to derive explicit conditions for determining an ultimate bound μ^* by which system (2.1) is ultimately stable for $\mu > \mu^*$. By utilizing the approach of [24], we derive delay-independent conditions in terms of some matrix inequalities ensuring ultimate exponential convergence of state trajectories of the system.

At first, we recall here some properties of M-matrix [26]. A matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is said to be M-matrix if $a_{ij} \leq 0$ whenever $i \neq j$ and all principal minors of A are positive. The following proposition is used in stating our main result.

Proposition 2.1. *Let $A \in \mathbb{R}^{n \times n}$ be an off-diagonal non-positive matrix, $a_{ii} > 0, i \in \underline{n}$. The following statements are equivalent.*

(i) A is a nonsingular M-matrix.

(ii) $Re\lambda_k(A) > 0$ for all eigenvalues $\lambda_k(A)$ of A.

(iii) There exists a vector $\xi \gg 0$ such that $A\xi \gg 0$.

(iv) There exists a vector $\eta \gg 0$ such that $A^T\eta \gg 0$.

From Proposition 2.1we obtain the following result.

Proposition 2.2. *Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular M-matrix, then there exists a vector $\xi \in int(\mathbb{R}^n_+)$, $\|\xi\|_\infty = 1$, such that $A\xi \gg 0$.*

3. Main results

To facilitate the statement of our results, we consider the following assumptions:

(A1) The matrices $\mathcal{A}(t) = \mathrm{diag}(a_i(t))$, $W_0(t) = (w^0_{ij}(t))$ and $W_1(t) = (w^1_{ij}(t))$ satisfy the following conditions

$$a_i(t) \leq \overline{a}_i,\ |w^0_{ij}(t)| \leq \overline{w}^0_{ij},\ |w^1_{ij}(t)| \leq \overline{w}^1_{ij}.$$

(A2) There exist constants $F_i \geq 0,\ G_i \geq 0$, such that

$$|f_i(u) - f_i(v)| \leq F_i|u - v|,\ |g_i(u) - g_i(v)| \leq G_i|u - v|$$

for all $u, v \in \mathbb{R}$ and $f_i(0) = 0, g_i(0) = 0, i \in \underline{n}$.

(A3) The disturbance vector $d(t) = (d_i(t))$ is bounded, that is, there exists a positive constant d_∞ such that

$$|d_i(t)| \leq d_\infty \text{ for all } t \geq 0,\ i \in \underline{n}.$$

Remark 3.1. *By assumptions* (A1)-(A3), *for each initial function $\phi(.) \in C([-\tau_{\max}, 0], \mathbb{R}^n)$, there exists a unique solution $x(t, \phi)$ of* (2.1) *defining on $[-\tau_{\max}, \infty)$* [1]. *On the other hand, although assumption* (A2) *implies $F(0) = 0, G(0) = 0$, system* (2.1) *may not have an equilibrium point. Particularly, $x = 0$ is neither an equilibrium point of* (2.1) *due to not vanished disturbance nor a necessarily stable motion.*

Let us denote the following matrices:

$$\begin{aligned}
&\mathcal{A} = \mathrm{diag}\{-\overline{a}_1, -\overline{a}_2, \ldots, -\overline{a}_n\},\ \overline{\mathcal{W}}_0 = (\overline{w}^0_{ij}),\ \overline{\mathcal{W}}_1 = (\overline{w}^1_{ij}),\\
&F = \mathrm{diag}\{F_1, F_2, \ldots, F_n\},\ G = \mathrm{diag}\{G_1, G_2, \ldots, G_n\},\\
&\mathcal{M} = \mathcal{A} - \overline{\mathcal{W}}_0 F - \overline{\mathcal{W}}_1 G.
\end{aligned}$$

The matrix $\mathcal{M}$ is obvious an M-matrix. Therefore, if $\mathcal{M}$ satisfies one of the equivalent conditions in Proposition 2.1 then, by Proposition 2.2, there exists a vector $\xi \in int(\mathbb{R}^n_+)$, $\|\xi\|_\infty = 1$, such that $\mathcal{M}\xi \gg 0$. Now, we are in the position to present our main result in the following theorem.

Theorem 3.1. *Let assumptions* (A1)-(A3) *hold. Assume that* $\mathcal{M}$ *is a nonsingular M-matrix. Then, system* (2.1) *is ultimately stable. More precisely, let* $\xi \in int(\mathbb{R}^n_+)$ *be a vector satisfying* $\|\xi\|_\infty = 1$ *and* $\mathcal{M}\xi \gg 0$, $m^* = (\mathcal{M}\xi)_{\min}$, $\delta^* = \dfrac{m^*}{\xi_{\min}}$ *and* $\sigma = \min_{i\in\underline{n}}\sigma_i$, *where* σ_i *is the unique positive solution of the scalar equation*

$$\sigma\xi_i + \sum_{j=1}^{n} G_j \overline{w}^1_{ij}\xi_j \left(e^{\sigma\tau_{\max}} - 1\right) - m^* = 0,\ i \in \underline{n}.$$

Then, every solution $x(t,\phi)$ *of system* (2.1) *satisfies the following bound*

$$\|x(t,\phi)\|_\infty \le \frac{d_\infty}{m^*} + \kappa^* \left(\|\phi\|_\infty - \frac{d_\infty}{\delta^*}\right)^+ e^{-\sigma t},\ t \ge 0,$$

where $\kappa^* = 1/\xi_{\min}$.

Proof. We divide the proof into several steps.
Step 1. By Proposition 2.2, there exists $\xi \in int(\mathbb{R}^n_+)$, $\|\xi\|_\infty = 1$, such that $\mathcal{M}\xi \gg 0$, and thus

$$\overline{a}_i\xi_i + \sum_{j=1}^{n}\left(F_j\overline{w}^0_{ij} + G_j\overline{w}^1_{ij}\right)\xi_j < 0,\ i \in \underline{n}. \tag{3.1}$$

Observe that,

$$m^* = (\mathcal{M}\xi)_{\min} = \min_{i\in\underline{n}}\left\{-\overline{a}_i\xi_i - \sum_{j=1}^{n}\left(F_j\overline{w}^0_{ij} + G_j\overline{w}^1_{ij}\right)\xi_j\right\}.$$

Hence $m^* > 0$ and from (3.1) we have

$$\overline{a}_i\xi_i + \sum_{j=1}^{n}\left(F_j\overline{w}^0_{ij} + G_j\overline{w}^1_{ij}\right)\xi_j \le -m^*. \tag{3.2}$$

Step 2. We will prove that $\|x(t,\phi)\| \le \dfrac{d_\infty}{m^*}$ for $t \ge 0$ if $\|\phi\|_\infty \le \dfrac{d_\infty}{\delta^*}$. In the following, we will use $x(t)$ to denote the solution $x(t,\phi)$ if it does not cause any confusion. Let $\|\phi\|_\infty \le \dfrac{d_\infty}{\delta^*}$ then we have $|x_i(t)| \le |\phi_i| \le \xi_i\dfrac{d_\infty}{m^*}$ for $t \in [-\tau_{\max}, 0]$, $i \in \underline{n}$. For any $q > 1$, assume that there exists an index $i \in \underline{n}$ and $\overline{t} > 0$ such that $|x_i(\overline{t})| = q\xi_i\dfrac{d_\infty}{m^*}$ and $|x_j(t)| \le q\xi_j\dfrac{d_\infty}{m^*}, \forall t \le \overline{t}, j \in \underline{n}$. Then $D^+\left|x_i(\overline{t})\right| \ge 0$. On the other hand, it follows from

(2.2) that

$$\begin{aligned} D^+|x_i(t)| &= \operatorname{sgn}(x_i(t))\dot{x}_i(t) \\ &\le a_i(t)|x_i(t)| + \sum_{j=1}^n |w_{ij}^0(t)||f_j(x_j(t))| \\ &\quad + \sum_{j=1}^n |w_{ij}^1(t)||g_j(x_j(t-\tau_{ij}(t)))| + |d_i(t)| \\ &\le \overline{a}_i|x_i(t)| + \sum_{j=1}^n F_j\overline{w}_{ij}^0|x_j(t)| \\ &\quad + \sum_{j=1}^n G_j\overline{w}_{ij}^1|x_j(t-\tau_{ij}(t))| + d_\infty,\ t\in[0,\overline{t}]. \end{aligned} \tag{3.3}$$

Thus,

$$\begin{aligned} D^+\left|x_i(\overline{t})\right| &\le \frac{qd_\infty}{m^*}\left(\overline{a}_i\xi_i + \sum_{j=1}^n\left(F_j\overline{w}_{ij}^0 + G_j\overline{w}_{ij}^1\right)\xi_j\right) + d_\infty \\ &\le (1-q)d_\infty < 0 \end{aligned} \tag{3.4}$$

which yields a contradiction. Therefore, $|x_i(t)| \le q\xi_i\dfrac{d_\infty}{m^*}$ for all $t\ge 0$. Let $q\to 1^+$ we obtain $|x_i(t)|\le \xi_i\dfrac{d_\infty}{m^*}$ for all $i\in \underline{n}$ and hence, $\|x(t)\|_\infty \le \dfrac{d_\infty}{m^*}\|\xi\|_\infty = \dfrac{d_\infty}{m^*}$.

Step 3. Now, assume that $\|\phi\|_\infty > \dfrac{d_\infty}{\delta^*}$. Then it is easy to verify that

$$|\phi_i| - \xi_i\frac{d_\infty}{m^*} \le \kappa^*\left(\|\phi\|_\infty - \frac{d_\infty}{\delta^*}\right)\xi_i,\ i\in\underline{n}.$$

For each $i\in\underline{n}$, consider the following scalar equation in $\sigma\in[0,\infty)$

$$H_i(\sigma) = \sigma\xi_i + \sum_{j=1}^n G_j\overline{w}_{ij}^1\xi_j\left(e^{\sigma\overline{\tau}}-1\right) - m^* = 0. \tag{3.5}$$

Since the function $H_i(\sigma)$ is continuous and strictly increasing on $[0,\infty)$, $H_i(0)<0$, $H_i(\sigma)\to\infty, \sigma\to\infty$, equation (3.5) has a unique positive solution σ_i. In addition, $H_i(\sigma)\le 0$ for all $\sigma\in(0,\sigma_i]$. Let $\sigma = \min_{i\in\underline{n}}\sigma_i$ then $H_i(\sigma)\le 0$ for all $i\in\underline{n}$.

Let us consider the functions $v_i(t), i\in\underline{n}$, as follows:

$$v_i(t) = \kappa^*\left(\|\phi\|_\infty - \frac{d_\infty}{\delta^*}\right)\xi_i e^{-\sigma t},\ t\in[-\tau_{\max},\infty). \tag{3.6}$$

Observing that, for $t \geq 0$ and $j \in \underline{n}$, we have

$$\begin{aligned} v_j(t-\tau_{ij}(t)) &= \kappa^* \left(\|\phi\|_\infty - \frac{d_\infty}{\delta^*} \right) \xi_j e^{-\sigma(t-\tau_{ij}(t))} \\ &\leq \kappa^* \left(\|\phi\|_\infty - \frac{d_\infty}{\delta^*} \right) \xi_j e^{-\sigma t} e^{\sigma \tau_{\max}} \\ &\leq e^{\sigma\tau_{\max}} v_j(t). \end{aligned}$$

Therefore, using (3.2) and (3.6), we have

$$\begin{aligned} &\overline{a}_i v_i(t) + \sum_{j=1}^n F_j \overline{w}_{ij}^0 v_j(t) + \sum_{j=1}^n G_j \overline{w}_{ij}^1 v_j(t-\tau_{ij}(t)) \\ &\quad \leq \beta e^{-\sigma t} \left(\overline{a}_i \xi_i + \sum_{j=1}^n F_j \overline{w}_{ij}^0 \xi_j + \sum_{j=1}^n G_j \overline{w}_{ij}^1 \xi_j e^{\sigma\tau_{\max}} \right) \\ &\quad \leq \beta e^{-\sigma t} \Big[\overline{a}_i \xi_i + \sum_{j=1}^n \left(F_j \overline{w}_{ij}^0 + G_j \overline{w}_{ij}^1 \right) \xi_j + \sum_{j=1}^n G_j \overline{w}_{ij}^1 \xi_j \left(e^{\sigma\tau_{\max}} - 1 \right) \Big] \\ &\quad \leq \beta e^{-\sigma t} \Big[-m^* + \sum_{j=1}^n G_j \overline{w}_{ij}^1 \xi_j \left(e^{\sigma\tau_{\max}} - 1 \right) \Big] \\ &\quad \leq -\beta\sigma\xi_i e^{-\sigma t}, \ t \geq 0, \ i \in \underline{n}, \end{aligned}$$

where $\beta = \kappa^* \left(\|\phi\|_\infty - \dfrac{d_\infty}{\delta^*} \right)$. This leads to

$$\dot{v}_i(t) \geq \overline{a}_i v_i(t) + \sum_{j=1}^n F_j \overline{w}_{ij}^0 v_j(t) + \sum_{j=1}^n G_j \overline{w}_{ij}^1 v_j(t-\tau_{ij}(t)). \tag{3.7}$$

Next, by using the following transformations:

$$u_i(t) = |x_i(t)| - \xi_i \frac{d_\infty}{m^*}, \ t \geq -\tau_{\max}, \ i \in \underline{n},$$

and by the same argument used in (3.3), we have

$$\begin{aligned} D^+ u_i(t) &\leq \overline{a}_i u_i(t) + \sum_{j=1}^n F_j \overline{w}_{ij}^0 u_j(t) + \sum_{j=1}^n G_j \overline{w}_{ij}^1 u_j(t-\tau_{ij}(t)) \\ &\quad + \frac{d_\infty}{m^*} \left[\overline{a}_i \xi_i + \sum_{j=1}^n \left(F_j \overline{w}_{ij}^0 + G_j \overline{w}_{ij}^1 \right) \xi_j \right] + d_\infty \\ &\leq \overline{a}_i u_i(t) + \sum_{j=1}^n F_j \overline{w}_{ij}^0 u_j(t) + \sum_{j=1}^n G_j \overline{w}_{ij}^1 u_j(t-\tau_{ij}(t)). \end{aligned} \tag{3.8}$$

We now prove that $u_i(t) \le v_i(t)$. Let $\rho_i(t) = u_i(t) - v_i(t)$, then, for $t \in [-\tau_{\max}, 0]$ we have

$$\begin{aligned} u_i(t) &\le |\phi_i| - \xi_i \frac{d_\infty}{m^*} \le \kappa^* \left(\|\phi\|_\infty - \frac{d_\infty}{\delta^*} \right) \xi_i \\ &\le \kappa^* \left(\|\phi\|_\infty - \frac{d_\infty}{\delta^*} \right) \xi_i e^{-\sigma t} = v_i(t). \end{aligned}$$

Thus, $\rho_i(t) \le 0$, for all $t \in [-\tau_{\max}, 0], i \in \underline{n}$. Assume that there exist an index $i \in \underline{n}$ and a $t_1 > 0$ such that $\rho_i(t_1) = 0$, $\rho_i(t) > 0$, $t \in (t_1, t_1 + \delta)$ for some $\delta > 0$ and $\rho_j(t) \le 0, \forall t \in [-\tau_{\max}, t_1]$. Then $D^+\rho_i(t_1) > 0$. However, for $t \in [0, t_1)$, it follows from (3.7) and (3.8) that

$$\begin{aligned} D^+\rho_i(t) &\le \overline{a}_i \rho_i(t) + \sum_{j=1}^n F_j \overline{w}_{ij}^0 \rho_j(t) \\ &\quad + \sum_{j=1}^n G_j \overline{w}_{ij}^1 \rho_j(t - \tau_{ij}(t)) \\ &\le \overline{a}_i \rho_i(t), \end{aligned}$$

and therefore, $D^+\rho_i(t_1) \le 0$ which yields a contradiction. This shows that $\rho_i(t) \le 0$ for all $t \ge 0, i \in \underline{n}$. Consequently,

$$\begin{aligned} |x_i(t)| &\le \xi_i \frac{d_\infty}{m^*} + \kappa^* \left(\|\phi\|_\infty - \frac{d_\infty}{\delta^*} \right) \xi_i e^{-\sigma t} \\ &\le \frac{d_\infty}{m^*} \|\xi\|_\infty + \kappa^* \left(\|\phi\|_\infty - \frac{d_\infty}{\delta^*} \right) \|\xi\|_\infty e^{-\sigma t} \\ &\le \frac{d_\infty}{m^*} + \kappa^* \left(\|\phi\|_\infty - \frac{d_\infty}{\delta^*} \right) e^{-\sigma t}, \ \forall t \ge 0, \ i \in \underline{n}. \end{aligned}$$

Finally, we obtain

$$\|x(t)\|_\infty \le \frac{d_\infty}{m^*} + \kappa^* \left(\|\phi\|_\infty - \frac{d_\infty}{\delta^*} \right)^+ e^{-\sigma t}, \ t \ge 0. \tag{3.9}$$

Step 4. Let $\mu > \dfrac{d_\infty}{m^*}$ and $x(t, \phi)$ be a solution of system (2.1). If $\|\phi\|_\infty \le \dfrac{d_\infty}{\delta^*}$ then, by *Step 2*, $\|x(t, \phi)\|_\infty \le \mu$ holds for all $t \ge T(\mu, \phi) = 0$. Assume $\|\phi\|_\infty > \dfrac{d_\infty}{\delta^*}$ then from (3.9) we have

$$\begin{aligned} \|x(t, \phi)\|_\infty &\le \frac{d_\infty}{m^*} + \left(\frac{\|\phi\|_\infty}{\xi_{\min}} - \frac{d_\infty}{m^*} \right) e^{-\sigma t} \\ &\le \frac{d_\infty}{m^*} \left(1 - e^{-\sigma t} \right) + \frac{\|\phi\|_\infty}{\xi_{\min}} e^{-\sigma t}. \end{aligned}$$

Therefore, if $\|\phi\|_\infty \le \mu\xi_{\min}$, note that $\mu > \dfrac{d_\infty}{m^*}$, then

$$\|x(t,\phi)\|_\infty \le \mu\left(1-e^{-\sigma t}\right) + \mu e^{-\sigma t} = \mu.$$

If $\|\phi\|_\infty > \mu\xi_{\min}$ then

$$T(\mu,\phi) := \frac{1}{\sigma}\ln\left(\frac{\frac{\|\phi\|_\infty}{\xi_{\min}} - \frac{d_\infty}{m^*}}{\mu - \frac{d_\infty}{m^*}}\right) > 0$$

and $\|x(t,\phi)\|_\infty \le \mu$ for $t \ge T(\mu,\phi)$. This shows that system (2.1) is ultimately stable. The proof is completed. □

Remark 3.2. *The result of Theorem* 3.1 *ensures that all state trajectories of system* (2.1) *will converge to a common threshold* $\mu_* = \dfrac{d_\infty}{m^*}$ *as the time tends to infinity. In other words, for any solution* $x(t,\phi)$ *of system* (2.1)*, it holds that*

$$\limsup_{t\to\infty}\|x(t,\phi)\|_\infty \le \frac{d_\infty}{m^*}.$$

Remark 3.3. *It can be seen in the proof of Theorem* 3.1 *that (using* (3.2) *and* (3.5)*), for a fixed vector* $\xi \in int(\mathbb{R}^n_+)$ *satisfying*

$$\left(\mathcal{A} - \overline{W}_0 F - \overline{W}_1 G\right)\xi \gg 0, \tag{3.10}$$

the exponential convergence rate σ *can be defined as* $\sigma = \min_{i\in\underline{n}}\sigma_i$*, where* σ_i *is the unique positive solution of the scalar equation*

$$(\overline{a}_i + \sigma)\xi_i + \sum_{j=1}^n \left(F_j\overline{w}^0_{ij} + G_j\overline{w}^1_{ij}e^{\sigma\tau_{\max}}\right)\xi_j = 0. \tag{3.11}$$

Thus, Theorem 3.1 *provides an explicit delay-independent criterion for the ultimately exponential convergence of system* (2.1)*. Moreover, the impact of delays on the decay rate is also explicit provided by computing the associated* σ *in* (3.11) *for any* $\xi \in int(\mathbb{R}^n_+)$ *satisfying* (3.10).

Remark 3.4. *As an application to the nonlinear time-varying system* (2.1) *without disturbances (i.e.* $d(t) = 0$*), the proposed conditions in Theorem* 3.1 *guarantee the Lyapunov exponential stability of the system.*

Corollary 3.1. *Let assumptions* (A1)-(A2) *hold. Assume that there exists a vector* $\xi \in int(\mathbb{R}^n_+)$ *satisfying* (3.10)*, then system* (2.1) *without disturbance is exponentially stable in the sense of Lyapunov. Moreover, every solution* $x(t,\phi)$ *of* (2.1) *satisfies*

$$\|x(t,\phi)\|_\infty \le \frac{\|\xi\|_\infty}{\xi_{\min}}\|\phi\|_\infty e^{-\sigma t},\ t \ge 0,$$

where $\sigma = \min_{i\in\underline{n}}\sigma_i$ *and* σ_i *is the unique positive solution of* (3.11).

From Corollary 3.1, we now discuss the global exponential stability of a special class of (2.1), namely the linear time-varying systems with time-varying delay

$$\begin{aligned} &\dot{x}(t) = A(t)x(t) + B(t)x(t-\tau(t)),\ t \geq 0, \\ &x(t) = \phi(t),\ t \in [-\tau_{\max}, 0], \end{aligned} \tag{3.12}$$

where $A(t) = (a_{ij}(t)) \in \mathbb{R}^{n\times n}, B(t) = (b_{ij}(t)) \in \mathbb{R}^{n\times n}$ are given continuous matrix functions, $0 \leq \tau(t) \leq \tau_{\max}$.

Corollary 3.2. *System* (3.12) *is globally exponentially stable if there exists a vector $\xi \in int(\mathbb{R}^n_+)$ such that*

$$(\mathcal{A}+\mathcal{B})\,\xi \ll 0,$$

where $a_{ii}(t) \leq \tilde{a}_{ii}$, $|a_{ij}(t)| \leq \tilde{a}_{ij}, i \neq j$, $|b_{ij}(t)| \leq \tilde{b}_{ij}$, $\mathcal{A} = (\tilde{a}_{ij})$ and $\mathcal{B} = (\tilde{b}_{ij})$. Moreover, every solution $x(t,\phi)$ of (3.12) *satisfies*

$$\|x(t,\phi)\|_\infty \leq \frac{\|\xi\|_\infty}{\xi_{\min}}\|\phi\|_\infty e^{-\sigma t},\ t \geq 0,$$

where $\sigma = \min_{i\in\underline{n}}\sigma_i$ and $\sigma_i, i \in \underline{n}$, be the unique positive solution of the equation

$$\left(\tilde{a}_{ii} + \sum_{j\neq i}\frac{1}{\xi_i}\tilde{a}_{ij}\xi_j\right) + \left(\sum_{j=1}^{n}\frac{1}{\xi_i}\tilde{b}_{ij}\xi_j\right)e^{\sigma\tau_{\max}} + \sigma = 0.$$

Remark 3.5. *Corollary* 3.2 *gives a delay-independent condition for the exponential stability of linear time-varying systems with delay. This corollary extends some recent results, for example, in* [27, 28], *to time-varying systems.*

As a brief discussion, we would like to mention here that, it is possible to derive the exponential decay rate σ, the μ-neighborhood and the transient time T by imposing in one condition is that the matrix $-\mathcal{M}_\sigma = -\mathcal{A} + \sigma I + \overline{W}_0 F + e^{\sigma\tau_{\max}}\overline{W}_1 G$ is Hurwitz for some $\sigma > 0$. Then μ and T can be determined as follows:

Step 1. Find a vector $\xi \in int(\mathbb{R}^n_+)$ such that $\mathcal{M}_\sigma\xi \gg 0$.

Step 2. Compute $m^* = (\mathcal{N}_\sigma\xi)_{\min}$ and $\delta^* = m^*/\xi_{\min}$, where

$$\mathcal{N}_\sigma = \sigma I + (e^{\sigma\tau_{\max}} - 1)\overline{W}_1 G.$$

Step 3. Transient time $T(\mu,\phi)$ for $\mu > \dfrac{d_\infty\|\xi\|_\infty}{m^*}$ is determined by

$$T(\mu,\phi) = \frac{1}{\sigma}\ln\left(\frac{\left[\dfrac{\|\phi\|_\infty}{\xi_{\min}} - \dfrac{d_\infty}{m^*}\right]\|\xi\|_\infty}{\mu - \dfrac{d_\infty}{m^*}\|\xi\|_\infty}\right)$$

if $\|\phi\|_\infty > \dfrac{\xi_{\min}}{\|\xi\|_\infty}\mu$.

4. An illustrative example

Consider the following nonlinear time-varying system

$$
\begin{aligned}
\dot{x}_i(t) = a_i(t)x_i(t) &+ \sum_{j=1}^{2} w_{ij}^0(t)f_j(x_j(t)) \\
&+ \sum_{j=1}^{2} w_{ij}^1(t)g_j(x_j(t-\tau_{ij}(t))) + d_i(t)
\end{aligned} \tag{4.1}
$$

where $a_1(t) = -5(1+|\sin t|)$, $a_2(t) = -6(1+e^{-t}\cos^2 t)$,

$$
\begin{aligned}
W^0(t) &= \begin{bmatrix} 2\sin 3t & \cos 2t \\ -e^{-t} & 0.5\cos^2 t \end{bmatrix}, \\
W^1(t) &= \begin{bmatrix} \cos 3t & 2\sin t \\ \dfrac{\sin t}{1+|\cos t|} & \dfrac{t\sin t}{1+t^2} \end{bmatrix}, \\
f_1(x_1) &= \sqrt{1+x_1^2} - 1,\ f_2(x_2) = \ln(1+|x_2|), \\
g_i(x_i) &= \tanh(x_i),\ \|d(t)\|_\infty \le 0.1, \tau_{ij}(t) = |\sin(\sqrt{t})|.
\end{aligned}
$$

Assumptions (A1) and (A2) are satisfied and we have

$$
\begin{aligned}
&\mathcal{A} = \mathrm{diag}\{5,6\},\ \overline{W}_0 = \begin{bmatrix} 2 & 1 \\ 1 & 0.5 \end{bmatrix},\ \overline{W}_1 = \begin{bmatrix} 1 & 2 \\ 1 & 0.5 \end{bmatrix}, \\
&F = G = I_2,\ \gamma = 0.1,\ \tau_{\max} = 1,
\end{aligned}
$$

and thus,

$$
\mathcal{M} = \mathcal{A} - \overline{W}_0 F - \overline{W}_1 G = \begin{bmatrix} 2 & -3 \\ -2 & 5 \end{bmatrix}.
$$

It is easy to verify that $\xi = (1 \quad 0.5)^T \in int(\mathbb{R}_+^2)$ satisfying $\mathcal{M}\xi \gg 0$. By Theorem 3.1, system (4.1) is practically stable. Taking (3.1) and (3.5) into account we obtain $m^* = 0.5, \delta^* = 1$, $\kappa^* = 2$ and $\sigma = 0.1579$. The disturbance $\|d(t)\|_\infty \le 0.1$. Every solution of system (4.1) satisfies the following exponential practical estimation

$$
\|x(t,\phi)\|_\infty \le 0.2 + 2\left(\|\phi\|_\infty - 0.1\right)^+ e^{-0.1579t},\ t \ge 0.
$$

State trajectories of system (4.1) with $d_1(t) = 0.1\sin^3 2t$ and $d_2(t) = 0.1\cos 4t$ are presented in Figure 1.

We also consider system (4.1) with time delay $\tau_{ij}(t) = \left|\sin(\omega\sqrt{t})\right|$ and conduct extensive simulation for large values of ω, i.e., $\tau_{ij}(t)$ is a fast time-varying delay. In our conducted simulation test, it was found that all the state trajectories of the system converged exponentially within the bound, for example, Figure 2 presents state trajectories of system (4.1) with $\omega = 10^6$. Moreover, system (4.1) without disturbance, i.e. $d(t) = 0$, is exponentially stable in the sense of Lyapunov as shown in Figure 3.

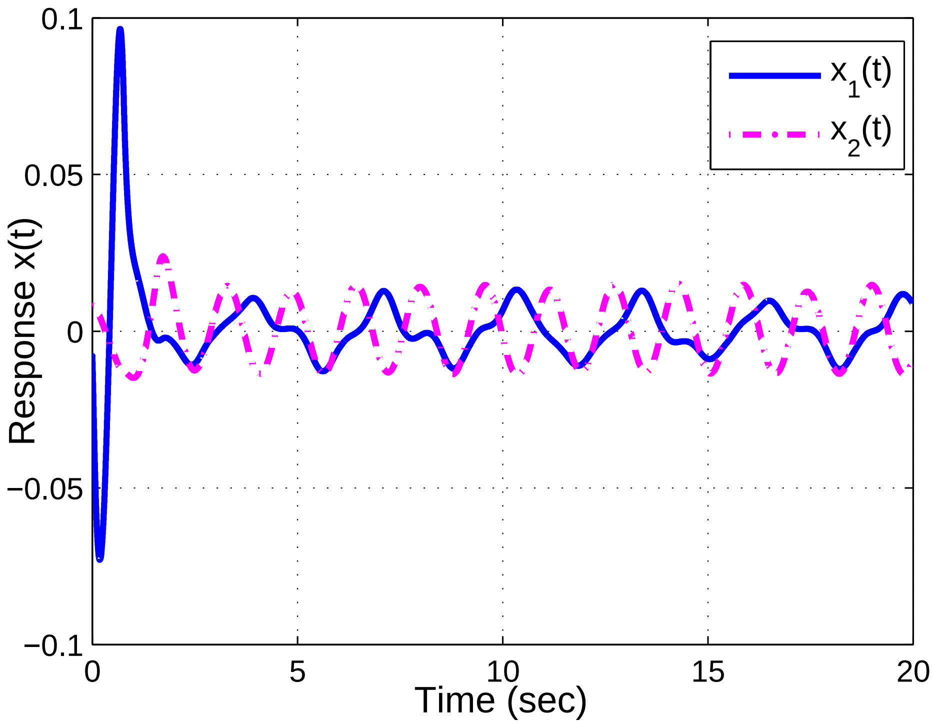

Figure 1. State trajectories of system (4.1)

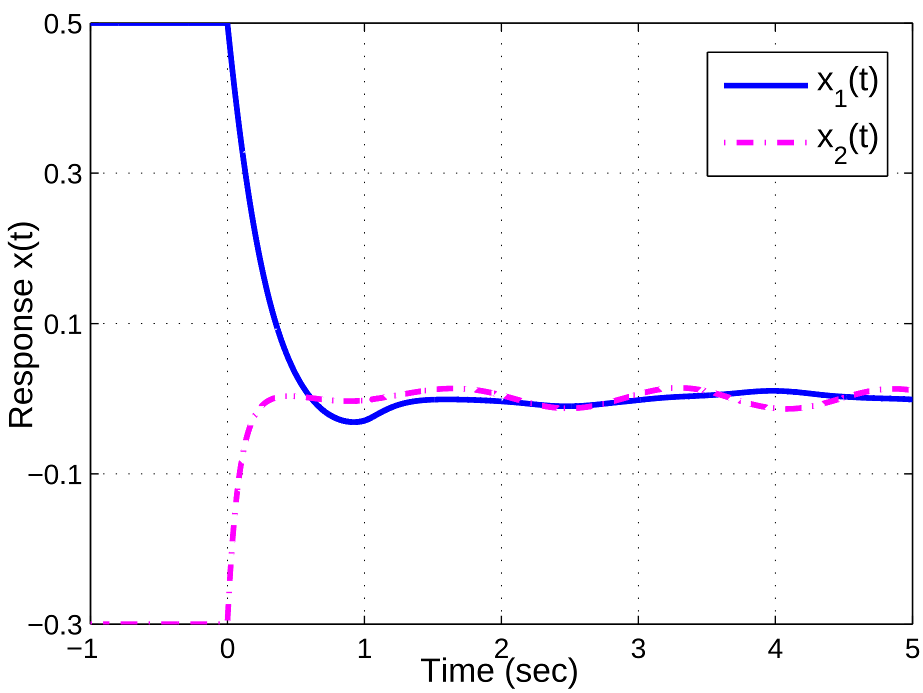

Figure 2. State trajectories of system (4.1) with $\tau_{ij}(t) = |\sin(10^6\sqrt{t})|$

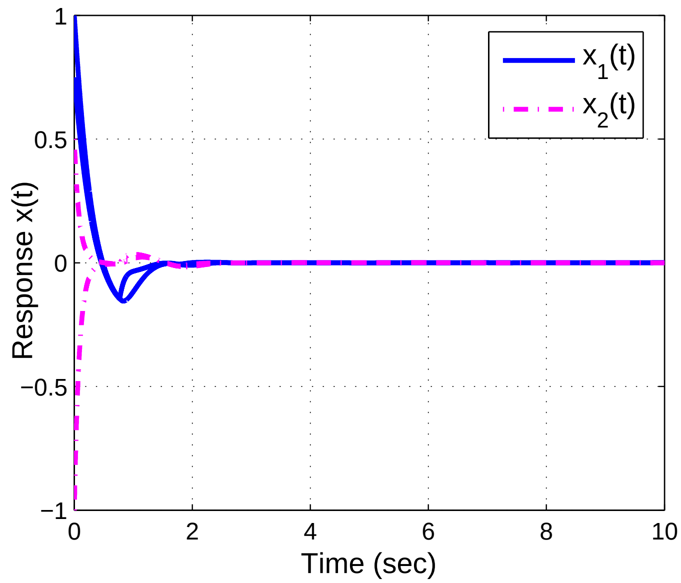

Figure 3. State trajectories of system (4.1) with $d(t) = 0$

5. Conclusions

This paper has addressed the ultimate stability of nonlinear time-varying systems with multiple delays and bounded disturbances. Explicit conditions have been derived for determining an ultimate bound and a finite transient time T that guarantee all the state trajectories of the system converge exponentially to the ultimate bound after a transient time T.

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