6 Exponential and Logarithmic Functions

Depreciation of Cars

You are ready to buy that first new car. You know that cars lose value over time due to depreciation and that different cars have different rates of depreciation. So you will research the depreciation rates for the cars you are thinking of buying. After all, for cars that sell for about the same price, the lower the depreciation rate, the more the car will be worth each year.

—See the Internet-based Chapter Project I—

••• A Look Back

Until now, our study of functions has concentrated on polynomial and rational functions. These functions belong to the class of **algebraic functions**—that is, functions that can be expressed in terms of sums, differences, products, quotients, powers, or roots of polynomials. Functions that are not algebraic are termed **transcendental** (they transcend, or go beyond, algebraic functions).

A Look Ahead •••

In this chapter, we study two transcendental functions: the exponential function and the logarithmic function. These functions occur frequently in a wide variety of applications, such as biology, chemistry, economics, and psychology.

The chapter begins with a discussion of composite, one-to-one, and inverse functions—concepts that are needed to explain the relationship between exponential and logarithmic functions.

Outline

6.1 Composite Functions

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Find the Value of a Function (Section 3.1, pp. 210–212)
- Domain of a Function (Section 3.1, pp. 214–216)

Now Work the **'Are You Prepared?'** problems on page 413.

OBJECTIVES
1. Form a Composite Function (p. 408)
2. Find the Domain of a Composite Function (p. 409)

1 Form a Composite Function

Suppose that an oil tanker is leaking oil and you want to determine the area of the circular oil patch around the ship. See Figure 1. It is determined that the oil is leaking from the tanker in such a way that the radius of the circular patch of oil around the ship is increasing at a rate of 3 feet per minute. Therefore, the radius r of the oil patch at any time t, in minutes, is given by $r(t) = 3t$. So after 20 minutes, the radius of the oil patch is $r(20) = 3(20) = 60$ feet.

The area A of a circle as a function of the radius r is given by $A(r) = \pi r^2$. The area of the circular patch of oil after 20 minutes is $A(60) = \pi(60)^2 = 3600\pi$ square feet. Note that $60 = r(20)$, so $A(60) = A(r(20))$. The argument of the function A is the output of the function r!

In general, the area of the oil patch can be expressed as a function of time t by evaluating $A(r(t))$ and obtaining $A(r(t)) = A(3t) = \pi(3t)^2 = 9\pi t^2$. The function $A(r(t))$ is a special type of function called a *composite function.*

As another example, consider the function $y = (2x + 3)^2$. Let $y = f(u) = u^2$ and $u = g(x) = 2x + 3$. Then by a substitution process, the original function is obtained as follows: $y = f(u) = f(g(x)) = (2x + 3)^2$.

In general, suppose that f and g are two functions and that x is a number in the domain of g. Evaluating g at x yields $g(x)$. If $g(x)$ is in the domain of f, then evaluating f at $g(x)$ yields the expression $f(g(x))$. The correspondence from x to $f(g(x))$ is called a *composite function* $f \circ g$.

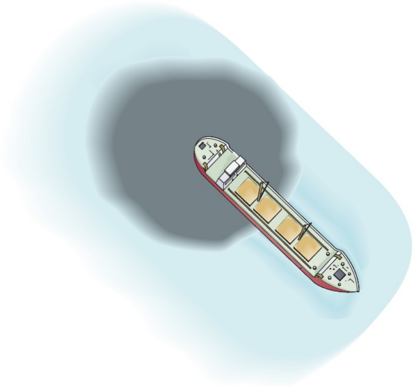


Figure 1

DEFINITION

Given two functions f and g, the **composite function**, denoted by $f \circ g$ (read as "f composed with g"), is defined by

$$\boxed{(f \circ g)(x) = f(g(x))}$$

The domain of $f \circ g$ is the set of all numbers x in the domain of g such that $g(x)$ is in the domain of f.

Look carefully at Figure 2. Only those values of x in the domain of g for which $g(x)$ is in the domain of f can be in the domain of $f \circ g$. The reason is that if $g(x)$ is not in the domain of f, then $f(g(x))$ is not defined. Because of this, the domain of $f \circ g$ is a subset of the domain of g; the range of $f \circ g$ is a subset of the range of f.

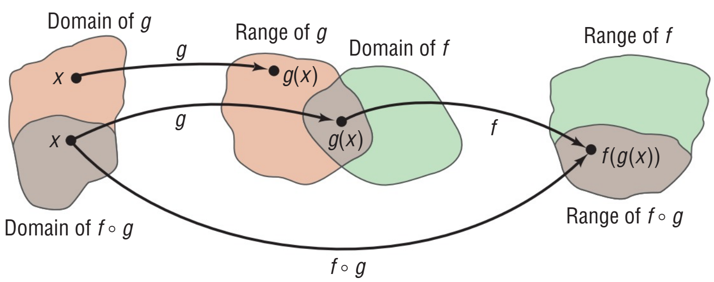


Figure 2

Figure 3 provides a second illustration of the definition. Here x is the input to the function g, yielding $g(x)$. Then $g(x)$ is the input to the function f, yielding $f(g(x))$. Note that the "inside" function g in $f(g(x))$ is "processed" first.

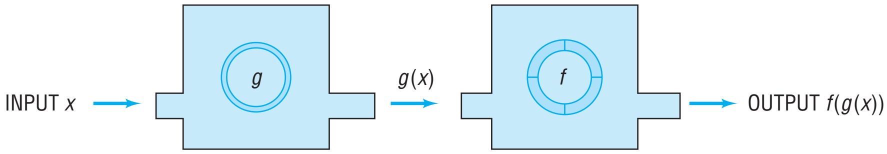


Figure 3

EXAMPLE 1 **Evaluating a Composite Function**

Suppose that $f(x) = 2x^2 - 3$ and $g(x) = 4x$. Find:

(a) $(f \circ g)(1)$ (b) $(g \circ f)(1)$ (c) $(f \circ f)(-2)$ (d) $(g \circ g)(-1)$

Solution

(a) $(f \circ g)(1) = f(g(1)) = f(4) = 2 \cdot 4^2 - 3 = 29$

$g(x) = 4x$ $\quad f(x) = 2x^2 - 3$

$g(1) = 4$

(b) $(g \circ f)(1) = g(f(1)) = g(-1) = 4 \cdot (-1) = -4$

$f(x) = 2x^2 - 3$ $\quad g(x) = 4x$

$f(1) = -1$

(c) $(f \circ f)(-2) = f(f(-2)) = f(5) = 2 \cdot 5^2 - 3 = 47$

$f(-2) = 2(-2)^2 - 3 = 5$

(d) $(g \circ g)(-1) = g(g(-1)) = g(-4) = 4 \cdot (-4) = -16$

$g(-1) = -4$

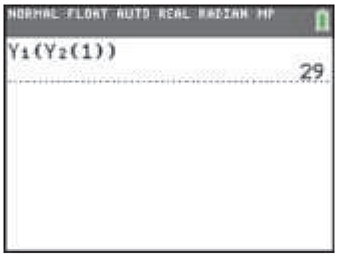


Figure 4

COMMENT Graphing calculators can be used to evaluate composite functions.* Let $Y_1 = f(x) = 2x^2 - 3$ and $Y_2 = g(x) = 4x$. Then, using a TI-84 Plus C graphing calculator, find $(f \circ g)(1)$ as shown in Figure 4. Note that this is the result obtained in Example 1(a).

Now Work PROBLEM 13

2 Find the Domain of a Composite Function

EXAMPLE 2 **Finding a Composite Function and Its Domain**

Suppose that $f(x) = x^2 + 3x - 1$ and $g(x) = 2x + 3$.

Find: (a) $f \circ g$ (b) $g \circ f$

Then find the domain of each composite function.

Solution The domain of f and the domain of g are the set of all real numbers.

(a) $(f \circ g)(x) = f(g(x)) = f(2x + 3) = (2x + 3)^2 + 3(2x + 3) - 1$

$f(x) = x^2 + 3x - 1$

$= 4x^2 + 12x + 9 + 6x + 9 - 1 = 4x^2 + 18x + 17$

Because the domains of both f and g are the set of all real numbers, the domain of $f \circ g$ is the set of all real numbers.

*Consult your owner's manual for the appropriate keystrokes.

(b) $(g \circ f)(x) = g(f(x)) = g(x^2 + 3x - 1) = 2(x^2 + 3x - 1) + 3$

$\uparrow$ $g(x) = 2x + 3$

$= 2x^2 + 6x - 2 + 3 = 2x^2 + 6x + 1$

Because the domains of both f and g are the set of all real numbers, the domain of $g \circ f$ is the set of all real numbers. ■

Example 2 illustrates that, in general, $f \circ g \neq g \circ f$. Sometimes $f \circ g$ does equal $g \circ f$, as we shall see in Example 5.

Look back at Figure 2 on page 408. In determining the domain of the composite function $(f \circ g)(x) = f(g(x))$, keep the following two thoughts in mind about the input x.

1. Any x not in the domain of g must be excluded.
2. Any x for which $g(x)$ is not in the domain of f must be excluded.

EXAMPLE 3 Finding the Domain of $f \circ g$

Find the domain of $f \circ g$ if $f(x) = \dfrac{1}{x+2}$ and $g(x) = \dfrac{4}{x-1}$.

Solution For $(f \circ g)(x) = f(g(x))$, first note that the domain of g is $\{x \mid x \neq 1\}$, so 1 is excluded from the domain of $f \circ g$. Next note that the domain of f is $\{x \mid x \neq -2\}$, which means that $g(x)$ cannot equal -2. Solve the equation $g(x) = -2$ to determine what additional value(s) of x to exclude.

$$\begin{aligned}
\frac{4}{x-1} &= -2 && g(x) = -2\\
4 &= -2(x-1) && \text{Multiply both sides by } x - 1.\\
4 &= -2x + 2 && \text{Apply the Distributive Property.}\\
2x &= -2 && \text{Add } 2x \text{ to both sides. Subtract 4 from both sides.}\\
x &= -1 && \text{Divide both sides by 2.}
\end{aligned}$$

Also exclude -1 from the domain of $f \circ g$.

The domain of $f \circ g$ is $\{x \mid x \neq -1, x \neq 1\}$.

✓ **Check:** For $x = 1$, $g(x) = \dfrac{4}{x-1}$ is not defined, so $(f \circ g)(x) = f(g(x))$ is not defined.

For $x = -1$, $g(-1) = -2$, and $(f \circ g)(-1) = f(g(-1)) = f(-2)$ is not defined. ■

EXAMPLE 4 Finding a Composite Function and Its Domain

Suppose that $f(x) = \dfrac{1}{x+2}$ and $g(x) = \dfrac{4}{x-1}$.

Find: (a) $f \circ g$ (b) $f \circ f$

Then find the domain of each composite function.

Solution The domain of f is $\{x \mid x \neq -2\}$ and the domain of g is $\{x \mid x \neq 1\}$.

(a) $(f \circ g)(x) = f(g(x)) = f\left(\dfrac{4}{x-1}\right) = \dfrac{1}{\dfrac{4}{x-1} + 2} = \dfrac{x-1}{4 + 2(x-1)} = \dfrac{x-1}{2x+2} = \dfrac{x-1}{2(x+1)}$

$\uparrow$ $f(x) = \dfrac{1}{x+2}$ $\quad$ $\uparrow$ Multiply by $\dfrac{x-1}{x-1}$.

In Example 3, the domain of $f \circ g$ was found to be $\{x \mid x \neq -1, x \neq 1\}$.

The domain of $f \circ g$ also can be found by first looking at the domain of g: $\{x \mid x \neq 1\}$. Exclude 1 from the domain of $f \circ g$ as a result. Then look at $f \circ g$ and note that x cannot equal -1, because $x = -1$ results in division by 0. So exclude -1 from the domain of $f \circ g$. Therefore, the domain of $f \circ g$ is $\{x \mid x \neq -1, x \neq 1\}$.

(b) $$(f \circ f)(x) = f(f(x)) = f\left(\frac{1}{x+2}\right) \underset{\substack{\uparrow \\ f(x) = \frac{1}{x+2}}}{=} \frac{1}{\frac{1}{x+2} + 2} \underset{\substack{\uparrow \\ \text{Multiply by } \frac{x+2}{x+2}.}}{=} \frac{x+2}{1 + 2(x+2)} = \frac{x+2}{2x+5}$$

The domain of $f \circ f$ consists of all values of x in the domain of f, $\{x \mid x \neq -2\}$, for which

$$f(x) = \frac{1}{x+2} \neq -2 \qquad \begin{aligned} \frac{1}{x+2} &= -2 \\ 1 &= -2(x+2) \\ 1 &= -2x - 4 \\ 2x &= -5 \\ x &= -\frac{5}{2} \end{aligned}$$

or, equivalently,

$$x \neq -\frac{5}{2}$$

The domain of $f \circ f$ is $\left\{x \,\middle|\, x \neq -\frac{5}{2}, x \neq -2\right\}$.

The domain of $f \circ f$ also can be found by recognizing that -2 is not in the domain of f and so should be excluded from the domain of $f \circ f$. Then, looking at $f \circ f$, note that x cannot equal $-\frac{5}{2}$. Do you see why? Therefore, the domain of $f \circ f$ is $\left\{x \,\middle|\, x \neq -\frac{5}{2}, x \neq -2\right\}$. ■

Now Work PROBLEMS 27 AND 29

EXAMPLE 5 Showing That Two Composite Functions Are Equal

If $f(x) = 3x - 4$ and $g(x) = \frac{1}{3}(x+4)$, show that

$$(f \circ g)(x) = (g \circ f)(x) = x$$

for every x in the domain of $f \circ g$ and $g \circ f$.

Solution

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f\left(\frac{x+4}{3}\right) && g(x) = \frac{1}{3}(x+4) = \frac{x+4}{3} \\ &= 3\left(\frac{x+4}{3}\right) - 4 && f(x) = 3x - 4 \\ &= x + 4 - 4 = x \end{aligned}$$

Seeing the Concept

Using a graphing calculator, let

$$Y_1 = f(x) = 3x - 4$$
$$Y_2 = g(x) = \frac{1}{3}(x + 4)$$
$$Y_3 = f \circ g, \quad Y_4 = g \circ f$$

Using the viewing window $-3 \le x \le 3$, $-2 \le y \le 2$, graph only Y_3 and Y_4. What do you see? TRACE to verify that $Y_3 = Y_4$. ■

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(3x - 4) && f(x) = 3x - 4 \\ &= \frac{1}{3}[(3x - 4) + 4] && g(x) = \frac{1}{3}(x + 4) \\ &= \frac{1}{3}(3x) = x\end{aligned}$$

We conclude that $(f \circ g)(x) = (g \circ f)(x) = x$. ■

In Section 6.2, we shall see that there is an important relationship between functions f and g for which $(f \circ g)(x) = (g \circ f)(x) = x$.

Now Work PROBLEM 39

Calculus Application

Some techniques in calculus require the ability to determine the components of a composite function. For example, the function $H(x) = \sqrt{x + 1}$ is the composition of the functions f and g, where $f(x) = \sqrt{x}$ and $g(x) = x + 1$, because $H(x) = (f \circ g)(x) = f(g(x)) = f(x + 1) = \sqrt{x + 1}$.

EXAMPLE 6 Finding the Components of a Composite Function

Find functions f and g such that $f \circ g = H$ if $H(x) = (x^2 + 1)^{50}$.

Solution The function H takes $x^2 + 1$ and raises it to the power 50. A natural way to decompose H is to raise the function $g(x) = x^2 + 1$ to the power 50. Let $f(x) = x^{50}$ and $g(x) = x^2 + 1$. Then

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(x^2 + 1) \\ &= (x^2 + 1)^{50} = H(x)\end{aligned}$$

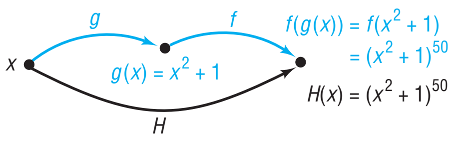


Figure 5

See Figure 5. ■

Other functions f and g may be found for which $f \circ g = H$ in Example 6. For instance, if $f(x) = x^2$ and $g(x) = (x^2 + 1)^{25}$, then

$$(f \circ g)(x) = f(g(x)) = f((x^2 + 1)^{25}) = [(x^2 + 1)^{25}]^2 = (x^2 + 1)^{50}$$

Although the functions f and g found as a solution to Example 6 are not unique, there is usually a "natural" selection for f and g that comes to mind first.

EXAMPLE 7 Finding the Components of a Composite Function

Find functions f and g such that $f \circ g = H$ if $H(x) = \dfrac{1}{x + 1}$.

Solution Here H is the reciprocal of $g(x) = x + 1$. Let $f(x) = \dfrac{1}{x}$ and $g(x) = x + 1$. Then

$$(f \circ g)(x) = f(g(x)) = f(x + 1) = \frac{1}{x + 1} = H(x)$$ ■

Now Work PROBLEM 47

6.1 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Find $f(3)$ if $f(x) = -4x^2 + 5x$. (pp. 210–212)

2. Find $f(3x)$ if $f(x) = 4 - 2x^2$. (pp. 210–212)

3. Find the domain of the function $f(x) = \dfrac{x^2 - 1}{x^2 - 25}$. (pp. 214–216)

Concepts and Vocabulary

4. Given two functions f and g, the __________ __________, denoted $f \circ g$, is defined by $(f \circ g)(x) =$ ________.

5. ***True or False*** If $f(x) = x^2$ and $g(x) = \sqrt{x + 9}$, then $(f \circ g)(4) = 5$.

6. If $f(x) = \sqrt{x + 2}$ and $g(x) = \dfrac{3}{x}$, which of the following does $(f \circ g)(x)$ equal?

(a) $\dfrac{3}{\sqrt{x + 2}}$ (b) $\dfrac{3}{\sqrt{x}} + 2$ (c) $\sqrt{\dfrac{3}{x} + 2}$ (d) $\sqrt{\dfrac{3}{x + 2}}$

7. If $H = f \circ g$ and $H(x) = \sqrt{25 - 4x^2}$, which of the following cannot be the component functions f and g?

(a) $f(x) = \sqrt{25 - x^2}$; $g(x) = 4x$

(b) $f(x) = \sqrt{x}$; $g(x) = 25 - 4x^2$

(c) $f(x) = \sqrt{25 - x}$; $g(x) = 4x^2$

(d) $f(x) = \sqrt{25 - 4x}$; $g(x) = x^2$

8. ***True or False*** The domain of the composite function $(f \circ g)(x)$ is the same as the domain of $g(x)$.

Skill Building

In Problems 9 and 10, evaluate each expression using the values given in the table.

9.

x	−3	−2	−1	0	1	2	3
$f(x)$	−7	−5	−3	−1	3	5	7
$g(x)$	8	3	0	−1	0	3	8

(a) $(f \circ g)(1)$ (b) $(f \circ g)(-1)$
(c) $(g \circ f)(-1)$ (d) $(g \circ f)(0)$
(e) $(g \circ g)(-2)$ (f) $(f \circ f)(-1)$

10.

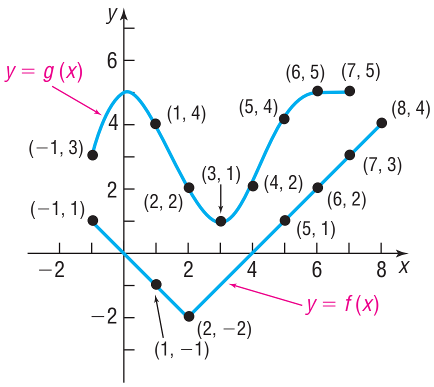
x	−3	−2	−1	0	1	2	3
$f(x)$	11	9	7	5	3	1	−1
$g(x)$	−8	−3	0	1	0	−3	−8

(a) $(f \circ g)(1)$ (b) $(f \circ g)(2)$
(c) $(g \circ f)(2)$ (d) $(g \circ f)(3)$
(e) $(g \circ g)(1)$ (f) $(f \circ f)(3)$

In Problems 11 and 12, evaluate each expression using the graphs of $y = f(x)$ and $y = g(x)$ shown in the figure.

11. (a) $(g \circ f)(-1)$ (b) $(g \circ f)(0)$
(c) $(f \circ g)(-1)$ (d) $(f \circ g)(4)$

12. (a) $(g \circ f)(1)$ (b) $(g \circ f)(5)$
(c) $(f \circ g)(0)$ (d) $(f \circ g)(2)$



In Problems 13–22, for the given functions f and g, find:

(a) $(f \circ g)(4)$ *(b)* $(g \circ f)(2)$ *(c)* $(f \circ f)(1)$ *(d)* $(g \circ g)(0)$

13. $f(x) = 2x$; $g(x) = 3x^2 + 1$

14. $f(x) = 3x + 2$; $g(x) = 2x^2 - 1$

15. $f(x) = 4x^2 - 3$; $g(x) = 3 - \dfrac{1}{2}x^2$

16. $f(x) = 2x^2$; $g(x) = 1 - 3x^2$

17. $f(x) = \sqrt{x}$; $g(x) = 2x$

18. $f(x) = \sqrt{x + 1}$; $g(x) = 3x$

19. $f(x) = |x|$; $g(x) = \dfrac{1}{x^2 + 1}$

20. $f(x) = |x - 2|$; $g(x) = \dfrac{3}{x^2 + 2}$

21. $f(x) = \dfrac{3}{x + 1}$; $g(x) = \sqrt[3]{x}$

22. $f(x) = x^{3/2}$; $g(x) = \dfrac{2}{x + 1}$

In Problems 23–38, for the given functions f and g, find:
(a) $f \circ g$ (b) $g \circ f$ (c) $f \circ f$ (d) $g \circ g$
State the domain of each composite function.

23. $f(x) = 2x + 3; \quad g(x) = 3x$

24. $f(x) = -x; \quad g(x) = 2x - 4$

25. $f(x) = 3x + 1; \quad g(x) = x^2$

26. $f(x) = x + 1; \quad g(x) = x^2 + 4$

27. $f(x) = x^2; \quad g(x) = x^2 + 4$

28. $f(x) = x^2 + 1; \quad g(x) = 2x^2 + 3$

29. $f(x) = \dfrac{3}{x-1}; \quad g(x) = \dfrac{2}{x}$

30. $f(x) = \dfrac{1}{x+3}; \quad g(x) = -\dfrac{2}{x}$

31. $f(x) = \dfrac{x}{x-1}; \quad g(x) = -\dfrac{4}{x}$

32. $f(x) = \dfrac{x}{x+3}; \quad g(x) = \dfrac{2}{x}$

33. $f(x) = \sqrt{x}; \quad g(x) = 2x + 3$

34. $f(x) = \sqrt{x-2}; \quad g(x) = 1 - 2x$

35. $f(x) = x^2 + 1; \quad g(x) = \sqrt{x-1}$

36. $f(x) = x^2 + 4; \quad g(x) = \sqrt{x-2}$

37. $f(x) = \dfrac{x-5}{x+1}; \quad g(x) = \dfrac{x+2}{x-3}$

38. $f(x) = \dfrac{2x-1}{x-2}; \quad g(x) = \dfrac{x+4}{2x-5}$

In Problems 39–46, show that $(f \circ g)(x) = (g \circ f)(x) = x$.

39. $f(x) = 2x; \quad g(x) = \dfrac{1}{2}x$

40. $f(x) = 4x; \quad g(x) = \dfrac{1}{4}x$

41. $f(x) = x^3; \quad g(x) = \sqrt[3]{x}$

42. $f(x) = x + 5; \quad g(x) = x - 5$

43. $f(x) = 2x - 6; \quad g(x) = \dfrac{1}{2}(x + 6)$

44. $f(x) = 4 - 3x; \quad g(x) = \dfrac{1}{3}(4 - x)$

45. $f(x) = ax + b; \quad g(x) = \dfrac{1}{a}(x - b) \quad a \neq 0$

46. $f(x) = \dfrac{1}{x}; \quad g(x) = \dfrac{1}{x}$

In Problems 47–52, find functions f and g so that $f \circ g = H$.

47. $H(x) = (2x + 3)^4$

48. $H(x) = (1 + x^2)^3$

49. $H(x) = \sqrt{x^2 + 1}$

50. $H(x) = \sqrt{1 - x^2}$

51. $H(x) = |2x + 1|$

52. $H(x) = |2x^2 + 3|$

Applications and Extensions

53. If $f(x) = 2x^3 - 3x^2 + 4x - 1$ and $g(x) = 2$, find $(f \circ g)(x)$ and $(g \circ f)(x)$.

54. If $f(x) = \dfrac{x+1}{x-1}$, find $(f \circ f)(x)$.

55. If $f(x) = 2x^2 + 5$ and $g(x) = 3x + a$, find a so that the graph of $f \circ g$ crosses the y-axis at 23.

56. If $f(x) = 3x^2 - 7$ and $g(x) = 2x + a$, find a so that the graph of $f \circ g$ crosses the y-axis at 68.

In Problems 57 and 58, use the functions f and g to find:
(a) $f \circ g$ (b) $g \circ f$
(c) the domain of $f \circ g$ and of $g \circ f$
(d) the conditions for which $f \circ g = g \circ f$

57. $f(x) = ax + b \quad g(x) = cx + d$

58. $f(x) = \dfrac{ax + b}{cx + d} \quad g(x) = mx$

59. Surface Area of a Balloon The surface area S (in square meters) of a hot-air balloon is given by

$$S(r) = 4\pi r^2$$

where r is the radius of the balloon (in meters). If the radius r is increasing with time t (in seconds) according to the formula $r(t) = \frac{2}{3}t^3$, $t \geq 0$, find the surface area S of the balloon as a function of the time t.

60. Volume of a Balloon The volume V (in cubic meters) of the hot-air balloon described in Problem 59 is given by $V(r) = \frac{4}{3}\pi r^3$. If the radius r is the same function of t as in Problem 59, find the volume V as a function of the time t.

61. Automobile Production The number N of cars produced at a certain factory in one day after t hours of operation is given by $N(t) = 100t - 5t^2$, $0 \leq t \leq 10$. If the cost C (in dollars) of producing N cars is $C(N) = 15{,}000 + 8000N$, find the cost C as a function of the time t of operation of the factory.

62. Environmental Concerns The spread of oil leaking from a tanker is in the shape of a circle. If the radius r (in feet) of the spread after t hours is $r(t) = 200\sqrt{t}$, find the area A of the oil slick as a function of the time t.

63. Production Cost The price p, in dollars, of a certain product and the quantity x sold obey the demand equation

$$p = -\frac{1}{4}x + 100 \quad 0 \leq x \leq 400$$

Suppose that the cost C, in dollars, of producing x units is

$$C = \frac{\sqrt{x}}{25} + 600$$

Assuming that all items produced are sold, find the cost C as a function of the price p.

[**Hint:** Solve for x in the demand equation and then form the composite function.]

64. Cost of a Commodity The price p, in dollars, of a certain commodity and the quantity x sold obey the demand equation

$$p = -\frac{1}{5}x + 200 \quad 0 \le x \le 1000$$

Suppose that the cost C, in dollars, of producing x units is

$$C = \frac{\sqrt{x}}{10} + 400$$

Assuming that all items produced are sold, find the cost C as a function of the price p.

65. Volume of a Cylinder The volume V of a right circular cylinder of height h and radius r is $V = \pi r^2 h$. If the height is twice the radius, express the volume V as a function of r.

66. Volume of a Cone The volume V of a right circular cone is $V = \frac{1}{3}\pi r^2 h$. If the height is twice the radius, express the volume V as a function of r.

67. Foreign Exchange Traders often buy foreign currency in the hope of making money when the currency's value changes. For example, on April 15, 2015, one U.S. dollar could purchase 0.9428 euro, and one euro could purchase 126.457 yen. Let $f(x)$ represent the number of euros you can buy with x dollars, and let $g(x)$ represent the number of yen you can buy with x euros.

(a) Find a function that relates dollars to euros.

(b) Find a function that relates euros to yen.

(c) Use the results of parts (a) and (b) to find a function that relates dollars to yen. That is, find $(g \circ f)(x) = g(f(x))$.

(d) What is $g(f(1000))$?

68. Temperature Conversion The function $C(F) = \frac{5}{9}(F - 32)$ converts a temperature in degrees Fahrenheit, F, to a temperature in degrees Celsius, C. The function $K(C) = C + 273$, converts a temperature in degrees Celsius to a temperature in kelvins, K.

(a) Find a function that converts a temperature in degrees Fahrenheit to a temperature in kelvins.

(b) Determine 80 degrees Fahrenheit in kelvins.

69. Discounts The manufacturer of a computer is offering two discounts on last year's model computer. The first discount is a \$200 rebate and the second discount is 20% off the regular price, p.

(a) Write a function f that represents the sale price if only the rebate applies.

(b) Write a function g that represents the sale price if only the 20% discount applies.

(c) Find $f \circ g$ and $g \circ f$. What does each of these functions represent? Which combination of discounts represents a better deal for the consumer? Why?

70. Taxes Suppose that you work for \$15 per hour. Write a function that represents gross salary G as a function of hours worked h. Your employer is required to withhold taxes (federal income tax, Social Security, Medicare) from your paycheck. Suppose your employer withholds 20% of your income for taxes. Write a function that represents net salary N as a function of gross salary G. Find and interpret $N \circ G$.

71. Let $f(x) = ax + b$ and $g(x) = bx + a$, where a and b are integers. If $f(1) = 8$ and $f(g(20)) - g(f(20)) = -14$, find the product of a and b.*

72. If f and g are odd functions, show that the composite function $f \circ g$ is also odd.

73. If f is an odd function and g is an even function, show that the composite functions $f \circ g$ and $g \circ f$ are both even.

*Courtesy of the Joliet Junior College Mathematics Department

Retain Your Knowledge

Problems 74–77 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

74. Given $f(x) = 3x + 8$ and $g(x) = x - 5$, find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$. State the domain of each.

75. Find the real zeros of $f(x) = 2x - 5\sqrt{x} + 2$.

76. Use a graphing utility to graph $f(x) = -x^3 + 4x - 2$ over the interval $[-3, 3]$. Approximate any local maxima and local minima. Determine where the function is increasing and where it is decreasing.

77. Find the domain of $R(x) = \frac{x^2 + 6x + 5}{x - 3}$. Find any horizontal, vertical, or oblique asymptotes.

'Are You Prepared?' Answers

1. -21 **2.** $4 - 18x^2$ **3.** $\{x \mid x \ne -5, x \ne 5\}$

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Functions (Section 3.1, pp. 207–218)
- Increasing/Decreasing Functions (Section 3.3, p. 234)
- Rational Expressions (Chapter R, Section R.7, pp. 63–71)

Now Work the **'Are You Prepared?'** problems on page 424.

OBJECTIVES
1. Determine Whether a Function Is One-to-One (p. 416)
2. Determine the Inverse of a Function Defined by a Map or a Set of Ordered Pairs (p. 418)
3. Obtain the Graph of the Inverse Function from the Graph of the Function (p. 421)
4. Find the Inverse of a Function Defined by an Equation (p. 422)

1 Determine Whether a Function Is One-to-One

Section 3.1 presented four different ways to represent a function: (1) a map, (2) a set of ordered pairs, (3) a graph, and (4) an equation. For example, Figures 6 and 7 illustrate two different functions represented as mappings. The function in Figure 6 shows the correspondence between states and their populations (in millions). The function in Figure 7 shows a correspondence between animals and life expectancies (in years).

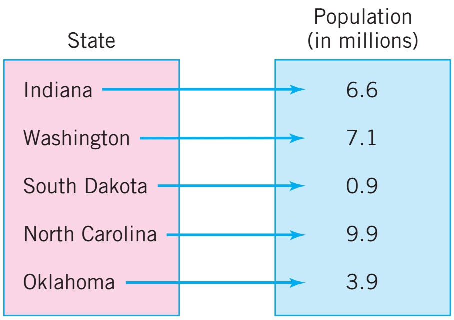


Figure 6

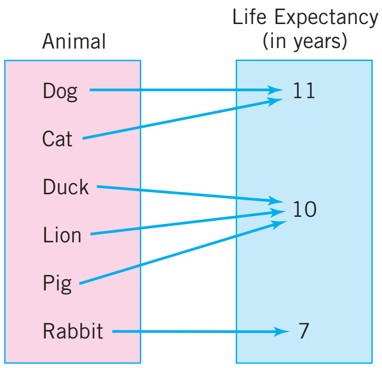


Figure 7

Suppose several people are asked to name a state that has a population of 0.9 million based on the function in Figure 6. Everyone will respond "South Dakota." Now, if the same people are asked to name an animal whose life expectancy is 11 years based on the function in Figure 7, some may respond "dog," while others may respond "cat." What is the difference between the functions in Figures 6 and 7? In Figure 6, no two elements in the domain correspond to the same element in the range. In Figure 7, this is not the case: Different elements in the domain correspond to the same element in the range. Functions such as the one in Figure 6 are given a special name.

DEFINITION

A function is **one-to-one** if any two different inputs in the domain correspond to two different outputs in the range. That is, if x_1 and x_2 are two different inputs of a function f, then f is one-to-one if $f(x_1) \neq f(x_2)$.

In Words
A function is not one-to-one if two different inputs correspond to the same output.

Put another way, a function f is one-to-one if no y in the range is the image of more than one x in the domain. A function is not one-to-one if any two (or more) different elements in the domain correspond to the same element in the range. So the function in Figure 7 is not one-to-one because two different elements in the domain, *dog* and *cat*, both correspond to 11 (and also because three different elements in the domain correspond to 10). Figure 8 illustrates the distinction among one-to-one functions, functions that are not one-to-one, and relations that are not functions.

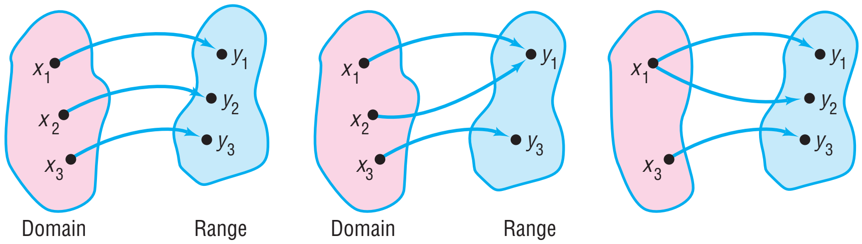
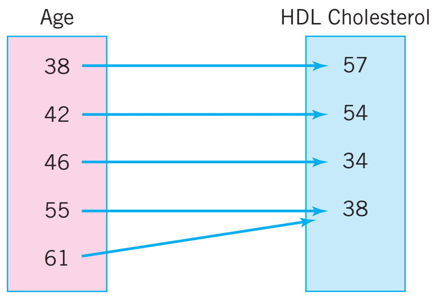


Figure 8 **(a)** One-to-one function: Each x in the domain has one and only one image in the range. **(b)** Not a one-to-one function: y_1 is the image of both x_1 and x_2. **(c)** Not a function: x_1 has two images, y_1 and y_2.

EXAMPLE 1 Determining Whether a Function Is One-to-One

Determine whether the following functions are one-to-one.

(a) For the following function, the domain represents the ages of five males, and the range represents their HDL (good) cholesterol scores (mg/dL).



(b) $\{(-2, 6), (-1, 3), (0, 2), (1, 5), (2, 8)\}$

Solution

(a) The function is not one-to-one because there are two different inputs, 55 and 61, that correspond to the same output, 38.

(b) The function is one-to-one because no two distinct inputs correspond to the same output.

Now Work PROBLEMS 13 AND 17

For functions defined by an equation $y = f(x)$ and for which the graph of f is known, there is a simple test, called the **horizontal-line test**, to determine whether f is one-to-one.

THEOREM

Horizontal-line Test

If every horizontal line intersects the graph of a function f in at most one point, then f is one-to-one.

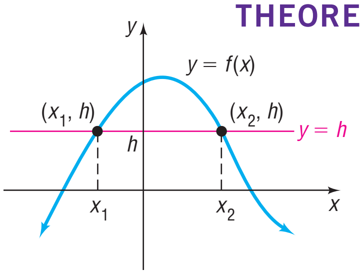


Figure 9 $f(x_1) = f(x_2) = h$ and $x_1 \neq x_2$; f is not a one-to-one function.

The reason why this test works can be seen in Figure 9, where the horizontal line $y = h$ intersects the graph at two distinct points, (x_1, h) and (x_2, h). Since h is the image of both x_1 and x_2 and $x_1 \neq x_2$, f is not one-to-one. Based on Figure 9, the horizontal-line test can be stated in another way: If the graph of any horizontal line intersects the graph of a function f at more than one point, then f is not one-to-one.

EXAMPLE 2 Using the Horizontal-line Test

For each function, use its graph to determine whether the function is one-to-one.

(a) $f(x) = x^2$ (b) $g(x) = x^3$

Solution (a) Figure 10(a) illustrates the horizontal-line test for $f(x) = x^2$. The horizontal line $y = 1$ intersects the graph of f twice, at $(1, 1)$ and at $(-1, 1)$, so f is not one-to-one.

(b) Figure 10(b) illustrates the horizontal-line test for $g(x) = x^3$. Because every horizontal line intersects the graph of g exactly once, it follows that g is one-to-one.

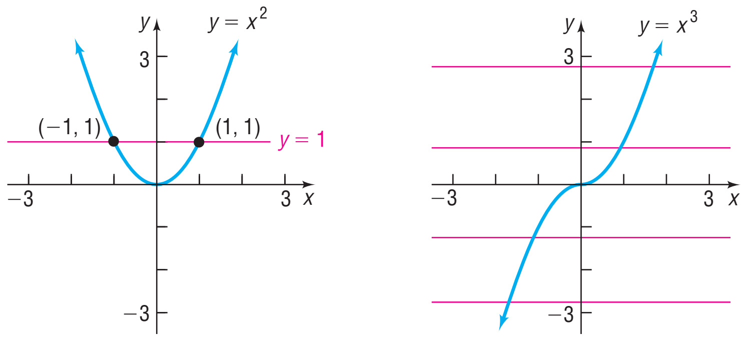


Figure 10 **(a)** A horizontal line intersects the graph twice; f is not one-to-one. **(b)** Every horizontal line intersects the graph exactly once; g is one-to-one.

Now Work PROBLEM 21

Look more closely at the one-to-one function $g(x) = x^3$. This function is an increasing function. Because an increasing (or decreasing) function will always have different y-values for unequal x-values, it follows that a function that is increasing (or decreasing) over its domain is also a one-to-one function.

THEOREM

A function that is increasing on an interval I is a one-to-one function on I.
A function that is decreasing on an interval I is a one-to-one function on I.

2 Determine the Inverse of a Function Defined by a Map or a Set of Ordered Pairs

DEFINITION

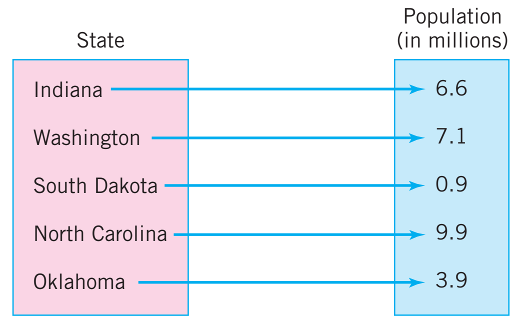
Suppose that f is a one-to-one function. Then, corresponding to each x in the domain of f, there is exactly one y in the range (because f is a function); and corresponding to each y in the range of f, there is exactly one x in the domain (because f is one-to-one). The correspondence from the range of f back to the domain of f is called the **inverse function of f**. The symbol f^{-1} is used to denote the inverse function of f.

In Words
Suppose that f is a one-to-one function so that the input 5 corresponds to the output 10. In the inverse function f^{-1}, the input 10 will correspond to the output 5.

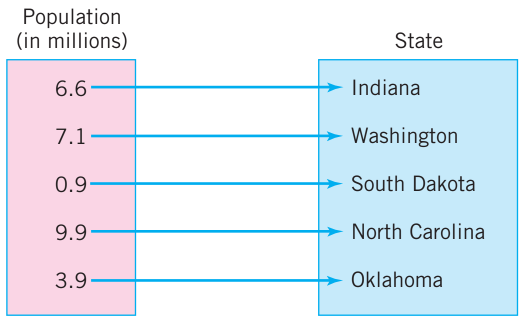
We will discuss how to find inverses for all four representations of functions: (1) maps, (2) sets of ordered pairs, (3) equations, and (4) graphs. We begin with finding inverses of functions represented by maps or sets of ordered pairs.

EXAMPLE 3 Finding the Inverse of a Function Defined by a Map

Find the inverse of the function defined by the map. Let the domain of the function represent certain states, and let the range represent the states' populations (in millions). Find the domain and the range of the inverse function.



Solution The function is one-to-one. To find the inverse function, interchange the elements in the domain with the elements in the range. For example, the function receives as input Indiana and outputs 6.6 million. So the inverse receives as input 6.6 million and outputs Indiana. The inverse function is shown next.



The domain of the inverse function is $\{6.6, 7.1, 0.9, 9.9, 3.9\}$. The range of the inverse function is {Indiana, Washington, South Dakota, North Carolina, Oklahoma}.

If the function f is a set of ordered pairs (x, y), then the inverse function of f, denoted f^{-1}, is the set of ordered pairs (y, x).

EXAMPLE 4 Finding the Inverse of a Function Defined by a Set of Ordered Pairs

Find the inverse of the following one-to-one function:

$$\{(-3, -27), (-2, -8), (-1, -1), (0, 0), (1, 1), (2, 8), (3, 27)\}$$

State the domain and the range of the function and its inverse.

Solution The inverse of the given function is found by interchanging the entries in each ordered pair and so is given by

$$\{(-27, -3), (-8, -2), (-1, -1), (0, 0), (1, 1), (8, 2), (27, 3)\}$$

The domain of the function is $\{-3, -2, -1, 0, 1, 2, 3\}$. The range of the function is $\{-27, -8, -1, 0, 1, 8, 27\}$. The domain of the inverse function is $\{-27, -8, -1, 0, 1, 8, 27\}$. The range of the inverse function is $\{-3, -2, -1, 0, 1, 2, 3\}$.

Now Work PROBLEMS 27 AND 31

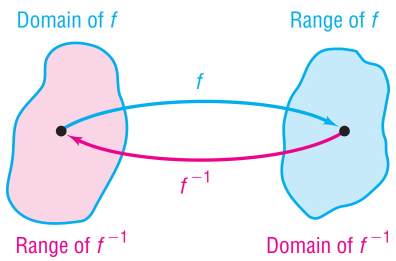


Figure 11

Remember, if f is a one-to-one function, it has an inverse function, f^{-1}. See Figure 11.

Based on the results of Example 4 and Figure 11, two facts are now apparent about a one-to-one function f and its inverse f^{-1}.

$$\boxed{\text{Domain of } f = \text{Range of } f^{-1} \qquad \text{Range of } f = \text{Domain of } f^{-1}}$$

Look again at Figure 11 to visualize the relationship. Starting with x, applying f, and then applying f^{-1} gets x back again. Starting with x, applying f^{-1}, and then applying f

WARNING Be careful! f^{-1} is a symbol for the inverse function of f. The -1 used in f^{-1} is not an exponent. That is, f^{-1} does *not* mean the reciprocal of f; $f^{-1}(x)$ is not equal to $\dfrac{1}{f(x)}$. ■

gets the number x back again. To put it simply, what f does, f^{-1} undoes, and vice versa. See the illustration that follows.

$$\boxed{\text{Input } x \text{ from domain of } f} \xrightarrow{\text{Apply } f} \boxed{f(x)} \xrightarrow{\text{Apply } f^{-1}} \boxed{f^{-1}(f(x)) = x}$$

$$\boxed{\text{Input } x \text{ from domain of } f^{-1}} \xrightarrow{\text{Apply } f^{-1}} \boxed{f^{-1}(x)} \xrightarrow{\text{Apply } f} \boxed{f(f^{-1}(x)) = x}$$

In other words,

$$f^{-1}(f(x)) = x \text{ where } x \text{ is in the domain of } f$$
$$f(f^{-1}(x)) = x \text{ where } x \text{ is in the domain of } f^{-1}$$

Consider the function $f(x) = 2x$, which multiplies the argument x by 2. The inverse function f^{-1} undoes whatever f does. So the inverse function of f is $f^{-1}(x) = \dfrac{1}{2}x$, which divides the argument by 2. For example, $f(3) = 2(3) = 6$ and $f^{-1}(6) = \dfrac{1}{2}(6) = 3$, so f^{-1} undoes what f did. This is verified by showing that

$$f^{-1}(f(x)) = f^{-1}(2x) = \frac{1}{2}(2x) = x \quad \text{and} \quad f(f^{-1}(x)) = f\left(\frac{1}{2}x\right) = 2\left(\frac{1}{2}x\right) = x$$

See Figure 12.

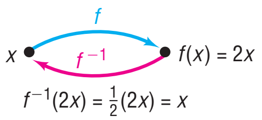


Figure 12

EXAMPLE 5 Verifying Inverse Functions

(a) Verify that the inverse of $g(x) = x^3$ is $g^{-1}(x) = \sqrt[3]{x}$.

(b) Verify that the inverse of $f(x) = 2x + 3$ is $f^{-1}(x) = \dfrac{1}{2}(x - 3)$.

Solution

(a) $g^{-1}(g(x)) = g^{-1}(x^3) = \sqrt[3]{x^3} = x$ for all x in the domain of g

$g(g^{-1}(x)) = g(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x$ for all x in the domain of g^{-1}

(b) $f^{-1}(f(x)) = f^{-1}(2x + 3) = \dfrac{1}{2}[(2x + 3) - 3] = \dfrac{1}{2}(2x) = x$ for all x in the domain of f

$f(f^{-1}(x)) = f\left(\dfrac{1}{2}(x - 3)\right) = 2\left[\dfrac{1}{2}(x - 3)\right] + 3 = (x - 3) + 3 = x$ for all x in the domain of f^{-1} ■

EXAMPLE 6 Verifying Inverse Functions

Verify that the inverse of $f(x) = \dfrac{1}{x - 1}$ is $f^{-1}(x) = \dfrac{1}{x} + 1$. For what values of x is $f^{-1}(f(x)) = x$? For what values of x is $f(f^{-1}(x)) = x$?

Solution The domain of f is $\{x \mid x \neq 1\}$ and the domain of f^{-1} is $\{x \mid x \neq 0\}$. Now

$$f^{-1}(f(x)) = f^{-1}\left(\frac{1}{x - 1}\right) = \frac{1}{\dfrac{1}{x - 1}} + 1 = x - 1 + 1 = x \quad \text{provided } x \neq 1$$

$$f(f^{-1}(x)) = f\left(\frac{1}{x} + 1\right) = \frac{1}{\left(\dfrac{1}{x} + 1\right) - 1} = \frac{1}{\dfrac{1}{x}} = x \quad \text{provided } x \neq 0$$

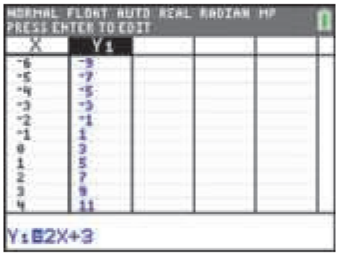
■

Now Work PROBLEMS 35 AND 39

3 Obtain the Graph of the Inverse Function from the Graph of the Function

For the functions in Example 5(b), we list points on the graph of $f = Y_1$ and on the graph of $f^{-1} = Y_2$ in Table 1. Note that whenever (a, b) is on the graph of f then (b, a) is on the graph of f^{-1}. Figure 13 shows these points plotted. Also shown is the graph of $y = x$, which you should observe is a line of symmetry of the points.

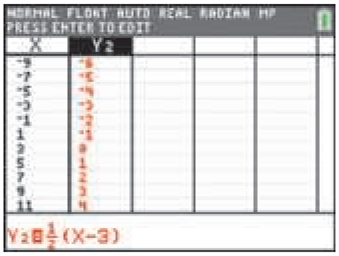
Table 1



NORMAL FLOAT AUTO REAL RADIAN MP
PRESS ENTER TO EDIT

X	Y_1
-6	-9
-5	-7
-4	-5
-3	-3
-2	-1
-1	1
0	3
1	5
2	7
3	9
4	11

$Y_1 \blacksquare 2X+3$



NORMAL FLOAT AUTO REAL RADIAN MP
PRESS ENTER TO EDIT

X	Y_2
-9	-6
-7	-5
-5	-4
-3	-3
-1	-2
1	-1
3	0
5	1
7	2
9	3
11	4

$Y_2 \blacksquare \frac{1}{2}(X-3)$

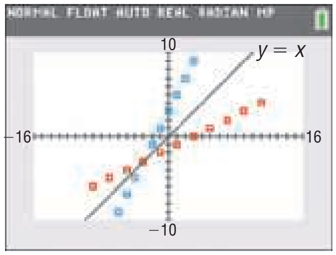


Figure 13

Exploration

Simultaneously graph $Y_1 = x$, $Y_2 = x^3$, and $Y_3 = \sqrt[3]{x}$ on a square screen with $-3 \le x \le 3$. What do you observe about the graphs of $Y_2 = x^3$, its inverse $Y_3 = \sqrt[3]{x}$, and the line $Y_1 = x$?

Repeat this experiment by simultaneously graphing $Y_1 = x$, $Y_2 = 2x + 3$, and $Y_3 = \dfrac{1}{2}(x - 3)$ on a square screen with $-6 \le x \le 3$. Do you see the symmetry of the graph of Y_2 and its inverse Y_3 with respect to the line $Y_1 = x$? ■

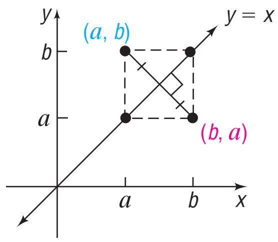


Figure 14

Suppose that (a, b) is a point on the graph of a one-to-one function f defined by $y = f(x)$. Then $b = f(a)$. This means that $a = f^{-1}(b)$, so (b, a) is a point on the graph of the inverse function f^{-1}. The relationship between the point (a, b) on f and the point (b, a) on f^{-1} is shown in Figure 14. The line segment with endpoints (a, b) and (b, a) is perpendicular to the line $y = x$ and is bisected by the line $y = x$. (Do you see why?) It follows that the point (b, a) on f^{-1} is the reflection about the line $y = x$ of the point (a, b) on f.

THEOREM

The graph of a one-to-one function f and the graph of its inverse function f^{-1} are symmetric with respect to the line $y = x$. ■

Figure 15 illustrates this result. Once the graph of f is known, the graph of f^{-1} may be obtained by reflecting the graph of f about the line $y = x$.

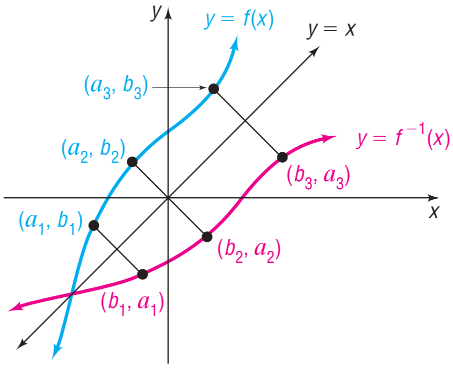


Figure 15

EXAMPLE 7 **Graphing the Inverse Function**

The graph in Figure 16(a) is that of a one-to-one function $y = f(x)$. Draw the graph of its inverse.

Solution Begin by adding the graph of $y = x$ to Figure 16(a). Since the points $(-2, -1)$, $(-1, 0)$, and $(2, 1)$ are on the graph of f, the points $(-1, -2)$, $(0, -1)$, and $(1, 2)$ must be on the graph of f^{-1}. Keeping in mind that the graph of f^{-1} is the reflection about the line $y = x$ of the graph of f, draw the graph of f^{-1}. See Figure 16(b).

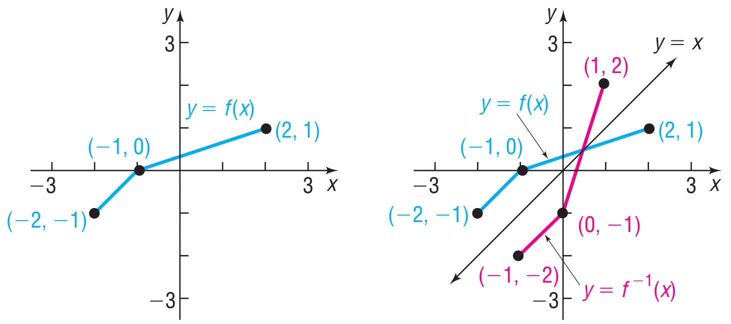


Figure 16 (a) (b) ■

Now Work PROBLEM 45

4 Find the Inverse of a Function Defined by an Equation

The fact that the graphs of a one-to-one function f and its inverse function f^{-1} are symmetric with respect to the line $y = x$ tells us more. It says that we can obtain f^{-1} by interchanging the roles of x and y in f. Look again at Figure 15. If f is defined by the equation

$$y = f(x)$$

then f^{-1} is defined by the equation

$$x = f(y)$$

The equation $x = f(y)$ defines f^{-1} *implicitly*. If we can solve this equation for y, we will have the *explicit* form of f^{-1}, that is,

$$y = f^{-1}(x)$$

Let's use this procedure to find the inverse of $f(x) = 2x + 3$. (Because f is a linear function and is increasing, f is one-to-one and so has an inverse function.)

EXAMPLE 8 How to Find the Inverse Function

Find the inverse of $f(x) = 2x + 3$. Graph f and f^{-1} on the same coordinate axes.

Step-by-Step Solution

Step 1: Replace $f(x)$ with y. In $y = f(x)$, interchange the variables x and y to obtain $x = f(y)$. This equation defines the inverse function f^{-1} implicitly.

Replace $f(x)$ with y in $f(x) = 2x + 3$ and obtain $y = 2x + 3$. Now interchange the variables x and y to obtain

$$x = 2y + 3$$

This equation defines the inverse function f^{-1} implicitly.

Step 2: If possible, solve the implicit equation for y in terms of x to obtain the explicit form of f^{-1}, $y = f^{-1}(x)$.

To find the explicit form of the inverse, solve $x = 2y + 3$ for y.

$$\begin{aligned} x &= 2y + 3 & \\ 2y + 3 &= x & &\text{Reflexive Property; If } a = b, \text{ then } b = a. \\ 2y &= x - 3 & &\text{Subtract 3 from both sides.} \\ y &= \frac{1}{2}(x - 3) & &\text{Multiply both sides by } \frac{1}{2}. \end{aligned}$$

The explicit form of the inverse function f^{-1} is

$$f^{-1}(x) = \frac{1}{2}(x - 3)$$

Step 3: Check the result by showing that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

We verified that f and f^{-1} are inverses in Example 5(b).

The graphs of $f(x) = 2x + 3$ and its inverse $f^{-1}(x) = \frac{1}{2}(x - 3)$ are shown in Figure 17. Note the symmetry of the graphs with respect to the line $y = x$. ■

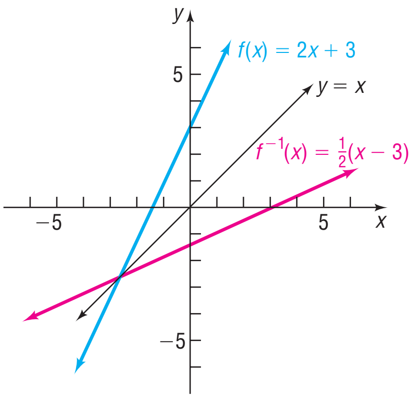


Figure 17

Procedure for Finding the Inverse of a One-to-One Function

Step 1: In $y = f(x)$, interchange the variables x and y to obtain

$$x = f(y)$$

This equation defines the inverse function f^{-1} implicitly.

Step 2: If possible, solve the implicit equation for y in terms of x to obtain the explicit form of f^{-1}:

$$y = f^{-1}(x)$$

Step 3: Check the result by showing that

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x$$

EXAMPLE 9 **Finding the Inverse Function**

The function

$$f(x) = \frac{2x+1}{x-1} \qquad x \neq 1$$

is one-to-one. Find its inverse function and check the result.

Solution **Step 1:** Replace $f(x)$ with y and interchange the variables x and y in

$$y = \frac{2x+1}{x-1}$$

to obtain

$$x = \frac{2y+1}{y-1}$$

Step 2: Solve for y.

$$\begin{aligned} x &= \frac{2y+1}{y-1} & \\ x(y-1) &= 2y+1 & &\text{Multiply both sides by } y-1. \\ xy - x &= 2y+1 & &\text{Apply the Distributive Property.} \\ xy - 2y &= x+1 & &\text{Subtract } 2y \text{ from both sides; add } x \text{ to both sides.} \\ (x-2)y &= x+1 & &\text{Factor.} \\ y &= \frac{x+1}{x-2} & &\text{Divide by } x-2. \end{aligned}$$

The inverse function is

$$f^{-1}(x) = \frac{x+1}{x-2} \quad x \neq 2 \qquad \text{Replace } y \text{ by } f^{-1}(x).$$

Step 3: ✓**Check:**

$$f^{-1}(f(x)) = f^{-1}\left(\frac{2x+1}{x-1}\right) = \frac{\frac{2x+1}{x-1}+1}{\frac{2x+1}{x-1}-2} = \frac{2x+1+x-1}{2x+1-2(x-1)} = \frac{3x}{3} = x, \quad x \neq 1$$

$$f(f^{-1}(x)) = f\left(\frac{x+1}{x-2}\right) = \frac{2\left(\frac{x+1}{x-2}\right)+1}{\frac{x+1}{x-2}-1} = \frac{2(x+1)+x-2}{x+1-(x-2)} = \frac{3x}{3} = x, \quad x \neq 2$$

■

Exploration

In Example 9, we found that if $f(x) = \frac{2x+1}{x-1}$, then $f^{-1}(x) = \frac{x+1}{x-2}$. Compare the vertical and horizontal asymptotes of f and f^{-1}.

Result The vertical asymptote of f is $x = 1$, and the horizontal asymptote is $y = 2$. The vertical asymptote of f^{-1} is $x = 2$, and the horizontal asymptote is $y = 1$. ■

Now Work PROBLEMS 53 AND 67

If a function is not one-to-one, it has no inverse function. Sometimes, though, an appropriate restriction on the domain of such a function will yield a new function that *is* one-to-one. Then the function defined on the restricted domain has an inverse function. Let's look at an example of this common practice.

EXAMPLE 10 **Finding the Inverse of a Domain-restricted Function**

Find the inverse of $y = f(x) = x^2$ if $x \geq 0$. Graph f and f^{-1}.

Solution The function $y = x^2$ is not one-to-one. [Refer to Example 2(a).] However, restricting the domain of this function to $x \geq 0$, as indicated, results in a new function that is increasing and therefore is one-to-one. Consequently, the function defined by $y = f(x) = x^2, x \geq 0$, has an inverse function, f^{-1}.

Follow the steps given previously to find f^{-1}.

Step 1: In the equation $y = x^2, x \geq 0$, interchange the variables x and y. The result is

$$x = y^2 \qquad y \geq 0$$

This equation defines the inverse function implicitly.

Step 2: Solve for y to get the explicit form of the inverse. Because $y \geq 0$, only one solution for y is obtained: $y = \sqrt{x}$. So $f^{-1}(x) = \sqrt{x}$.

Step 3: **Check:** $f^{-1}(f(x)) = f^{-1}(x^2) = \sqrt{x^2} = |x| = x$ because $x \geq 0$

$$f(f^{-1}(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$$

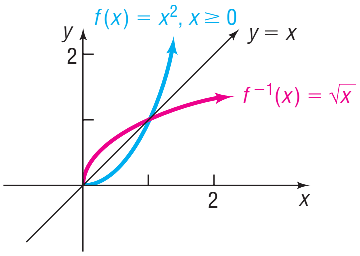


Figure 18

Figure 18 illustrates the graphs of $f(x) = x^2, x \geq 0$, and $f^{-1}(x) = \sqrt{x}$. Note that the domain of f = range of $f^{-1} = [0, \infty)$, and the domain of f^{-1} = range of $f = [0, \infty)$. ■

SUMMARY

1. If a function f is one-to-one, then it has an inverse function f^{-1}.
2. Domain of f = Range of f^{-1}; Range of f = Domain of f^{-1}.
3. To verify that f^{-1} is the inverse of f, show that $f^{-1}(f(x)) = x$ for every x in the domain of f and that $f(f^{-1}(x)) = x$ for every x in the domain of f^{-1}.
4. The graphs of f and f^{-1} are symmetric with respect to the line $y = x$.

6.2 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

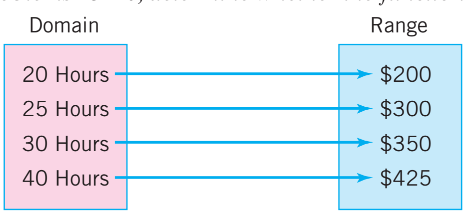
1. Is the set of ordered pairs $\{(1, 3), (2, 3), (-1, 2)\}$ a function? Why or why not? (pp. 207–210)
2. Where is the function $f(x) = x^2$ increasing? Where is it decreasing? (p. 234)
3. What is the domain of $f(x) = \dfrac{x + 5}{x^2 + 3x - 18}$? (pp. 214–216)
4. Simplify: $\dfrac{\dfrac{1}{x} + 1}{\dfrac{1}{x^2} - 1}$ (pp. 69–70)

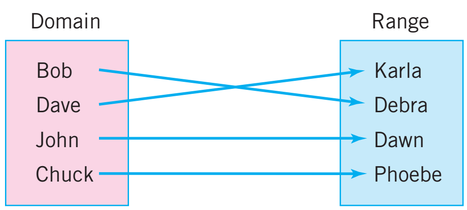
Concepts and Vocabulary

5. If x_1 and x_2 are two different inputs of a function f, then f is one-to-one if ______________.
6. If every horizontal line intersects the graph of a function f at no more than one point, then f is a(n) ______________ function.
7. If f is a one-to-one function and $f(3) = 8$, then $f^{-1}(8) =$ ______________.
8. If f^{-1} denotes the inverse of a function f, then the graphs of f and f^{-1} are symmetric with respect to the line ______________.
9. If the domain of a one-to-one function f is $[4, \infty)$, then the range of its inverse function f^{-1} is ______________.
10. ***True or False*** If f and g are inverse functions, then the domain of f is the same as the range of g.
11. If $(-2, 3)$ is a point on the graph of a one-to-one function f, which of the following points is on the graph of f^{-1}?
(a) $(3, -2)$ (b) $(2, -3)$ (c) $(-3, 2)$ (d) $(-2, -3)$
12. Suppose f is a one-to-one function with a domain of $\{x | x \neq 3\}$ and a range of $\left\{x \middle| x \neq \dfrac{2}{3}\right\}$. Which of the following is the domain of f^{-1}?
(a) $\{x | x \neq 3\}$
(b) All real numbers
(c) $\left\{x \middle| x \neq \dfrac{2}{3}, x \neq 3\right\}$
(d) $\left\{x \middle| x \neq \dfrac{2}{3}\right\}$

Skill Building

In Problems 13–20, determine whether the function is one-to-one.

13. 

14. 

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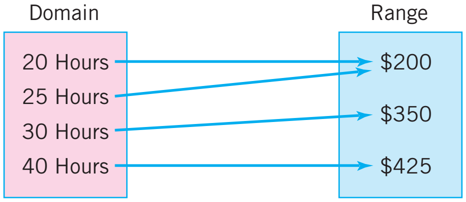

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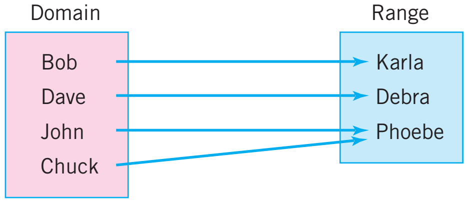


17. $\{(2, 6), (-3, 6), (4, 9), (1, 10)\}$

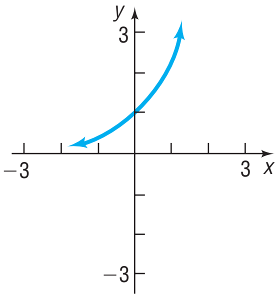
18. $\{(-2, 5), (-1, 3), (3, 7), (4, 12)\}$

19. $\{(0, 0), (1, 1), (2, 16), (3, 81)\}$

20. $\{(1, 2), (2, 8), (3, 18), (4, 32)\}$

In Problems 21–26, the graph of a function f is given. Use the horizontal-line test to determine whether f is one-to-one.

21.



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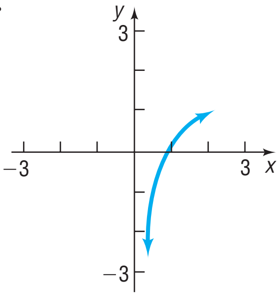

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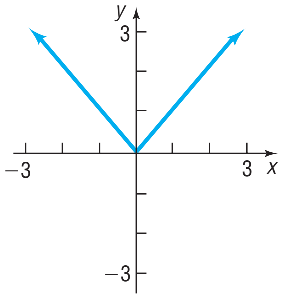

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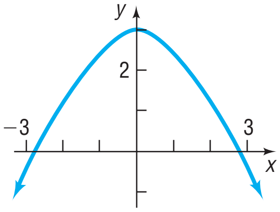

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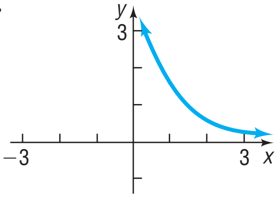

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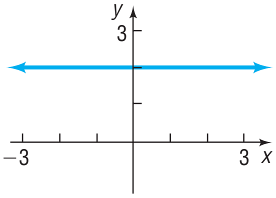
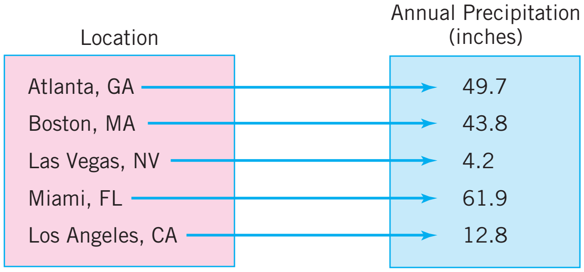


In Problems 27–34, find the inverse of each one-to-one function. State the domain and the range of each inverse function.

27.



Source: *currentresults.com*

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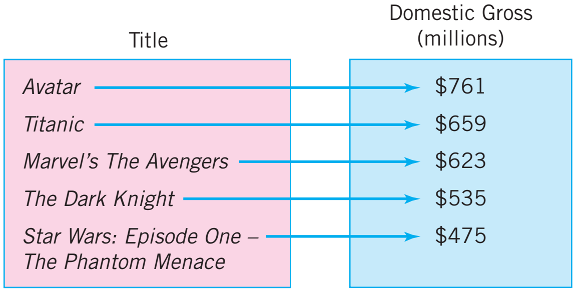

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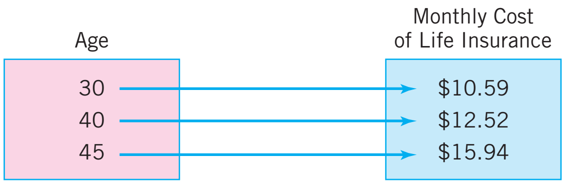
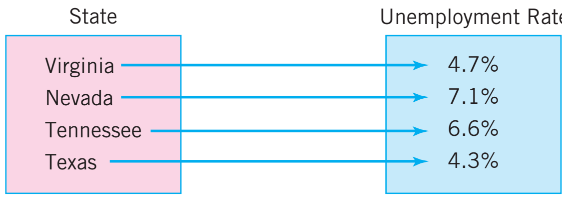


Source: *tiaa-cref.org*

30.



Source: *United States Bureau of Labor Statistics, March 2015*

31. $\{(-3, 5), (-2, 9), (-1, 2), (0, 11), (1, -5)\}$

32. $\{(-2, 2), (-1, 6), (0, 8), (1, -3), (2, 9)\}$

33. $\{(-2, 1), (-3, 2), (-10, 0), (1, 9), (2, 4)\}$

34. $\{(-2, -8), (-1, -1), (0, 0), (1, 1), (2, 8)\}$

In Problems 35–44, verify that the functions f and g are inverses of each other by showing that $f(g(x)) = x$ and $g(f(x)) = x$. Give any values of x that need to be excluded from the domain of f and the domain of g.

35. $f(x) = 3x + 4; \quad g(x) = \frac{1}{3}(x - 4)$

36. $f(x) = 3 - 2x; \quad g(x) = -\frac{1}{2}(x - 3)$

37. $f(x) = 4x - 8; \quad g(x) = \frac{x}{4} + 2$

38. $f(x) = 2x + 6; \quad g(x) = \frac{1}{2}x - 3$

39. $f(x) = x^3 - 8; \quad g(x) = \sqrt[3]{x + 8}$

40. $f(x) = (x - 2)^2, x \geq 2; \quad g(x) = \sqrt{x} + 2$

41. $f(x) = \dfrac{1}{x}; \quad g(x) = \dfrac{1}{x}$

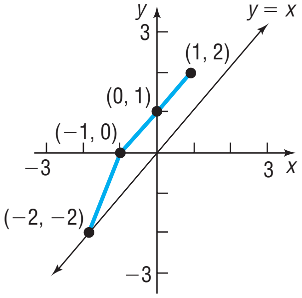
42. $f(x) = x; \quad g(x) = x$

43. $f(x) = \dfrac{2x+3}{x+4}; \quad g(x) = \dfrac{4x-3}{2-x}$

44. $f(x) = \dfrac{x-5}{2x+3}; \quad g(x) = \dfrac{3x+5}{1-2x}$

In Problems 45–50, the graph of a one-to-one function f is given. Draw the graph of the inverse function f^{-1}.

45.



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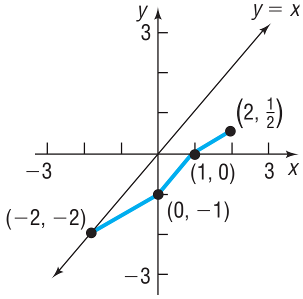

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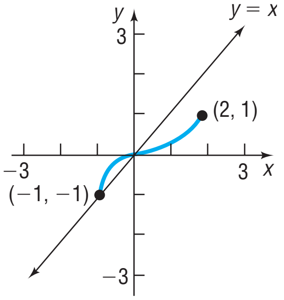
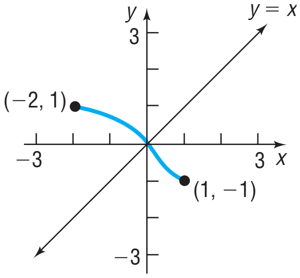


48.



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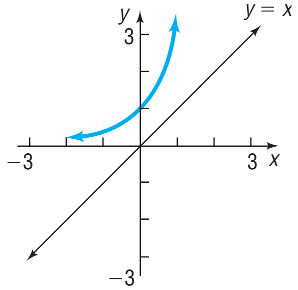

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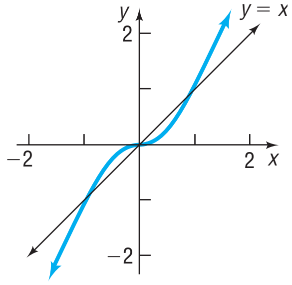


In Problems 51–62, the function f is one-to-one. (a) Find its inverse function f^{-1} and check your answer. (b) Find the domain and the range of f and f^{-1}. (c) Graph f, f^{-1}, and $y = x$ on the same coordinate axes.

51. $f(x) = 3x$

52. $f(x) = -4x$

53. $f(x) = 4x + 2$

54. $f(x) = 1 - 3x$

55. $f(x) = x^3 - 1$

56. $f(x) = x^3 + 1$

57. $f(x) = x^2 + 4, \quad x \ge 0$

58. $f(x) = x^2 + 9, \quad x \ge 0$

59. $f(x) = \dfrac{4}{x}$

60. $f(x) = -\dfrac{3}{x}$

61. $f(x) = \dfrac{1}{x-2}$

62. $f(x) = \dfrac{4}{x+2}$

In Problems 63–74, the function f is one-to-one. (a) Find its inverse function f^{-1} and check your answer. (b) Find the domain and the range of f and f^{-1}.

63. $f(x) = \dfrac{2}{3+x}$

64. $f(x) = \dfrac{4}{2-x}$

65. $f(x) = \dfrac{3x}{x+2}$

66. $f(x) = -\dfrac{2x}{x-1}$

67. $f(x) = \dfrac{2x}{3x-1}$

68. $f(x) = -\dfrac{3x+1}{x}$

69. $f(x) = \dfrac{3x+4}{2x-3}$

70. $f(x) = \dfrac{2x-3}{x+4}$

71. $f(x) = \dfrac{2x+3}{x+2}$

72. $f(x) = \dfrac{-3x-4}{x-2}$

73. $f(x) = \dfrac{x^2-4}{2x^2}, \quad x > 0$

74. $f(x) = \dfrac{x^2+3}{3x^2} \quad x > 0$

Applications and Extensions

75. Use the graph of $y = f(x)$ given in Problem 45 to evaluate the following:
(a) $f(-1)$ (b) $f(1)$ (c) $f^{-1}(1)$ (d) $f^{-1}(2)$

76. Use the graph of $y = f(x)$ given in Problem 46 to evaluate the following:
(a) $f(2)$ (b) $f(1)$ (c) $f^{-1}(0)$ (d) $f^{-1}(-1)$

77. If $f(7) = 13$ and f is one-to-one, what is $f^{-1}(13)$?

78. If $g(-5) = 3$ and g is one-to-one, what is $g^{-1}(3)$?

79. The domain of a one-to-one function f is $[5, \infty)$, and its range is $[-2, \infty)$. State the domain and the range of f^{-1}.

80. The domain of a one-to-one function f is $[0, \infty)$, and its range is $[5, \infty)$. State the domain and the range of f^{-1}.

81. The domain of a one-to-one function g is $(-\infty, 0]$, and its range is $[0, \infty)$. State the domain and the range of g^{-1}.

82. The domain of a one-to-one function g is $[0, 15]$, and its range is $(0, 8)$. State the domain and the range of g^{-1}.

83. A function $y = f(x)$ is increasing on the interval $[0, 5]$. What conclusions can you draw about the graph of $y = f^{-1}(x)$?

84. A function $y = f(x)$ is decreasing on the interval $[0, 5]$. What conclusions can you draw about the graph of $y = f^{-1}(x)$?

85. Find the inverse of the linear function

$$f(x) = mx + b, \quad m \neq 0$$

86. Find the inverse of the function

$$f(x) = \sqrt{r^2 - x^2}, \quad 0 \leq x \leq r$$

87. A function f has an inverse function f^{-1}. If the graph of f lies in quadrant I, in which quadrant does the graph of f^{-1} lie?

88. A function f has an inverse function f^{-1}. If the graph of f lies in quadrant II, in which quadrant does the graph of f^{-1} lie?

89. The function $f(x) = |x|$ is not one-to-one. Find a suitable restriction on the domain of f so that the new function that results is one-to-one. Then find the inverse of the new function.

90. The function $f(x) = x^4$ is not one-to-one. Find a suitable restriction on the domain of f so that the new function that results is one-to-one. Then find the inverse of the new function.

In applications, the symbols used for the independent and dependent variables are often based on common usage. So, rather than using $y = f(x)$ to represent a function, an applied problem might use $C = C(q)$ to represent the cost C of manufacturing q units of a good. Because of this, the inverse notation f^{-1} used in a pure mathematics problem is not used when finding inverses of applied problems. Rather, the inverse of a function such as $C = C(q)$ will be $q = q(C)$. So $C = C(q)$ is a function that represents the cost C as a function of the number q of units manufactured, and $q = q(C)$ is a function that represents the number q as a function of the cost C. Problems 91–94 illustrate this idea.

91. Vehicle Stopping Distance Taking into account reaction time, the distance d (in feet) that a car requires to come to a complete stop while traveling r miles per hour is given by the function

$$d(r) = 6.97r - 90.39$$

(a) Express the speed r at which the car is traveling as a function of the distance d required to come to a complete stop.
(b) Verify that $r = r(d)$ is the inverse of $d = d(r)$ by showing that $r(d(r)) = r$ and $d(r(d)) = d$.
(c) Predict the speed that a car was traveling if the distance required to stop was 300 feet.

92. Height and Head Circumference The head circumference C of a child is related to the height H of the child (both in inches) through the function

$$H(C) = 2.15C - 10.53$$

(a) Express the head circumference C as a function of height H.
(b) Verify that $C = C(H)$ is the inverse of $H = H(C)$ by showing that $H(C(H)) = H$ and $C(H(C)) = C$.
(c) Predict the head circumference of a child who is 26 inches tall.

93. Ideal Body Weight One model for the ideal body weight W for men (in kilograms) as a function of height h (in inches) is given by the function

$$W(h) = 50 + 2.3(h - 60)$$

(a) What is the ideal weight of a 6-foot male?
(b) Express the height h as a function of weight W.
(c) Verify that $h = h(W)$ is the inverse of $W = W(h)$ by showing that $h(W(h)) = h$ and $W(h(W)) = W$.
(d) What is the height of a male who is at his ideal weight of 80 kilograms?
[**Note:** The ideal body weight W for women (in kilograms) as a function of height h (in inches) is given by $W(h) = 45.5 + 2.3(h - 60)$.]

94. Temperature Conversion The function $F(C) = \frac{9}{5}C + 32$ converts a temperature from C degrees Celsius to F degrees Fahrenheit.

(a) Express the temperature in degrees Celsius C as a function of the temperature in degrees Fahrenheit F.
(b) Verify that $C = C(F)$ is the inverse of $F = F(C)$ by showing that $C(F(C)) = C$ and $F(C(F)) = F$.
(c) What is the temperature in degrees Celsius if it is 70 degrees Fahrenheit?

95. Income Taxes The function

$$T(g) = 5156.25 + 0.25(g - 37{,}450)$$

represents the 2015 federal income tax T (in dollars) due for a "single" filer whose modified adjusted gross income is g dollars, where $37{,}450 \leq g \leq 90{,}750$.
(a) What is the domain of the function T?
(b) Given that the tax due T is an increasing linear function of modified adjusted gross income g, find the range of the function T.
(c) Find adjusted gross income g as a function of federal income tax T. What are the domain and the range of this function?

96. Income Taxes The function

$$T(g) = 1845 + 0.15(g - 18{,}450)$$

represents the 2015 federal income tax T (in dollars) due for a "married filing jointly" filer whose modified adjusted gross income is g dollars, where $18{,}450 \leq g \leq 74{,}900$.
(a) What is the domain of the function T?
(b) Given that the tax due T is an increasing linear function of modified adjusted gross income g, find the range of the function T.
(c) Find adjusted gross income g as a function of federal income tax T. What are the domain and the range of this function?

97. Gravity on Earth If a rock falls from a height of 100 meters on Earth, the height H (in meters) after t seconds is approximately

$$H(t) = 100 - 4.9t^2$$

(a) In general, quadratic functions are not one-to-one. However, the function H is one-to-one. Why?
(b) Find the inverse of H and verify your result.
(c) How long will it take a rock to fall 80 meters?

98. Period of a Pendulum The period T (in seconds) of a simple pendulum as a function of its length l (in feet) is given by

$$T(l) = 2\pi\sqrt{\frac{l}{32.2}}$$

(a) Express the length l as a function of the period T.
(a) How long is a pendulum whose period is 3 seconds?

99. Given

$$f(x) = \frac{ax + b}{cx + d}$$

find $f^{-1}(x)$. If $c \neq 0$, under what conditions on a, b, c, and d is $f = f^{-1}$?

Explaining Concepts: Discussion and Writing

100. Can a one-to-one function and its inverse be equal? What must be true about the graph of f for this to happen? Give some examples to support your conclusion.

101. Draw the graph of a one-to-one function that contains the points $(-2, -3)$, $(0, 0)$, and $(1, 5)$. Now draw the graph of its inverse. Compare your graph to those of other students. Discuss any similarities. What differences do you see?

102. Give an example of a function whose domain is the set of real numbers and that is neither increasing nor decreasing on its domain, but is one-to-one.
[**Hint:** Use a piecewise-defined function.]

103. Is every odd function one-to-one? Explain.

104. Suppose that $C(g)$ represents the cost C, in dollars, of manufacturing g cars. Explain what $C^{-1}(800{,}000)$ represents.

105. Explain why the horizontal-line test can be used to identify one-to-one functions from a graph.

106. Explain why a function must be one-to-one in order to have an inverse that is a function. Use the function $y = x^2$ to support your explanation.

Retain Your Knowledge

Problems 107–110 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

107. Use the techniques of shifting, compressing or stretching, and reflections to graph $f(x) = -|x + 2| + 3$.

108. Find the zeros of the quadratic function $f(x) = 3x^2 + 5x + 1$. What are the x-intercepts, if any, of the graph of the function?

109. Find the domain of $R(x) = \dfrac{6x^2 - 11x - 2}{2x^2 - x - 6}$. Find any horizontal, vertical, or oblique asymptotes.

110. If $f(x) = 3x^2 - 7x$, find $f(x + h) - f(x)$.

'Are You Prepared?' Answers

1. Yes; for each input x there is one output y.

2. Increasing on $[0, \infty)$; decreasing on $(-\infty, 0]$

3. $\{x \mid x \neq -6, x \neq 3\}$

4. $\dfrac{x}{1 - x}, x \neq 0, x \neq -1$

6.3 Exponential Functions

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Exponents (Chapter R, Section R.2, pp. 22–24, and Section R.8, pp. 77–78)
- Graphing Techniques: Transformations (Section 3.5, pp. 256–264)
- Solving Linear and Quadratic Equations (Section 1.2, pp. 102–103, and Section 1.3, pp. 110–115)
- Average Rate of Change (Section 3.3, pp. 238–239)
- Quadratic Functions (Section 4.3, pp. 298–306)
- Linear Functions (Section 4.1, pp. 281–287)
- Horizontal Asymptotes (Section 5.4, pp. 377–379)

Now Work the **'Are You Prepared?'** problems on page 439.

OBJECTIVES
1. Evaluate Exponential Functions (p. 428)
2. Graph Exponential Functions (p. 432)
3. Define the Number e (p. 436)
4. Solve Exponential Equations (p. 437)

1 Evaluate Exponential Functions

Chapter R, Section R.8, gives a definition for raising a real number a to a rational power. That discussion provides meaning to expressions of the form

$$a^r$$

where the base a is a positive real number and the exponent r is a rational number.

But what is the meaning of a^x, where the base a is a positive real number and the exponent x is an irrational number? Although a rigorous definition requires methods discussed in calculus, the basis for the definition is easy to follow: Select a rational number r that is formed by truncating (removing) all but a finite number of digits from the irrational number x. Then it is reasonable to expect that

$$a^x \approx a^r$$

For example, take the irrational number $\pi = 3.14159\ldots$ Then an approximation to a^π is

$$a^\pi \approx a^{3.14}$$

where the digits after the hundredths position have been removed from the value for π. A better approximation would be

$$a^\pi \approx a^{3.14159}$$

where the digits after the hundred-thousandths position have been removed. Continuing in this way, we can obtain approximations to a^π to any desired degree of accuracy.

Most calculators have an $\boxed{x^y}$ key or a caret key $\boxed{\wedge}$ for working with exponents. To evaluate expressions of the form a^x, enter the base a, then press the $\boxed{x^y}$ key (or the $\boxed{\wedge}$ key), enter the exponent x, and press $\boxed{=}$ (or $\boxed{\text{ENTER}}$).

EXAMPLE 1 **Using a Calculator to Evaluate Powers of 2**

Using a calculator, evaluate:

(a) $2^{1.4}$ (b) $2^{1.41}$ (c) $2^{1.414}$ (d) $2^{1.4142}$ (e) $2^{\sqrt{2}}$

Solution Figure 19 shows the solution to parts (a) and (e) using a TI-84 Plus C graphing calculator.

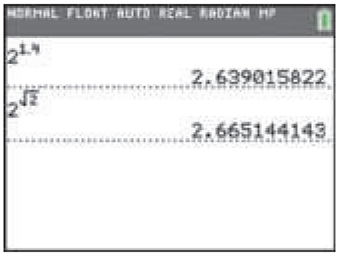


Figure 19

(a) $2^{1.4} \approx 2.639015822$ (b) $2^{1.41} \approx 2.657371628$

(c) $2^{1.414} \approx 2.66474965$ (d) $2^{1.4142} \approx 2.665119089$

(e) $2^{\sqrt{2}} \approx 2.665144143$

Now Work PROBLEM 15

It can be shown that the familiar laws for rational exponents hold for real exponents.

THEOREM **Laws of Exponents**

If $s, t, a,$ and b are real numbers with $a > 0$ and $b > 0$, then

$$a^s \cdot a^t = a^{s+t} \qquad (a^s)^t = a^{st} \qquad (ab)^s = a^s \cdot b^s$$
$$1^s = 1 \qquad a^{-s} = \frac{1}{a^s} = \left(\frac{1}{a}\right)^s \qquad a^0 = 1 \qquad \textbf{(1)}$$

Introduction to Exponential Growth

Suppose a function f has the following two properties:

1. The value of f doubles with every 1-unit increase in the independent variable x.
2. The value of f at $x = 0$ is 5, so $f(0) = 5$.

Table 2

x	$f(x)$
0	5
1	10
2	20
3	40
4	80

Table 2 shows values of the function f for $x = 0, 1, 2, 3,$ and 4.

Let's find an equation $y = f(x)$ that describes this function f. The key fact is that the value of f doubles for every 1-unit increase in x.

$f(0) = 5$

$f(1) = 2f(0) = 2 \cdot 5 = 5 \cdot 2^1$ **Double the value of f at 0 to get the value at 1.**

$f(2) = 2f(1) = 2(5 \cdot 2) = 5 \cdot 2^2$ **Double the value of f at 1 to get the value at 2.**

$$f(3) = 2f(2) = 2(5 \cdot 2^2) = 5 \cdot 2^3$$
$$f(4) = 2f(3) = 2(5 \cdot 2^3) = 5 \cdot 2^4$$

The pattern leads to

$$f(x) = 2f(x-1) = 2(5 \cdot 2^{x-1}) = 5 \cdot 2^x$$

DEFINITION

An **exponential function** is a function of the form

$$\boxed{f(x) = Ca^x}$$

where a is a positive real number $(a > 0)$, $a \neq 1$, and $C \neq 0$ is a real number. The domain of f is the set of all real numbers. The base a is the **growth factor**, and because $f(0) = Ca^0 = C$, C is called the **initial value**.

WARNING It is important to distinguish a power function, $g(x) = ax^n$, $n \geq 2$, an integer, from an exponential function, $f(x) = C \cdot a^x$, $a \neq 1$, $a > 0$. In a power function, the base is a variable and the exponent is a constant. In an exponential function, the base is a constant and the exponent is a variable.

In the definition of an exponential function, the base $a = 1$ is excluded because this function is simply the constant function $f(x) = C \cdot 1^x = C$. Bases that are negative are also excluded, otherwise, many values of x would have to be excluded from the domain, such as $x = \frac{1}{2}$ and $x = \frac{3}{4}$. [Recall that $(-2)^{1/2} = \sqrt{-2}$, $(-3)^{3/4} = \sqrt[4]{(-3)^3} = \sqrt[4]{-27}$, and so on, are not defined in the set of real numbers.]

Transformations (vertical shifts, horizontal shifts, reflections, and so on) of a function of the form $f(x) = Ca^x$ also represent exponential functions. Some examples of exponential functions are

$$f(x) = 2^x \qquad F(x) = \left(\frac{1}{3}\right)^x + 5 \qquad G(x) = 2 \cdot 3^{x-3}$$

For each function, note that the base of the exponential expression is a constant and the exponent contains a variable.

In the function $f(x) = 5 \cdot 2^x$, notice that the ratio of consecutive outputs is constant for 1-unit increases in the input. This ratio equals the constant 2, the base of the exponential function. In other words,

$$\frac{f(1)}{f(0)} = \frac{5 \cdot 2^1}{5} = 2 \qquad \frac{f(2)}{f(1)} = \frac{5 \cdot 2^2}{5 \cdot 2^1} = 2 \qquad \frac{f(3)}{f(2)} = \frac{5 \cdot 2^3}{5 \cdot 2^2} = 2 \quad \text{and so on}$$

This leads to the following result.

THEOREM

For an exponential function $f(x) = Ca^x$, where $a > 0$ and $a \neq 1$, if x is any real number, then

$$\boxed{\frac{f(x+1)}{f(x)} = a \quad \text{or} \quad f(x+1) = af(x)}$$

In Words
For 1-unit changes in the input x of an exponential function $f(x) = C \cdot a^x$, the ratio of consecutive outputs is the constant a.

Proof

$$\frac{f(x+1)}{f(x)} = \frac{Ca^{x+1}}{Ca^x} = a^{x+1-x} = a^1 = a$$

EXAMPLE 2 Identifying Linear or Exponential Functions

Determine whether the given function is linear, exponential, or neither. For those that are linear, find a linear function that models the data. For those that are exponential, find an exponential function that models the data.

(a)

x	y
−1	5
0	2
1	−1
2	−4
3	−7

(b)

x	y
−1	32
0	16
1	8
2	4
3	2

(c)

x	y
−1	2
0	4
1	7
2	11
3	16

Solution For each function, compute the average rate of change of y with respect to x and the ratio of consecutive outputs. If the average rate of change is constant, then the function is linear. If the ratio of consecutive outputs is constant, then the function is exponential.

Table 3 (a)

x	y	Average Rate of Change	Ratio of Consecutive Outputs
−1	5		
		$\frac{\Delta y}{\Delta x} = \frac{2-5}{0-(-1)} = -3$	$\frac{2}{5}$
0	2		
		$\frac{-1-2}{1-0} = -3$	$\frac{-1}{2} = -\frac{1}{2}$
1	−1		
		$\frac{-4-(-1)}{2-1} = -3$	$\frac{-4}{-1} = 4$
2	−4		
		$\frac{-7-(-4)}{3-2} = -3$	$\frac{-7}{-4} = \frac{7}{4}$
3	−7		

(b)

x	y	Average Rate of Change	Ratio of Consecutive Outputs
−1	32		
		$\frac{\Delta y}{\Delta x} = \frac{16-32}{0-(-1)} = -16$	$\frac{16}{32} = \frac{1}{2}$
0	16		
		−8	$\frac{8}{16} = \frac{1}{2}$
1	8		
		−4	$\frac{4}{8} = \frac{1}{2}$
2	4		
		−2	$\frac{2}{4} = \frac{1}{2}$
3	2		

(c)

x	y	Average Rate of Change	Ratio of Consecutive Outputs
−1	2		
		$\frac{\Delta y}{\Delta x} = \frac{4-2}{0-(-1)} = 2$	2
0	4		
		3	$\frac{7}{4}$
1	7		
		4	$\frac{11}{7}$
2	11		
		5	$\frac{16}{11}$
3	16		

(a) See Table 3(a) on the previous page. The average rate of change for every 1-unit increase in x is -3. Therefore, the function is a linear function. In a linear function the average rate of change is the slope m, so $m = -3$. The y-intercept b is the value of the function at $x = 0$, so $b = 2$. The linear function that models the data is $f(x) = mx + b = -3x + 2$.

(b) See Table 3(b) on the previous page. For this function, the average rate of change from -1 to 0 is -16, and the average rate of change from 0 to 1 is -8. Because the average rate of change is not constant, the function is not a linear function. The ratio of consecutive outputs for a 1-unit increase in the inputs is a constant, $\frac{1}{2}$. Because the ratio of consecutive outputs is constant, the function is an exponential function with growth factor $a = \frac{1}{2}$. The initial value of the exponential function is $C = 16$. Therefore, the exponential function that models the data is $g(x) = Ca^x = 16 \cdot \left(\frac{1}{2}\right)^x$.

(c) See Table 3(c) on the previous page. For this function, the average rate of change from -1 to 0 is 2, and the average rate of change from 0 to 1 is 3. Because the average rate of change is not constant, the function is not a linear function. The ratio of consecutive outputs from -1 to 0 is 2, and the ratio of consecutive outputs from 0 to 1 is $\frac{7}{4}$. Because the ratio of consecutive outputs is not a constant, the function is not an exponential function. ■

Now Work PROBLEM 27

2 Graph Exponential Functions

If we know how to graph an exponential function of the form $f(x) = a^x$, then we could use transformations (shifting, stretching, and so on) to obtain the graph of any exponential function.

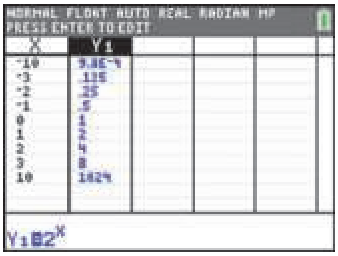
First, let's graph the exponential function $f(x) = 2^x$.

EXAMPLE 3 Graphing an Exponential Function

Graph the exponential function: $f(x) = 2^x$

Solution

Table 4



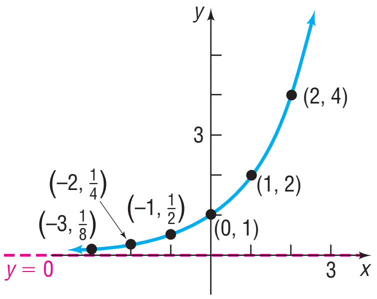
X	Y_1
-10	9.8E-4
-3	.125
-2	.25
-1	.5
0	1
1	2
2	4
3	8
10	1024

The domain of $f(x) = 2^x$ is the set of all real numbers. Begin by locating some points on the graph of $f(x) = 2^x$, as listed in Table 4.

Because $2^x > 0$ for all x, the range of f is $(0, \infty)$. Therefore, the graph has no x-intercepts, and in fact the graph will lie above the x-axis for all x. As Table 4 indicates, the y-intercept is 1. Table 4 also indicates that as $x \to -\infty$, the values of $f(x) = 2^x$ get closer and closer to 0. Therefore, the x-axis $(y = 0)$ is a horizontal asymptote to the graph as $x \to -\infty$. This provides us the end behavior for x large and negative.

To determine the end behavior for x large and positive, look again at Table 4. As $x \to \infty$, $f(x) = 2^x$ grows very quickly, causing the graph of $f(x) = 2^x$ to rise very rapidly. It is apparent that f is an increasing function and so is one-to-one.

Using all this information, plot some of the points from Table 4 and connect them with a smooth, continuous curve, as shown in Figure 20.



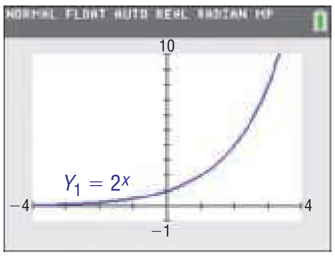


Figure 20 $f(x) = 2^x$

■

Graphs that look like the one shown in Figure 20 occur very frequently in a variety of situations. For example, the graph in Figure 21 shows the total monthly data used globally by mobile devices (uploads and downloads) from the first quarter of 2010 through the fourth quarter of 2014. One might conclude from this graph that global mobile data usage is growing *exponentially*.

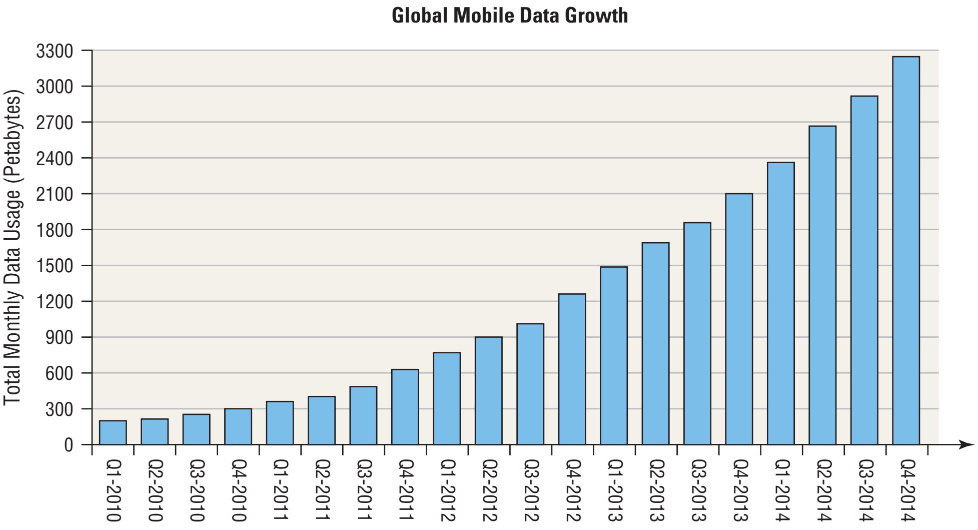


Figure 21 *Source: Ericsson Mobility Report, February 2015*

Later in this chapter, more will be said about situations that lead to exponential growth. For now, let's continue to explore the properties of exponential functions.

The graph of $f(x) = 2^x$ in Figure 20 is typical of all exponential functions of the form $f(x) = a^x$ with $a > 1$. Such functions are increasing functions and, hence, are one-to-one. Their graphs lie above the x-axis, pass through the point $(0, 1)$, and thereafter rise rapidly as $x \to \infty$. As $x \to -\infty$, the x-axis $(y = 0)$ is a horizontal asymptote. There are no vertical asymptotes. Finally, the graphs are smooth and continuous with no corners or gaps.

Figures 22 and 23 illustrate the graphs of two more exponential functions whose bases are larger than 1. Notice that the larger the base, the steeper the graph is when $x > 0$, and when $x < 0$, the larger the base, the closer the graph of the equation is to the x-axis.

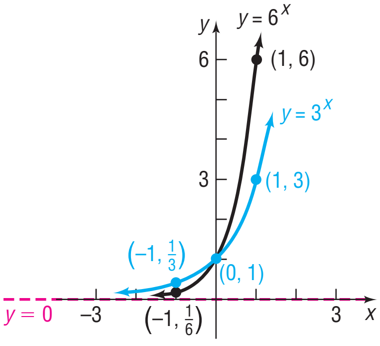


Figure 22

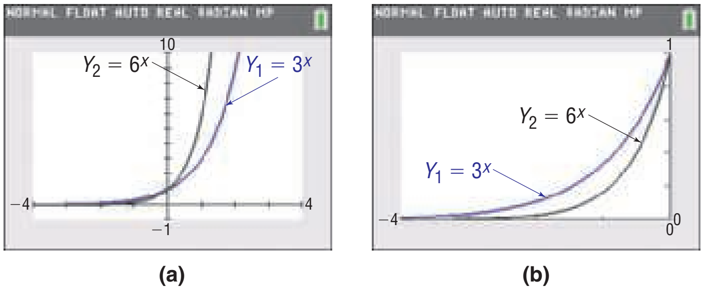


Figure 23 (a) (b)

Properties of the Exponential Function $f(x) = a^x$, $a > 1$

1. The domain is the set of all real numbers, or $(-\infty, \infty)$ using interval notation; the range is the set of positive real numbers, or $(0, \infty)$ using interval notation.
2. There are no x-intercepts; the y-intercept is 1.
3. The x-axis $(y = 0)$ is a horizontal asymptote as $x \to -\infty$ $\left[\lim_{x \to -\infty} a^x = 0\right]$
4. $f(x) = a^x$, $a > 1$, is an increasing function and is one-to-one.

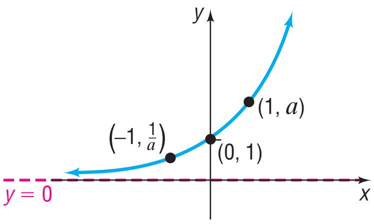


Figure 24 $f(x) = a^x$, $a > 1$

5. The graph of f contains the points $(0, 1)$, $(1, a)$, and $\left(-1, \dfrac{1}{a}\right)$.
6. The graph of f is smooth and continuous, with no corners or gaps. See Figure 24.

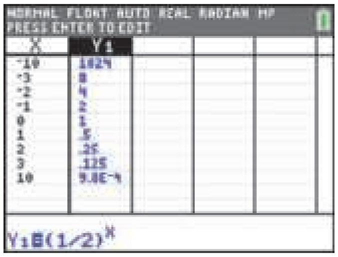
Now consider $f(x) = a^x$ when $0 < a < 1$.

EXAMPLE 4 Graphing an Exponential Function

Graph the exponential function: $f(x) = \left(\dfrac{1}{2}\right)^x$

Solution The domain of $f(x) = \left(\dfrac{1}{2}\right)^x$ consists of all real numbers. As before, locate some points on the graph, as shown in Table 5. Because $\left(\dfrac{1}{2}\right)^x > 0$ for all x, the range of f is the interval $(0, \infty)$. The graph lies above the x-axis and has no x-intercepts. The y-intercept is 1. As $x \to -\infty$, $f(x) = \left(\dfrac{1}{2}\right)^x$ grows very quickly. As $x \to \infty$, the values of $f(x)$ approach 0. The x-axis $(y = 0)$ is a horizontal asymptote as $x \to \infty$. It is apparent that f is a decreasing function and so is one-to-one. Figure 25 illustrates the graph.

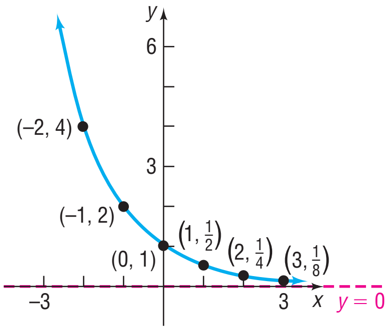
Table 5



NORMAL FLOAT AUTO REAL RADIAN MP
PRESS ENTER TO EDIT

X	Y1
-10	1024
-3	8
-2	4
-1	2
0	1
1	.5
2	.25
3	.125
10	9.8E-4

Y1=(1/2)^X



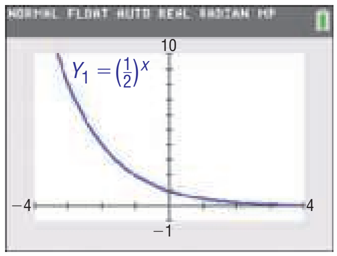


Figure 25 $f(x) = \left(\dfrac{1}{2}\right)^x$

The graph of $y = \left(\dfrac{1}{2}\right)^x$ also can be obtained from the graph of $y = 2^x$ using transformations. The graph of $y = \left(\dfrac{1}{2}\right)^x = 2^{-x}$ is a reflection about the y-axis of the graph of $y = 2^x$ (replace x by $-x$). See Figures 26(a) and 26(b).

Seeing the Concept

Using a graphing utility, simultaneously graph:

(a) $Y_1 = 3^x$, $Y_2 = \left(\dfrac{1}{3}\right)^x$

(b) $Y_1 = 6^x$, $Y_2 = \left(\dfrac{1}{6}\right)^x$

Conclude that the graph of $Y_2 = \left(\dfrac{1}{a}\right)^x$, for $a > 0$, is the reflection about the y-axis of the graph of $Y_1 = a^x$. ■

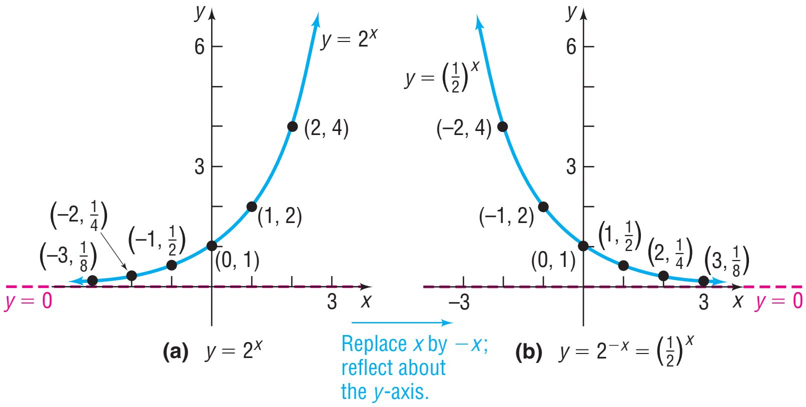


Figure 26

The graph of $f(x) = \left(\dfrac{1}{2}\right)^x$ in Figure 25 is typical of all exponential functions of the form $f(x) = a^x$ with $0 < a < 1$. Such functions are decreasing and one-to-one.

Their graphs lie above the x-axis and pass through the point $(0, 1)$. The graphs rise rapidly as $x \to -\infty$. As $x \to \infty$, the x-axis $(y = 0)$ is a horizontal asymptote. There are no vertical asymptotes. Finally, the graphs are smooth and continuous, with no corners or gaps.

Figures 27 and 28 illustrate the graphs of two more exponential functions whose bases are between 0 and 1. Notice that the smaller base results in a graph that is steeper when $x < 0$. When $x > 0$, the graph of the equation with the smaller base is closer to the x-axis.

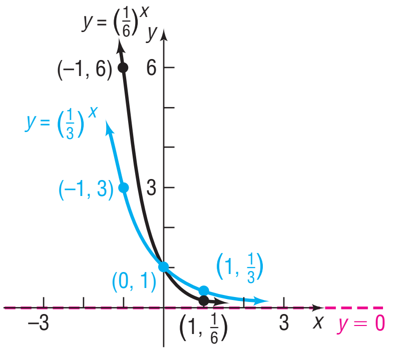


Figure 27

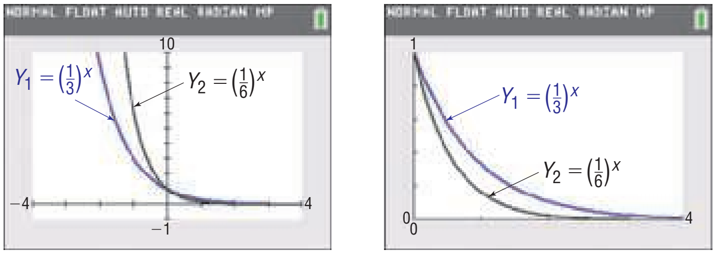


Figure 28 **(a)** **(b)**

Properties of the Exponential Function $f(x) = a^x$, $0 < a < 1$

1. The domain is the set of all real numbers, or $(-\infty, \infty)$ using interval notation; the range is the set of positive real numbers, or $(0, \infty)$ using interval notation.
2. There are no x-intercepts; the y-intercept is 1.
3. The x-axis $(y = 0)$ is a horizontal asymptote as $x \to \infty$ $\left[\lim_{x\to\infty} a^x = 0\right]$.
4. $f(x) = a^x$, $0 < a < 1$, is a decreasing function and is one-to-one.
5. The graph of f contains the points $\left(-1, \dfrac{1}{a}\right)$, $(0, 1)$, and $(1, a)$.
6. The graph of f is smooth and continuous, with no corners or gaps. See Figure 29.

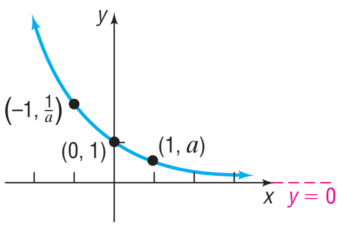


Figure 29 $f(x) = a^x, 0 < a < 1$

EXAMPLE 5 **Graphing Exponential Functions Using Transformations**

Graph $f(x) = 2^{-x} - 3$, and determine the domain, range, and horizontal asymptote of f.

Solution Begin with the graph of $y = 2^x$. Figure 30 shows the stages.

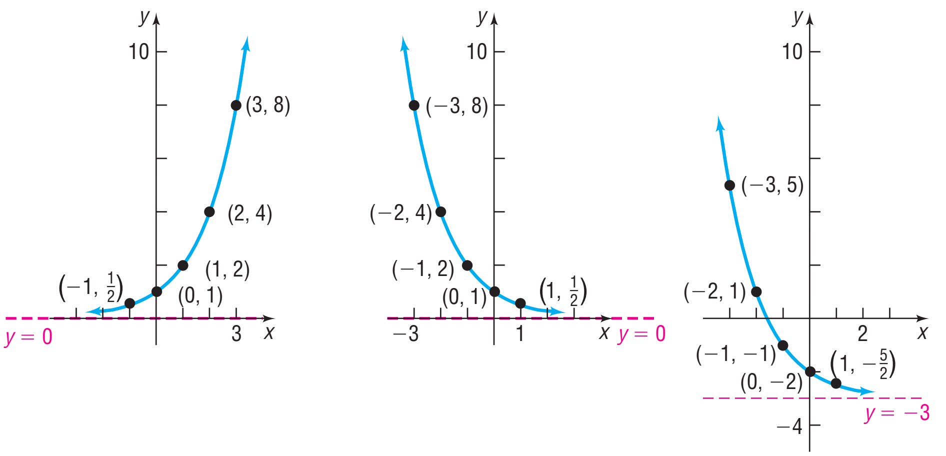


Figure 30 **(a)** $y = 2^x$ Replace x by $-x$; reflect about the y-axis. **(b)** $y = 2^{-x}$ Subtract 3; shift down 3 units. **(c)** $y = 2^{-x} - 3$

Historical Feature

The number e is named in honor of the Swiss mathematician Leonhard Euler (1707–1783).

As Figure 30(c) on the previous page illustrates, the domain of $f(x) = 2^{-x} - 3$ is the interval $(-\infty, \infty)$ and the range is the interval $(-3, \infty)$. The horizontal asymptote of f is the line $y = -3$.

Check: Graph $Y_1 = 2^{-x} - 3$ to verify the graph obtained in Figure 30(c).

Now Work PROBLEM 43

3 Define the Number e

Many problems that occur in nature require the use of an exponential function whose base is a certain irrational number, symbolized by the letter e.

One way of arriving at this important number e is given next.

DEFINITION

The **number *e*** is defined as the number that the expression

$$\left(1 + \frac{1}{n}\right)^n \tag{2}$$

approaches as $n \to \infty$. In calculus, this is expressed using limit notation as

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

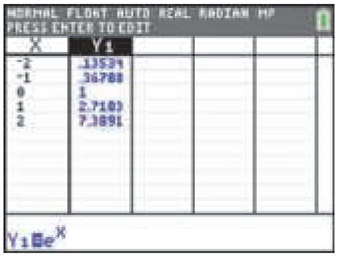
Table 6 illustrates what happens to the defining expression (2) as n takes on increasingly large values. The last number in the right column in the table approximates e correct to nine decimal places. That is, $e = 2.718281828\ldots$. Remember, the three dots indicate that the decimal places continue. Because these decimal places continue but do not repeat, e is an irrational number. The number e is often expressed as a decimal rounded to a specific number of places. For example, $e \approx 2.71828$ is rounded to five decimal places.

The exponential function $f(x) = e^x$, whose base is the number e, occurs with such frequency in applications that it is usually referred to as *the* exponential function. Indeed, most calculators have the key $\boxed{e^x}$ or $\boxed{\exp(x)}$, which may be used to evaluate the exponential function for a given value of x.

Table 6

n	$\frac{1}{n}$	$1 + \frac{1}{n}$	$\left(1 + \frac{1}{n}\right)^n$
1	1	2	2
2	0.5	1.5	2.25
5	0.2	1.2	2.48832
10	0.1	1.1	2.59374246
100	0.01	1.01	2.704813829
1,000	0.001	1.001	2.716923932
10,000	0.0001	1.0001	2.718145927
100,000	0.00001	1.00001	2.718268237
1,000,000	0.000001	1.000001	2.718280469
10,000,000,000	10^{-10}	$1 + 10^{-10}$	2.718281828

Table 7



Now use your calculator to approximate e^x for $x = -2, x = -1, x = 0, x = 1$, and $x = 2$. See Table 7. The graph of the exponential function $f(x) = e^x$ is given in Figures 31(a) and (b). Since $2 < e < 3$, the graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$. See Figure 31(c).

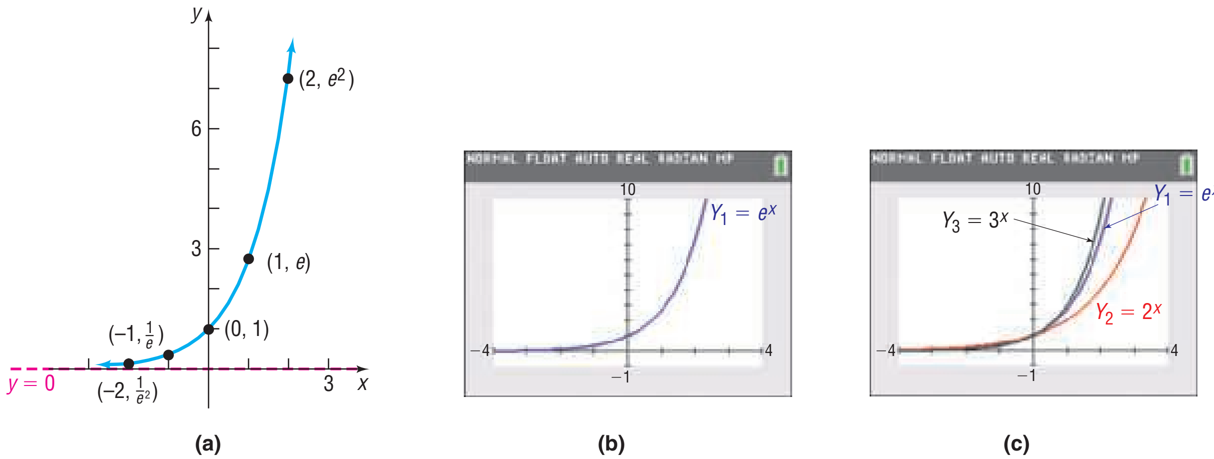


Figure 31 $y = e^x$

(a) (b) (c)

EXAMPLE 6 **Graphing Exponential Functions Using Transformations**

Graph $f(x) = -e^{x-3}$ and determine the domain, range, and horizontal asymptote of f.

Solution Begin with the graph of $y = e^x$. Figure 32 shows the stages.

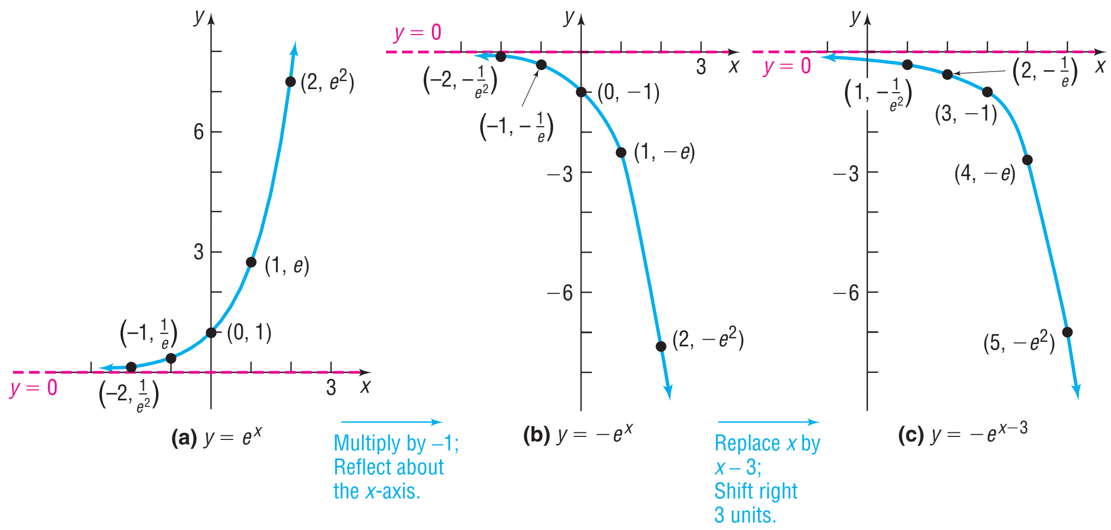


Figure 32 **(a)** $y = e^x$ **(b)** $y = -e^x$ **(c)** $y = -e^{x-3}$

As Figure 32(c) illustrates, the domain of $f(x) = -e^{x-3}$ is the interval $(-\infty, \infty)$, and the range is the interval $(-\infty, 0)$. The horizontal asymptote is the line $y = 0$.

✓**Check:** Graph $Y_1 = -e^{x-3}$ to verify the graph obtained in Figure 32(c). ■

Now Work PROBLEM 55

4 Solve Exponential Equations

Equations that involve terms of the form a^x, where $a > 0$ and $a \neq 1$, are referred to as **exponential equations**. Such equations can sometimes be solved by appropriately applying the Laws of Exponents and property (3):

$$\text{If} \quad a^u = a^v, \quad \text{then} \quad u = v. \qquad \textbf{(3)}$$

In Words
When two exponential expressions with the same base are equal, then their exponents are equal.

Property (3) is a consequence of the fact that exponential functions are one-to-one. To use property (3), each side of the equality must be written with the same base.

EXAMPLE 7 Solving an Exponential Equation

Solve: $4^{2x-1} = 8^x$

Algebraic Solution

Write each exponential expression so each has the same base.

$$4^{2x-1} = 8^x$$

$$(2^2)^{(2x-1)} = (2^3)^x \qquad \mathbf{4 = 2^2;\ 8 = 2^3}$$

$$2^{2(2x-1)} = 2^{3x} \qquad \mathbf{(a^r)^s = a^{rs}}$$

$$2(2x-1) = 3x \qquad \textbf{If } \mathbf{a^u = a^v}\textbf{, then } \mathbf{u = v.}$$

$$4x - 2 = 3x$$

$$x = 2$$

The solution set is $\{2\}$. ■

Graphing Solution

Graph $Y_1 = 4^{2x-1}$ and $Y_2 = 8^x$. Use INTERSECT to determine the point of intersection. See Figure 33.

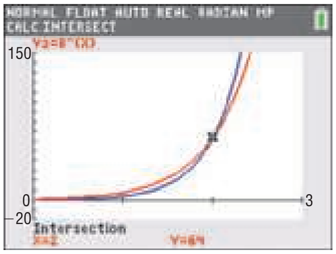


Figure 33

The graphs intersect at $(2, 64)$, so the solution set is $\{2\}$. ■

Now Work PROBLEMS 65 AND 75

EXAMPLE 8 Solving an Exponential Equation

Solve: $e^{-x^2} = (e^x)^2 \cdot \dfrac{1}{e^3}$

Solution Use the Laws of Exponents first to get a single expression with the base e on the right side.

$$(e^x)^2 \cdot \frac{1}{e^3} = e^{2x} \cdot e^{-3} = e^{2x-3}$$

As a result,

$$e^{-x^2} = e^{2x-3}$$

$$-x^2 = 2x - 3 \qquad \textbf{Apply property (3).}$$

$$x^2 + 2x - 3 = 0 \qquad \textbf{Place the quadratic equation in standard form.}$$

$$(x+3)(x-1) = 0 \qquad \textbf{Factor.}$$

$$x = -3 \quad \text{or} \quad x = 1 \qquad \textbf{Use the Zero-Product Property.}$$

The solution set is $\{-3, 1\}$. ■

Now Work PROBLEM 81

EXAMPLE 9 Exponential Probability

Between 9:00 PM and 10:00 PM, cars arrive at Burger King's drive-thru at the rate of 12 cars per hour (0.2 car per minute). The following formula from statistics can be used to determine the probability that a car will arrive within t minutes of 9:00 PM.

$$F(t) = 1 - e^{-0.2t}$$

(a) Determine the probability that a car will arrive within 5 minutes of 9 PM (that is, before 9:05 PM).

(b) Determine the probability that a car will arrive within 30 minutes of 9 PM (before 9:30 PM).

(c) Graph F using your graphing utility.

(d) What value does F approach as t increases without bound in the positive direction?

Solution (a) The probability that a car will arrive within 5 minutes is found by evaluating $F(t)$ at $t = 5$.

$$F(5) = 1 - e^{-0.2(5)} \approx 0.6321$$

$\uparrow$ Use a calculator.

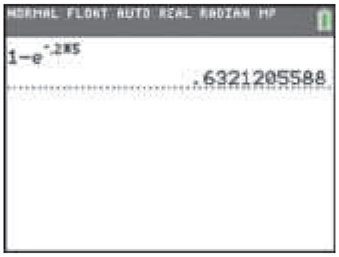


Figure 34 $F(5)$

See Figure 34. There is a 63.21% probability that a car will arrive within 5 minutes.

(b) The probability that a car will arrive within 30 minutes is found by evaluating $F(t)$ at $t = 30$.

$$F(30) = 1 - e^{-0.2(30)} \approx 0.9975$$

$\uparrow$ Use a calculator.

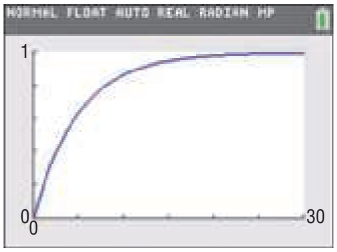


Figure 35 $F(t) = 1 - e^{-0.2t}$

There is a 99.75% probability that a car will arrive within 30 minutes.

(c) See Figure 35 for the graph of F.

(d) As time passes, the probability that a car will arrive increases. The value that F approaches can be found by letting $t \to \infty$. Since $e^{-0.2t} = \dfrac{1}{e^{0.2t}}$, it follows that $e^{-0.2t} \to 0$ as $t \to \infty$. Therefore, F approaches 1 as t gets large. The algebraic analysis is confirmed by Figure 35. ■

Now Work PROBLEM 113

SUMMARY

Properties of the Exponential Function

$f(x) = a^x, \quad a > 1$	Domain: the interval $(-\infty, \infty)$; range: the interval $(0, \infty)$ x-intercepts: none; y-intercept: 1 Horizontal asymptote: x-axis $(y = 0)$ as $x \to -\infty$ Increasing; one-to-one; smooth; continuous See Figure 24 for a typical graph.
$f(x) = a^x, \quad 0 < a < 1$	Domain: the interval $(-\infty, \infty)$; range: the interval $(0, \infty)$ x-intercepts: none; y-intercept: 1 Horizontal asymptote: x-axis $(y = 0)$ as $x \to \infty$ Decreasing; one-to-one; smooth; continuous See Figure 29 for a typical graph.

If $a^u = a^v$, then $u = v$.

6.3 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. $4^3 =$ ____; $8^{2/3} =$ ____; $3^{-2} =$ ____. (pp. 22–24 and pp. 77–78)

2. Solve: $x^2 + 3x = 4$ (pp. 110–115)

3. ***True or False*** To graph $y = (x - 2)^3$, shift the graph of $y = x^3$ to the left 2 units. (pp. 256–264)

4. Find the average rate of change of $f(x) = 3x - 5$ from $x = 0$ to $x = 4$. (pp. 238–239; 281–284)

5. ***True or False*** The function $f(x) = \dfrac{2x}{x - 3}$ has $y = 2$ as a horizontal asymptote. (pp. 377–379)

Concepts and Vocabulary

6. A(n) __________ ________ is a function of the form $f(x) = Ca^x$, where $a > 0$, $a \neq 1$, and $C \neq 0$ are real numbers. The base a is the ________ ______ and C is the ______ ______.

7. For an exponential function $f(x) = Ca^x$, $\dfrac{f(x+1)}{f(x)} =$ ___.

8. ***True or False*** The domain of the exponential function $f(x) = a^x$, where $a > 0$ and $a \neq 1$, is the set of all real numbers.

9. ***True or False*** The graph of the exponential function $f(x) = a^x$, where $a > 0$ and $a \neq 1$, has no x-intercept.

10. The graph of every exponential function $f(x) = a^x$, where $a > 0$ and $a \neq 1$, passes through three points: _____, _____, and _____.

11. If $3^x = 3^4$, then $x =$ __.

12. ***True or False*** The graphs of $y = 3^x$ and $y = \left(\dfrac{1}{3}\right)^x$ are identical.

13. Which of the following exponential functions is an increasing function?

(a) $f(x) = 0.5^x$ (b) $f(x) = \left(\dfrac{5}{2}\right)^x$

(c) $f(x) = \left(\dfrac{2}{3}\right)^x$ (d) $f(x) = 0.9^x$

14. Which of the following is the range of the exponential function $f(x) = a^x$, $a > 0$ and $a \neq 1$?
(a) $(-\infty, \infty)$ (b) $(-\infty, 0)$
(c) $(0, \infty)$ (d) $(-\infty, 0) \cup (0, \infty)$

Skill Building

In Problems 15–26, approximate each number using a calculator. Express your answer rounded to three decimal places.

15. (a) $2^{3.14}$ (b) $2^{3.141}$ (c) $2^{3.1415}$ (d) 2^{π}

16. (a) $2^{2.7}$ (b) $2^{2.71}$ (c) $2^{2.718}$ (d) 2^{e}

17. (a) $3.1^{2.7}$ (b) $3.14^{2.71}$ (c) $3.141^{2.718}$ (d) π^{e}

18. (a) $2.7^{3.1}$ (b) $2.71^{3.14}$ (c) $2.718^{3.141}$ (d) e^{π}

19. $(1 + 0.04)^6$

20. $\left(1 + \dfrac{0.09}{12}\right)^{24}$

21. $8.4\left(\dfrac{1}{3}\right)^{2.9}$

22. $158\left(\dfrac{5}{6}\right)^{8.63}$

23. $e^{1.2}$

24. $e^{-1.3}$

25. $125e^{0.026(7)}$

26. $83.6e^{-0.157(9.5)}$

In Problems 27–34, determine whether the given function is linear, exponential, or neither. For those that are linear functions, find a linear function that models the data; for those that are exponential, find an exponential function that models the data.

27.

x	$f(x)$
−1	3
0	6
1	12
2	18
3	30

28.

x	$g(x)$
−1	2
0	5
1	8
2	11
3	14

29.

x	$H(x)$
−1	$\frac{1}{4}$
0	1
1	4
2	16
3	64

30.

x	$F(x)$
−1	$\frac{2}{3}$
0	1
1	$\frac{3}{2}$
2	$\frac{9}{4}$
3	$\frac{27}{8}$

31.

x	$f(x)$
−1	$\frac{3}{2}$
0	3
1	6
2	12
3	24

32.

x	$g(x)$
−1	6
0	1
1	0
2	3
3	10

33.

x	$H(x)$
−1	2
0	4
1	6
2	8
3	10

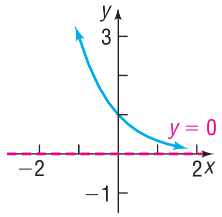
34.

x	$F(x)$
−1	$\frac{1}{2}$
0	$\frac{1}{4}$
1	$\frac{1}{8}$
2	$\frac{1}{16}$
3	$\frac{1}{32}$

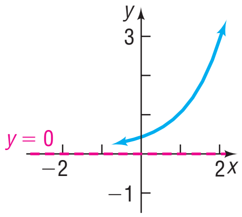
In Problems 35–42, the graph of an exponential function is given. Match each graph to one of the following functions:

(A) $y = 3^x$ (B) $y = 3^{-x}$ (C) $y = -3^x$ (D) $y = -3^{-x}$
(E) $y = 3^x - 1$ (F) $y = 3^{x-1}$ (G) $y = 3^{1-x}$ (H) $y = 1 - 3^x$

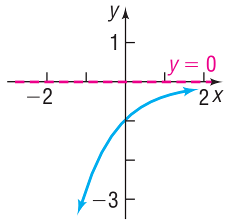
35.



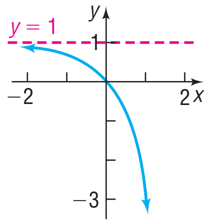
36.



37.



38.



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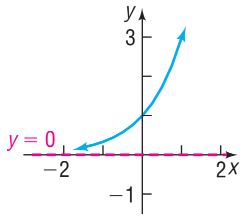

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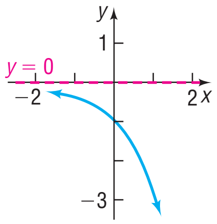

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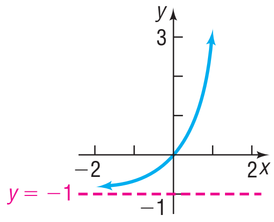

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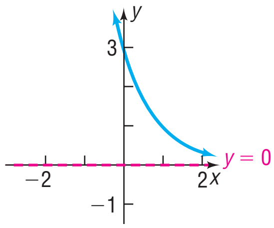


In Problems 43–54, use transformations to graph each function. Determine the domain, range, and horizontal asymptote of each function.

43. $f(x) = 2^x + 1$ **44.** $f(x) = 3^x - 2$ **45.** $f(x) = 3^{x-1}$ **46.** $f(x) = 2^{x+2}$

47. $f(x) = 3 \cdot \left(\frac{1}{2}\right)^x$ **48.** $f(x) = 4 \cdot \left(\frac{1}{3}\right)^x$ **49.** $f(x) = 3^{-x} - 2$ **50.** $f(x) = -3^x + 1$

51. $f(x) = 2 + 4^{x-1}$ **52.** $f(x) = 1 - 2^{x+3}$ **53.** $f(x) = 2 + 3^{x/2}$ **54.** $f(x) = 1 - 2^{-x/3}$

In Problems 55–62, begin with the graph of $y = e^x$ (Figure 31) and use transformations to graph each function. Determine the domain, range, and horizontal asymptote of each function.

55. $f(x) = e^{-x}$ **56.** $f(x) = -e^x$ **57.** $f(x) = e^{x+2}$ **58.** $f(x) = e^x - 1$

59. $f(x) = 5 - e^{-x}$ **60.** $f(x) = 9 - 3e^{-x}$ **61.** $f(x) = 2 - e^{-x/2}$ **62.** $f(x) = 7 - 3e^{2x}$

In Problems 63–82, solve each equation.

63. $7^x = 7^3$ **64.** $5^x = 5^{-6}$ **65.** $2^{-x} = 16$ **66.** $3^{-x} = 81$

67. $\left(\frac{1}{5}\right)^x = \frac{1}{25}$ **68.** $\left(\frac{1}{4}\right)^x = \frac{1}{64}$ **69.** $2^{2x-1} = 4$ **70.** $5^{x+3} = \frac{1}{5}$

71. $3^{x^3} = 9^x$ **72.** $4^{x^2} = 2^x$ **73.** $8^{-x+14} = 16^x$ **74.** $9^{-x+15} = 27^x$

75. $3^{x^2-7} = 27^{2x}$ **76.** $5^{x^2+8} = 125^{2x}$ **77.** $4^x \cdot 2^{x^2} = 16^2$ **78.** $9^{2x} \cdot 27^{x^2} = 3^{-1}$

79. $e^x = e^{3x+8}$ **80.** $e^{3x} = e^{2-x}$ **81.** $e^{x^2} = e^{3x} \cdot \frac{1}{e^2}$ **82.** $(e^4)^x \cdot e^{x^2} = e^{12}$

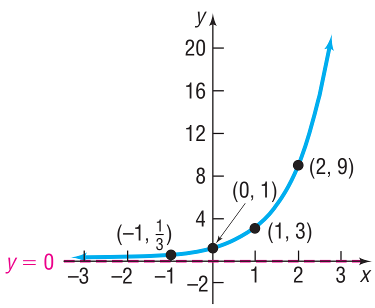
83. If $4^x = 7$, what does 4^{-2x} equal? **84.** If $2^x = 3$, what does 4^{-x} equal?

85. If $3^{-x} = 2$, what does 3^{2x} equal? **86.** If $5^{-x} = 3$, what does 5^{3x} equal?

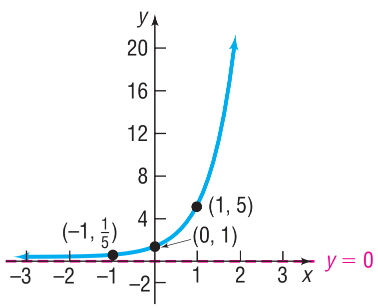
87. If $9^x = 25$, what does 3^x equal? **88.** If $2^{-3x} = \frac{1}{1000}$, what does 2^x equal?

In Problems 89–92, determine the exponential function whose graph is given.

89.



90.



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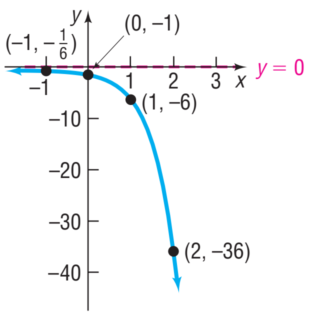
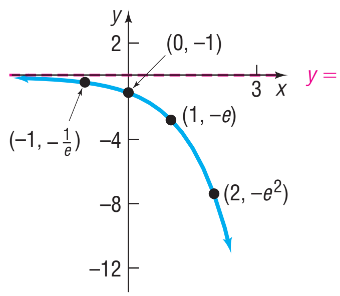


92.



93. Find an exponential function with horizontal asymptote $y = 2$ whose graph contains the points $(0, 3)$ and $(1, 5)$.

94. Find an exponential function with horizontal asymptote $y = -3$ whose graph contains the points $(0, -2)$ and $(-2, 1)$.

Mixed Practice

95. Suppose that $f(x) = 2^x$.
(a) What is $f(4)$? What point is on the graph of f?
(b) If $f(x) = \frac{1}{16}$, what is x? What point is on the graph of f?

96. Suppose that $f(x) = 3^x$.
(a) What is $f(4)$? What point is on the graph of f?
(b) If $f(x) = \frac{1}{9}$, what is x? What point is on the graph of f?

97. Suppose that $g(x) = 4^x + 2$.
(a) What is $g(-1)$? What point is on the graph of g?
(b) If $g(x) = 66$, what is x? What point is on the graph of g?

98. Suppose that $g(x) = 5^x - 3$.
(a) What is $g(-1)$? What point is on the graph of g?
(b) If $g(x) = 122$, what is x? What point is on the graph of g?

99. Suppose that $H(x) = \left(\frac{1}{2}\right)^x - 4$.
(a) What is $H(-6)$? What point is on the graph of H?
(b) If $H(x) = 12$, what is x? What point is on the graph of H?
(c) Find the zero of H.

100. Suppose that $F(x) = \left(\frac{1}{3}\right)^x - 3$.
(a) What is $F(-5)$? What point is on the graph of F?
(b) If $F(x) = 24$, what is x? What point is on the graph of F?
(c) Find the zero of F.

In Problems 101–104, graph each function. Based on the graph, state the domain and the range, and find any intercepts.

101. $f(x) = \begin{cases} e^{-x} & \text{if } x < 0 \\ e^{x} & \text{if } x \geq 0 \end{cases}$

102. $f(x) = \begin{cases} e^{x} & \text{if } x < 0 \\ e^{-x} & \text{if } x \geq 0 \end{cases}$

103. $f(x) = \begin{cases} -e^{x} & \text{if } x < 0 \\ -e^{-x} & \text{if } x \geq 0 \end{cases}$

104. $f(x) = \begin{cases} -e^{-x} & \text{if } x < 0 \\ -e^{x} & \text{if } x \geq 0 \end{cases}$

Applications and Extensions

105. Optics If a single pane of glass obliterates 3% of the light passing through it, the percent p of light that passes through n successive panes is given approximately by the function

$$p(n) = 100(0.97)^n$$

(a) What percent of light will pass through 10 panes?
(b) What percent of light will pass through 25 panes?
(c) Explain the meaning of the base 0.97 in this problem.

106. Atmospheric Pressure The atmospheric pressure p on a balloon or airplane decreases with increasing height. This pressure, measured in millimeters of mercury, is related to the height h (in kilometers) above sea level by the function

$$p(h) = 760e^{-0.145h}$$

(a) Find the atmospheric pressure at a height of 2 km (over a mile).
(b) What is it at a height of 10 kilometers (over 30,000 feet)?

107. Depreciation The price p, in dollars, of a Honda Civic EX-L sedan that is x years old is modeled by

$$p(x) = 22{,}265(0.90)^x$$

(a) How much should a 3-year-old Civic EX-L sedan cost?
(b) How much should a 9-year-old Civic EX-L sedan cost?
(c) Explain the meaning of the base 0.90 in this problem.

108. Healing of Wounds The normal healing of wounds can be modeled by an exponential function. If A_0 represents the original area of the wound and if A equals the area of the wound, then the function

$$A(n) = A_0 e^{-0.35n}$$

describes the area of a wound after n days following an injury when no infection is present to retard the healing. Suppose that a wound initially had an area of 100 square millimeters.
(a) If healing is taking place, how large will the area of the wound be after 3 days?
(b) How large will it be after 10 days?

109. Advanced-Stage Pancreatic Cancer The percentage of patients P who have survived t years after initial diagnosis of advanced-stage pancreatic cancer is modeled by the function

$$P(t) = 100(0.3)^t$$

Source: *Cancer Treatment Centers of America*

(a) According to the model, what percent of patients survive 1 year after initial diagnosis?
(b) What percent of patients survive 2 years after initial diagnosis?
(c) Explain the meaning of the base 0.3 in the context of this problem.

110. Endangered Species In a protected environment, the population P of a certain endangered species recovers over time t (in years) according to the model

$$P(t) = 30(1.149)^t$$

(a) What is the size of the initial population of the species?
(b) According to the model, what will be the population of the species in 5 years?
(c) According to the model, what will be the population of the species in 10 years?
(d) According to the model, what will be the population of the species in 15 years?
(e) What is happening to the population every 5 years?

111. Drug Medication The function

$$D(h) = 5e^{-0.4h}$$

can be used to find the number of milligrams D of a certain drug that is in a patient's bloodstream h hours after the drug has been administered. How many milligrams will be present after 1 hour? After 6 hours?

112. Spreading of Rumors A model for the number N of people in a college community who have heard a certain rumor is

$$N = P(1 - e^{-0.15d})$$

where P is the total population of the community and d is the number of days that have elapsed since the rumor began. In a community of 1000 students, how many students will have heard the rumor after 3 days?

113. Exponential Probability Between 12:00 PM and 1:00 PM, cars arrive at Citibank's drive-thru at the rate of 6 cars per hour (0.1 car per minute). The following formula from probability can be used to determine the probability that a car will arrive within t minutes of 12:00 PM.

$$F(t) = 1 - e^{-0.1t}$$

(a) Determine the probability that a car will arrive within 10 minutes of 12:00 PM (that is, before 12:10 PM).
(b) Determine the probability that a car will arrive within 40 minutes of 12:00 PM (before 12:40 PM).
(c) What value does F approach as t becomes unbounded in the positive direction?
(d) Graph F using a graphing utility.
(e) Using INTERSECT, determine how many minutes are needed for the probability to reach 50%.

114. Exponential Probability Between 5:00 PM and 6:00 PM, cars arrive at Jiffy Lube at the rate of 9 cars per hour (0.15 car per minute). This formula from probability can be used to determine the probability that a car will arrive within t minutes of 5:00 PM:

$$F(t) = 1 - e^{-0.15t}$$

(a) Determine the probability that a car will arrive within 15 minutes of 5:00 PM (that is, before 5:15 PM).
(b) Determine the probability that a car will arrive within 30 minutes of 5:00 PM (before 5:30 PM).
(c) What value does F approach as t becomes unbounded in the positive direction?
(d) Graph F using a graphing utility.
(e) Using INTERSECT, determine how many minutes are needed for the probability to reach 60%.

115. Poisson Probability Between 5:00 PM and 6:00 PM, cars arrive at a McDonald's drive-thru at the rate of 20 cars per hour. The following formula from probability can be used to determine the probability that x cars will arrive between 5:00 PM and 6:00 PM.

$$P(x) = \frac{20^x e^{-20}}{x!}$$

where

$$x! = x \cdot (x-1) \cdot (x-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$$

(a) Determine the probability that $x = 15$ cars will arrive between 5:00 PM and 6:00 PM.
(b) Determine the probability that $x = 20$ cars will arrive between 5:00 PM and 6:00 PM.

116. Poisson Probability People enter a line for the *Demon Roller Coaster* at the rate of 4 per minute. The following formula from probability can be used to determine the probability that x people will arrive within the next minute.

$$P(x) = \frac{4^x e^{-4}}{x!}$$

where

$$x! = x \cdot (x-1) \cdot (x-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$$

(a) Determine the probability that $x = 5$ people will arrive within the next minute.
(b) Determine the probability that $x = 8$ people will arrive within the next minute.

117. Relative Humidity The relative humidity is the ratio (expressed as a percent) of the amount of water vapor in the air to the maximum amount that the air can hold at a specific temperature. The relative humidity, R, is found using the following formula:

$$R = 10^{\left(\frac{4221}{T+459.4} - \frac{4221}{D+459.4} + 2\right)}$$

where T is the air temperature (in °F) and D is the dew point temperature (in °F).

(a) Determine the relative humidity if the air temperature is 50° Fahrenheit and the dew point temperature is 41° Fahrenheit.
(b) Determine the relative humidity if the air temperature is 68° Fahrenheit and the dew point temperature is 59° Fahrenheit.
(c) What is the relative humidity if the air temperature and the dew point temperature are the same?

118. Learning Curve Suppose that a student has 500 vocabulary words to learn. If the student learns 15 words after 5 minutes, the function

$$L(t) = 500(1 - e^{-0.0061t})$$

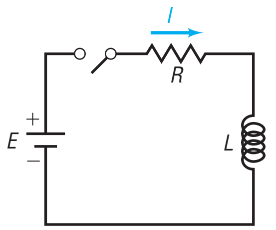
approximates the number of words L that the student will have learned after t minutes.

(a) How many words will the student have learned after 30 minutes?

(b) How many words will the student have learned after 60 minutes?

119. Current in an RL Circuit The equation governing the amount of current I (in amperes) after time t (in seconds) in a single RL circuit consisting of a resistance R (in ohms), an inductance L (in henrys), and an electromotive force E (in volts) is

$$I = \frac{E}{R}\left[1 - e^{-(R/L)t}\right]$$



(a) If $E = 120$ volts, $R = 10$ ohms, and $L = 5$ henrys, how much current I_1 is flowing after 0.3 second? After 0.5 second? After 1 second?

(b) What is the maximum current?

(c) Graph this function $I = I_1(t)$, measuring I along the y-axis and t along the x-axis.

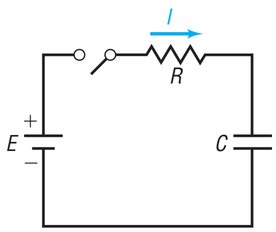
(d) If $E = 120$ volts, $R = 5$ ohms, and $L = 10$ henrys, how much current I_2 is flowing after 0.3 second? After 0.5 second? After 1 second?

(e) What is the maximum current?

(f) Graph the function $I = I_2(t)$ on the same coordinate axes as $I_1(t)$.

120. Current in an RC Circuit The equation governing the amount of current I (in amperes) after time t (in microseconds) in a single RC circuit consisting of a resistance R (in ohms), a capacitance C (in microfarads), and an electromotive force E (in volts) is

$$I = \frac{E}{R}e^{-t/(RC)}$$



(a) If $E = 120$ volts, $R = 2000$ ohms, and $C = 1.0$ microfarad, how much current I_1 is flowing initially $(t = 0)$? After 1000 microseconds? After 3000 microseconds?

(b) What is the maximum current?

(c) Graph the function $I = I_1(t)$, measuring I along the y-axis and t along the x-axis.

(d) If $E = 120$ volts, $R = 1000$ ohms, and $C = 2.0$ microfarads, how much current I_2 is flowing initially? After 1000 microseconds? After 3000 microseconds?

(e) What is the maximum current?

(f) Graph the function $I = I_2(t)$ on the same coordinate axes as $I_1(t)$.

121. If f is an exponential function of the form $f(x) = Ca^x$ with growth factor 3, and if $f(6) = 12$, what is $f(7)$?

122. Another Formula for *e* Use a calculator to compute the values of

$$2 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

for $n = 4, 6, 8$, and 10. Compare each result with e.

[**Hint:** $1! = 1, 2! = 2 \cdot 1, 3! = 3 \cdot 2 \cdot 1,$
$n! = n(n-1) \cdot \cdots \cdot (3)(2)(1)$.]

123. Another Formula for *e* Use a calculator to compute the various values of the expression. Compare the values to e.

$$2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{\text{etc.}}}}}}$$

124. Difference Quotient If $f(x) = a^x$, show that

$$\frac{f(x+h) - f(x)}{h} = a^x \cdot \frac{a^h - 1}{h} \qquad h \neq 0$$

125. If $f(x) = a^x$, show that $f(A + B) = f(A) \cdot f(B)$.

126. If $f(x) = a^x$, show that $f(-x) = \dfrac{1}{f(x)}$.

127. If $f(x) = a^x$, show that $f(\alpha x) = [f(x)]^\alpha$.

Problems 128 and 129 provide definitions for two other transcendental functions.

128. The **hyperbolic sine function**, designated by $\sinh x$, is defined as

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

(a) Show that $f(x) = \sinh x$ is an odd function.

(b) Graph $f(x) = \sinh x$ using a graphing utility.

129. The **hyperbolic cosine function**, designated by $\cosh x$, is defined as

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

(a) Show that $f(x) = \cosh x$ is an even function.

(b) Graph $f(x) = \cosh x$ using a graphing utility.

(c) Refer to Problem 128. Show that, for every x,

$$(\cosh x)^2 - (\sinh x)^2 = 1$$

130. Historical Problem Pierre de Fermat (1601–1665) conjectured that the function

$$f(x) = 2^{(2^x)} + 1$$

for $x = 1, 2, 3, \ldots$, would always have a value equal to a prime number. But Leonhard Euler (1707–1783) showed that this formula fails for $x = 5$. Use a calculator to determine the prime numbers produced by f for $x = 1, 2, 3, 4$. Then show that $f(5) = 641 \times 6{,}700{,}417$, which is not prime.

Explaining Concepts: Discussion and Writing

131. The bacteria in a 4-liter container double every minute. After 60 minutes the container is full. How long did it take to fill half the container?

132. Explain in your own words what the number e is. Provide at least two applications that use this number.

133. Do you think that there is a power function that increases more rapidly than an exponential function whose base is greater than 1? Explain.

134. As the base a of an exponential function $f(x) = a^x$, where $a > 1$, increases, what happens to the behavior of its graph for $x > 0$? What happens to the behavior of its graph for $x < 0$?

135. The graphs of $y = a^{-x}$ and $y = \left(\frac{1}{a}\right)^x$ are identical. Why?

Retain Your Knowledge

Problems 136–139 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

136. Solve the inequality: $x^3 + 5x^2 \leq 4x + 20$.

137. Solve the inequality: $\dfrac{x+1}{x-2} \geq 1$.

138. Find the equation of the quadratic function f that has its vertex at $(3, 5)$ and contains the point $(2, 3)$.

139. Consider the quadratic function $f(x) = x^2 + 2x - 3$.
(a) Graph f by determining whether its graph opens up or down and by finding its vertex, axis of symmetry, y-intercept, and x-intercepts, if any.
(b) Determine the domain and range of f.
(c) Determine where f is increasing and where it is decreasing.

'Are You Prepared?' Answers

1. $64; 4; \frac{1}{9}$ **2.** $\{-4, 1\}$ **3.** False **4.** 3 **5.** True

6.4 Logarithmic Functions

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Solve Linear Inequalities (Section 1.7, pp. 150–151)
- Solve Quadratic Inequalities (Section 4.5, pp. 320–322)
- Polynomial and Rational Inequalities (Section 5.6, pp. 393–397)
- Solve Linear Equations (Section 1.2, pp. 102–103)

Now Work the **'Are You Prepared?'** problems on page 454.

OBJECTIVES
1 Change Exponential Statements to Logarithmic Statements and Logarithmic Statements to Exponential Statements (p. 446)
2 Evaluate Logarithmic Expressions (p. 446)
3 Determine the Domain of a Logarithmic Function (p. 447)
4 Graph Logarithmic Functions (p. 448)
5 Solve Logarithmic Equations (p. 452)

Recall that a one-to-one function $y = f(x)$ has an inverse function that is defined (implicitly) by the equation $x = f(y)$. In particular, the exponential function $y = f(x) = a^x$, where $a > 0$ and $a \neq 1$, is one-to-one and, hence, has an inverse function that is defined implicitly by the equation

$$x = a^y, \quad a > 0, \quad a \neq 1$$

This inverse function is so important that it is given a name, the *logarithmic function*.

DEFINITION

The **logarithmic function with base *a***, where $a > 0$ and $a \neq 1$, is denoted by $y = \log_a x$ (read as "y is the logarithm with base a of x") and is defined by

$$\boxed{y = \log_a x \quad \text{if and only if} \quad x = a^y}$$

The domain of the logarithmic function $y = \log_a x$ is $x > 0$.

In Words
When you read $\log_a x$, think to yourself "a raised to what power gives me x."

As this definition illustrates, **a logarithm is a name for a certain exponent.** So $\log_a x$ represents the exponent to which a must be raised to obtain x.

EXAMPLE 1 **Relating Logarithms to Exponents**

(a) If $y = \log_3 x$, then $x = 3^y$. For example, the logarithmic statement $4 = \log_3 81$ is equivalent to the exponential statement $81 = 3^4$.

(b) If $y = \log_5 x$, then $x = 5^y$. For example, $-1 = \log_5\left(\frac{1}{5}\right)$ is equivalent to $\frac{1}{5} = 5^{-1}$.

1 Change Exponential Statements to Logarithmic Statements and Logarithmic Statements to Exponential Statements

The definition of a logarithm can be used to convert from exponential form to logarithmic form, and vice versa, as the following two examples illustrate.

EXAMPLE 2 **Changing Exponential Statements to Logarithmic Statements**

Change each exponential statement to an equivalent statement involving a logarithm.

(a) $1.2^3 = m$ (b) $e^b = 9$ (c) $a^4 = 24$

Solution Use the fact that $y = \log_a x$ and $x = a^y$, where $a > 0$ and $a \neq 1$, are equivalent.

(a) If $1.2^3 = m$, then $3 = \log_{1.2} m$. (b) If $e^b = 9$, then $b = \log_e 9$.
(c) If $a^4 = 24$, then $4 = \log_a 24$.

Now Work PROBLEM 11

EXAMPLE 3 **Changing Logarithmic Statements to Exponential Statements**

Change each logarithmic statement to an equivalent statement involving an exponent.

(a) $\log_a 4 = 5$ (b) $\log_e b = -3$ (c) $\log_3 5 = c$

Solution (a) If $\log_a 4 = 5$, then $a^5 = 4$. (b) If $\log_e b = -3$, then $e^{-3} = b$.
(c) If $\log_3 5 = c$, then $3^c = 5$.

Now Work PROBLEM 19

2 Evaluate Logarithmic Expressions

To find the exact value of a logarithm, write the logarithm in exponential notation using the fact that $y = \log_a x$ is equivalent to $a^y = x$, and use the fact that if $a^u = a^v$, then $u = v$.

EXAMPLE 4 **Finding the Exact Value of a Logarithmic Expression**

Find the exact value of:

(a) $\log_2 16$ (b) $\log_3 \frac{1}{27}$

Solution (a) To evaluate $\log_2 16$, think "2 raised to what power yields 16?" So, $y = \log_2 16$

$2^y = 16$ **Change to exponential form.**

$2^y = 2^4$ **$16 = 2^4$**

$y = 4$ **Equate exponents.**

Therefore, $\log_2 16 = 4$.

(b) To evaluate $\log_3 \frac{1}{27}$, think "3 raised to what power yields $\frac{1}{27}$?" So,

$y = \log_3 \frac{1}{27}$

$3^y = \frac{1}{27}$ **Change to exponential form.**

$3^y = 3^{-3}$ **$\frac{1}{27} = \frac{1}{3^3} = 3^{-3}$**

$y = -3$ **Equate exponents.**

Therefore, $\log_3 \frac{1}{27} = -3$. ■

Now Work PROBLEM **27**

3 Determine the Domain of a Logarithmic Function

The logarithmic function $y = \log_a x$ has been defined as the inverse of the exponential function $y = a^x$. That is, if $f(x) = a^x$, then $f^{-1}(x) = \log_a x$. Based on the discussion in Section 6.2 on inverse functions, for a function f and its inverse f^{-1},

$$\text{Domain of } f^{-1} = \text{Range of } f \quad \text{and} \quad \text{Range of } f^{-1} = \text{Domain of } f$$

Consequently, it follows that

Domain of the logarithmic function = Range of the exponential function = $(0, \infty)$

Range of the logarithmic function = Domain of the exponential function = $(-\infty, \infty)$

The next box summarizes some properties of the logarithmic function.

$y = \log_a x$ (defining equation: $x = a^y$)

Domain: $(0, \infty)$ Range: $(-\infty, \infty)$

The domain of a logarithmic function consists of the *positive* real numbers, so the argument of a logarithmic function must be greater than zero.

EXAMPLE 5 **Finding the Domain of a Logarithmic Function**

Find the domain of each logarithmic function.

(a) $F(x) = \log_2(x + 3)$ (b) $g(x) = \log_5\left(\frac{1 + x}{1 - x}\right)$ (c) $h(x) = \log_{1/2}|x|$

Solution (a) The domain of F consists of all x for which $x + 3 > 0$, that is, $x > -3$. Using interval notation, the domain of F is $(-3, \infty)$.

(b) The domain of g is restricted to

$$\frac{1 + x}{1 - x} > 0$$

Solve this inequality to find that the domain of g consists of all x between -1 and 1, that is, $-1 < x < 1$ or, using interval notation, $(-1, 1)$.

(c) Since $|x| > 0$, provided that $x \neq 0$, the domain of h consists of all real numbers except zero or, using interval notation, $(-\infty, 0) \cup (0, \infty)$. ■

Now Work PROBLEMS 41 AND 47

4 Graph Logarithmic Functions

Because exponential functions and logarithmic functions are inverses of each other, the graph of the logarithmic function $y = \log_a x$ is the reflection about the line $y = x$ of the graph of the exponential function $y = a^x$, as shown in Figure 36.

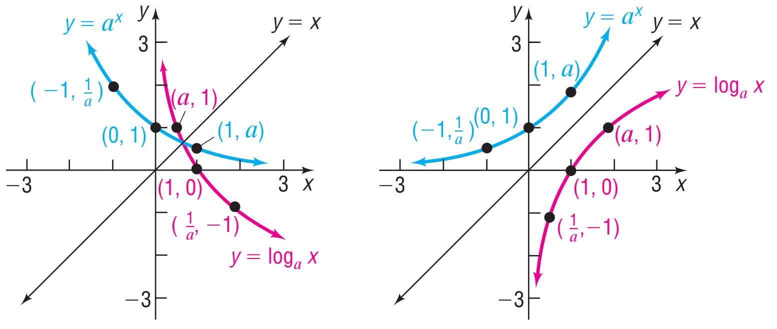


Figure 36 **(a)** $0 < a < 1$ **(b)** $a > 1$

For example, to graph $y = \log_2 x$, graph $y = 2^x$ and reflect it about the line $y = x$. See Figure 37. To graph $y = \log_{1/3} x$, graph $y = \left(\frac{1}{3}\right)^x$ and reflect it about the line $y = x$. See Figure 38.

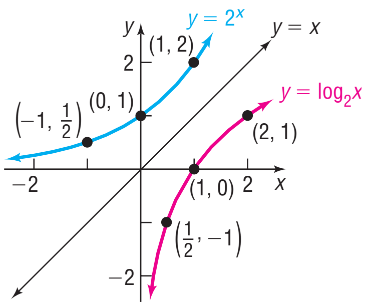


Figure 37

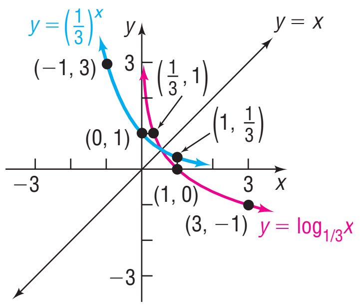


Figure 38

Now Work PROBLEM 61

The graphs of $y = \log_a x$ in Figures 36(a) and (b) lead to the following properties.

Properties of the Logarithmic Function $f(x) = \log_a x$; $a > 0$, $a \neq 1$

1. The domain is the set of positive real numbers, or $(0, \infty)$ using interval notation; the range is the set of all real numbers, or $(-\infty, \infty)$ using interval notation.
2. The x-intercept of the graph is 1. There is no y-intercept.
3. The y-axis $(x = 0)$ is a vertical asymptote of the graph.
4. A logarithmic function is decreasing if $0 < a < 1$ and is increasing if $a > 1$.
5. The graph of f contains the points $(1, 0)$, $(a, 1)$, and $\left(\frac{1}{a}, -1\right)$.
6. The graph is smooth and continuous, with no corners or gaps.

In Words
$y = \log_e x$ is written $y = \ln x$.

If the base of a logarithmic function is the number e, the result is the **natural logarithm function**. This function occurs so frequently in applications that it is given a special symbol, **ln** (from the Latin, *logarithmus naturalis*). That is,

$$y = \ln x \quad \text{if and only if} \quad x = e^y \qquad \textbf{(1)}$$

Because $y = \ln x$ and the exponential function $y = e^x$ are inverse functions, the graph of $y = \ln x$ can be obtained by reflecting the graph of $y = e^x$ about the line $y = x$. See Figure 39.

Using a calculator with an $\boxed{\ln}$ key, we can obtain other points on the graph of $f(x) = \ln x$. See Table 8.

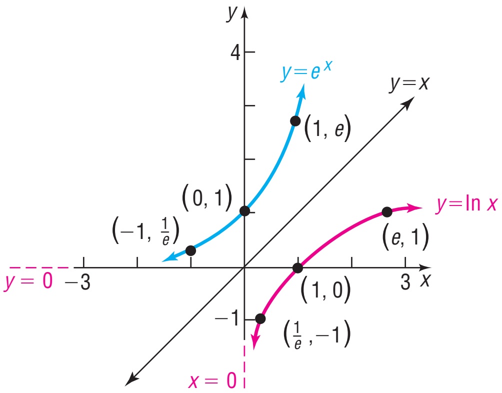
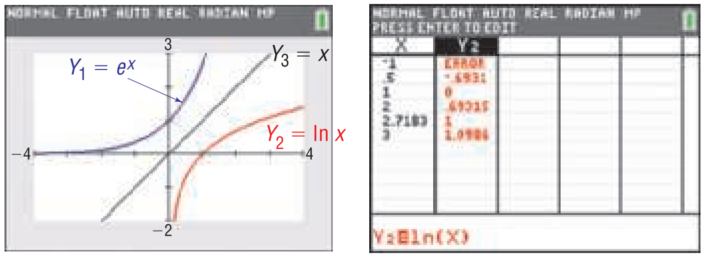


Figure 39

Table 8



NORMAL FLOAT AUTO REAL RADIAN MP
PRESS ENTER TO EDIT

X	Y2
-1	ERROR
.5	-.6931
1	0
2	.69315
2.7183	1
3	1.0986

Y2=ln(X)

EXAMPLE 6 Graphing a Logarithmic Function and Its Inverse

(a) Find the domain of the logarithmic function $f(x) = -\ln(x - 2)$.
(b) Graph f.
(c) From the graph, determine the range and vertical asymptote of f.
(d) Find f^{-1}, the inverse of f.
(e) Find the domain and the range of f^{-1}.
(f) Graph f^{-1}.

Solution

(a) The domain of f consists of all x for which $x - 2 > 0$, or equivalently, $x > 2$. The domain of f is $\{x | x > 2\}$, or $(2, \infty)$ in interval notation.

(b) To obtain the graph of $y = -\ln(x - 2)$, begin with the graph of $y = \ln x$ and use transformations. See Figure 40.

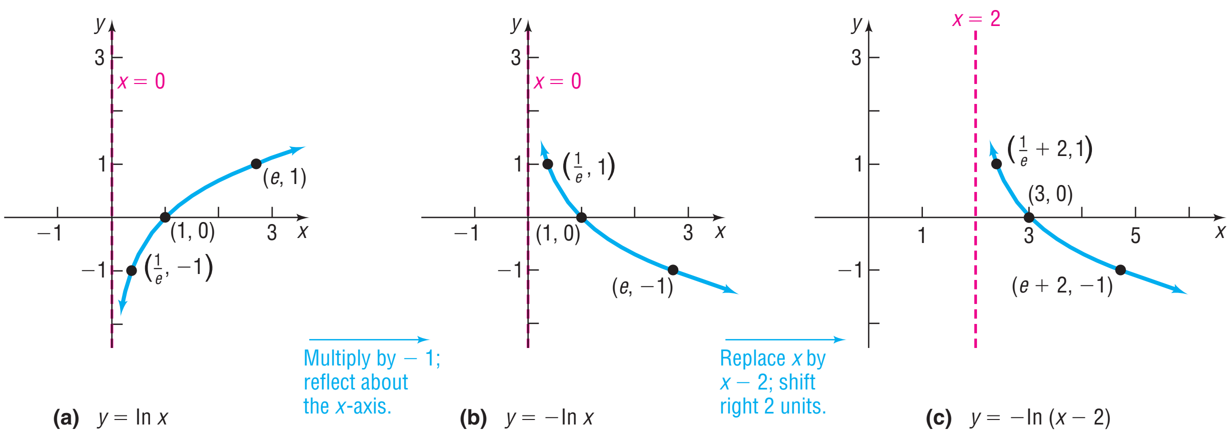


Figure 40 (a) $y = \ln x$ (b) $y = -\ln x$ (c) $y = -\ln(x - 2)$

(c) The range of $f(x) = -\ln(x - 2)$ is the set of all real numbers. The vertical asymptote is $x = 2$. [Do you see why? The original asymptote $(x = 0)$ is shifted to the right 2 units.]

(d) To find f^{-1}, begin with $y = -\ln(x - 2)$. The inverse function is defined (implicitly) by the equation

$$x = -\ln(y - 2)$$

Now solve for y.

$$\begin{aligned} -x &= \ln(y - 2) && \text{Isolate the logarithm.} \\ e^{-x} &= y - 2 && \text{Change to exponential form.} \\ y &= e^{-x} + 2 && \text{Solve for } y. \end{aligned}$$

The inverse of f is $f^{-1}(x) = e^{-x} + 2$.

(e) The domain of f^{-1} equals the range of f, which is the set of all real numbers, from part (c). The range of f^{-1} is the domain of f, which is $(2, \infty)$ in interval notation.

(f) To graph f^{-1}, use the graph of f in Figure 40(c) and reflect it about the line $y = x$. See Figure 41. We could also graph $f^{-1}(x) = e^{-x} + 2$ using transformations.

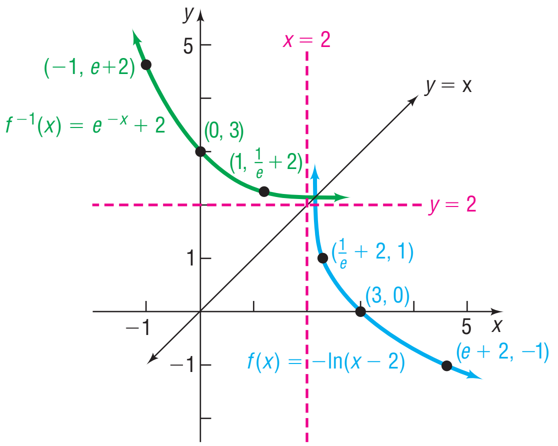


Figure 41

Now Work PROBLEM 73

If the base of a logarithmic function is the number 10, the result is the **common logarithm function**. If the base a of the logarithmic function is not indicated, it is understood to be 10. That is,

$$\boxed{y = \log x \quad \text{if and only if} \quad x = 10^y}$$

Because $y = \log x$ and the exponential function $y = 10^x$ are inverse functions, the graph of $y = \log x$ can be obtained by reflecting the graph of $y = 10^x$ about the line $y = x$. See Figure 42.

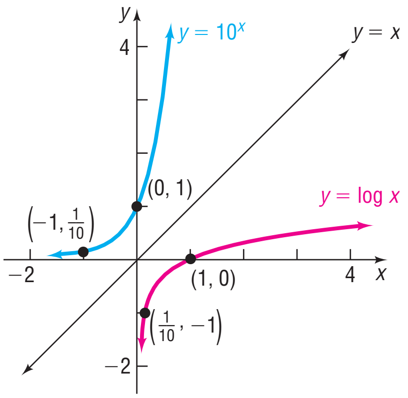


Figure 42

EXAMPLE 7 Graphing a Logarithmic Function and Its Inverse

(a) Find the domain of the logarithmic function $f(x) = 3\log(x - 1)$.
(b) Graph f.
(c) From the graph, determine the range and vertical asymptote of f.
(d) Find f^{-1}, the inverse of f.
(e) Find the domain and the range of f^{-1}.
(f) Graph f^{-1}.

Solution

(a) The domain of f consists of all x for which $x - 1 > 0$, or equivalently, $x > 1$. The domain of f is $\{x \mid x > 1\}$, or $(1, \infty)$ in interval notation.

(b) To obtain the graph of $y = 3\log(x - 1)$, begin with the graph of $y = \log x$ and use transformations. See Figure 43.

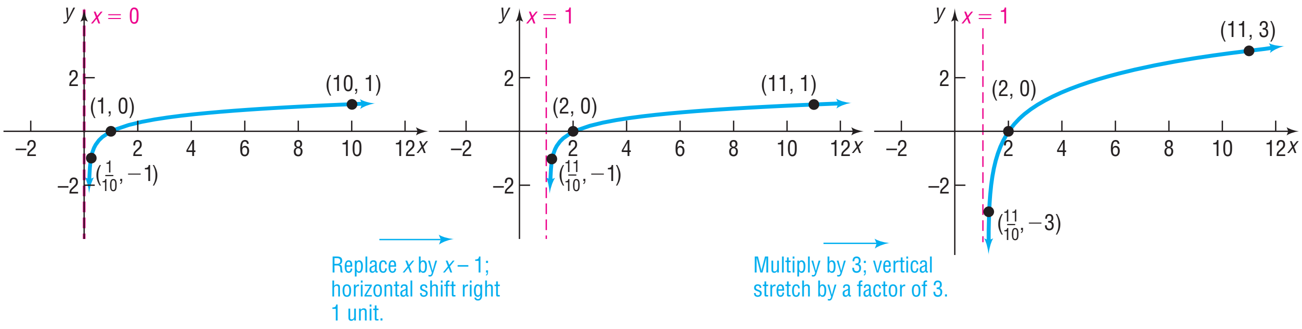


Figure 43 **(a)** $y = \log x$ **(b)** $y = \log(x - 1)$ **(c)** $y = 3\log(x - 1)$

(c) The range of $f(x) = 3\log(x - 1)$ is the set of all real numbers. The vertical asymptote is $x = 1$.

(d) Begin with $y = 3\log(x - 1)$. The inverse function is defined implicitly by the equation

$$x = 3\log(y - 1)$$

Proceed to solve for y.

$$\frac{x}{3} = \log(y - 1) \quad \text{Isolate the logarithm.}$$

$$10^{x/3} = y - 1 \quad \text{Change to exponential form.}$$

$$y = 10^{x/3} + 1 \quad \text{Solve for } y.$$

The inverse of f is $f^{-1}(x) = 10^{x/3} + 1$.

(e) The domain of f^{-1} is the range of f, which is the set of all real numbers, from part (c). The range of f^{-1} is the domain of f, which is $(1, \infty)$ in interval notation.

(f) To graph f^{-1}, we use the graph of f in Figure 43(c) and reflect it about the line $y = x$. See Figure 44. We could also graph $f^{-1}(x) = 10^{x/3} + 1$ using transformations.

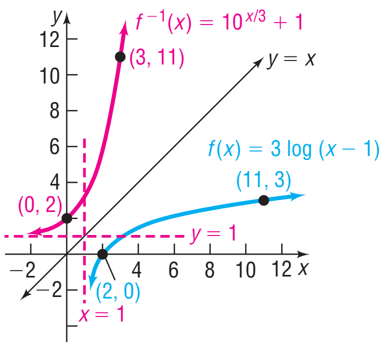


Figure 44

Now Work PROBLEM 81

5 Solve Logarithmic Equations

Equations that contain logarithms are called **logarithmic equations**. Care must be taken when solving logarithmic equations algebraically. In the expression $\log_a M$, remember that a and M are positive and $a \neq 1$. Be sure to check each apparent solution in the original equation and discard any that are extraneous.

Some logarithmic equations can be solved by changing the logarithmic equation to exponential form using the fact that $y = \log_a x$ means $a^y = x$.

EXAMPLE 8 Solving Logarithmic Equations

Solve:

(a) $\log_3(4x - 7) = 2$ (b) $\log_x 64 = 2$

Solution (a) To solve, change the logarithmic equation to exponential form.

$$\log_3(4x-7) = 2$$
$$4x - 7 = 3^2 \quad \text{Change to exponential form.}$$
$$4x - 7 = 9$$
$$4x = 16$$
$$x = 4$$

Check: $\log_3(4x-7) = \log_3(4 \cdot 4 - 7) = \log_3 9 = 2$ $\quad 3^2 = 9$

The solution set is $\{4\}$.

(b) To solve, change the logarithmic equation to exponential form.

$$\log_x 64 = 2$$
$$x^2 = 64 \quad \text{Change to exponential form.}$$
$$x = \pm\sqrt{64} = \pm 8 \quad \text{Square Root Method}$$

Because the base of a logarithm must be positive, discard -8. Check the potential solution 8.

Check: $\log_8 64 = 2$ $\quad 8^2 = 64$

The solution set is $\{8\}$. ■

EXAMPLE 9 Using Logarithms to Solve an Exponential Equation

Solve: $e^{2x} = 5$

Solution To solve, change the exponential equation to logarithmic form.

$$e^{2x} = 5$$
$$\ln 5 = 2x \quad \text{Change to logarithmic form.}$$
$$x = \frac{\ln 5}{2} \quad \text{Exact solution}$$
$$\approx 0.805 \quad \text{Approximate solution}$$

The solution set is $\left\{\frac{\ln 5}{2}\right\}$. ■

Now Work PROBLEMS 89 AND 101

EXAMPLE 10 **Alcohol and Driving**

Blood alcohol concentration (BAC) is a measure of the amount of alcohol in a person's bloodstream. A BAC of 0.04% means that a person has 4 parts alcohol per 10,000 parts blood in the body. Relative risk is defined as the likelihood of one event occurring divided by the likelihood of a second event occurring. For example, if an individual with a BAC of 0.02% is 1.4 times as likely to have a car accident as an individual who has not been drinking, the relative risk of an accident with a BAC of 0.02% is 1.4. Recent medical research suggests that the relative risk R of having an accident while driving a car can be modeled by an equation of the form

$$R = e^{kx}$$

where x is the percent of concentration of alcohol in the bloodstream and k is a constant.

(a) Research indicates that the relative risk of a person having an accident with a BAC of 0.02% is 1.4. Find the constant k in the equation.

(b) Using this value of k, what is the relative risk if the concentration is 0.17%?

(c) Using this same value of k, what BAC corresponds to a relative risk of 100?

(d) If the law asserts that anyone with a relative risk of 4 or more should not have driving privileges, at what concentration of alcohol in the bloodstream should a driver be arrested and charged with DUI (driving under the influence)?

Solution

(a) For a concentration of alcohol in the blood of 0.02% and a relative risk of 1.4, let $x = 0.02$ and $R = 1.4$ in the equation and solve for k.

$$\begin{aligned} R &= e^{kx} & \\ 1.4 &= e^{k(0.02)} & \quad R = 1.4;\ x = 0.02 \\ 0.02k &= \ln 1.4 & \quad \text{Change to logarithmic form.} \\ k &= \frac{\ln 1.4}{0.02} \approx 16.82 & \quad \text{Solve for } k. \end{aligned}$$

(b) A concentration of 0.17% means $x = 0.17$. Use $k = 16.82$ in the equation to find the relative risk R:

$$R = e^{kx} = e^{(16.82)(0.17)} \approx 17.5$$

For a concentration of alcohol in the blood of 0.17%, the relative risk of an accident is about 17.5. That is, a person with a BAC of 0.17% is 17.5 times as likely to have a car accident as a person with no alcohol in the bloodstream.

(c) A relative risk of 100 means $R = 100$. Use $k = 16.82$ in the equation $R = e^{kx}$. The concentration x of alcohol in the blood obeys

$$\begin{aligned} 100 &= e^{16.82x} & \quad R = e^{kx};\ R = 100;\ k = 16.82 \\ 16.82x &= \ln 100 & \quad \text{Change to logarithmic form.} \\ x &= \frac{\ln 100}{16.82} \approx 0.27 & \quad \text{Solve for } x. \end{aligned}$$

Note: A BAC of 0.30% results in a loss of consciousness in most people. ■

For a concentration of alcohol in the blood of 0.27%, the relative risk of an accident is 100.

(d) A relative risk of 4 means $R = 4$. Use $k = 16.82$ in the equation $R = e^{kx}$. The concentration x of alcohol in the bloodstream obeys

$$\begin{aligned} 4 &= e^{16.82x} \\ 16.82x &= \ln 4 \\ x &= \frac{\ln 4}{16.82} \approx 0.082 \end{aligned}$$

Note: The blood alcohol content at which a DUI citation is given is 0.08%. ■

A driver with a BAC of 0.082% or more should be arrested and charged with DUI. ■

SUMMARY

Properties of the Logarithmic Function

$f(x) = \log_a x, \quad a > 1$ $(y = \log_a x$ means $x = a^y)$	Domain: the interval $(0, \infty)$; Range: the interval $(-\infty, \infty)$ x-intercept: 1; y-intercept: none; vertical asymptote: $x = 0$ (y-axis); increasing; one-to-one See Figure 45(a) for a typical graph.
$f(x) = \log_a x, \quad 0 < a < 1$ $(y = \log_a x$ means $x = a^y)$	Domain: the interval $(0, \infty)$; Range: the interval $(-\infty, \infty)$ x-intercept: 1; y-intercept: none; vertical asymptote: $x = 0$ (y-axis); decreasing; one-to-one See Figure 45(b) for a typical graph.

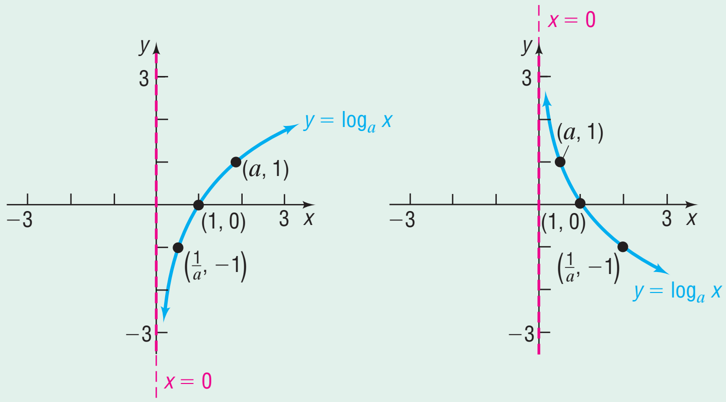


Figure 45 **(a)** $a > 1$ **(b)** $0 < a < 1$

6.4 Assess Your Understanding

'Are You Prepared?'

Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.

1. Solve each inequality:
(a) $3x - 7 \le 8 - 2x$ (pp. 150–151)
(b) $x^2 - x - 6 > 0$ (pp. 320–322)

2. Solve the inequality: $\dfrac{x-1}{x+4} > 0$ (pp. 393–397)

3. Solve: $2x + 3 = 9$ (pp. 102–103)

Concepts and Vocabulary

4. The domain of the logarithmic function $f(x) = \log_a x$ is ________.

5. The graph of every logarithmic function $f(x) = \log_a x$, where $a > 0$ and $a \ne 1$, passes through three points: ________, ________, and ________.

6. If the graph of a logarithmic function $f(x) = \log_a x$, where $a > 0$ and $a \ne 1$, is increasing, then its base must be larger than _____.

7. ***True or False*** If $y = \log_a x$, then $y = a^x$.

8. ***True or False*** The graph of $f(x) = \log_a x$, where $a > 0$ and $a \ne 1$, has an x-intercept equal to 1 and no y-intercept.

9. Select the answer that completes the statement: $y = \ln x$ if and only if ________.
(a) $x = e^y$ (b) $y = e^x$ (c) $x = 10^y$ (d) $y = 10^x$

10. Choose the domain of $f(x) = \log_3(x + 2)$.
(a) $(-\infty, \infty)$ (b) $(2, \infty)$ (c) $(-2, \infty)$ (d) $(0, \infty)$

Skill Building

In Problems 11–18, change each exponential statement to an equivalent statement involving a logarithm.

11. $9 = 3^2$ **12.** $16 = 4^2$ **13.** $a^2 = 1.6$ **14.** $a^3 = 2.1$

15. $2^x = 7.2$ **16.** $3^x = 4.6$ **17.** $e^x = 8$ **18.** $e^{2.2} = M$

In Problems 19–26, change each logarithmic statement to an equivalent statement involving an exponent.

19. $\log_2 8 = 3$ **20.** $\log_3\left(\dfrac{1}{9}\right) = -2$ **21.** $\log_a 3 = 6$ **22.** $\log_b 4 = 2$

23. $\log_3 2 = x$ **24.** $\log_2 6 = x$ **25.** $\ln 4 = x$ **26.** $\ln x = 4$

In Problems 27–38, find the exact value of each logarithm without using a calculator.

27. $\log_2 1$ **28.** $\log_8 8$ **29.** $\log_5 25$ **30.** $\log_3\left(\frac{1}{9}\right)$

31. $\log_{1/2} 16$ **32.** $\log_{1/3} 9$ **33.** $\log_{10} \sqrt{10}$ **34.** $\log_5 \sqrt[3]{25}$

35. $\log_{\sqrt{2}} 4$ **36.** $\log_{\sqrt{3}} 9$ **37.** $\ln \sqrt{e}$ **38.** $\ln e^3$

In Problems 39–50, find the domain of each function.

39. $f(x) = \ln(x - 3)$ **40.** $g(x) = \ln(x - 1)$ **41.** $F(x) = \log_2 x^2$

42. $H(x) = \log_5 x^3$ **43.** $f(x) = 3 - 2\log_4\left(\frac{x}{2} - 5\right)$ **44.** $g(x) = 8 + 5\ln(2x + 3)$

45. $f(x) = \ln\left(\frac{1}{x + 1}\right)$ **46.** $g(x) = \ln\left(\frac{1}{x - 5}\right)$ **47.** $g(x) = \log_5\left(\frac{x + 1}{x}\right)$

48. $h(x) = \log_3\left(\frac{x}{x - 1}\right)$ **49.** $f(x) = \sqrt{\ln x}$ **50.** $g(x) = \frac{1}{\ln x}$

In Problems 51–58, use a calculator to evaluate each expression. Round your answer to three decimal places.

51. $\ln \frac{5}{3}$ **52.** $\frac{\ln 5}{3}$ **53.** $\frac{\ln \frac{10}{3}}{0.04}$ **54.** $\frac{\ln \frac{2}{3}}{-0.1}$

55. $\frac{\ln 4 + \ln 2}{\log 4 + \log 2}$ **56.** $\frac{\log 15 + \log 20}{\ln 15 + \ln 20}$ **57.** $\frac{2\ln 5 + \log 50}{\log 4 - \ln 2}$ **58.** $\frac{3\log 80 - \ln 5}{\log 5 + \ln 20}$

59. Find a so that the graph of $f(x) = \log_a x$ contains the point $(2, 2)$.

60. Find a so that the graph of $f(x) = \log_a x$ contains the point $\left(\frac{1}{2}, -4\right)$.

In Problems 61–64, graph each function and its inverse on the same set of axes.

61. $f(x) = 3^x;\ f^{-1}(x) = \log_3 x$ **62.** $f(x) = 4^x;\ f^{-1}(x) = \log_4 x$

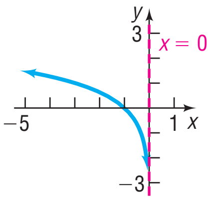
63. $f(x) = \left(\frac{1}{2}\right)^x;\ f^{-1}(x) = \log_{1/2} x$ **64.** $f(x) = \left(\frac{1}{3}\right)^x;\ f^{-1}(x) = \log_{1/3} x$

In Problems 65–72, the graph of a logarithmic function is given. Match each graph to one of the following functions:

(A) $y = \log_3 x$ (B) $y = \log_3(-x)$ (C) $y = -\log_3 x$ (D) $y = -\log_3(-x)$

(E) $y = \log_3 x - 1$ (F) $y = \log_3(x - 1)$ (G) $y = \log_3(1 - x)$ (H) $y = 1 - \log_3 x$

65.



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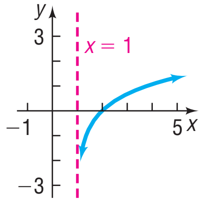

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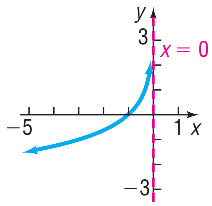

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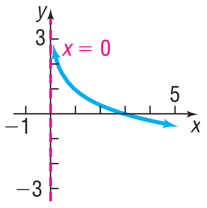

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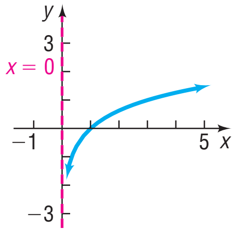

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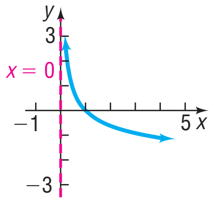

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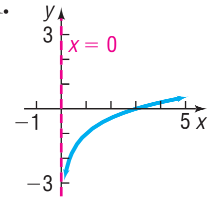

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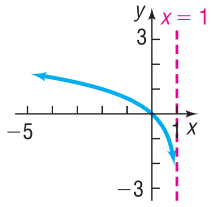


In Problems 73–88, use the given function f.

(a) Find the domain of f. (b) Graph f. (c) From the graph, determine the range and any asymptotes of f.
(d) Find f^{-1}, the inverse of f. (e) Find the domain and the range of f^{-1}. (f) Graph f^{-1}.

73. $f(x) = \ln(x + 4)$ **74.** $f(x) = \ln(x - 3)$ **75.** $f(x) = 2 + \ln x$ **76.** $f(x) = -\ln(-x)$

77. $f(x) = \ln(2x) - 3$ **78.** $f(x) = -2\ln(x + 1)$ **79.** $f(x) = \log(x - 4) + 2$ **80.** $f(x) = \frac{1}{2}\log x - 5$

81. $f(x) = \frac{1}{2}\log(2x)$ **82.** $f(x) = \log(-2x)$ **83.** $f(x) = 3 + \log_3(x + 2)$ **84.** $f(x) = 2 - \log_3(x + 1)$

85. $f(x) = e^{x+2} - 3$ **86.** $f(x) = 3e^x + 2$ **87.** $f(x) = 2^{x/3} + 4$ **88.** $f(x) = -3^{x+1}$

In Problems 89–112, solve each equation.

89. $\log_3 x = 2$ **90.** $\log_5 x = 3$ **91.** $\log_2(2x+1) = 3$ **92.** $\log_3(3x-2) = 2$

93. $\log_x 4 = 2$ **94.** $\log_x\left(\frac{1}{8}\right) = 3$ **95.** $\ln e^x = 5$ **96.** $\ln e^{-2x} = 8$

97. $\log_4 64 = x$ **98.** $\log_5 625 = x$ **99.** $\log_3 243 = 2x+1$ **100.** $\log_6 36 = 5x+3$

101. $e^{3x} = 10$ **102.** $e^{-2x} = \frac{1}{3}$ **103.** $e^{2x+5} = 8$ **104.** $e^{-2x+1} = 13$

105. $\log_3(x^2+1) = 2$ **106.** $\log_5(x^2+x+4) = 2$ **107.** $\log_2 8^x = -3$ **108.** $\log_3 3^x = -1$

109. $5e^{0.2x} = 7$ **110.** $8 \cdot 10^{2x-7} = 3$ **111.** $2 \cdot 10^{2-x} = 5$ **112.** $4e^{x+1} = 5$

Mixed Practice

113. Suppose that $G(x) = \log_3(2x+1) - 2$.
(a) What is the domain of G?
(b) What is $G(40)$? What point is on the graph of G?
(c) If $G(x) = 3$, what is x? What point is on the graph of G?
(d) What is the zero of G?

114. Suppose that $F(x) = \log_2(x+1) - 3$.
(a) What is the domain of F?
(b) What is $F(7)$? What point is on the graph of F?
(c) If $F(x) = -1$, what is x? What point is on the graph of F?
(d) What is the zero of F?

In Problems 115–118, graph each function. Based on the graph, state the domain and the range, and find any intercepts.

115. $f(x) = \begin{cases} \ln(-x) & \text{if } x < 0 \\ \ln x & \text{if } x > 0 \end{cases}$

116. $f(x) = \begin{cases} \ln(-x) & \text{if } x \le -1 \\ -\ln(-x) & \text{if } -1 < x < 0 \end{cases}$

117. $f(x) = \begin{cases} -\ln x & \text{if } 0 < x < 1 \\ \ln x & \text{if } x \ge 1 \end{cases}$

118. $f(x) = \begin{cases} \ln x & \text{if } 0 < x < 1 \\ -\ln x & \text{if } x \ge 1 \end{cases}$

Applications and Extensions

119. Chemistry The pH of a chemical solution is given by the formula

$$\text{pH} = -\log_{10}[H^+]$$

where $[H^+]$ is the concentration of hydrogen ions in moles per liter. Values of pH range from 0 (acidic) to 14 (alkaline).
(a) What is the pH of a solution for which $[H^+]$ is 0.1?
(b) What is the pH of a solution for which $[H^+]$ is 0.01?
(c) What is the pH of a solution for which $[H^+]$ is 0.001?
(d) What happens to pH as the hydrogen ion concentration decreases?
(e) Determine the hydrogen ion concentration of an orange $(\text{pH} = 3.5)$.
(f) Determine the hydrogen ion concentration of human blood $(\text{pH} = 7.4)$.

120. Diversity Index Shannon's diversity index is a measure of the diversity of a population. The diversity index is given by the formula

$$H = -(p_1 \log p_1 + p_2 \log p_2 + \cdots + p_n \log p_n)$$

where p_1 is the proportion of the population that is species 1, p_2 is the proportion of the population that is species 2, and so on. In this problem, the population is people in the United States and the species is race.
(a) According to the U.S. Census Bureau, the distribution of race in the United States in 2015 was as follows:

Race	Proportion
White	0.617
Black or African American	0.124
American Indian and Alaskan Native	0.007
Asian	0.053
Native Hawaiian and Other Pacific Islander	0.002
Hispanic	0.177
Two or More Races	0.020

Source: *U.S. Census Bureau*

Compute the diversity index of the United States in 2015.
(b) The largest value of the diversity index is given by $H_{max} = \log(S)$, where S is the number of categories of race. Compute H_{max}.
(c) The **evenness ratio** is given by $E_H = \frac{H}{H_{max}}$, where $0 \le E_H \le 1$. If $E_H = 1$, there is complete evenness. Compute the evenness ratio for the United States.
(d) Obtain the distribution of race for the United States in 2010 from the Census Bureau. Compute Shannon's diversity index. Is the United States becoming more diverse? Why?

121. Atmospheric Pressure The atmospheric pressure p on an object decreases with increasing height. This pressure, measured in millimeters of mercury, is related to the height h (in kilometers) above sea level by the function

$$p(h) = 760e^{-0.145h}$$

(a) Find the height of an aircraft if the atmospheric pressure is 320 millimeters of mercury.
(b) Find the height of a mountain if the atmospheric pressure is 667 millimeters of mercury.

122. Healing of Wounds The normal healing of wounds can be modeled by an exponential function. If A_0 represents the original area of the wound, and if A equals the area of the wound, then the function

$$A(n) = A_0e^{-0.35n}$$

describes the area of a wound after n days following an injury when no infection is present to retard the healing. Suppose that a wound initially had an area of 100 square millimeters.
(a) If healing is taking place, after how many days will the wound be one-half its original size?
(b) How long before the wound is 10% of its original size?

123. Exponential Probability Between 12:00 PM and 1:00 PM, cars arrive at Citibank's drive-thru at the rate of 6 cars per hour (0.1 car per minute). The following formula from statistics can be used to determine the probability that a car will arrive within t minutes of 12:00 PM.

$$F(t) = 1 - e^{-0.1t}$$

(a) Determine how many minutes are needed for the probability to reach 50%.
(b) Determine how many minutes are needed for the probability to reach 80%.
(c) Is it possible for the probability to equal 100%? Explain.

124. Exponential Probability Between 5:00 PM and 6:00 PM, cars arrive at Jiffy Lube at the rate of 9 cars per hour (0.15 car per minute). The following formula from statistics can be used to determine the probability that a car will arrive within t minutes of 5:00 PM.

$$F(t) = 1 - e^{-0.15t}$$

(a) Determine how many minutes are needed for the probability to reach 50%.
(b) Determine how many minutes are needed for the probability to reach 80%.

125. Drug Medication The formula

$$D = 5e^{-0.4h}$$

can be used to find the number of milligrams D of a certain drug that is in a patient's bloodstream h hours after the drug was administered. When the number of milligrams reaches 2, the drug is to be administered again. What is the time between injections?

126. Spreading of Rumors A model for the number N of people in a college community who have heard a certain rumor is

$$N = P(1 - e^{-0.15d})$$

where P is the total population of the community and d is the number of days that have elapsed since the rumor began. In a community of 1000 students, how many days will elapse before 450 students have heard the rumor?

127. Current in an RL Circuit The equation governing the amount of current I (in amperes) after time t (in seconds) in a simple RL circuit consisting of a resistance R (in ohms), an inductance L (in henrys), and an electromotive force E (in volts) is

$$I = \frac{E}{R}\left[1 - e^{-(R/L)t}\right]$$

If $E = 12$ volts, $R = 10$ ohms, and $L = 5$ henrys, how long does it take to obtain a current of 0.5 ampere? Of 1.0 ampere? Graph the equation.

128. Learning Curve Psychologists sometimes use the function

$$L(t) = A(1 - e^{-kt})$$

to measure the amount L learned at time t. Here A represents the amount to be learned, and the number k measures the rate of learning. Suppose that a student has an amount A of 200 vocabulary words to learn. A psychologist determines that the student has learned 20 vocabulary words after 5 minutes.
(a) Determine the rate of learning k.
(b) Approximately how many words will the student have learned after 10 minutes?
(c) After 15 minutes?
(d) How long does it take for the student to learn 180 words?

Loudness of Sound *Problems 129–132 use the following discussion: The* ***loudness*** $L(x)$*, measured in decibels (dB), of a sound of intensity x, measured in watts per square meter, is defined as* $L(x) = 10 \log \frac{x}{I_0}$*, where* $I_0 = 10^{-12}$ *watt per square meter is the least intense sound that a human ear can detect. Determine the loudness, in decibels, of each of the following sounds.*

129. Normal conversation: intensity of $x = 10^{-7}$ watt per square meter.

130. Amplified rock music: intensity of 10^{-1} watt per square meter.

131. Heavy city traffic: intensity of $x = 10^{-3}$ watt per square meter.

132. Diesel truck traveling 40 miles per hour 50 feet away: intensity 10 times that of a passenger car traveling 50 miles per hour 50 feet away, whose loudness is 70 decibels.

The Richter Scale *Problems 133 and 134 on the next page use the following discussion: The* ***Richter scale*** *is one way of converting seismographic readings into numbers that provide an easy reference for measuring the magnitude M of an earthquake. All earthquakes are compared to a* ***zero-level earthquake*** *whose seismographic reading measures 0.001 millimeter at a distance of 100 kilometers from the epicenter. An earthquake whose seismographic reading measures x millimeters has* ***magnitude*** $M(x)$*, given by*

$$M(x) = \log\left(\frac{x}{x_0}\right)$$

where $x_0 = 10^{-3}$ *is the reading of a zero-level earthquake the same distance from its epicenter. In Problems 133 and 134, determine the magnitude of each earthquake.*

133. Magnitude of an Earthquake Mexico City in 1985: seismographic reading of 125,892 millimeters 100 kilometers from the center

134. Magnitude of an Earthquake San Francisco in 1906: seismographic reading of 50,119 millimeters 100 kilometers from the center

135. Alcohol and Driving The concentration of alcohol in a person's bloodstream is measurable. Suppose that the relative risk R of having an accident while driving a car can be modeled by an equation of the form

$$R = e^{kx}$$

where x is the percent concentration of alcohol in the bloodstream and k is a constant.

(a) Suppose that a concentration of alcohol in the bloodstream of 0.03 percent results in a relative risk of an accident of 1.4. Find the constant k in the equation.

(b) Using this value of k, what is the relative risk if the concentration is 0.17 percent?

(c) Using the same value of k, what concentration of alcohol corresponds to a relative risk of 100?

(d) If the law asserts that anyone with a relative risk of having an accident of 5 or more should not have driving privileges, at what concentration of alcohol in the bloodstream should a driver be arrested and charged with a DUI?

(e) Compare this situation with that of Example 10. If you were a lawmaker, which situation would you support? Give your reasons.

Explaining Concepts: Discussion and Writing

136. Is there any function of the form $y = x^{\alpha}, 0 < \alpha < 1$, that increases more slowly than a logarithmic function whose base is greater than 1? Explain.

137. In the definition of the logarithmic function, the base a is not allowed to equal 1. Why?

138. Critical Thinking In buying a new car, one consideration might be how well the price of the car holds up over time. Different makes of cars have different depreciation rates. One way to compute a depreciation rate for a car is given here. Suppose that the current prices of a certain automobile are as shown in the table.

Age in Years					
New	1	2	3	4	5
\$38,000	\$36,600	\$32,400	\$28,750	\$25,400	\$21,200

Use the formula $\text{New} = \text{Old}(e^{Rt})$ to find R, the annual depreciation rate, for a specific time t. When might be the best time to trade in the car? Consult the NADA ("blue") book and compare two like models that you are interested in. Which has the better depreciation rate?

Retain Your Knowledge

Problems 139–142 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

139. Find the real zeros of $g(x) = 4x^4 - 37x^2 + 9$. What are the x-intercepts of the graph of g?

140. Find the average rate of change of $f(x) = 9^x$ from $\frac{1}{2}$ to 1.

141. Use the Intermediate Value Theorem to show that the function $f(x) = 4x^3 - 2x^2 - 7$ has a real zero in the interval $[1, 2]$.

142. A complex polynomial function f of degree 4 with real coefficients has the zeros $-1, 2$, and $3 - i$. Find the remaining zero(s) of f. Then find a polynomial function that has the zeros.

'Are You Prepared?' Answers

1. (a) $x \le 3$ (b) $x < -2$ or $x > 3$ **2.** $x < -4$ or $x > 1$ **3.** $\{3\}$

6.5 Properties of Logarithms

OBJECTIVES
1. Work with the Properties of Logarithms (p. 458)
2. Write a Logarithmic Expression as a Sum or Difference of Logarithms (p. 460)
3. Write a Logarithmic Expression as a Single Logarithm (p. 461)
4. Evaluate a Logarithm Whose Base Is Neither 10 Nor e (p. 462)
5. Graph a Logarithmic Function Whose Base Is Neither 10 Nor e (p. 464)

1 Work with the Properties of Logarithms

Logarithms have some very useful properties that can be derived directly from the definition and the laws of exponents.

EXAMPLE 1 **Establishing Properties of Logarithms**

(a) Show that $\log_a 1 = 0$. (b) Show that $\log_a a = 1$.

Solution (a) This fact was established when we graphed $y = \log_a x$ (see Figure 36 on page 448). To show the result algebraically, let $y = \log_a 1$. Then

$$\begin{aligned} y &= \log_a 1 \\ a^y &= 1 && \text{Change to exponential form.} \\ a^y &= a^0 && a^0 = 1 \text{ since } a > 0,\ a \neq 1 \\ y &= 0 && \text{Solve for } y. \\ \log_a 1 &= 0 && y = \log_a 1 \end{aligned}$$

(b) Let $y = \log_a a$. Then

$$\begin{aligned} y &= \log_a a \\ a^y &= a && \text{Change to exponential form.} \\ a^y &= a^1 && a = a^1 \\ y &= 1 && \text{Solve for } y. \\ \log_a a &= 1 && y = \log_a a \end{aligned}$$

To summarize:

$$\boxed{\log_a 1 = 0 \qquad \log_a a = 1}$$

THEOREM **Properties of Logarithms**

In the properties given next, M and a are positive real numbers, $a \neq 1$, and r is any real number.

The number $\log_a M$ is the exponent to which a must be raised to obtain M. That is,

$$a^{\log_a M} = M \qquad \textbf{(1)}$$

The logarithm with base a of a raised to a power equals that power. That is,

$$\log_a a^r = r \qquad \textbf{(2)}$$

The proof uses the fact that $y = a^x$ and $y = \log_a x$ are inverses.

Proof of Property (1) For inverse functions,

$$f(f^{-1}(x)) = x \quad \text{for all } x \text{ in the domain of } f^{-1}$$

Use $f(x) = a^x$ and $f^{-1}(x) = \log_a x$ to find

$$f(f^{-1}(x)) = a^{\log_a x} = x \quad \text{for } x > 0$$

Now let $x = M$ to obtain $a^{\log_a M} = M$, where $M > 0$. ■

Proof of Property (2) For inverse functions,

$$f^{-1}(f(x)) = x \quad \text{for all } x \text{ in the domain of } f$$

Use $f(x) = a^x$ and $f^{-1}(x) = \log_a x$ to find

$$f^{-1}(f(x)) = \log_a a^x = x \quad \text{for all real numbers } x$$

Now let $x = r$ to obtain $\log_a a^r = r$, where r is any real number. ■

EXAMPLE 2 **Using Properties (1) and (2)**

(a) $2^{\log_2 \pi} = \pi$ (b) $\log_{0.2} 0.2^{-\sqrt{2}} = -\sqrt{2}$ (c) $\ln e^{kt} = kt$

Now Work PROBLEM 15

Other useful properties of logarithms are given next.

THEOREM

Properties of Logarithms

In the following properties, M, N, and a are positive real numbers, $a \neq 1$, and r is any real number.

The Log of a Product Equals the Sum of the Logs

$$\log_a(MN) = \log_a M + \log_a N \tag{3}$$

The Log of a Quotient Equals the Difference of the Logs

$$\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N \tag{4}$$

The Log of a Power Equals the Product of the Power and the Log

$$\log_a M^r = r\log_a M \tag{5}$$

$$a^r = e^{r \ln a} \tag{6}$$

We shall derive properties (3), (5), and (6) and leave the derivation of property (4) as an exercise (see Problem 109).

Proof of Property (3) Let $A = \log_a M$ and let $B = \log_a N$. These expressions are equivalent to the exponential expressions

$$a^A = M \quad \text{and} \quad a^B = N$$

Now

$$\begin{aligned} \log_a(MN) = \log_a(a^A a^B) &= \log_a a^{A+B} && \textbf{Law of Exponents} \\ &= A + B && \textbf{Property (2) of logarithms} \\ &= \log_a M + \log_a N \end{aligned}$$

■

Proof of Property (5) Let $A = \log_a M$. This expression is equivalent to

$$a^A = M$$

Now

$$\begin{aligned} \log_a M^r = \log_a(a^A)^r &= \log_a a^{rA} && \textbf{Law of Exponents} \\ &= rA && \textbf{Property (2) of logarithms} \\ &= r\log_a M \end{aligned}$$

■

Proof of Property (6) Property (1), with $a = e$, gives

$$e^{\ln M} = M$$

Now let $M = a^r$ and apply property (5).

$$e^{\ln a^r} = e^{r\ln a} = (e^{\ln a})^r = a^r$$

■

Now Work PROBLEM 19

2 Write a Logarithmic Expression as a Sum or Difference of Logarithms

Logarithms can be used to transform products into sums, quotients into differences, and powers into factors. Such transformations prove useful in certain types of calculus problems.

EXAMPLE 3 **Writing a Logarithmic Expression as a Sum of Logarithms**

Write $\log_a(x\sqrt{x^2+1})$, $x > 0$, as a sum of logarithms. Express all powers as factors.

Solution

$$\begin{aligned}\log_a\left(x\sqrt{x^2+1}\right) &= \log_a x + \log_a\sqrt{x^2+1} && \log_a(M\cdot N) = \log_a M + \log_a N\\ &= \log_a x + \log_a(x^2+1)^{1/2}\\ &= \log_a x + \frac{1}{2}\log_a(x^2+1) && \log_a M^r = r\log_a M\end{aligned}$$

EXAMPLE 4 Writing a Logarithmic Expression as a Difference of Logarithms

Write

$$\ln\frac{x^2}{(x-1)^3}\qquad x>1$$

as a difference of logarithms. Express all powers as factors.

Solution

$$\ln\frac{x^2}{(x-1)^3} \underset{\uparrow}{=} \ln x^2 - \ln(x-1)^3 \underset{\uparrow}{=} 2\ln x - 3\ln(x-1)$$

$\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$ $\quad$ $\log_a M^r = r\log_a M$

EXAMPLE 5 Writing a Logarithmic Expression as a Sum and Difference of Logarithms

Write

$$\log_a\frac{\sqrt{x^2+1}}{x^3(x+1)^4}\qquad x>0$$

as a sum and difference of logarithms. Express all powers as factors.

Solution

$$\begin{aligned}\log_a\frac{\sqrt{x^2+1}}{x^3(x+1)^4} &= \log_a\sqrt{x^2+1} - \log_a\left[x^3(x+1)^4\right] && \text{Property (4)}\\ &= \log_a\sqrt{x^2+1} - \left[\log_a x^3 + \log_a(x+1)^4\right] && \text{Property (3)}\\ &= \log_a(x^2+1)^{1/2} - \log_a x^3 - \log_a(x+1)^4\\ &= \frac{1}{2}\log_a(x^2+1) - 3\log_a x - 4\log_a(x+1) && \text{Property (5)}\end{aligned}$$

WARNING In using properties (3) through (5), be careful about the values that the variable may assume. For example, the domain of the variable for $\log_a x$ is $x>0$ and for $\log_a(x-1)$ is $x>1$. If these functions are added, the domain is $x>1$. That is, the equality $\log_a x + \log_a(x-1) = \log_a[x(x-1)]$ is true only for $x>1$.

Now Work PROBLEM 51

3 Write a Logarithmic Expression as a Single Logarithm

Another use of properties (3) through (5) is to write sums and/or differences of logarithms with the same base as a single logarithm. This skill will be needed to solve certain logarithmic equations discussed in the next section.

EXAMPLE 6 Writing Expressions as a Single Logarithm

Write each of the following as a single logarithm.

(a) $\log_a 7 + 4\log_a 3$ $\qquad$ (b) $\frac{2}{3}\ln 8 - \ln(5^2-1)$

(c) $\log_a x + \log_a 9 + \log_a(x^2+1) - \log_a 5$

Solution

(a)
$$\begin{aligned}\log_a 7 + 4\log_a 3 &= \log_a 7 + \log_a 3^4 && r\log_a M = \log_a M^r\\ &= \log_a 7 + \log_a 81\\ &= \log_a(7\cdot 81) && \log_a M + \log_a N = \log_a(M\cdot N)\\ &= \log_a 567\end{aligned}$$

(b)
$$
\begin{aligned}
\frac{2}{3}\ln 8 - \ln(5^2 - 1) &= \ln 8^{2/3} - \ln(25 - 1) && r\log_a M = \log_a M^r \\
&= \ln 4 - \ln 24 && 8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4 \\
&= \ln\left(\frac{4}{24}\right) && \log_a M - \log_a N = \log_a\left(\frac{M}{N}\right) \\
&= \ln\left(\frac{1}{6}\right) \\
&= \ln 1 - \ln 6 \\
&= -\ln 6 && \ln 1 = 0
\end{aligned}
$$

(c)
$$
\begin{aligned}
\log_a x + \log_a 9 + \log_a(x^2 + 1) - \log_a 5 &= \log_a(9x) + \log_a(x^2 + 1) - \log_a 5 \\
&= \log_a[9x(x^2 + 1)] - \log_a 5 \\
&= \log_a\left[\frac{9x(x^2 + 1)}{5}\right]
\end{aligned}
$$

WARNING A common error made by some students is to express the logarithm of a sum as the sum of logarithms.

$$\log_a(M + N) \quad \text{is not equal to} \quad \log_a M + \log_a N$$

Correct statement $\log_a(MN) = \log_a M + \log_a N$ **Property (3)**

Another common error is to express the difference of logarithms as the quotient of logarithms.

$$\log_a M - \log_a N \quad \text{is not equal to} \quad \frac{\log_a M}{\log_a N}$$

Correct statement $\log_a M - \log_a N = \log_a\left(\frac{M}{N}\right)$ **Property (4)**

A third common error is to express a logarithm raised to a power as the product of the power times the logarithm.

$$(\log_a M)^r \quad \text{is not equal to} \quad r\log_a M$$

Correct statement $\log_a M^r = r\log_a M$ **Property (5)**

Now Work PROBLEMS 57 AND 63

Two other important properties of logarithms are consequences of the fact that the logarithmic function $y = \log_a x$ is a one-to-one function.

THEOREM

Properties of Logarithms

In the following properties, M, N, and a are positive real numbers, $a \neq 1$.

If $M = N$, then $\log_a M = \log_a N$. **(7)**

If $\log_a M = \log_a N$, then $M = N$. **(8)**

Property (7) is used as follows: Starting with the equation $M = N$, "take the logarithm of both sides" to obtain $\log_a M = \log_a N$.

Properties (7) and (8) are useful for solving *exponential and logarithmic equations*, a topic discussed in the next section.

4 Evaluate a Logarithm Whose Base Is Neither 10 Nor *e*

Logarithms with base 10—common logarithms—were used to facilitate arithmetic computations before the widespread use of calculators. (See the Historical Feature at the end of this section.) Natural logarithms—that is, logarithms whose base is the number e—remain very important because they arise frequently in the study of natural phenomena.

Common logarithms are usually abbreviated by writing **log**, with the base understood to be 10, just as natural logarithms are abbreviated by **ln**, with the base understood to be e.

Most calculators have both $\boxed{\text{log}}$ and $\boxed{\text{ln}}$ keys to calculate the common logarithm and the natural logarithm of a number, respectively. Let's look at an example to see how to approximate logarithms having a base other than 10 or e.

EXAMPLE 7 Approximating a Logarithm Whose Base Is Neither 10 Nor *e*

Approximate $\log_2 7$. Round the answer to four decimal places.

Solution Remember, $\log_2 7$ means "2 raised to what exponent equals 7." Let $y = \log_2 7$. Then $2^y = 7$. Because $2^2 = 4$ and $2^3 = 8$, the value of $\log_2 7$ is between 2 and 3.

$$
\begin{aligned}
2^y &= 7 \\
\ln 2^y &= \ln 7 && \text{Property (7)} \\
y \ln 2 &= \ln 7 && \text{Property (5)} \\
y &= \frac{\ln 7}{\ln 2} && \text{Exact value} \\
y &\approx 2.8074 && \text{Approximate value rounded to four decimal places}
\end{aligned}
$$

Example 7 shows how to approximate a logarithm whose base is 2 by changing to logarithms involving the base *e*. In general, the **Change-of-Base Formula** is used.

THEOREM

Change-of-Base Formula

If $a \neq 1$, $b \neq 1$, and M are positive real numbers, then

$$\log_a M = \frac{\log_b M}{\log_b a} \tag{9}$$

Proof Let $y = \log_a M$. Then

$$
\begin{aligned}
a^y &= M \\
\log_b a^y &= \log_b M && \text{Property (7)} \\
y \log_b a &= \log_b M && \text{Property (5)} \\
y &= \frac{\log_b M}{\log_b a} && \text{Solve for } y. \\
\log_a M &= \frac{\log_b M}{\log_b a} && y = \log_a M
\end{aligned}
$$

Technology Note

Some calculators have features for evaluating logarithms with bases other than 10 or *e*. For example, the TI-84 Plus C has the logBASE function (under Math > Math > A: logBASE). Consult the user's manual for your calculator.

Because most calculators have keys only for $\boxed{\log}$ and $\boxed{\ln}$, in practice, the Change-of-Base Formula uses either $b = 10$ or $b = e$. That is,

$$\log_a M = \frac{\log M}{\log a} \quad \text{and} \quad \log_a M = \frac{\ln M}{\ln a} \tag{10}$$

EXAMPLE 8 Using the Change-of-Base Formula

Approximate:

(a) $\log_5 89$ (b) $\log_{\sqrt{2}} \sqrt{5}$

Round answers to four decimal places.

Solution

(a)
$$\log_5 89 = \frac{\log 89}{\log 5} \approx \frac{1.949390007}{0.6989700043} \approx 2.7889$$

or

$$\log_5 89 = \frac{\ln 89}{\ln 5} \approx \frac{4.48863637}{1.609437912} \approx 2.7889$$

$$\text{(b)}\ \log_{\sqrt{2}} \sqrt{5} = \frac{\log \sqrt{5}}{\log \sqrt{2}} = \frac{\frac{1}{2}\log 5}{\frac{1}{2}\log 2} = \frac{\log 5}{\log 2} \approx 2.3219$$

or

$$\log_{\sqrt{2}} \sqrt{5} = \frac{\ln \sqrt{5}}{\ln \sqrt{2}} = \frac{\frac{1}{2}\ln 5}{\frac{1}{2}\ln 2} = \frac{\ln 5}{\ln 2} \approx 2.3219$$

Now Work PROBLEMS 23 AND 71

5 Graph a Logarithmic Function Whose Base Is Neither 10 Nor *e*

The Change-of-Base Formula also can be used to graph logarithmic functions whose bases are neither 10 nor *e*.

EXAMPLE 9 Graphing a Logarithmic Function Whose Base Is Neither 10 Nor *e*

Use a graphing utility to graph $y = \log_2 x$.

Solution Let's use the Change-of-Base Formula to express $y = \log_2 x$ in terms of logarithms with base 10 or base *e*. Graph either $y = \dfrac{\ln x}{\ln 2}$ or $y = \dfrac{\log x}{\log 2}$ to obtain the graph of $y = \log_2 x$. See Figure 46.

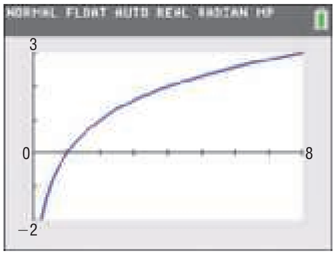


Figure 46 $y = \log_2 x$

✓**Check:** Verify that $y = \dfrac{\ln x}{\ln 2}$ and $y = \dfrac{\log x}{\log 2}$ result in the same graph by graphing each on the same screen. ■

Now Work PROBLEM 79

SUMMARY

Properties of Logarithms

In the list that follows, a, b, M, N, and r are real numbers. Also, $a > 0$, $a \neq 1$, $b > 0$, $b \neq 1$, $M > 0$, and $N > 0$.

Definition $y = \log_a x$ means $x = a^y$

Properties of logarithms

$\log_a 1 = 0$; $\log_a a = 1$

$a^{\log_a M} = M$; $\log_a a^r = r$

$\log_a (MN) = \log_a M + \log_a N$

$\log_a \left(\dfrac{M}{N}\right) = \log_a M - \log_a N$

$\log_a M^r = r \log_a M$

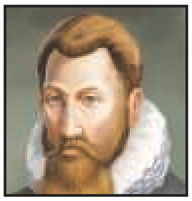
$a^r = e^{r \ln a}$

If $M = N$, then $\log_a M = \log_a N$.

If $\log_a M = \log_a N$, then $M = N$.

Change-of-Base Formula $\log_a M = \dfrac{\log_b M}{\log_b a}$

Historical Feature



John Napier (1550–1617)

Logarithms were invented about 1590 by John Napier (1550–1617) and Joost Bürgi (1552–1632), working independently. Napier, whose work had the greater influence, was a Scottish lord, a secretive man whose neighbors were inclined to believe him to be in league with the devil. His approach to logarithms was very different from ours; it was based on the relationship between arithmetic and geometric sequences—discussed in a later chapter—and not on the inverse function relationship of logarithms to exponential functions (described in Section 6.4). Napier's tables, published in 1614, listed what would now be called *natural logarithms* of sines and were rather difficult to use. A London professor, Henry Briggs, became interested in the tables and visited Napier. In their conversations, they developed the idea of common logarithms, which were published in 1617. Their importance for calculation was immediately recognized, and by 1650 they were being printed as far away as China. They remained an important calculation tool until the advent of the inexpensive handheld calculator about 1972, which has decreased their calculation—but not their theoretical—importance.

A side effect of the invention of logarithms was the popularization of the decimal system of notation for real numbers.

6.5 Assess Your Understanding

Concepts and Vocabulary

1. $\log_a 1 =$ _____

2. $a^{\log_a M} =$ _____

3. $\log_a a^r =$ _____

4. $\log_a (MN) =$ _________ + _________

5. $\log_a\left(\dfrac{M}{N}\right) =$ _________ − _________

6. $\log_a M^r =$ _________

7. If $\log_8 M = \dfrac{\log_5 7}{\log_5 8}$, then $M =$ _____.

8. ***True or False*** $\ln(x+3) - \ln(2x) = \dfrac{\ln(x+3)}{\ln(2x)}$

9. ***True or False*** $\log_2(3x^4) = 4\log_2(3x)$

10. ***True or False*** $\log\left(\dfrac{2}{3}\right) = \dfrac{\log 2}{\log 3}$

11. Choose the expression equivalent to 2^x.
(a) e^{2x} (b) $e^{x\ln 2}$ (c) $e^{\log_2 x}$ (d) $e^{2\ln x}$

12. Writing $\log_a x - \log_a y + 2\log_a z$ as a single logarithm results in which of the following?
(a) $\log_a(x - y + 2z)$ (b) $\log_a\left(\dfrac{xz^2}{y}\right)$
(c) $\log_a\left(\dfrac{2xz}{y}\right)$ (d) $\log_a\left(\dfrac{x}{yz^2}\right)$

Skill Building

In Problems 13–28, use properties of logarithms to find the exact value of each expression. Do not use a calculator.

13. $\log_3 3^{71}$ **14.** $\log_2 2^{-13}$ **15.** $\ln e^{-4}$ **16.** $\ln e^{\sqrt{2}}$

17. $2^{\log_2 7}$ **18.** $e^{\ln 8}$ **19.** $\log_8 2 + \log_8 4$ **20.** $\log_6 9 + \log_6 4$

21. $\log_6 18 - \log_6 3$ **22.** $\log_8 16 - \log_8 2$ **23.** $\log_2 6 \cdot \log_6 8$ **24.** $\log_3 8 \cdot \log_8 9$

25. $3^{\log_3 5 - \log_3 4}$ **26.** $5^{\log_5 6 + \log_5 7}$ **27.** $e^{\log_{e^2} 16}$ **28.** $e^{\log_{e^2} 9}$

In Problems 29–36, suppose that $\ln 2 = a$ *and* $\ln 3 = b$. *Use properties of logarithms to write each logarithm in terms of a and b.*

29. $\ln 6$ **30.** $\ln\dfrac{2}{3}$ **31.** $\ln 1.5$ **32.** $\ln 0.5$

33. $\ln 8$ **34.** $\ln 27$ **35.** $\ln\sqrt[5]{6}$ **36.** $\ln\sqrt[4]{\dfrac{2}{3}}$

In Problems 37–56, write each expression as a sum and/or difference of logarithms. Express powers as factors.

37. $\log_5(25x)$ **38.** $\log_3\dfrac{x}{9}$ **39.** $\log_2 z^3$ **40.** $\log_7 x^5$

41. $\ln(ex)$ **42.** $\ln\dfrac{e}{x}$ **43.** $\ln\dfrac{x}{e^x}$ **44.** $\ln(xe^x)$

45. $\log_a(u^2v^3)$ $\quad u > 0, v > 0$ **46.** $\log_2\left(\dfrac{a}{b^2}\right)$ $\quad a > 0, b > 0$ **47.** $\ln(x^2\sqrt{1-x})$ $\quad 0 < x < 1$

48. $\ln(x\sqrt{1+x^2})$ $\quad x > 0$ **49.** $\log_2\left(\dfrac{x^3}{x-3}\right)$ $\quad x > 3$ **50.** $\log_5\left(\dfrac{\sqrt[3]{x^2+1}}{x^2-1}\right)$ $\quad x > 1$

51. $\log\left[\dfrac{x(x+2)}{(x+3)^2}\right]$ $\quad x > 0$ **52.** $\log\left[\dfrac{x^3\sqrt{x+1}}{(x-2)^2}\right]$ $\quad x > 2$ **53.** $\ln\left[\dfrac{x^2-x-2}{(x+4)^2}\right]^{1/3}$ $\quad x > 2$

54. $\ln\left[\dfrac{(x-4)^2}{x^2-1}\right]^{2/3}$ $\quad x > 4$ **55.** $\ln\dfrac{5x\sqrt{1+3x}}{(x-4)^3}$ $\quad x > 4$ **56.** $\ln\left[\dfrac{5x^2\sqrt[3]{1-x}}{4(x+1)^2}\right]$ $\quad 0 < x < 1$

In Problems 57–70, write each expression as a single logarithm.

57. $3\log_5 u + 4\log_5 v$ **58.** $2\log_3 u - \log_3 v$ **59.** $\log_3\sqrt{x} - \log_3 x^3$

60. $\log_2\left(\dfrac{1}{x}\right) + \log_2\left(\dfrac{1}{x^2}\right)$ **61.** $\log_4(x^2-1) - 5\log_4(x+1)$ **62.** $\log(x^2+3x+2) - 2\log(x+1)$

63. $\ln\left(\dfrac{x}{x-1}\right) + \ln\left(\dfrac{x+1}{x}\right) - \ln(x^2-1)$ **64.** $\log\left(\dfrac{x^2+2x-3}{x^2-4}\right) - \log\left(\dfrac{x^2+7x+6}{x+2}\right)$ **65.** $8\log_2\sqrt{3x-2} - \log_2\left(\dfrac{4}{x}\right) + \log_2 4$

66. $21\log_3\sqrt[3]{x} + \log_3(9x^2) - \log_3 9$ **67.** $2\log_a(5x^3) - \dfrac{1}{2}\log_a(2x+3)$ **68.** $\dfrac{1}{3}\log(x^3+1) + \dfrac{1}{2}\log(x^2+1)$

69. $2\log_2(x+1) - \log_2(x+3) - \log_2(x-1)$ **70.** $3\log_5(3x+1) - 2\log_5(2x-1) - \log_5 x$

In Problems 71–78, use the Change-of-Base Formula and a calculator to evaluate each logarithm. Round your answer to three decimal places.

71. $\log_3 21$ **72.** $\log_5 18$ **73.** $\log_{1/3} 71$ **74.** $\log_{1/2} 15$

75. $\log_{\sqrt{2}} 7$ **76.** $\log_{\sqrt{5}} 8$ **77.** $\log_{\pi} e$ **78.** $\log_{\pi} \sqrt{2}$

In Problems 79–84, graph each function using a graphing utility and the Change-of-Base Formula.

79. $y = \log_4 x$ **80.** $y = \log_5 x$ **81.** $y = \log_2(x + 2)$

82. $y = \log_4(x - 3)$ **83.** $y = \log_{x-1}(x + 1)$ **84.** $y = \log_{x+2}(x - 2)$

Mixed Practice

85. If $f(x) = \ln x$, $g(x) = e^x$, and $h(x) = x^2$, find:
(a) $(f \circ g)(x)$. What is the domain of $f \circ g$?
(b) $(g \circ f)(x)$. What is the domain of $g \circ f$?
(c) $(f \circ g)(5)$
(d) $(f \circ h)(x)$. What is the domain of $f \circ h$?
(e) $(f \circ h)(e)$

86. If $f(x) = \log_2 x$, $g(x) = 2^x$, and $h(x) = 4x$, find:
(a) $(f \circ g)(x)$. What is the domain of $f \circ g$?
(b) $(g \circ f)(x)$. What is the domain of $g \circ f$?
(c) $(f \circ g)(3)$
(d) $(f \circ h)(x)$. What is the domain of $f \circ h$?
(e) $(f \circ h)(8)$

Applications and Extensions

In Problems 87–96, express y as a function of x. The constant C is a positive number.

87. $\ln y = \ln x + \ln C$

88. $\ln y = \ln(x + C)$

89. $\ln y = \ln x + \ln(x + 1) + \ln C$

90. $\ln y = 2 \ln x - \ln(x + 1) + \ln C$

91. $\ln y = 3x + \ln C$

92. $\ln y = -2x + \ln C$

93. $\ln(y - 3) = -4x + \ln C$

94. $\ln(y + 4) = 5x + \ln C$

95. $3 \ln y = \frac{1}{2}\ln(2x + 1) - \frac{1}{3}\ln(x + 4) + \ln C$

96. $2 \ln y = -\frac{1}{2}\ln x + \frac{1}{3}\ln(x^2 + 1) + \ln C$

97. Find the value of $\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdot \log_5 6 \cdot \log_6 7 \cdot \log_7 8$.

98. Find the value of $\log_2 4 \cdot \log_4 6 \cdot \log_6 8$.

99. Find the value of $\log_2 3 \cdot \log_3 4 \cdot \cdots \cdot \log_n(n + 1) \cdot \log_{n+1} 2$.

100. Find the value of $\log_2 2 \cdot \log_2 4 \cdot \cdots \cdot \log_2 2^n$.

101. Show that $\log_a(x + \sqrt{x^2 - 1}) + \log_a(x - \sqrt{x^2 - 1}) = 0$.

102. Show that $\log_a(\sqrt{x} + \sqrt{x - 1}) + \log_a(\sqrt{x} - \sqrt{x - 1}) = 0$.

103. Show that $\ln(1 + e^{2x}) = 2x + \ln(1 + e^{-2x})$.

104. Difference Quotient If $f(x) = \log_a x$, show that $\dfrac{f(x + h) - f(x)}{h} = \log_a\left(1 + \dfrac{h}{x}\right)^{1/h}$, $h \neq 0$.

105. If $f(x) = \log_a x$, show that $-f(x) = \log_{1/a} x$.

106. If $f(x) = \log_a x$, show that $f(AB) = f(A) + f(B)$.

107. If $f(x) = \log_a x$, show that $f\left(\dfrac{1}{x}\right) = -f(x)$.

108. If $f(x) = \log_a x$, show that $f(x^\alpha) = \alpha f(x)$.

109. Show that $\log_a\left(\dfrac{M}{N}\right) = \log_a M - \log_a N$, where a, M, and N are positive real numbers and $a \neq 1$.

110. Show that $\log_a\left(\dfrac{1}{N}\right) = -\log_a N$, where a and N are positive real numbers and $a \neq 1$.

Explaining Concepts: Discussion and Writing

111. Graph $Y_1 = \log(x^2)$ and $Y_2 = 2\log(x)$ using a graphing utility. Are they equivalent? What might account for any differences in the two functions?

112. Write an example that illustrates why $(\log_a x)^r \neq r \log_a x$.

113. Write an example that illustrates why

$$\log_2(x + y) \neq \log_2 x + \log_2 y.$$

114. Does $3^{\log_3(-5)} = -5$? Why or why not?

Retain Your Knowledge

Problems 115–118 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

115. Use a graphing utility to solve $x^3 - 3x^2 - 4x + 8 = 0$. Round answers to two decimal places.

116. Without solving, determine the character of the solution of the quadratic equation $4x^2 - 28x + 49 = 0$ in the complex number system.

117. Find the real zeros of

$$f(x) = 5x^5 + 44x^4 + 116x^3 + 95x^2 - 4x - 4$$

118. Graph $f(x) = \sqrt{2 - x}$ using the techniques of shifting, compressing or stretching, and reflecting. State the domain and the range of f.

6.6 Logarithmic and Exponential Equations

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Solve Equations Using a Graphing Utility (Section 1.2, pp. 100–101)
- Solve Quadratic Equations (Section 1.3, pp. 110–115)
- Solve Equations Quadratic in Form (Section 1.5, pp. 131–133)

Now Work the **'Are You Prepared?'** problems on page 472.

OBJECTIVES
1 Solve Logarithmic Equations (p. 467)
2 Solve Exponential Equations (p. 469)
3 Solve Logarithmic and Exponential Equations Using a Graphing Utility (p. 471)

1 Solve Logarithmic Equations

In Section 6.4 we solved logarithmic equations by changing a logarithmic equation to an exponential equation. That is, we used the definition of a logarithm:

$$y = \log_a x \quad \text{is equivalent to} \quad x = a^y \qquad a > 0, a \neq 1$$

For example, to solve the equation $\log_2(1 - 2x) = 3$, use the equivalent exponential equation $1 - 2x = 2^3$ and solve for x.

$$\begin{aligned} \log_2(1 - 2x) &= 3 \\ 1 - 2x &= 2^3 && \text{Change to exponential form.} \\ -2x &= 7 && \text{Simplify} \\ x &= -\frac{7}{2} && \text{Divide both sides by } -2. \end{aligned}$$

You should check this solution for yourself.

For most logarithmic equations, some manipulation of the equation (usually using properties of logarithms) is required to obtain a solution. Also, to avoid extraneous solutions with logarithmic equations, determine the domain of the variable first.

Our practice will be to solve equations, whenever possible, by finding exact solutions using algebraic methods and exact or approximate solutions using a graphing utility. When algebraic methods cannot be used, approximate solutions will be obtained using a graphing utility. The reader is encouraged to pay particular attention to the form of equations for which exact solutions are possible.

Let's begin with an example of a logarithmic equation that requires using the fact that a logarithmic function is a one-to-one function.

$$\text{If } \log_a M = \log_a N, \text{ then } M = N \qquad M, N, \text{ and } a \text{ are positive and } a \neq 1$$

EXAMPLE 1 **Solving a Logarithmic Equation**

Solve: $2 \log_5 x = \log_5 9$

Algebraic Solution

The domain of the variable in this equation is $x > 0$. Note that each logarithm is to the same base, 5. Find the exact solution as follows:

$$
\begin{aligned}
2\log_5 x &= \log_5 9 \\
\log_5 x^2 &= \log_5 9 && \log_a M^r = r\log_a M \\
x^2 &= 9 && \text{If } \log_a M = \log_a N, \text{ then } M = N. \\
x = 3 \quad &\text{or} \quad \cancel{x = -3} && \text{The domain of the variable is } x > 0. \text{ Therefore, } -3 \text{ is extraneous and must be discarded.}
\end{aligned}
$$

Check:

$$
\begin{aligned}
2\log_5 3 &\stackrel{?}{=} \log_5 9 \\
\log_5 3^2 &\stackrel{?}{=} \log_5 9 && r\log_a M = \log_a M^r \\
\log_5 9 &= \log_5 9
\end{aligned}
$$

The solution set is $\{3\}$.

Graphing Solution

To solve the equation using a graphing utility, graph $Y_1 = 2\log_5 x = \frac{2\log x}{\log 5}$ and $Y_2 = \log_5 9 = \frac{\log 9}{\log 5}$, and determine the point of intersection. See Figure 47. The point of intersection is $(3, 1.3652124)$; so $x = 3$ is the only solution. The solution set is $\{3\}$.

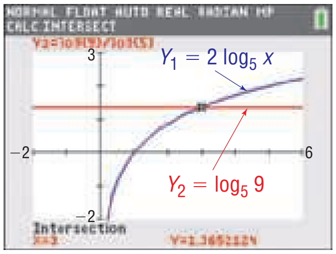


Figure 47

Now Work PROBLEM 13

EXAMPLE 2 Solving a Logarithmic Equation

Solve: $\log_5(x+6) + \log_5(x+2) = 1$

Algebraic Solution

The domain of the variable requires that $x + 6 > 0$ and $x + 2 > 0$, so $x > -6$ and $x > -2$. This means any solution must satisfy $x > -2$. To obtain an exact solution, first express the left side as a single logarithm. Then change the equation to exponential form.

$$
\begin{aligned}
\log_5(x+6) + \log_5(x+2) &= 1 \\
\log_5[(x+6)(x+2)] &= 1 && \log_a M + \log_a N = \log_a(MN) \\
(x+6)(x+2) &= 5^1 = 5 && \text{Change to exponential form.} \\
x^2 + 8x + 12 &= 5 && \text{Multiply out.} \\
x^2 + 8x + 7 &= 0 && \text{Place the quadratic equation in standard form.} \\
(x+7)(x+1) &= 0 && \text{Factor.} \\
x = -7 \quad \text{or} \quad x &= -1 && \text{Zero-Product Property}
\end{aligned}
$$

Only $x = -1$ satisfies the restriction that $x > -2$, so $x = -7$ is extraneous. The solution set is $\{-1\}$, which you should check.

Graphing Solution

Graph $Y_1 = \log_5(x+6) + \log_5(x+2) = \frac{\log(x+6)}{\log 5} + \frac{\log(x+2)}{\log 5}$ and $Y_2 = 1$, and determine the point(s) of intersection. See Figure 48. The point of intersection is $(-1, 1)$, so $x = -1$ is the only solution. The solution set is $\{-1\}$.

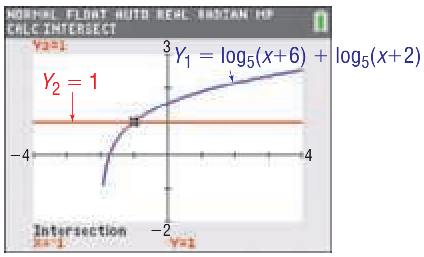


Figure 48

WARNING A negative solution is not automatically extraneous. You must determine whether the potential solution causes the argument of any logarithmic expression in the equation to be negative or 0.

Now Work PROBLEM 21

EXAMPLE 3 Solving a Logarithmic Equation

Solve: $\ln x + \ln(x - 4) = \ln(x + 6)$

Algebraic Solution

The domain of the variable requires that $x > 0$, $x - 4 > 0$, and $x + 6 > 0$. As a result, the domain of the variable is $x > 4$. Begin the solution using the log of a product property.

$$
\begin{aligned}
\ln x + \ln(x-4) &= \ln(x+6) & & \\
\ln[x(x-4)] &= \ln(x+6) & & \ln M + \ln N = \ln(MN) \\
x(x-4) &= x+6 & & \text{If } \ln M = \ln N \text{, then } M = N. \\
x^2 - 4x &= x + 6 & & \text{Multiply out.} \\
x^2 - 5x - 6 &= 0 & & \text{Place the quadratic equation in standard form.} \\
(x-6)(x+1) &= 0 & & \text{Factor.} \\
x = 6 \quad \text{or} \quad x &= -1 & & \text{Zero-Product Property}
\end{aligned}
$$

Because the domain of the variable is $x > 4$, discard -1 as extraneous. The solution set is $\{6\}$, which you should check. ■

Graphing Solution

Graph $Y_1 = \ln x + \ln(x - 4)$ and $Y_2 = \ln(x + 6)$, and determine the point(s) of intersection. See Figure 49. The x-coordinate of the point of intersection is 6, so the solution set is $\{6\}$.

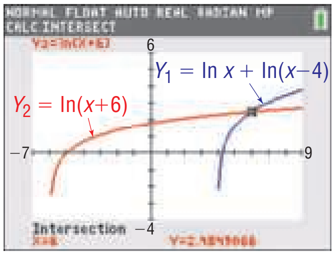


Figure 49 ■

WARNING In using properties of logarithms to solve logarithmic equations, avoid using the property $\log_a x^r = r \log_a x$, when r is even. The reason can be seen in this example:

Solve: $\log_3 x^2 = 4$

Solution: The domain of the variable x is all real numbers except 0.

(a)
$$
\begin{aligned}
\log_3 x^2 &= 4 & & \\
x^2 &= 3^4 = 81 & & \text{Change to exponential form.} \\
x &= -9 \text{ or } x = 9 & &
\end{aligned}
$$

(b)
$$
\begin{aligned}
\log_3 x^2 &= 4 & & \log_a x^r = r \log_a x \\
2 \log_3 x &= 4 & & \text{Domain of variable is } x > 0. \\
\log_3 x &= 2 & & \\
x &= 9 & &
\end{aligned}
$$

Both -9 and 9 are solutions of $\log_3 x^2 = 4$ (as you can verify). The process in part (b) does not find the solution -9 because the domain of the variable was further restricted to $x > 0$ due to the application of the property $\log_a x^r = r \log_a x$. ■

Now Work PROBLEM 31

2 Solve Exponential Equations

In Sections 6.3 and 6.4, we solved exponential equations algebraically by expressing each side of the equation using the same base. That is, we used the one-to-one property of the exponential function:

$$\text{If } a^u = a^v \text{, then } u = v \qquad a > 0, a \neq 1$$

For example, to solve the exponential equation $4^{2x+1} = 16$, notice that $16 = 4^2$ and apply the property above to obtain the equation $2x + 1 = 2$, from which we find $x = \frac{1}{2}$.

Not all exponential equations can be readily expressed so that each side of the equation has the same base. For such equations, algebraic techniques often can be used to obtain exact solutions. When algebraic techniques cannot be used, a graphing utility can be used to obtain approximate solutions. You should pay particular attention to the form of equations for which exact solutions are obtained.

EXAMPLE 4 Solving an Exponential Equation

Solve: $2^x = 5$

Algebraic Solution

Because 5 cannot be written as an integer power of 2 ($2^2 = 4$ and $2^3 = 8$), write the exponential equation as the equivalent logarithmic equation.

$$2^x = 5$$
$$x = \log_2 5 = \frac{\ln 5}{\ln 2}$$

Change-of-Base Formula (10), Section 6.5

Alternatively, the equation $2^x = 5$ can be solved by taking the natural logarithm (or common logarithm) of each side.

$$2^x = 5$$
$$\ln 2^x = \ln 5 \qquad \textbf{If } M = N, \textbf{ then } \ln M = \ln N.$$
$$x \ln 2 = \ln 5 \qquad \ln M^r = r \ln M$$
$$x = \frac{\ln 5}{\ln 2} \qquad \text{Exact solution}$$
$$\approx 2.322 \qquad \text{Approximate solution}$$

The solution set is $\left\{\frac{\ln 5}{\ln 2}\right\}$.

Graphing Solution

Graph $Y_1 = 2^x$ and $Y_2 = 5$, and determine the x-coordinate of the point of intersection. See Figure 50.

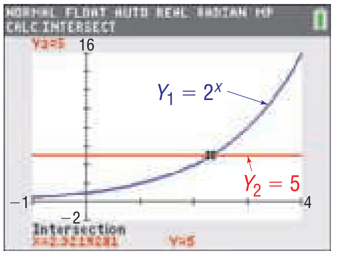


Figure 50

The approximate solution, rounded to three decimal places, is 2.322.

Now Work PROBLEM 43

EXAMPLE 5 Solving an Exponential Equation

Solve: $8 \cdot 3^x = 5$

Algebraic Solution

Isolate the exponential expression and then rewrite the statement as an equivalent logarithm.

$$8 \cdot 3^x = 5$$
$$3^x = \frac{5}{8} \qquad \text{Solve for } 3^x.$$
$$x = \log_3\left(\frac{5}{8}\right) = \frac{\ln\left(\frac{5}{8}\right)}{\ln 3} \qquad \text{Exact solution}$$
$$\approx -0.428 \qquad \text{Approximate solution}$$

The solution set is $\left\{\log_3\left(\frac{5}{8}\right)\right\}$.

Graphing Solution

Graph $Y_1 = 8 \cdot 3^x$ and $Y_2 = 5$, and determine the x-coordinate of the point of intersection. See Figure 51.

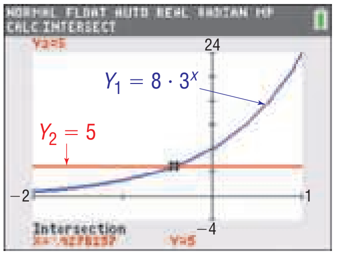


Figure 51

The approximate solution, rounded to three decimal places, is −0.428.

EXAMPLE 6 Solving an Exponential Equation

Solve: $5^{x-2} = 3^{3x+2}$

Algebraic Solution

Because the bases are different, first apply property (7), Section 6.5 (take the natural logarithm of each side), and then use appropriate properties of logarithms. The result is a linear equation in x that can be solved.

Graphing Solution

Graph $Y_1 = 5^{x-2}$ and $Y_2 = 3^{3x+2}$, and determine the x-coordinate of the point of intersection. See Figure 52.

$$5^{x-2} = 3^{3x+2}$$

$$\ln 5^{x-2} = \ln 3^{3x+2} \qquad \textbf{If } M = N, \ln M = \ln N.$$

$$(x-2)\ln 5 = (3x+2)\ln 3 \qquad \ln M^r = r\ln M$$

$$(\ln 5)x - 2\ln 5 = (3\ln 3)x + 2\ln 3 \qquad \text{Distribute. The equation is now linear in } x.$$

$$(\ln 5)x - (3\ln 3)x = 2\ln 3 + 2\ln 5 \qquad \text{Place terms involving } x \text{ on the left.}$$

$$(\ln 5 - 3\ln 3)x = 2(\ln 3 + \ln 5) \qquad \text{Factor.}$$

$$x = \frac{2(\ln 3 + \ln 5)}{\ln 5 - 3\ln 3} \qquad \text{Exact solution}$$

$$\approx -3.212 \qquad \text{Approximate solution}$$

The solution set is $\left\{\dfrac{2(\ln 3 + \ln 5)}{\ln 5 - 3\ln 3}\right\}$.

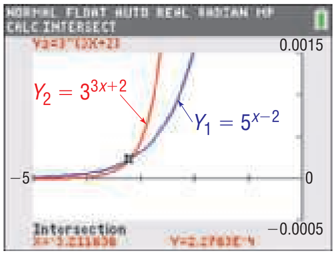


Figure 52

The approximate solution, rounded to three decimal places, is -3.212. Note that the y-coordinate, 2.2763E–4, is in scientific notation and means $2.2763 \times 10^{-4} = 0.0002763$.

Now Work PROBLEM 53

Note: Because of the properties of logarithms, exact solutions involving logarithms often can be expressed in multiple ways. For example, the solution to $5^{x-2} = 3^{3x+2}$ from Example 6 can be expressed equivalently as $\dfrac{2\ln 15}{\ln 5 - \ln 27}$ or as $\dfrac{\ln 225}{\ln(5/27)}$, among others. Do you see why?

The next example deals with an exponential equation that is quadratic in form.

EXAMPLE 7 Solving an Exponential Equation That Is Quadratic in Form

Solve: $4^x - 2^x - 12 = 0$

Algebraic Solution

Note that $4^x = (2^2)^x = 2^{2x} = (2^x)^2$, so the equation is quadratic in form and can be written as

$$(2^x)^2 - 2^x - 12 = 0 \qquad \text{Let } u = 2^x\text{; then } u^2 - u - 12 = 0.$$

Now we can factor as usual.

$$(2^x - 4)(2^x + 3) = 0 \qquad (u-4)(u+3) = 0$$

$$2^x - 4 = 0 \quad \text{or} \quad 2^x + 3 = 0 \qquad u - 4 = 0 \quad \text{or} \quad u + 3 = 0$$

$$2^x = 4 \qquad 2^x = -3 \qquad u = 2^x = 4 \quad \text{or} \quad u = 2^x = -3$$

The equation on the left has the solution $x = 2$, since $2^x = 4 = 2^2$; the equation on the right has no solution, since $2^x > 0$ for all x. The only solution is 2. The solution set is $\{2\}$.

Graphing Solution

Graph $Y_1 = 4^x - 2^x - 12$, and determine the x-intercept. See Figure 53. The x-intercept is 2, so the solution set is $\{2\}$.

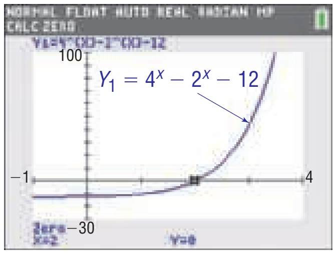


Figure 53

Now Work PROBLEM 61

3 Solve Logarithmic and Exponential Equations Using a Graphing Utility

The algebraic techniques introduced in this section to obtain exact solutions apply only to certain types of logarithmic and exponential equations. Solutions for other types are usually studied in calculus, using numerical methods. For such types, we can use a graphing utility to approximate the solution.

EXAMPLE 8 **Solving Equations Using a Graphing Utility**

Solve: $x + e^x = 2$

Express the solution(s) rounded to two decimal places.

Solution The solution is found by graphing $Y_1 = x + e^x$ and $Y_2 = 2$. Since Y_1 is an increasing function (do you know why?), there is only one point of intersection for Y_1 and Y_2. Figure 54 shows the graphs of Y_1 and Y_2. Using the INTERSECT command reveals that the solution is 0.44, rounded to two decimal places. ■

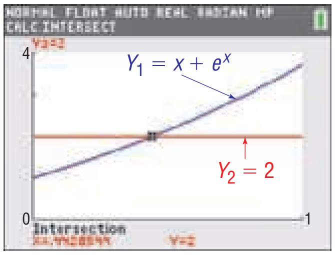


Figure 54

Now Work PROBLEM 71

6.6 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Solve $x^2 - 7x - 30 = 0$. (pp. 110–115)

2. Solve $(x+3)^2 - 4(x+3) + 3 = 0$. (pp. 131–133)

3. Approximate the solution(s) to $x^3 = x^2 - 5$ using a graphing utility. (pp. 100–101)

4. Approximate the solution(s) to $x^3 - 2x + 2 = 0$ using a graphing utility. (pp. 100–101)

Skill Building

In Problems 5–40, solve each logarithmic equation. Express irrational solutions in exact form and as a decimal rounded to three decimal places.

5. $\log_4 x = 2$ **6.** $\log(x+6) = 1$ **7.** $\log_2(5x) = 4$

8. $\log_3(3x-1) = 2$ **9.** $\log_4(x+2) = \log_4 8$ **10.** $\log_5(2x+3) = \log_5 3$

11. $\frac{1}{2}\log_3 x = 2\log_3 2$ **12.** $-2\log_4 x = \log_4 9$ **13.** $3\log_2 x = -\log_2 27$

14. $2\log_5 x = 3\log_5 4$ **15.** $3\log_2(x-1) + \log_2 4 = 5$ **16.** $2\log_3(x+4) - \log_3 9 = 2$

17. $\log x + \log(x+15) = 2$ **18.** $\log x + \log(x-21) = 2$ **19.** $\log(2x+1) = 1 + \log(x-2)$

20. $\log(2x) - \log(x-3) = 1$ **21.** $\log_2(x+7) + \log_2(x+8) = 1$ **22.** $\log_6(x+4) + \log_6(x+3) = 1$

23. $\log_8(x+6) = 1 - \log_8(x+4)$ **24.** $\log_5(x+3) = 1 - \log_5(x-1)$ **25.** $\ln x + \ln(x+2) = 4$

26. $\ln(x+1) - \ln x = 2$ **27.** $\log_3(x+1) + \log_3(x+4) = 2$ **28.** $\log_2(x+1) + \log_2(x+7) = 3$

29. $\log_{1/3}(x^2+x) - \log_{1/3}(x^2-x) = -1$ **30.** $\log_4(x^2-9) - \log_4(x+3) = 3$

31. $\log_a(x-1) - \log_a(x+6) = \log_a(x-2) - \log_a(x+3)$ **32.** $\log_a x + \log_a(x-2) = \log_a(x+4)$

33. $2\log_5(x-3) - \log_5 8 = \log_5 2$ **34.** $\log_3 x - 2\log_3 5 = \log_3(x+1) - 2\log_3 10$

35. $2\log_6(x+2) = 3\log_6 2 + \log_6 4$ **36.** $3(\log_7 x - \log_7 2) = 2\log_7 4$

37. $2\log_{13}(x+2) = \log_{13}(4x+7)$ **38.** $\log(x-1) = \frac{1}{3}\log 2$

39. $(\log_3 x)^2 - 5(\log_3 x) = 6$ **40.** $\ln x - 3\sqrt{\ln x} + 2 = 0$

In Problems 41–68, solve each exponential equation. Express irrational solutions in exact form and as a decimal rounded to three decimal places.

41. $2^{x-5} = 8$ **42.** $5^{-x} = 25$ **43.** $2^x = 10$ **44.** $3^x = 14$

45. $8^{-x} = 1.2$ **46.** $2^{-x} = 1.5$ **47.** $5(2^{3x}) = 8$ **48.** $0.3(4^{0.2x}) = 0.2$

49. $3^{1-2x} = 4^x$ **50.** $2^{x+1} = 5^{1-2x}$ **51.** $\left(\frac{3}{5}\right)^x = 7^{1-x}$ **52.** $\left(\frac{4}{3}\right)^{1-x} = 5^x$

53. $1.2^x = (0.5)^{-x}$ **54.** $0.3^{1+x} = 1.7^{2x-1}$ **55.** $\pi^{1-x} = e^x$ **56.** $e^{x+3} = \pi^x$

57. $2^{2x} + 2^x - 12 = 0$

58. $3^{2x} + 3^x - 2 = 0$

59. $3^{2x} + 3^{x+1} - 4 = 0$

60. $2^{2x} + 2^{x+2} - 12 = 0$

61. $16^x + 4^{x+1} - 3 = 0$

62. $9^x - 3^{x-1} + 1 = 0$

63. $25^x - 8 \cdot 5^x = -16$

64. $36^x - 6 \cdot 6^x = -9$

65. $3 \cdot 4^x + 4 \cdot 2^x + 8 = 0$

66. $2 \cdot 49^x + 11 \cdot 7^x + 5 = 0$

67. $4^x - 10 \cdot 4^{-x} = 3$

68. $3^x - 14 \cdot 3^{-x} = 5$

In Problems 69–82, use a graphing utility to solve each equation. Express your answer rounded to two decimal places.

69. $\log_5(x+1) - \log_4(x-2) = 1$

70. $\log_2(x-1) - \log_6(x+2) = 2$

71. $e^x = -x$

72. $e^{2x} = x + 2$

73. $e^x = x^2$

74. $e^x = x^3$

75. $\ln x = -x$

76. $\ln(2x) = -x + 2$

77. $\ln x = x^3 - 1$

78. $\ln x = -x^2$

79. $e^x + \ln x = 4$

80. $e^x - \ln x = 4$

81. $e^{-x} = \ln x$

82. $e^{-x} = -\ln x$

Mixed Practice

In Problems 83–94, solve each equation. Express irrational solutions in exact form and as a decimal rounded to three decimal places.

83. $\log_2(x+1) - \log_4 x = 1$
[**Hint:** Change $\log_4 x$ to base 2.]

84. $\log_2(3x+2) - \log_4 x = 3$

85. $\log_{16} x + \log_4 x + \log_2 x = 7$

86. $\log_9 x + 3\log_3 x = 14$

87. $(\sqrt[3]{2})^{2-x} = 2^{x^2}$

88. $\log_2 x^{\log_2 x} = 4$

89. $\dfrac{e^x + e^{-x}}{2} = 1$
[**Hint:** Multiply each side by e^x.]

90. $\dfrac{e^x + e^{-x}}{2} = 3$

91. $\dfrac{e^x - e^{-x}}{2} = 2$

92. $\dfrac{e^x - e^{-x}}{2} = -2$

93. $\log_5 x + \log_3 x = 1$
[**Hint:** Use the Change-of-Base Formula.]

94. $\log_2 x - \log_6 x = 3$

---

95. $f(x) = \log_2(x+3)$ and $g(x) = \log_2(3x+1)$.
(a) Solve $f(x) = 3$. What point is on the graph of f?
(b) Solve $g(x) = 4$. What point is on the graph of g?
(c) Solve $f(x) = g(x)$. Do the graphs of f and g intersect? If so, where?
(d) Solve $(f+g)(x) = 7$.
(e) Solve $(f-g)(x) = 2$.

96. $f(x) = \log_3(x+5)$ and $g(x) = \log_3(x-1)$.
(a) Solve $f(x) = 2$. What point is on the graph of f?
(b) Solve $g(x) = 3$. What point is on the graph of g?
(c) Solve $f(x) = g(x)$. Do the graphs of f and g intersect? If so, where?
(d) Solve $(f+g)(x) = 3$.
(e) Solve $(f-g)(x) = 2$.

97. (a) If $f(x) = 3^{x+1}$ and $g(x) = 2^{x+2}$, graph f and g on the same Cartesian plane.
(b) Find the point(s) of intersection of the graphs of f and g by solving $f(x) = g(x)$. Round answers to three decimal places. Label any intersection points on the graph drawn in part (a).
(c) Based on the graph, solve $f(x) > g(x)$.

98. (a) If $f(x) = 5^{x-1}$ and $g(x) = 2^{x+1}$, graph f and g on the same Cartesian plane.
(b) Find the point(s) of intersection of the graphs of f and g by solving $f(x) = g(x)$. Label any intersection points on the graph drawn in part (a).
(c) Based on the graph, solve $f(x) > g(x)$.

99. (a) Graph $f(x) = 3^x$ and $g(x) = 10$ on the same Cartesian plane.
(b) Shade the region bounded by the y-axis, $f(x) = 3^x$, and $g(x) = 10$ on the graph drawn in part (a).
(c) Solve $f(x) = g(x)$ and label the point of intersection on the graph drawn in part (a).

100. (a) Graph $f(x) = 2^x$ and $g(x) = 12$ on the same Cartesian plane.
(b) Shade the region bounded by the y-axis, $f(x) = 2^x$, and $g(x) = 12$ on the graph drawn in part (a).
(c) Solve $f(x) = g(x)$ and label the point of intersection on the graph drawn in part (a).

101. (a) Graph $f(x) = 2^{x+1}$ and $g(x) = 2^{-x+2}$ on the same Cartesian plane.
(b) Shade the region bounded by the y-axis, $f(x) = 2^{x+1}$, and $g(x) = 2^{-x+2}$ on the graph drawn in part (a).
(c) Solve $f(x) = g(x)$ and label the point of intersection on the graph drawn in part (a).

102. (a) Graph $f(x) = 3^{-x+1}$ and $g(x) = 3^{x-2}$ on the same Cartesian plane.
(b) Shade the region bounded by the y-axis, $f(x) = 3^{-x+1}$, and $g(x) = 3^{x-2}$ on the graph drawn in part (a).
(c) Solve $f(x) = g(x)$ and label the point of intersection on the graph drawn in part (a).

103. (a) Graph $f(x) = 2^x - 4$.
(b) Find the zero of f.
(c) Based on the graph, solve $f(x) < 0$.

104. (a) Graph $g(x) = 3^x - 9$.
(b) Find the zero of g.
(c) Based on the graph, solve $g(x) > 0$.

Applications and Extensions

105. A Population Model The resident population of the United States in 2015 was 320 million people and was growing at a rate of 0.7% per year. Assuming that this growth rate continues, the model $P(t) = 320(1.007)^{t-2015}$ represents the population P (in millions of people) in year t.

(a) According to this model, when will the population of the United States be 400 million people?

(b) According to this model, when will the population of the United States be 435 million people?

Source: *U.S. Census Bureau*

106. A Population Model The population of the world in 2015 was 7.21 billion people and was growing at a rate of 1.1% per year. Assuming that this growth rate continues, the model $P(t) = 7.21(1.011)^{t-2015}$ represents the population P (in billions of people) in year t.

(a) According to this model, when will the population of the world be 9 billion people?

(b) According to this model, when will the population of the world be 12.5 billion people?

Source: *U.S. Census Bureau*

107. Depreciation The value V of a Chevy Cruze LS that is t years old can be modeled by $V(t) = 18{,}700(0.84)^t$.

(a) According to the model, when will the car be worth \$9000?

(b) According to the model, when will the car be worth \$6000?

(c) According to the model, when will the car be worth \$2000?

Source: *Kelley Blue Book*

108. Depreciation The value V of a Honda Civic SE that is t years old can be modeled by $V(t) = 18{,}955(0.905)^t$.

(a) According to the model, when will the car be worth \$16,000?

(b) According to the model, when will the car be worth \$10,000?

(c) According to the model, when will the car be worth \$7500?

Source: *Kelley Blue Book*

Explaining Concepts: Discussion and Writing

109. Fill in the reason for each step in the following two solutions.

Solve: $\log_3(x-1)^2 = 2$

Solution A

$\log_3(x-1)^2 = 2$

$(x-1)^2 = 3^2 = 9$______

$(x-1) = \pm 3$______

$x - 1 = -3$ or $x - 1 = 3$______

$x = -2$ or $x = 4$______

Solution B

$\log_3(x-1)^2 = 2$

$2\log_3(x-1) = 2$______

$\log_3(x-1) = 1$______

$x - 1 = 3^1 = 3$______

$x = 4$______

Both solutions given in Solution A check. Explain what caused the solution $x = -2$ to be lost in Solution B.

Retain Your Knowledge

Problems 110–113 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

110. Solve: $4x^3 + 3x^2 - 25x + 6 = 0$

111. Determine whether the function $\{(0,-4), (2,-2), (4,0), (6,2)\}$ is one-to-one.

112. For $f(x) = \dfrac{x}{x-2}$ and $g(x) = \dfrac{x+5}{x-3}$, find $f \circ g$. Then find the domain of $f \circ g$.

113. Find the domain of $f(x) = \sqrt{x+3} + \sqrt{x-1}$.

'Are You Prepared?' Answers

1. $\{-3, 10\}$ **2.** $\{-2, 0\}$ **3.** $\{-1.43\}$ **4.** $\{-1.77\}$

6.7 Financial Models

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Simple Interest (Section 1.6, pp. 139–140)

Now Work the **'Are You Prepared?'** problems on page 481.

OBJECTIVES

1 Determine the Future Value of a Lump Sum of Money (p. 475)
2 Calculate Effective Rates of Return (p. 478)
3 Determine the Present Value of a Lump Sum of Money (p. 479)
4 Determine the Rate of Interest or the Time Required to Double a Lump Sum of Money (p. 480)

1 Determine the Future Value of a Lump Sum of Money

Interest is money paid for the use of money. The total amount borrowed (whether by an individual from a bank in the form of a loan or by a bank from an individual in the form of a savings account) is called the **principal**. The **rate of interest**, expressed as a percent, is the amount charged for the use of the principal for a given period of time, usually on a yearly (that is, per annum) basis.

THEOREM

Simple Interest Formula

If a principal of P dollars is borrowed for a period of t years at a per annum interest rate r, expressed as a decimal, the interest I charged is

$$I = Prt \qquad (1)$$

Interest charged according to formula (1) is called **simple interest**.

In problems involving interest, the term **payment period** is defined as follows.

Annually:	Once per year	**Monthly:**	12 times per year
Semiannually:	Twice per year	**Daily:**	365 times per year*
Quarterly:	Four times per year		

When the interest due at the end of a payment period is added to the principal so that the interest computed at the end of the next payment period is based on this new principal amount (old principal + interest), the interest is said to have been **compounded**. **Compound interest** is interest paid on the principal and on previously earned interest.

EXAMPLE 1 **Computing Compound Interest**

A credit union pays interest of 2% per annum compounded quarterly on a certain savings plan. If \$1000 is deposited in such a plan and the interest is left to accumulate, how much is in the account after 1 year?

Solution Use the simple interest formula, $I = Prt$. The principal P is \$1000 and the rate of interest is $2\% = 0.02$. After the first quarter of a year, the time t is $\frac{1}{4}$ year, so the interest earned is

$$I = Prt = (\$1000)(0.02)\left(\frac{1}{4}\right) = \$5$$

The new principal is $P + I = \$1000 + \$5 = \$1005$. At the end of the second quarter, the interest on this principal is

$$I = (\$1005)(0.02)\left(\frac{1}{4}\right) = \$5.03$$

*Most banks use a 360-day "year." Why do you think they do?

At the end of the third quarter, the interest on the new principal of $\$1005 + \$5.03 = \$1010.03$ is

$$I = (\$1010.03)(0.02)\left(\frac{1}{4}\right) = \$5.05$$

Finally, after the fourth quarter, the interest is

$$I = (\$1015.08)(0.02)\left(\frac{1}{4}\right) = \$5.08$$

After 1 year the account contains $\$1015.08 + \$5.08 = \$1020.16$.

The pattern of the calculations performed in Example 1 leads to a general formula for compound interest. For this purpose, let P represent the principal to be invested at a per annum interest rate r that is compounded n times per year, so the time of each compounding period is $\frac{1}{n}$ years. (For computing purposes, r is expressed as a decimal.) The interest earned after each compounding period is given by formula (1).

$$\text{Interest} = \text{principal} \times \text{rate} \times \text{time} = P \cdot r \cdot \frac{1}{n} = P \cdot \left(\frac{r}{n}\right)$$

The amount A after one compounding period is

$$A = P + P \cdot \left(\frac{r}{n}\right) = P \cdot \left(1 + \frac{r}{n}\right)$$

After two compounding periods, the amount A, based on the new principal $P \cdot \left(1 + \frac{r}{n}\right)$, is

$$A = \underbrace{P\cdot\left(1 + \frac{r}{n}\right)}_{\text{New principal}} + \underbrace{P\cdot\left(1 + \frac{r}{n}\right)\left(\frac{r}{n}\right)}_{\text{Interest on new principal}} \underset{\substack{\uparrow \\ \text{Factor out } P\cdot\left(1 + \frac{r}{n}\right)}}{=} P\cdot\left(1 + \frac{r}{n}\right)\left(1 + \frac{r}{n}\right) = P\cdot\left(1 + \frac{r}{n}\right)^2$$

After three compounding periods, the amount A is

$$A = P \cdot \left(1 + \frac{r}{n}\right)^2 + P \cdot \left(1 + \frac{r}{n}\right)^2\left(\frac{r}{n}\right) = P \cdot \left(1 + \frac{r}{n}\right)^2 \cdot \left(1 + \frac{r}{n}\right) = P \cdot \left(1 + \frac{r}{n}\right)^3$$

Continuing this way, after n compounding periods (1 year), the amount A is

$$A = P \cdot \left(1 + \frac{r}{n}\right)^n$$

Because t years will contain $n \cdot t$ compounding periods, the amount after t years is

$$A = P \cdot \left(1 + \frac{r}{n}\right)^{nt}$$

THEOREM

Compound Interest Formula

The amount A after t years due to a principal P invested at an annual interest rate r, expressed as a decimal, compounded n times per year is

$$A = P \cdot \left(1 + \frac{r}{n}\right)^{nt} \qquad (2)$$

Exploration

To see the effects of compounding interest monthly on an initial deposit of \$1, graph $Y_1 = \left(1 + \frac{r}{12}\right)^{12x}$ with $r = 0.06$ and $r = 0.12$ for $0 \leq x \leq 30$. What is the future value of \$1 in 30 years when the interest rate per annum is $r = 0.06$ (6%)? What is the future value of \$1 in 30 years when the interest rate per annum is $r = 0.12$ (12%)? Does doubling the interest rate double the future value?

For example, to rework Example 1, use $P = \$1000$, $r = 0.02$, $n = 4$ (quarterly compounding), and $t = 1$ year to obtain

$$A = P\cdot\left(1 + \frac{r}{n}\right)^{nt} = 1000\left(1 + \frac{0.02}{4}\right)^{4\cdot 1} = \$1020.15^*$$

In equation (2), the amount A is typically referred to as the **future value** of the account, while P is called the **present value**.

Now Work PROBLEM 7

*The result shown here differs from Example 1 due to rounding.

EXAMPLE 2 **Comparing Investments Using Different Compounding Periods**

Investing \$1000 at an annual rate of 10% compounded annually, semiannually, quarterly, monthly, and daily will yield the following amounts after 1 year:

Annual compounding $(n = 1)$: $$A = P \cdot (1 + r) = (\$1000)(1 + 0.10) = \$1100.00$$

Semiannual compounding $(n = 2)$: $$A = P \cdot \left(1 + \frac{r}{2}\right)^2 = (\$1000)(1 + 0.05)^2 = \$1102.50$$

Quarterly compounding $(n = 4)$: $$A = P \cdot \left(1 + \frac{r}{4}\right)^4 = (\$1000)(1 + 0.025)^4 = \$1103.81$$

Monthly compounding $(n = 12)$: $$A = P \cdot \left(1 + \frac{r}{12}\right)^{12} = (\$1000)\left(1 + \frac{0.10}{12}\right)^{12} = \$1104.71$$

Daily compounding $(n = 365)$: $$A = P \cdot \left(1 + \frac{r}{365}\right)^{365} = (\$1000)\left(1 + \frac{0.10}{365}\right)^{365} = \$1105.16$$

From Example 2, note that the effect of compounding more frequently is that the amount after 1 year is higher: \$1000 compounded 4 times a year at 10% results in \$1103.81, \$1000 compounded 12 times a year at 10% results in \$1104.71, and \$1000 compounded 365 times a year at 10% results in \$1105.16. This leads to the following question: What would happen to the amount after 1 year if the number of times that the interest is compounded were increased without bound?

Let's find the answer. Suppose that P is the principal, r is the per annum interest rate, and n is the number of times that the interest is compounded each year. The amount after 1 year is

$$A = P \cdot \left(1 + \frac{r}{n}\right)^n$$

Rewrite this expression as follows:

$$A = P \cdot \left(1 + \frac{r}{n}\right)^n = P \cdot \left(1 + \frac{1}{\frac{n}{r}}\right)^n = P \cdot \left[\left(1 + \frac{1}{\frac{n}{r}}\right)^{n/r}\right]^r \underset{h = \frac{n}{r}}{=} P \cdot \left[\left(1 + \frac{1}{h}\right)^h\right]^r \qquad \textbf{(3)}$$

Now suppose that the number n of times that the interest is compounded per year gets larger and larger; that is, suppose that $n \to \infty$. Then $h = \frac{n}{r} \to \infty$, and the expression in brackets in equation (3) equals e. That is, $A \to Pe^r$.

Table 9 compares $\left(1 + \frac{r}{n}\right)^n$, for large values of n, to e^r for $r = 0.05$, $r = 0.10$, $r = 0.15$, and $r = 1$. The larger that n gets, the closer $\left(1 + \frac{r}{n}\right)^n$ gets to e^r. No matter how frequent the compounding, the amount after 1 year has the definite ceiling Pe^r.

Table 9

	$\left(1 + \frac{r}{n}\right)^n$			
	$n = 100$	$n = 1000$	$n = 10{,}000$	e^r
$r = 0.05$	1.0512580	1.0512698	1.051271	1.0512711
$r = 0.10$	1.1051157	1.1051654	1.1051704	1.1051709
$r = 0.15$	1.1617037	1.1618212	1.1618329	1.1618342
$r = 1$	2.7048138	2.7169239	2.7181459	2.7182818

When interest is compounded so that the amount after 1 year is Pe^r, the interest is said to be **compounded continuously**.

THEOREM

Continuous Compounding

The amount A after t years due to a principal P invested at an annual interest rate r compounded continuously is

$$A = Pe^{rt} \tag{4}$$

EXAMPLE 3 **Using Continuous Compounding**

The amount A that results from investing a principal P of \$1000 at an annual rate r of 10% compounded continuously for a time t of 1 year is

$$A = \$1000e^{0.10} = (\$1000)(1.10517) = \$1105.17$$

Now Work PROBLEM 13

2 Calculate Effective Rates of Return

Suppose that you have \$1000 and a bank offers to pay you 3% annual interest on a savings account with interest compounded monthly. What annual interest rate must be earned for you to have the same amount at the end of the year as you would have if the interest had been compounded annually (once per year)? To answer this question, first determine the value of the \$1000 in the account that earns 3% compounded monthly.

$$A = \$1000\left(1 + \frac{0.03}{12}\right)^{12} \quad \text{Use } A = P\left(1 + \frac{r}{n}\right)^n \text{ with } P = \$1000,\ r = 0.03,\ n = 12.$$
$$= \$1030.42$$

So the interest earned is \$30.42. Using $I = Prt$ with $t = 1$, $I = \$30.42$, and $P = \$1000$, the annual simple interest rate is $0.03042 = 3.042\%$. This interest rate is known as the *effective rate of interest*.

The **effective rate of interest** is the annual simple interest rate that would yield the same amount as compounding n times per year, or continuously, after 1 year.

THEOREM

Effective Rate of Interest

The effective rate of interest r_e of an investment earning an annual interest rate r is given by

$$\text{Compounding } n \text{ times per year:} \quad r_e = \left(1 + \frac{r}{n}\right)^n - 1$$

$$\text{Continuous compounding:} \quad r_e = e^r - 1$$

EXAMPLE 4 **Computing the Effective Rate of Interest—Which Is the Best Deal?**

Suppose you want to buy a 5-year certificate of deposit (CD). You visit three banks to determine their CD rates. American Express offers you 2.15% annual interest compounded monthly, and First Internet Bank offers you 2.20% compounded quarterly. Discover offers 2.12% compounded daily. Determine which bank is offering the best deal.

Solution The bank that offers the best deal is the one with the highest effective interest rate.

American Express

$$\begin{aligned} r_e &= \left(1 + \frac{0.0215}{12}\right)^{12} - 1 \\ &\approx 1.02171 - 1 \\ &= 0.02171 \\ &= 2.171\% \end{aligned}$$

First Internet Bank

$$\begin{aligned} r_e &= \left(1 + \frac{0.022}{4}\right)^{4} - 1 \\ &\approx 1.02218 - 1 \\ &= 0.02218 \\ &= 2.218\% \end{aligned}$$

Discover

$$\begin{aligned} r_e &= \left(1 + \frac{0.0212}{365}\right)^{365} - 1 \\ &\approx 1.02143 - 1 \\ &= 0.02143 \\ &= 2.143\% \end{aligned}$$

The effective rate of interest is highest for First Internet Bank, so First Internet Bank is offering the best deal.

Now Work PROBLEM 23

3 Determine the Present Value of a Lump Sum of Money

When people in finance speak of the "time value of money," they are usually referring to the *present value* of money. The **present value** of A dollars to be received at a future date is the principal that you would need to invest now so that it will grow to A dollars in the specified time period. The present value of money to be received at a future date is always less than the amount to be received, since the amount to be received will equal the present value (money invested now) *plus* the interest accrued over the time period.

The compound interest formula (2) is used to develop a formula for present value. If P is the present value of A dollars to be received after t years at a per annum interest rate r compounded n times per year, then, by formula (2),

$$A = P \cdot \left(1 + \frac{r}{n}\right)^{nt}$$

To solve for P, divide both sides by $\left(1 + \frac{r}{n}\right)^{nt}$. The result is

$$\frac{A}{\left(1 + \frac{r}{n}\right)^{nt}} = P \quad \text{or} \quad P = A \cdot \left(1 + \frac{r}{n}\right)^{-nt}$$

THEOREM

Present Value Formulas

The present value P of A dollars to be received after t years, assuming a per annum interest rate r compounded n times per year, is

$$P = A \cdot \left(1 + \frac{r}{n}\right)^{-nt} \tag{5}$$

If the interest is compounded continuously, then

$$P = Ae^{-rt} \tag{6}$$

To derive formula (6), solve formula (4) for P.

EXAMPLE 5 **Computing the Value of a Zero-Coupon Bond**

A zero-coupon (noninterest-bearing) bond can be redeemed in 10 years for \$1000. How much should you be willing to pay for it now if you want a return of

(a) 8% compounded monthly? (b) 7% compounded continuously?

Solution (a) To find the present value of \$1000, use formula (5) with $A = \$1000$, $n = 12$, $r = 0.08$, and $t = 10$.

$$P = A \cdot \left(1 + \frac{r}{n}\right)^{-nt} = \$1000\left(1 + \frac{0.08}{12}\right)^{-12(10)} = \$450.52$$

For a return of 8% compounded monthly, pay \$450.52 for the bond.

(b) Here use formula (6) with $A = \$1000$, $r = 0.07$, and $t = 10$.

$$P = Ae^{-rt} = \$1000e^{-(0.07)(10)} = \$496.59$$

For a return of 7% compounded continuously, pay \$496.59 for the bond.

Now Work PROBLEM 15

4 Determine the Rate of Interest or the Time Required to Double a Lump Sum of Money

EXAMPLE 6 **Rate of Interest Required to Double an Investment**

What annual rate of interest compounded annually is needed in order to double an investment in 5 years?

Solution If P is the principal and P is to double, then the amount A will be $2P$. Use the compound interest formula with $n = 1$ and $t = 5$ to find r.

$$A = P \cdot \left(1 + \frac{r}{n}\right)^{nt}$$

$$2P = P \cdot (1 + r)^5 \qquad A = 2P, n = 1, t = 5$$

$$2 = (1 + r)^5 \qquad \text{Divide both sides by } P.$$

$$1 + r = \sqrt[5]{2} \qquad \text{Take the fifth root of each side.}$$

$$r = \sqrt[5]{2} - 1 \approx 1.148698 - 1 = 0.148698$$

The annual rate of interest needed to double the principal in 5 years is 14.87%. ■

Now Work PROBLEM 31

EXAMPLE 7 **Time Required to Double or Triple an Investment**

(a) How long will it take for an investment to double in value if it earns 5% compounded continuously?
(b) How long will it take to triple at this rate?

Solution (a) If P is the initial investment and P is to double, then the amount A will be $2P$. Use formula (4) for continuously compounded interest with $r = 0.05$. Then

$$A = Pe^{rt}$$

$$2P = Pe^{0.05t} \qquad A = 2P, r = 0.05$$

$$2 = e^{0.05t} \qquad \text{Divide out the } P\text{'s.}$$

$$0.05t = \ln 2 \qquad \text{Rewrite as a logarithm.}$$

$$t = \frac{\ln 2}{0.05} \approx 13.86 \qquad \text{Solve for } t.$$

It will take about 14 years to double the investment.

(b) To triple the investment, let $A = 3P$ in formula (4).

$$A = Pe^{rt}$$

$$3P = Pe^{0.05t} \qquad A = 3P, r = 0.05$$

$$3 = e^{0.05t} \qquad \text{Divide out the } P\text{'s.}$$

$$0.05t = \ln 3 \qquad \text{Rewrite as a logarithm.}$$

$$t = \frac{\ln 3}{0.05} \approx 21.97 \qquad \text{Solve for } t.$$

It will take about 22 years to triple the investment. ■

Now Work PROBLEM 35

6.7 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the page listed in red.*

1. What is the interest due if \$500 is borrowed for 6 months at a simple interest rate of 6% per annum? (pp. 139–140)

2. If you borrow \$5000 and, after 9 months, pay off the loan in the amount of \$5500, what per annum rate of interest was charged? (pp. 139–140)

Concepts and Vocabulary

3. The total amount borrowed (whether by an individual from a bank in the form of a loan or by a bank from an individual in the form of a savings account) is called the ________.

4. If a principal of P dollars is borrowed for a period of t years at a per annum interest rate r, expressed as a decimal, the interest I charged is _____ = _______. Interest charged according to this formula is called ______ _______.

5. In working problems involving interest, if the payment period of the interest is quarterly, then interest is paid _____ times per year.

6. The ________ _____ ___ _______ is the equivalent annual simple interest rate that would yield the same amount as compounding n times per year, or continuously, after 1 year.

Skill Building

In Problems 7–14, find the amount that results from each investment.

7. \$100 invested at 4% compounded quarterly after a period of 2 years

8. \$50 invested at 6% compounded monthly after a period of 3 years

9. \$500 invested at 8% compounded quarterly after a period of $2\frac{1}{2}$ years

10. \$300 invested at 12% compounded monthly after a period of $1\frac{1}{2}$ years

11. \$600 invested at 5% compounded daily after a period of 3 years

12. \$700 invested at 6% compounded daily after a period of 2 years

13. \$1000 invested at 11% compounded continuously after a period of 2 years

14. \$400 invested at 7% compounded continuously after a period of 3 years

In Problems 15–22, find the principal needed now to get each amount; that is, find the present value.

15. To get \$100 after 2 years at 6% compounded monthly

16. To get \$75 after 3 years at 8% compounded quarterly

17. To get \$1000 after $2\frac{1}{2}$ years at 6% compounded daily

18. To get \$800 after $3\frac{1}{2}$ years at 7% compounded monthly

19. To get \$600 after 2 years at 4% compounded quarterly

20. To get \$300 after 4 years at 3% compounded daily

21. To get \$80 after $3\frac{1}{4}$ years at 9% compounded continuously

22. To get \$800 after $2\frac{1}{2}$ years at 8% compounded continuously

In Problems 23–26, find the effective rate of interest.

23. For 5% compounded quarterly

24. For 6% compounded monthly

25. For 5% compounded continuously

26. For 6% compounded continuously

In Problems 27–30, determine the rate that represents the better deal.

27. 6% compounded quarterly or $6\frac{1}{4}$% compounded annually

28. 9% compounded quarterly or $9\frac{1}{4}$% compounded annually

29. 9% compounded monthly or 8.8% compounded daily

30. 8% compounded semiannually or 7.9% compounded daily

31. What rate of interest compounded annually is required to double an investment in 3 years?

32. What rate of interest compounded annually is required to double an investment in 6 years?

33. What rate of interest compounded annually is required to triple an investment in 5 years?

34. What rate of interest compounded annually is required to triple an investment in 10 years?

35. (a) How long does it take for an investment to double in value if it is invested at 8% compounded monthly?
    (b) How long does it take if the interest is compounded continuously?

36. (a) How long does it take for an investment to triple in value if it is invested at 6% compounded monthly?
    (b) How long does it take if the interest is compounded continuously?

37. What rate of interest compounded quarterly will yield an effective interest rate of 7%?

38. What rate of interest compounded continuously will yield an effective interest rate of 6%?

Applications and Extensions

39. Time Required to Reach a Goal If Tanisha has \$100 to invest at 4% per annum compounded monthly, how long will it be before she has \$150? If the compounding is continuous, how long will it be?

40. Time Required to Reach a Goal If Angela has \$100 to invest at 2.5% per annum compounded monthly, how long will it be before she has \$175? If the compounding is continuous, how long will it be?

41. Time Required to Reach a Goal How many years will it take for an initial investment of \$10,000 to grow to \$25,000? Assume a rate of interest of 6% compounded continuously.

42. Time Required to Reach a Goal How many years will it take for an initial investment of \$25,000 to grow to \$80,000? Assume a rate of interest of 7% compounded continuously.

43. Price Appreciation of Homes What will a \$90,000 condominium cost 5 years from now if the price appreciation for condos over that period averages 3% compounded annually?

44. Credit Card Interest A department store charges 1.25% per month on the unpaid balance for customers with charge accounts (interest is compounded monthly). A customer charges \$200 and does not pay her bill for 6 months. What is the bill at that time?

45. Saving for a Car Jerome will be buying a used car for \$15,000 in 3 years. How much money should he ask his parents for now so that, if he invests it at 5% compounded continuously, he will have enough to buy the car?

46. Paying off a Loan John requires \$3000 in 6 months to pay off a loan that has no prepayment privileges. If he has the \$3000 now, how much of it should he save in an account paying 3% compounded monthly so that in 6 months he will have exactly \$3000?

47. Return on a Stock George contemplates the purchase of 100 shares of a stock selling for \$15 per share. The stock pays no dividends. The history of the stock indicates that it should grow at an annual rate of 15% per year. How much should the 100 shares of stock be worth in 5 years?

48. Return on an Investment A business purchased for \$650,000 in 2012 is sold in 2015 for \$850,000. What is the annual rate of return for this investment?

49. Comparing Savings Plans Jim places \$1000 in a bank account that pays 5.6% compounded continuously. After 1 year, will he have enough money to buy a computer system that costs \$1060? If another bank will pay Jim 5.9% compounded monthly, is this a better deal?

50. Savings Plans On January 1, Kim places \$1000 in a certificate of deposit that pays 6.8% compounded continuously and matures in 3 months. Then Kim places the \$1000 and the interest in a passbook account that pays 5.25% compounded monthly. How much does Kim have in the passbook account on May 1?

51. Comparing IRA Investments Will invests \$2000 in his IRA in a bond trust that pays 9% interest compounded semiannually. His friend Henry invests \$2000 in his IRA in a certificate of deposit that pays $8\frac{1}{2}\%$ compounded continuously. Who has more money after 20 years, Will or Henry?

52. Comparing Two Alternatives Suppose that April has access to an investment that will pay 10% interest compounded continuously. Which is better: to be given \$1000 now so that she can take advantage of this investment opportunity or to be given \$1325 after 3 years?

53. College Costs The average annual cost of college at 4-year private colleges was \$31,231 in the 2014–2015 academic year. This was a 3.7% increase from the previous year.

Source: *The College Board*

(a) If the cost of college increases by 3.7% each year, what will be the average cost of college at a 4-year private college for the 2034–2035 academic year?

(b) College savings plans, such as a 529 plan, allow individuals to put money aside now to help pay for college later. If one such plan offers a rate of 2% compounded continuously, how much should be put in a college savings plan in 2016 to pay for 1 year of the cost of college at a 4-year private college for an incoming freshman in 2034?

54. Analyzing Interest Rates on a Mortgage Colleen and Bill have just purchased a house for \$650,000, with the seller holding a second mortgage of \$100,000. They promise to pay the seller \$100,000 plus all accrued interest 5 years from now. The seller offers them three interest options on the second mortgage:

(a) Simple interest at 6% per annum
(b) 5.5% interest compounded monthly
(c) 5.25% interest compounded continuously

Which option is best? That is, which results in paying the least interest on the loan?

55. 2009 Federal Stimulus Package In February 2009, President Obama signed into law a \$787 billion federal stimulus package. At that time, 20-year Series EE bonds had a fixed rate of 1.3% compounded semiannually. If the federal government financed the stimulus through EE bonds, how much would it have to pay back in 2029? How much interest was paid to finance the stimulus?

Source: *U.S. Treasury Department*

56. Per Capita Federal Debt In 2015, the federal debt was about \$18 trillion. In 2015, the U.S. population was about 320 million. Assuming that the federal debt is increasing about 4.5% per year and the U.S. population is increasing about 0.7% per year, determine the per capita debt (total debt divided by population) in 2030.

Inflation *Problems 57–62 require the following discussion.* ***Inflation*** *is a term used to describe the erosion of the purchasing power of money. For example, if the annual inflation rate is 3%, then \$1000 worth of purchasing power now will have only \$970 worth of purchasing power in 1 year because 3% of the original \$1000 (0.03 × 1000 = 30) has been eroded due to inflation. In general, if the rate of inflation averages r per annum over n years, the amount A that \$P will purchase after n years is*

$$A = P \cdot (1 - r)^n$$

where r is expressed as a decimal.

57. **Inflation** If the inflation rate averages 3%, how much will \$1000 purchase in 2 years?

58. **Inflation** If the inflation rate averages 2%, how much will \$1000 purchase in 3 years?

59. **Inflation** If the amount that \$1000 will purchase is only \$950 after 2 years, what was the average inflation rate?

60. **Inflation** If the amount that \$1000 will purchase is only \$930 after 2 years, what was the average inflation rate?

61. **Inflation** If the average inflation rate is 2%, how long is it until purchasing power is cut in half?

62. **Inflation** If the average inflation rate is 4%, how long is it until purchasing power is cut in half?

*Problems 63–66 involve zero-coupon bonds. A **zero-coupon bond** is a bond that is sold now at a discount and will pay its face value at the time when it matures; no interest payments are made.*

63. **Zero-Coupon Bonds** A zero-coupon bond can be redeemed in 20 years for \$10,000. How much should you be willing to pay for it now if you want a return of:
    (a) 5% compounded monthly?
    (b) 5% compounded continuously?

64. **Zero-Coupon Bonds** A child's grandparents are considering buying a \$80,000 face-value, zero-coupon bond at birth so that she will have money for her college education 17 years later. If they want a rate of return of 6% compounded annually, what should they pay for the bond?

65. **Zero-Coupon Bonds** How much should a \$10,000 face-value, zero-coupon bond, maturing in 10 years, be sold for now if its rate of return is to be 4.5% compounded annually?

66. **Zero-Coupon Bonds** If Pat pays \$15,334.65 for a \$25,000 face-value, zero-coupon bond that matures in 8 years, what is his annual rate of return?

67. **Time to Double or Triple an Investment** The formula

$$t = \frac{\ln m}{n \ln\left(1 + \frac{r}{n}\right)}$$

can be used to find the number of years t required to multiply an investment m times when r is the per annum interest rate compounded n times a year.
    (a) How many years will it take to double the value of an IRA that compounds annually at the rate of 6%?
    (b) How many years will it take to triple the value of a savings account that compounds quarterly at an annual rate of 5%?
    (c) Give a derivation of this formula.

68. **Time to Reach an Investment Goal** The formula

$$t = \frac{\ln A - \ln P}{r}$$

can be used to find the number of years t required for an investment P to grow to a value A when compounded continuously at an annual rate r.
    (a) How long will it take to increase an initial investment of \$1000 to \$4500 at an annual rate of 5.75%?
    (b) What annual rate is required to increase the value of a \$2000 IRA to \$30,000 in 35 years?
    (c) Give a derivation of this formula.

*Problems 69–72 require the following discussion. The **consumer price index (CPI)** indicates the relative change in price over time for a fixed basket of goods and services. It is a cost of living index that helps measure the effect of inflation on the cost of goods and services. The CPI uses the base period 1982–1984 for comparison (the CPI for this period is 100). The CPI for March 2015 was 236.12. This means that \$100 in the period 1982–1984 had the same purchasing power as \$236.12 in March 2015. In general, if the rate of inflation averages r percent per annum over n years, then the CPI index after n years is*

$$\text{CPI} = \text{CPI}_0\left(1 + \frac{r}{100}\right)^n$$

where CPI_0 is the CPI index at the beginning of the n-year period.
***Source:** U.S. Bureau of Labor Statistics*

69. **Consumer Price Index**
    (a) The CPI was 214.5 for 2009 and 236.7 for 2014. Assuming that annual inflation remained constant for this time period, determine the average annual inflation rate.
    (b) Using the inflation rate from part (a), in what year will the CPI reach 300?

70. **Consumer Price Index** If the current CPI is 234.2 and the average annual inflation rate is 2.8%, what will be the CPI in 5 years?

71. **Consumer Price Index** If the average annual inflation rate is 3.1%, how long will it take for the CPI index to double? (A doubling of the CPI index means purchasing power is cut in half.)

72. **Consumer Price Index** The base period for the CPI changed in 1998. Under the previous weight and item structure, the CPI for 1995 was 456.5. If the average annual inflation rate was 5.57%, what year was used as the base period for the CPI?

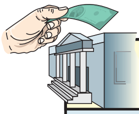
Explaining Concepts: Discussion and Writing

73. Explain in your own words what the term *compound interest* means. What does *continuous compounding* mean?

74. Explain in your own words the meaning of *present value*.

75. **Critical Thinking** You have just contracted to buy a house and will seek financing in the amount of \$100,000. You go to several banks. Bank 1 will lend you \$100,000 at the rate of 4.125% amortized over 30 years with a loan origination fee of 0.45%. Bank 2 will lend you \$100,000 at the rate of 3.375% amortized over 15 years with a loan origination fee of 0.95%. Bank 3 will lend you \$100,000 at the rate of 4.25% amortized over 30 years with no loan origination fee. Bank 4 will lend you \$100,000 at the rate of 3.625% amortized over

15 years with no loan origination fee. Which loan would you take? Why? Be sure to have sound reasons for your choice. Use the information in the table to assist you. If the amount of the monthly payment does not matter to you, which loan would you take? Again, have sound reasons for your choice. Compare your final decision with others in the class. Discuss.



	Monthly Payment	Loan Origination Fee
Bank 1	\$485.00	\$450.00
Bank 2	\$709.00	\$950.00
Bank 3	\$492.00	\$0.00
Bank 4	\$721.00	\$0.00

Retain Your Knowledge

Problems 76–79 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

76. Find the remainder R when $f(x) = 6x^3 + 3x^2 + 2x - 11$ is divided by $g(x) = x - 1$. Is g a factor of f?

77. The function $f(x) = \dfrac{x}{x-2}$ is one-to-one. Find f^{-1}.

78. Find the real zeros of
$$f(x) = x^5 - x^4 - 15x^3 - 21x^2 - 16x - 20.$$
Then write f in factored form.

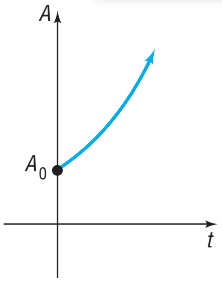
79. Solve: $\log_2(x+3) = 2\log_2(x-3)$

'Are You Prepared?' Answers

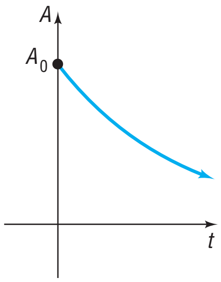
1. \$15 **2.** $13\frac{1}{3}\%$

6.8 Exponential Growth and Decay Models; Newton's Law; Logistic Growth and Decay Models

OBJECTIVES
1. Find Equations of Populations That Obey the Law of Uninhibited Growth (p. 484)
2. Find Equations of Populations That Obey the Law of Decay (p. 487)
3. Use Newton's Law of Cooling (p. 488)
4. Use Logistic Models (p. 489)



(a) $A(t) = A_0e^{kt}, k > 0$
Exponential growth



(b) $A(t) = A_0e^{kt}, k < 0$
Exponential decay

Figure 55

1 Find Equations of Populations That Obey the Law of Uninhibited Growth

Many natural phenomena have been found to follow the law that an amount A varies with time t according to the function

$$A(t) = A_0e^{kt} \qquad (1)$$

Here A_0 is the original amount $(t = 0)$ and $k \neq 0$ is a constant.

If $k > 0$, then equation (1) states that the amount A is increasing over time; if $k < 0$, the amount A is decreasing over time. In either case, when an amount A varies over time according to equation (1), it is said to follow the **exponential law** or the **law of uninhibited growth** $(k > 0)$ **or decay** $(k < 0)$. See Figure 55.

For example, in Section 6.7, continuously compounded interest was shown to follow the law of uninhibited growth. In this section we shall look at some additional phenomena that follow the exponential law.

Cell division is the growth process of many living organisms, such as amoebas, plants, and human skin cells. Based on an ideal situation in which no cells die and no by-products are produced, the number of cells present at a given time follows the law of uninhibited growth. Actually, however, after enough time has passed, growth at an exponential rate will cease due to the influence of factors such as lack of living space and dwindling food supply. The law of uninhibited growth accurately models only the early stages of the cell division process.

The cell division process begins with a culture containing N_0 cells. Each cell in the culture grows for a certain period of time and then divides into two identical cells. Assume that the time needed for each cell to divide in two is constant and does not change as the number of cells increases. These new cells then grow, and eventually each divides in two, and so on.

Uninhibited Growth of Cells

A model that gives the number N of cells in a culture after a time t has passed (in the early stages of growth) is

$$N(t) = N_0 e^{kt} \qquad k > 0 \tag{2}$$

where N_0 is the initial number of cells and k is a positive constant that represents the growth rate of the cells.

Using formula (2) to model the growth of cells employs a function that yields positive real numbers, even though the number of cells being counted must be an integer. This is a common practice in many applications.

EXAMPLE 1 Bacterial Growth

A colony of bacteria that grows according to the law of uninhibited growth is modeled by the function $N(t) = 100e^{0.045t}$, where N is measured in grams and t is measured in days.

(a) Determine the initial amount of bacteria.
(b) What is the growth rate of the bacteria?
(c) Graph the function using a graphing utility.
(d) What is the population after 5 days?
(e) How long will it take for the population to reach 140 grams?
(f) What is the doubling time for the population?

Solution

(a) The initial amount of bacteria, N_0, is obtained when $t = 0$, so

$$N_0 = N(0) = 100e^{0.045(0)} = 100 \text{ grams}$$

(b) Compare $N(t) = 100e^{0.045t}$ to $N(t) = N_0 e^{kt}$. The value of k, 0.045, indicates a growth rate of 4.5%.

(c) Figure 56 shows the graph of $N(t) = 100e^{0.045t}$.

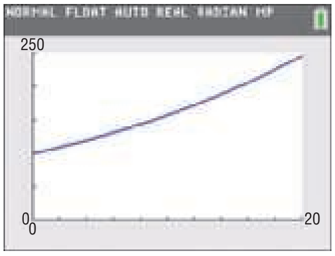


Figure 56 $Y_1 = 100e^{0.045x}$

(d) The population after 5 days is $N(5) = 100e^{0.045(5)} \approx 125.2$ grams.

(e) To find how long it takes for the population to reach 140 grams, solve the equation $N(t) = 140$.

$$\begin{aligned} 100e^{0.045t} &= 140 \\ e^{0.045t} &= 1.4 && \text{Divide both sides of the equation by 100.} \\ 0.045t &= \ln 1.4 && \text{Rewrite as a logarithm.} \\ t &= \frac{\ln 1.4}{0.045} && \text{Divide both sides of the equation by 0.045.} \\ &\approx 7.5 \text{ days} \end{aligned}$$

The population reaches 140 grams in about 7.5 days.

(f) The population doubles when $N(t) = 200$ grams, so the doubling time is found by solving the equation $200 = 100e^{0.045t}$ for t.

$$\begin{aligned} 200 &= 100e^{0.045t} \\ 2 &= e^{0.045t} && \text{Divide both sides of the equation by 100.} \\ \ln 2 &= 0.045t && \text{Rewrite as a logarithm.} \\ t &= \frac{\ln 2}{0.045} && \text{Divide both sides of the equation by 0.045.} \\ &\approx 15.4 \text{ days} \end{aligned}$$

The population doubles approximately every 15.4 days. ■

Now Work PROBLEM 1

EXAMPLE 2 Bacterial Growth

A colony of bacteria increases according to the law of uninhibited growth.

(a) If N is the number of cells and t is the time in hours, express N as a function of t.
(b) If the number of bacteria doubles in 3 hours, find the function that gives the number of cells in the culture.
(c) How long will it take for the size of the colony to triple?
(d) How long will it take for the population to double a second time (that is, increase four times)?

Solution (a) Using formula (2), the number N of cells at time t is

$$N(t) = N_0e^{kt}$$

where N_0 is the initial number of bacteria present and k is a positive number.

(b) To find the growth rate k, note that the number of cells doubles in 3 hours, so

$$N(3) = 2N_0$$

But $N(3) = N_0e^{k(3)}$, so

$$\begin{aligned} N_0e^{k(3)} &= 2N_0 \\ e^{3k} &= 2 && \text{Divide both sides by } N_0. \\ 3k &= \ln 2 && \text{Rewrite as a logarithm.} \\ k &= \frac{1}{3}\ln 2 \approx 0.23105 \end{aligned}$$

The function that models this growth process is, therefore,

$$N(t) = N_0e^{0.23105t}$$

(c) The time t needed for the size of the colony to triple requires that $N = 3N_0$. Substitute $3N_0$ for N to get

$$\begin{aligned} 3N_0 &= N_0e^{0.23105t} \\ 3 &= e^{0.23105t} && \text{Divide both sides by } N_0. \\ 0.23105t &= \ln 3 && \text{Rewrite as a logarithm.} \\ t &= \frac{\ln 3}{0.23105} \approx 4.755 \text{ hours} \end{aligned}$$

It will take about 4.755 hours, or 4 hours and 45 minutes, for the size of the colony to triple.

(d) If a population doubles in 3 hours, it will double a second time in 3 more hours, for a total time of 6 hours. ■

2 Find Equations of Populations That Obey the Law of Decay

Radioactive materials follow the law of uninhibited decay.

Uninhibited Radioactive Decay

The amount A of a radioactive material present at time t is given by

$$A(t) = A_0 e^{kt} \qquad k < 0 \tag{3}$$

where A_0 is the original amount of radioactive material and k is a negative number that represents the rate of decay.

All radioactive substances have a specific **half-life**, which is the time required for half of the radioactive substance to decay. **Carbon dating** uses the fact that all living organisms contain two kinds of carbon, carbon-12 (a stable carbon) and carbon-14 (a radioactive carbon with a half-life of 5730 years). While an organism is living, the ratio of carbon-12 to carbon-14 is constant. But when an organism dies, the original amount of carbon-12 present remains unchanged, whereas the amount of carbon-14 begins to decrease. This change in the amount of carbon-14 present relative to the amount of carbon-12 present makes it possible to calculate when the organism died.

EXAMPLE 3 Estimating the Age of Ancient Tools

Traces of burned wood along with ancient stone tools in an archeological dig in Chile were found to contain approximately 1.67% of the original amount of carbon-14.

(a) If the half-life of carbon-14 is 5730 years, approximately when was the tree cut and burned?
(b) Using a graphing utility, graph the relation between the percentage of carbon-14 remaining and time.
(c) Use a graphing utility to determine the time that elapses until half of the carbon-14 remains. This answer should equal the half-life of carbon-14.
(d) Use a graphing utility to verify the answer found in part (a).

Solution (a) Using formula (3), the amount A of carbon-14 present at time t is

$$A(t) = A_0 e^{kt}$$

where A_0 is the original amount of carbon-14 present and k is a negative number. We first seek the number k. To find it, we use the fact that after 5730 years half of the original amount of carbon-14 remains, so $A(5730) = \frac{1}{2}A_0$. Then

$$\frac{1}{2}A_0 = A_0 e^{k(5730)}$$

$$\frac{1}{2} = e^{5730k} \qquad \text{Divide both sides of the equation by } A_0.$$

$$5730k = \ln\frac{1}{2} \qquad \text{Rewrite as a logarithm.}$$

$$k = \frac{1}{5730}\ln\frac{1}{2} \approx -0.000120968$$

Formula (3), therefore, becomes

$$A(t) = A_0 e^{-0.000120968t}$$

If the amount A of carbon-14 now present is 1.67% of the original amount, it follows that

$$0.0167A_0 = A_0 e^{-0.000120968t}$$

$$0.0167 = e^{-0.000120968t} \qquad \text{Divide both sides of the equation by } A_0.$$

$$-0.000120968t = \ln 0.0167 \qquad \text{Rewrite as a logarithm.}$$

$$t = \frac{\ln 0.0167}{-0.000120968} \approx 33{,}830 \text{ years}$$

The tree was cut and burned about 33,830 years ago. Some archeologists use this conclusion to argue that humans lived in the Americas nearly 34,000 years ago, much earlier than is generally accepted.

(b) Figure 57 shows the graph of $y = e^{-0.000120968x}$, where y is the fraction of carbon-14 present and x is the time.

(c) Graph $Y_1 = e^{-0.000120968x}$ and $Y_2 = 0.5$, where x is time. Use INTERSECT to find that it takes 5730 years until half the carbon-14 remains. The half-life of carbon-14 is 5730 years.

(d) Graph $Y_1 = e^{-0.000120968x}$ and $Y_2 = 0.0167$ where x is time. Use INTERSECT to find that it takes 33,830 years until 1.67% of the carbon-14 remains. ■

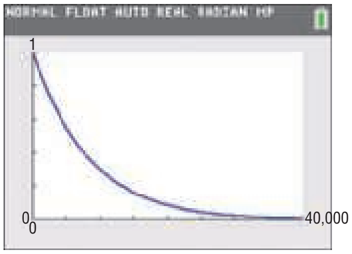


Figure 57 $Y_1 = e^{-0.000120968x}$

Now Work PROBLEM 3

3 Use Newton's Law of Cooling

Newton's Law of Cooling* states that the temperature of a heated object decreases exponentially over time toward the temperature of the surrounding medium.

Newton's Law of Cooling

The temperature u of a heated object at a given time t can be modeled by the following function:

$$u(t) = T + (u_0 - T)e^{kt} \qquad k < 0 \tag{4}$$

where T is the constant temperature of the surrounding medium, u_0 is the initial temperature of the heated object, and k is a negative constant.

EXAMPLE 4 **Using Newton's Law of Cooling**

An object is heated to 100°C (degrees Celsius) and is then allowed to cool in a room whose air temperature is 30°C.

(a) If the temperature of the object is 80°C after 5 minutes, when will its temperature be 50°C?
(b) Using a graphing utility, graph the relation found between the temperature and time.
(c) Using a graphing utility, verify the results from part (a).
(d) Using a graphing utility, determine the elapsed time before the object is 35°C.
(e) What do you notice about the temperature as time passes?

Solution (a) Using formula (4) with $T = 30$ and $u_0 = 100$, the temperature (in degrees Celsius) of the object at time t (in minutes) is

$$u(t) = 30 + (100 - 30)e^{kt} = 30 + 70e^{kt}$$

where k is a negative constant. To find k, use the fact that $u = 80$ when $t = 5$ [that is, $u(5) = 80$]. Then

$$\begin{aligned}
80 &= 30 + 70e^{k(5)} && u(5) = 80 \\
50 &= 70e^{5k} && \text{Simplify.} \\
e^{5k} &= \frac{50}{70} && \text{Solve for } e^{5k}. \\
5k &= \ln\frac{5}{7} && \text{Rewrite as a logarithm.} \\
k &= \frac{1}{5}\ln\frac{5}{7} \approx -0.0673 && \text{Solve for } k.
\end{aligned}$$

Formula (4), therefore, becomes

$$u(t) = 30 + 70e^{-0.0673t}$$

*Named after Sir Isaac Newton (1642–1727), one of the cofounders of calculus.

To find t when $u = 50°\text{C}$, solve the equation

$$50 = 30 + 70e^{-0.0673t}$$

$$20 = 70e^{-0.0673t} \qquad \text{Subtract 30 from both sides.}$$

$$e^{-0.0673t} = \frac{20}{70} \qquad \text{Solve for } e^{-0.0673t}$$

$$-0.0673t = \ln\frac{2}{7} \qquad \text{Rewrite as a logarithm.}$$

$$t = \frac{\ln\frac{2}{7}}{-0.0673} \approx 18.6 \text{ minutes} \qquad \text{Solve for } t.$$

The temperature of the object will be 50°C after about 18.6 minutes, or 18 minutes and 37 seconds.

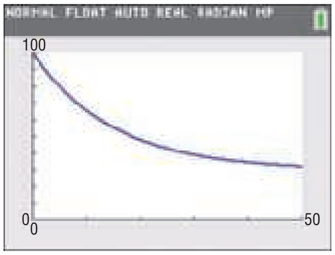


Figure 58 $Y_1 = 30 + 70e^{-0.0673x}$

(b) Figure 58 shows the graph of $y = 30 + 70e^{-0.0673x}$, where y is the temperature and x is the time.

(c) Graph $Y_1 = 30 + 70e^{-0.0673x}$ and $Y_2 = 50$, where x is time. Use INTERSECT to find that it takes $x = 18.6$ minutes (18 minutes, 37 seconds) for the temperature to cool to 50°C.

(d) Graph $Y_1 = 30 + 70e^{-0.0673x}$ and $Y_2 = 35$, where x is time. Use INTERSECT to find that it takes $x = 39.21$ minutes (39 minutes, 13 seconds) for the temperature to cool to 35°C.

(e) As t increases, the value of $e^{-0.0673t}$ approaches zero, so the value of u, the temperature of the object, approaches 30°C, the air temperature of the room. ■

Now Work PROBLEM 13

4 Use Logistic Models

The exponential growth model $A(t) = A_0e^{kt}$, $k > 0$, assumes uninhibited growth, meaning that the value of the function grows without limit. Recall that cell division can be modeled using this function, assuming that no cells die and no by-products are produced. However, cell division eventually is limited by factors such as living space and food supply. The **logistic model**, given next, can describe situations where the growth or decay of the dependent variable is limited.

Logistic Model

In a logistic model, the population P after time t is given by the function

$$\boxed{P(t) = \frac{c}{1 + ae^{-bt}}} \qquad \textbf{(5)}$$

where a, b, and c are constants with $a > 0$ and $c > 0$. The model is a growth model if $b > 0$; the model is a decay model if $b < 0$.

The number c is called the **carrying capacity** (for growth models) because the value $P(t)$ approaches c as t approaches infinity; that is, $\lim_{t\to\infty} P(t) = c$. The number $|b|$ is the growth rate for $b > 0$ and the decay rate for $b < 0$. Figure 59(a) on the next page shows the graph of a typical logistic growth function, and Figure 59(b) shows the graph of a typical logistic decay function.

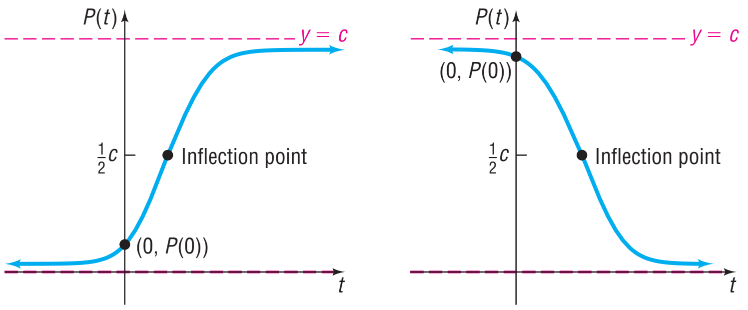


Figure 59 (a) $P(t) = \dfrac{c}{1 + ae^{-bt}}$, $b > 0$ Logistic growth (b) $P(t) = \dfrac{c}{1 + ae^{-bt}}$, $b < 0$ Logistic decay

Based on the figures, the following properties of logistic functions emerge.

Properties of the Logistic Model, Equation (5)

1. The domain is the set of all real numbers. The range is the interval $(0, c)$, where c is the carrying capacity.
2. There are no x-intercepts; the y-intercept is $P(0)$.
3. There are two horizontal asymptotes: $y = 0$ and $y = c$.
4. $P(t)$ is an increasing function if $b > 0$ and a decreasing function if $b < 0$.
5. There is an **inflection point** where $P(t)$ equals $\frac{1}{2}$ of the carrying capacity. The inflection point is the point on the graph where the graph changes from being curved upward to being curved downward for growth functions, and the point where the graph changes from being curved downward to being curved upward for decay functions.
6. The graph is smooth and continuous, with no corners or gaps.

EXAMPLE 5 Fruit Fly Population

Fruit flies are placed in a half-pint milk bottle with a banana (for food) and yeast plants (for food and to provide a stimulus to lay eggs). Suppose that the fruit fly population after t days is given by

$$P(t) = \frac{230}{1 + 56.5e^{-0.37t}}$$

(a) State the carrying capacity and the growth rate.
(b) Determine the initial population.
(c) What is the population after 5 days?
(d) How long does it take for the population to reach 180?
(e) Use a graphing utility to determine how long it takes for the population to reach one-half of the carrying capacity.

Solution (a) As $t \to \infty$, $e^{-0.37t} \to 0$ and $P(t) \to \frac{230}{1}$. The carrying capacity of the half-pint bottle is 230 fruit flies. The growth rate is $|b| = |0.37| = 37\%$ per day.

(b) To find the initial number of fruit flies in the half-pint bottle, evaluate $P(0)$.

$$P(0) = \frac{230}{1 + 56.5e^{-0.37(0)}} = \frac{230}{1 + 56.5} = 4$$

So, initially, there were 4 fruit flies in the half-pint bottle.

(c) To find the number of fruit flies in the half-pint bottle after 5 days, evaluate $P(5)$.

$$P(5) = \frac{230}{1 + 56.5e^{-0.37(5)}} \approx 23 \text{ fruit flies}$$

After 5 days, there are approximately 23 fruit flies in the bottle.

(d) To determine when the population of fruit flies will be 180, solve the equation $P(t) = 180$.

$$\frac{230}{1 + 56.5e^{-0.37t}} = 180$$

$$230 = 180(1 + 56.5e^{-0.37t})$$

$$1.2778 = 1 + 56.5e^{-0.37t} \qquad \text{Divide both sides by 180.}$$

$$0.2778 = 56.5e^{-0.37t} \qquad \text{Subtract 1 from both sides.}$$

$$0.0049 = e^{-0.37t} \qquad \text{Divide both sides by 56.5.}$$

$$\ln(0.0049) = -0.37t \qquad \text{Rewrite as a logarithmic expression.}$$

$$t \approx 14.4 \text{ days} \qquad \text{Divide both sides by } -0.37.$$

It will take approximately 14.4 days (14 days, 10 hours) for the population to reach 180 fruit flies.

(e) One-half of the carrying capacity is 115 fruit flies. Solve $P(t) = 115$ by graphing $Y_1 = \dfrac{230}{1 + 56.5e^{-0.37x}}$ and $Y_2 = 115$ and using INTERSECT. See Figure 60. The population will reach one-half of the carrying capacity in about 10.9 days (10 days, 22 hours). ■

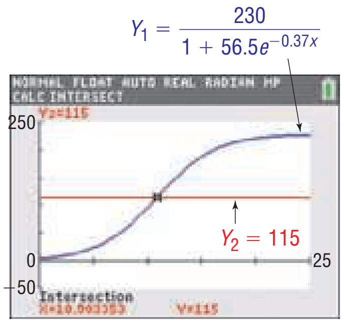


Figure 60

Look at Figure 60. Notice the point where the graph reaches 115 fruit flies (one-half of the carrying capacity): The graph changes from being curved upward to being curved downward. Using the language of calculus, we say the graph changes from increasing at an increasing rate to increasing at a decreasing rate. For any logistic growth function, when the population reaches one-half the carrying capacity, the population growth starts to slow down.

Exploration

On the same viewing rectangle, graph

$$Y_1 = \frac{500}{1 + 24e^{-0.03x}} \text{ and } Y_2 = \frac{500}{1 + 24e^{-0.08x}}$$

What effect does the growth rate $|b|$ have on the logistic growth function? ■

Now Work PROBLEM 23

EXAMPLE 6 Wood Products

The EFISCEN wood product model classifies wood products according to their life-span. There are four classifications: short (1 year), medium short (4 years), medium long (16 years), and long (50 years). Based on data obtained from the European Forest Institute, the percentage of remaining wood products after t years for wood products with long life-spans (such as those used in the building industry) is given by

$$P(t) = \frac{100.3952}{1 + 0.0316e^{0.0581t}}$$

(a) What is the decay rate?
(b) What is the percentage of remaining wood products after 10 years?
(c) How long does it take for the percentage of remaining wood products to reach 50%?
(d) Explain why the numerator given in the model is reasonable.

Solution

(a) The decay rate is $|b| = |-0.0581| = 5.81\%$.

(b) Evaluate $P(10)$.

$$P(10) = \frac{100.3952}{1 + 0.0316e^{0.0581(10)}} \approx 95.0$$

So 95% of long-life-span wood products remain after 10 years.

(c) Solve the equation $P(t) = 50$.

$$\frac{100.3952}{1 + 0.0316e^{0.0581t}} = 50$$

$$100.3952 = 50(1 + 0.0316e^{0.0581t})$$

$$2.0079 = 1 + 0.0316e^{0.0581t} \quad \text{Divide both sides by 50.}$$

$$1.0079 = 0.0316e^{0.0581t} \quad \text{Subtract 1 from both sides.}$$

$$31.8956 = e^{0.0581t} \quad \text{Divide both sides by 0.0316.}$$

$$\ln(31.8956) = 0.0581t \quad \text{Rewrite as a logarithmic expression.}$$

$$t \approx 59.6 \text{ years} \quad \text{Divide both sides by 0.0581.}$$

It will take approximately 59.6 years for the percentage of long-life-span wood products remaining to reach 50%.

(d) The numerator of 100.3952 is reasonable because the maximum percentage of wood products remaining that is possible is 100%.

Now Work PROBLEM 27

6.8 Assess Your Understanding

Applications and Extensions

1. **Growth of an Insect Population** The size P of a certain insect population at time t (in days) obeys the model $P(t) = 500e^{0.02t}$.
   (a) Determine the number of insects at $t = 0$ days.
   (b) What is the growth rate of the insect population?
   (c) Graph the function using a graphing utility.
   (d) What is the population after 10 days?
   (e) When will the insect population reach 800?
   (f) When will the insect population double?

2. **Growth of Bacteria** The number N of bacteria present in a culture at time t (in hours) obeys the model $N(t) = 1000e^{0.01t}$.
   (a) Determine the number of bacteria at $t = 0$ hours.
   (b) What is the growth rate of the bacteria?
   (c) Graph the function using a graphing utility.
   (d) What is the population after 4 hours?
   (e) When will the number of bacteria reach 1700?
   (f) When will the number of bacteria double?

3. **Radioactive Decay** Strontium-90 is a radioactive material that decays according to the function $A(t) = A_0e^{-0.0244t}$, where A_0 is the initial amount present and A is the amount present at time t (in years). Assume that a scientist has a sample of 500 grams of strontium-90.
   (a) What is the decay rate of strontium-90?
   (b) Graph the function using a graphing utility.
   (c) How much strontium-90 is left after 10 years?
   (d) When will 400 grams of strontium-90 be left?
   (e) What is the half-life of strontium-90?

4. **Radioactive Decay** Iodine-131 is a radioactive material that decays according to the function $A(t) = A_0e^{-0.087t}$, where A_0 is the initial amount present and A is the amount present at time t (in days). Assume that a scientist has a sample of 100 grams of iodine-131.
   (a) What is the decay rate of iodine-131?
   (b) Graph the function using a graphing utility.
   (c) How much iodine-131 is left after 9 days?
   (d) When will 70 grams of iodine-131 be left?
   (e) What is the half-life of iodine-131?

5. **Growth of a Colony of Mosquitoes** The population of a colony of mosquitoes obeys the law of uninhibited growth.
   (a) If N is the population of the colony and t is the time in days, express N as a function of t.
   (b) If there are 1000 mosquitoes initially and there are 1800 after 1 day, what is the size of the colony after 3 days?
   (c) How long is it until there are 10,000 mosquitoes?

6. **Bacterial Growth** A culture of bacteria obeys the law of uninhibited growth.
   (a) If N is the number of bacteria in the culture and t is the time in hours, express N as a function of t.
   (b) If 500 bacteria are present initially and there are 800 after 1 hour, how many will be present in the culture after 5 hours?
   (c) How long is it until there are 20,000 bacteria?

7. **Population Growth** The population of a southern city follows the exponential law.
   (a) If N is the population of the city and t is the time in years, express N as a function of t.
   (b) If the population doubled in size over an 18-month period and the current population is 10,000, what will the population be 2 years from now?

8. **Population Decline** The population of a midwestern city follows the exponential law.
   (a) If N is the population of the city and t is the time in years, express N as a function of t.

(b) If the population decreased from 900,000 to 800,000 from 2013 to 2015, what will the population be in 2017?

9. **Radioactive Decay** The half-life of radium is 1690 years. If 10 grams is present now, how much will be present in 50 years?

10. **Radioactive Decay** The half-life of radioactive potassium is 1.3 billion years. If 10 grams is present now, how much will be present in 100 years? In 1000 years?

11. **Estimating the Age of a Tree** A piece of charcoal is found to contain 30% of the carbon-14 that it originally had.
(a) When did the tree from which the charcoal came die? Use 5730 years as the half-life of carbon-14.
(b) Using a graphing utility, graph the relation between the percentage of carbon-14 remaining and time.
(c) Using INTERSECT, determine the time that elapses until half of the carbon-14 remains.
(d) Verify the answer found in part (a).

12. **Estimating the Age of a Fossil** A fossilized leaf contains 70% of its normal amount of carbon-14.
(a) How old is the fossil? Use 5730 years as the half-life of carbon-14.
(b) Using a graphing utility, graph the relation between the percentage of carbon-14 remaining and time.
(c) Using INTERSECT, determine the time that elapses until one-fourth of the carbon-14 remains.
(d) Verify the answer found in part (a).

13. **Cooling Time of a Pizza** A pizza baked at 450°F is removed from the oven at 5:00 PM and placed in a room that is a constant 70°F. After 5 minutes, the pizza is at 300°F.
(a) At what time can you begin eating the pizza if you want its temperature to be 135°F?
(b) Using a graphing utility, graph the relation between temperature and time.
(c) Using INTERSECT, determine the time that needs to elapse before the pizza is 160°F.
(d) TRACE the function for large values of time. What do you notice about y, the temperature?

14. **Newton's Law of Cooling** A thermometer reading 72°F is placed in a refrigerator where the temperature is a constant 38°F.
(a) If the thermometer reads 60°F after 2 minutes, what will it read after 7 minutes?
(b) How long will it take before the thermometer reads 39°F?
(c) Using a graphing utility, graph the relation between temperature and time.
(d) Using INTERSECT, determine the time that must elapse before the thermometer reads 45°F.
(e) TRACE the function for large values of time. What do you notice about y, the temperature?

15. **Newton's Law of Heating** A thermometer reading 8°C is brought into a room with a constant temperature of 35°C.
(a) If the thermometer reads 15°C after 3 minutes, what will it read after being in the room for 5 minutes? For 10 minutes?
(b) Graph the relation between temperature and time. TRACE to verify that your answers are correct.

[**Hint:** You need to construct a formula similar to equation (4).]

16. **Warming Time of a Beer Stein** A beer stein has a temperature of 28°F. It is placed in a room with a constant temperature of 70°F. After 10 minutes, the temperature of the stein has risen to 35°F. What will the temperature of the stein be after 30 minutes? How long will it take the stein to reach a temperature of 45°F? (See the hint given for Problem 15.)

17. **Decomposition of Chlorine in a Pool** Under certain water conditions, the free chlorine (hypochlorous acid, HOCl) in a swimming pool decomposes according to the law of uninhibited decay. After shocking his pool, Ben tested the water and found the amount of free chlorine to be 2.5 parts per million (ppm). Twenty-four hours later, Ben tested the water again and found the amount of free chlorine to be 2.2 ppm. What will be the reading after 3 days (that is, 72 hours)? When the chlorine level reaches 1.0 ppm, Ben must shock the pool again. How long can Ben go before he must shock the pool again?

18. **Decomposition of Dinitrogen Pentoxide** At 45°C, dinitrogen pentoxide (N_2O_5) decomposes into nitrous dioxide (NO_2) and oxygen (O_2) according to the law of uninhibited decay. An initial amount of 0.25 mole of dinitrogen pentoxide decomposes to 0.15 mole in 17 minutes. How much dinitrogen pentoxide will remain after 30 minutes? How long will it take until 0.01 mole of dinitrogen pentoxide remains?

19. **Decomposition of Sucrose** Reacting with water in an acidic solution at 35°C, sucrose ($C_{12}H_{22}O_{11}$) decomposes into glucose ($C_6H_{12}O_6$) and fructose ($C_6H_{12}O_6$)* according to the law of uninhibited decay. An initial amount of 0.40 mole of sucrose decomposes to 0.36 mole in 30 minutes. How much sucrose will remain after 2 hours? How long will it take until 0.10 mole of sucrose remains?

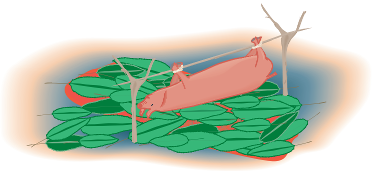
20. **Decomposition of Salt in Water** Salt (NaCl) decomposes in water into sodium (Na^-) and chloride (Cl^-) ions according to the law of uninhibited decay. If the initial amount of salt is 25 kilograms and, after 10 hours, 15 kilograms of salt is left, how much salt is left after 1 day? How long does it take until $\frac{1}{2}$ kilogram of salt is left?

21. **Radioactivity from Chernobyl** After the release of radioactive material into the atmosphere from a nuclear power plant at Chernobyl (Ukraine) in 1986, the hay in Austria was contaminated by iodine-131 (half-life 8 days). If it is safe to feed the hay to cows when 10% of the iodine-131 remains, how long did the farmers need to wait to use this hay?

22. **Pig Roasts** The hotel Bora-Bora is having a pig roast. At noon, the chef put the pig in a large earthen oven. The pig's original temperature was 75°F. At 2:00 PM the chef checked

*Author's Note: Surprisingly, the chemical formulas for glucose and fructose are the same: This is not a typo.

the pig's temperature and was upset because it had reached only 100°F. If the oven's temperature remains a constant 325°F, at what time may the hotel serve its guests, assuming that pork is done when it reaches 175°F?



23. Population of a Bacteria Culture The logistic growth model

$$P(t) = \frac{1000}{1 + 32.33e^{-0.439t}}$$

represents the population (in grams) of a bacterium after t hours.

(a) Determine the carrying capacity of the environment.
(b) What is the growth rate of the bacteria?
(c) Determine the initial population size.
(d) Use a graphing utility to graph $P = P(t)$.
(e) What is the population after 9 hours?
(f) When will the population be 700 grams?
(g) How long does it take for the population to reach one-half the carrying capacity?

24. Population of an Endangered Species Environmentalists often capture an endangered species and transport the species to a controlled environment where the species can produce offspring and regenerate its population. Suppose that six American bald eagles are captured, transported to Montana, and set free. Based on experience, the environmentalists expect the population to grow according to the model

$$P(t) = \frac{500}{1 + 82.33e^{-0.162t}}$$

where t is measured in years.

(a) Determine the carrying capacity of the environment.
(b) What is the growth rate of the bald eagle?
(c) Use a graphing utility to graph $P = P(t)$.
(d) What is the population after 3 years?
(e) When will the population be 300 eagles?
(f) How long does it take for the population to reach one-half of the carrying capacity?

25. Invasive Species A habitat can be altered by invasive species that crowd out or replace native species. The logistic model

$$P(t) = \frac{431}{1 + 7.91e^{-0.017t}}$$

represents the number of invasive species present in the Great Lakes t years after 1900.

(a) Evaluate and interpret $P(0)$.
(b) What is the growth rate of invasive species?
(c) Use a graphing utility to graph $P = P(t)$.
(d) How many invasive species were present in the Great Lakes in 2000?
(e) In what year was the number of invasive species 175?

Source: *NOAA*

26. Word Users According to a survey by Olsten Staffing Services, the percentage of companies reporting usage of Microsoft Word t years since 1984 is given by

$$P(t) = \frac{99.744}{1 + 3.014e^{-0.799t}}$$

(a) What is the growth rate in the percentage of Microsoft Word users?
(b) Use a graphing utility to graph $P = P(t)$.
(c) What was the percentage of Microsoft Word users in 1990?
(d) During what year did the percentage of Microsoft Word users reach 90%?
(e) Explain why the numerator given in the model is reasonable. What does it imply?

27. Home Computers The logistic model

$$P(t) = \frac{95.4993}{1 + 0.0405e^{0.1968t}}$$

represents the percentage of households that do not own a personal computer t years since 1984.

(a) Evaluate and interpret $P(0)$.
(b) Use a graphing utility to graph $P = P(t)$.
(c) What percentage of households did not own a personal computer in 1995?
(d) In what year did the percentage of households that do not own a personal computer reach 10%?

Source: *U.S. Department of Commerce*

28. Farmers The logistic model

$$W(t) = \frac{14{,}656{,}248}{1 + 0.059e^{0.057t}}$$

represents the number of farm workers in the United States t years after 1910.

(a) Evaluate and interpret $W(0)$.
(b) Use a graphing utility to graph $W = W(t)$.
(c) How many farm workers were there in the United States in 2010?
(d) When did the number of farm workers in the United States reach 10,000,000?
(e) According to this model, what happens to the number of farm workers in the United States as t approaches ∞? Based on this result, do you think that it is reasonable to use this model to predict the number of farm workers in the United States in 2060? Why?

Source: *U.S. Department of Agriculture*

29. Birthdays The logistic model

$$P(n) = \frac{113.3198}{1 + 0.115e^{0.0912n}}$$

models the probability that, in a room of n people, no two people share the same birthday.

(a) Use a graphing utility to graph $P = P(n)$.
(b) In a room of $n = 15$ people, what is the probability that no two share the same birthday?

(c) How many people must be in a room before the probability that no two people share the same birthday falls below 10%?
(d) What happens to the probability as n increases? Explain what this result means.

30. Social Networking The logistic model

$$P(t) = \frac{30.3}{1 + 5.31e^{-0.703t}}$$

gives the percentage of Americans who report using social network websites "several times per day," where t represents the number of years after 2008.
(a) Evaluate and interpret $P(0)$.
(b) What is the growth rate?
(c) Use a graphing utility to graph $P = P(t)$.
(d) During 2012, what percentage of Americans visited social network websites several times per day?
(e) In what year did 28.2% of Americans visit social network websites several times per day?
Source: *Edison Research, 2014*

Problems 31 and 32 refer to the following discussion: Uninhibited growth can be modeled by exponential functions other than $A(t) = A_0e^{kt}$. For example, if an initial population P_0 requires n units of time to double, then the function $P(t) = P_0 \cdot 2^{t/n}$ models the size of the population at time t. Likewise, a population requiring n units of time to triple can be modeled by $P(t) = P_0 \cdot 3^{t/n}$.

31. Growth of a Human Population The population of a town is growing exponentially.
(a) If its population doubled in size over an 8-year period and the current population is 25,000, write an exponential function of the form $P(t) = P_0 \cdot 2^{t/n}$ that models the population.
(b) Graph the function using a graphing utility.
(c) What will the population be in 3 years?
(d) When will the population reach 80,000?
(e) Express the model from part (a) in the form $A(t) = A_0e^{kt}$.

32. Growth of an Insect Population An insect population grows exponentially.
(a) If the population triples in 20 days, and 50 insects are present initially, write an exponential function of the form $P(t) = P_0 \cdot 3^{t/n}$ that models the population.
(b) Graph the function using a graphing utility.
(c) What will the population be in 47 days?
(d) When will the population reach 700?
(e) Express the model from part (a) in the form $A(t) = A_0e^{kt}$.

Retain Your Knowledge

Problems 33–36 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

33. Find the equation of the linear function f that passes through the points $(4, 1)$ and $(8, -5)$.

34. Determine whether the graphs of the linear functions $f(x) = 5x - 1$ and $g(x) = \frac{1}{5}x + 1$ are parallel, perpendicular, or neither.

35. Write the logarithmic expression $\ln\left(\frac{x^2\sqrt{y}}{z}\right)$ as the sum and/or difference of logarithms. Express powers as factors.

36. Rationalize the denominator of $\frac{10}{\sqrt[3]{25}}$.

6.9 Building Exponential, Logarithmic, and Logistic Models from Data

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Building Linear Models from Data (Section 4.2, pp. 291–294)
- Building Cubic Models from Data (Section 5.1, pp. 345–346)
- Building Quadratic Models from Data (Section 4.4, pp. 314–315)

OBJECTIVES

1 Build an Exponential Model from Data (p. 496)
2 Build a Logarithmic Model from Data (p. 498)
3 Build a Logistic Model from Data (p. 498)

Finding the linear function of best fit $(y = ax + b)$ for a set of data was discussed in Section 4.2. Likewise, finding the quadratic function of best fit $(y = ax^2 + bx + c)$ and finding the cubic function of best fit $(y = ax^3 + bx^2 + cx + d)$ were discussed in Sections 4.4 and 5.1, respectively.

In this section we discuss how to use a graphing utility to find equations of best fit that describe the relation between two variables when the relation is thought to be exponential $(y = ab^x)$, logarithmic $(y = a + b \ln x)$, or logistic $\left(y = \dfrac{c}{1 + ae^{-bx}}\right)$. As before, a scatter diagram of the data is drawn to help to determine the appropriate model to use.

Figure 61 shows scatter diagrams that will typically be observed for the three models. Below each scatter diagram are any restrictions on the values of the parameters.

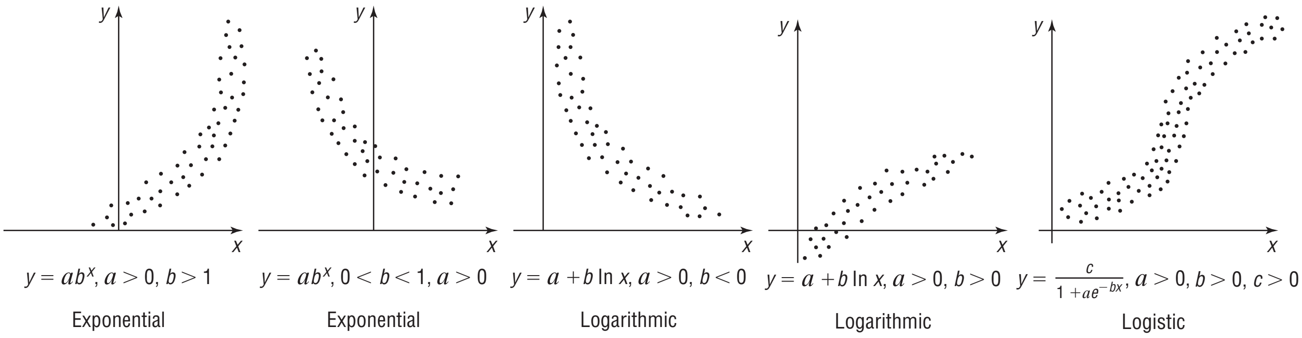


Figure 61

Most graphing utilities have REGression options that fit data to a specific type of curve. Once the data have been entered and a scatter diagram obtained, the type of curve that you want to fit to the data is selected. Then that REGression option is used to obtain the curve of *best fit* of the type selected.

The correlation coefficient r will appear only if the model can be written as a linear expression. As it turns out, r will appear for the linear, power, exponential, and logarithmic models, since these models can be written as a linear expression. Remember, the closer $|r|$ is to 1, the better the fit.

1 Build an Exponential Model from Data

We saw in Section 6.7 that the future value of money behaves exponentially, and we saw in Section 6.8 that growth and decay models also behave exponentially. The next example shows how data can lead to an exponential model.

EXAMPLE 1

Fitting an Exponential Function to Data

Mariah deposited \$20,000 in a well-diversified mutual fund 6 years ago. The data in Table 10 represent the value of the account at the beginning of each year for the last 7 years.

Table 10

Year, x	Account Value, y
0	20,000
1	21,516
2	23,355
3	24,885
4	27,484
5	30,053
6	32,622

(a) Using a graphing utility, draw a scatter diagram with year as the independent variable.

(b) Using a graphing utility, build an exponential model from the data.

(c) Express the function found in part (b) in the form $A = A_0e^{kt}$.

(d) Graph the exponential function found in part (b) or (c) on the scatter diagram.

(e) Using the solution to part (b) or (c), predict the value of the account after 10 years.

(f) Interpret the value of k found in part (c).

Solution (a) Enter the data into the graphing utility and draw the scatter diagram as shown in Figure 62.

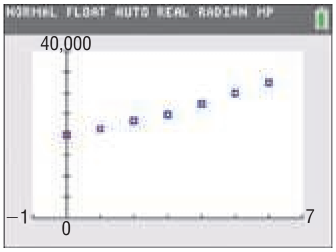


Figure 62

(b) A graphing utility fits the data in Table 10 to an exponential model of the form $y = ab^x$ using the EXPonential REGression option. Figure 63 shows that $y = ab^x = 19{,}820.43(1.085568)^x$. Notice that $|r| = 0.999$, which is close to 1, indicating a good fit.

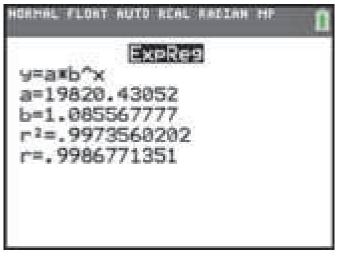


Figure 63

(c) To express $y = ab^x$ in the form $A = A_0e^{kt}$, where $x = t$ and $y = A$, proceed as follows:

$$ab^x = A_0e^{kt}$$

If $x = t = 0$, then $a = A_0$. This leads to

$$b^x = e^{kt}$$
$$b^x = (e^k)^t$$
$$b = e^k \qquad x = t$$

Because $y = ab^x = 19{,}820.43(1.085568)^x$, this means that $a = 19{,}820.43$ and $b = 1.085568$.

$$a = A_0 = 19{,}820.43 \quad \text{and} \quad b = e^k = 1.085568$$

To find k, rewrite $e^k = 1.085568$ as a logarithm to obtain

$$k = \ln(1.085568) \approx 0.08210$$

As a result, $A = A_0e^{kt} = 19{,}820.43e^{0.08210t}$.

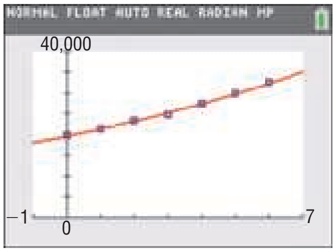


Figure 64

(d) See Figure 64 for the graph of the exponential function of best fit.

(e) Let $t = 10$ in the function found in part (c). The predicted value of the account after 10 years is

$$A = A_0e^{kt} = 19{,}820.43e^{0.08210(10)} \approx \$45{,}047$$

(f) The value of $k = 0.08210 = 8.210\%$ represents the annual growth rate of the account. It represents the rate of interest earned, assuming the account is growing continuously. ■

Now Work PROBLEM 1

2 Build a Logarithmic Model from Data

Some relations between variables follow a logarithmic model.

EXAMPLE 2 Fitting a Logarithmic Function to Data

Jodi, a meteorologist, is interested in finding a function that explains the relation between the height of a weather balloon (in kilometers) and the atmospheric pressure (measured in millimeters of mercury) on the balloon. She collects the data shown in Table 11.

(a) Using a graphing utility, draw a scatter diagram of the data with atmospheric pressure as the independent variable.

(b) It is known that the relation between atmospheric pressure and height follows a logarithmic model. Using a graphing utility, build a logarithmic model from the data.

(c) Draw the logarithmic function found in part (b) on the scatter diagram.

(d) Use the function found in part (b) to predict the height of the weather balloon if the atmospheric pressure is 560 millimeters of mercury.

Table 11

Atmospheric Pressure, p	Height, h
760	0
740	0.184
725	0.328
700	0.565
650	1.079
630	1.291
600	1.634
580	1.862
550	2.235

Solution

(a) Enter the data into the graphing utility, and draw the scatter diagram. See Figure 65.

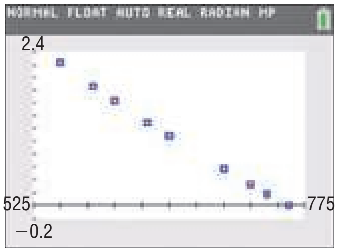


Figure 65

(b) A graphing utility fits the data in Table 11 to a logarithmic function of the form $y = a + b \ln x$ by using the LOGarithm REGression option. See Figure 66. The logarithmic model from the data is

$$h(p) = 45.7863 - 6.9025 \ln p$$

where h is the height of the weather balloon and p is the atmospheric pressure. Notice that $|r|$ is close to 1, indicating a good fit.

(c) Figure 67 shows the graph of $h(p) = 45.7863 - 6.9025 \ln p$ on the scatter diagram.

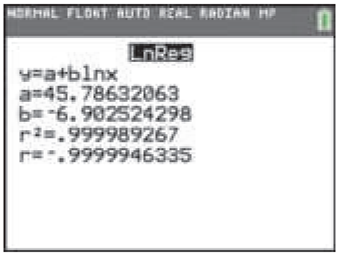


Figure 66

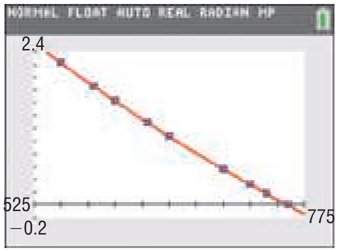


Figure 67

(d) Using the function found in part (b), Jodi predicts the height of the weather balloon when the atmospheric pressure is 560 to be

$$h(560) = 45.7863 - 6.9025 \ln 560$$
$$\approx 2.108 \text{ kilometers}$$

Now Work PROBLEM 5

3 Build a Logistic Model from Data

Logistic growth models can be used to model situations for which the value of the dependent variable is limited. Many real-world situations conform to this scenario. For example, the population of the human race is limited by the availability of natural resources such as food and shelter. When the value of the dependent variable is limited, a logistic growth model is often appropriate.

EXAMPLE 3 Fitting a Logistic Function to Data

The data in Table 12 represent the amount of yeast biomass in a culture after t hours.

Table 12

Time (hours)	Yeast Biomass	Time (hours)	Yeast Biomass	Time (hours)	Yeast Biomass
0	9.6	7	257.3	14	640.8
1	18.3	8	350.7	15	651.1
2	29.0	9	441.0	16	655.9
3	47.2	10	513.3	17	659.6
4	71.1	11	559.7	18	661.8
5	119.1	12	594.8		
6	174.6	13	629.4		

Source: *Tor Carlson (Über Geschwindigkeit and Grösse der Hefevermehrung in Würze, Biochemische Zeitschrift, Bd. 57, pp. 313–334, 1913)*

(a) Using a graphing utility, draw a scatter diagram of the data with time as the independent variable.
(b) Using a graphing utility, build a logistic model from the data.
(c) Using a graphing utility, graph the function found in part (b) on the scatter diagram.
(d) What is the predicted carrying capacity of the culture?
(e) Use the function found in part (b) to predict the population of the culture at $t = 19$ hours.

Solution

(a) See Figure 68 for a scatter diagram of the data.

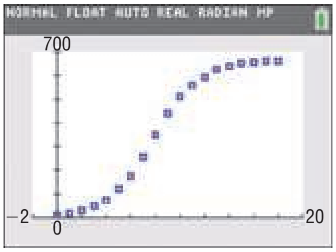


Figure 68

(b) A graphing utility fits the data in Table 12 to a logistic growth model of the form $y = \dfrac{c}{1 + ae^{-bx}}$ by using the LOGISTIC regression option. See Figure 69. The logistic model from the data is

$$y = \frac{663.0}{1 + 71.6e^{-0.5470x}}$$

where y is the amount of yeast biomass in the culture and x is the time.

(c) See Figure 70 for the graph of the logistic model.

Figure 69

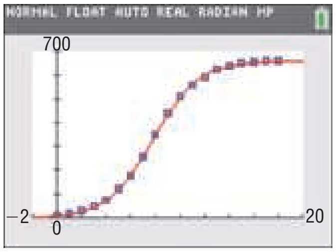


Figure 70

(d) Based on the logistic growth model found in part (b), the carrying capacity of the culture is 663.

(e) Using the logistic growth model found in part (b), the predicted amount of yeast biomass at $t = 19$ hours is

$$y = \frac{663.0}{1 + 71.6e^{-0.5470(19)}} \approx 661.5$$

Now Work PROBLEM 7

6.9 Assess Your Understanding

Applications and Extensions

1. **Biology** A strain of *E. coli*, Beu 397-recA441, is placed into a nutrient broth at 30° Celsius and allowed to grow. The following data are collected. Theory states that the number of bacteria in the petri dish will initially grow according to the law of uninhibited growth. The population is measured using an optical device in which the amount of light that passes through the petri dish is measured.

Time (hours), x	Population, y
0	0.09
2.5	0.18
3.5	0.26
4.5	0.35
6	0.50

Source: *Dr. Polly Lavery, Joliet Junior College*

(a) Draw a scatter diagram treating time as the independent variable.
(b) Using a graphing utility, build an exponential model from the data.
(c) Express the function found in part (b) in the form $N(t) = N_0e^{kt}$.
(d) Graph the exponential function found in part (b) or (c) on the scatter diagram.
(e) Use the exponential function from part (b) or (c) to predict the population at $x = 7$ hours.
(f) Use the exponential function from part (b) or (c) to predict when the population will reach 0.75.

2. **Ethanol Production** The data in the table below represent ethanol production (in billions of gallons) in the United States from 2000 to 2014.

Year	Ethanol Produced (billion gallons)	Year	Ethanol Produced (billion gallons)
2000 $(x = 0)$	1.6	2008 $(x = 8)$	9.2
2001 $(x = 1)$	1.8	2009 $(x = 9)$	10.8
2002 $(x = 2)$	2.1	2010 $(x = 10)$	13.2
2003 $(x = 3)$	2.8	2011 $(x = 11)$	13.9
2004 $(x = 4)$	3.4	2012 $(x = 12)$	13.3
2005 $(x = 5)$	3.9	2013 $(x = 13)$	13.3
2006 $(x = 6)$	4.9	2014 $(x = 14)$	14.3
2007 $(x = 7)$	6.5		

Source: *Renewable Fuels Association, 2015*

(a) Using a graphing utility, draw a scatter diagram of the data using 0 for 2000, 1 for 2001, and so on, as the independent variable.
(b) Using a graphing utility, build an exponential model from the data.
(c) Express the function found in part (b) in the form $A(t) = A_0e^{kt}$.
(d) Graph the exponential function found in part (b) or (c) on the scatter diagram.
(e) Use the model to predict the amount of ethanol that will be produced in 2016.
(f) Interpret the meaning of k in the function found in part (c).

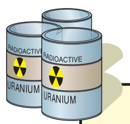
3. **Advanced-Stage Breast Cancer** The data in the table below represent the percentage of patients who have survived after diagnosis of advanced-stage breast cancer at 6-month intervals of time.

Time after Diagnosis (years)	Percentage Surviving
0.5	95.7
1	83.6
1.5	74.0
2	58.6
2.5	47.4
3	41.9
3.5	33.6

Source: *Cancer Treatment Centers of America*

(a) Using a graphing utility, draw a scatter diagram of the data with time after diagnosis as the independent variable.
(b) Using a graphing utility, build an exponential model from the data.
(c) Express the function found in part (b) in the form $A(t) = A_0e^{kt}$.
(d) Graph the exponential function found in part (b) or (c) on the scatter diagram.
(e) What percentage of patients diagnosed with advanced-stage cancer are expected to survive for 4 years after initial diagnosis?
(f) Interpret the meaning of k in the function found in part (c).

4. **Chemistry** A chemist has a 100-gram sample of a radioactive material. He records the amount of radioactive material every week for 7 weeks and obtains the following data:



Week	Weight (grams)
0	100.0
1	88.3
2	75.9
3	69.4
4	59.1
5	51.8
6	45.5

(a) Using a graphing utility, draw a scatter diagram with week as the independent variable.
(b) Using a graphing utility, build an exponential model from the data.
(c) Express the function found in part (b) in the form $A(t) = A_0 e^{kt}$.
(d) Graph the exponential function found in part (b) or (c) on the scatter diagram.
(e) From the result found in part (b), determine the half-life of the radioactive material.
(f) How much radioactive material will be left after 50 weeks?
(g) When will there be 20 grams of radioactive material?

5. Milk Production The data in the table below represent the number of dairy farms (in thousands) and the amount of milk produced (in billions of pounds) in the United States for various years.

Year	Dairy Farms (thousands)	Milk Produced (billion pounds)
1980	334	128
1985	269	143
1990	193	148
1995	140	155
2000	105	167
2005	78	177
2010	63	193

Source: *Statistical Abstract of the United States, 2012*

(a) Using a graphing utility, draw a scatter diagram of the data with the number of dairy farms as the independent variable.
(b) Using a graphing utility, build a logarithmic model from the data.
(c) Graph the logarithmic function found in part (b) on the scatter diagram.
(d) In 2008, there were 67 thousand dairy farms in the United States. Use the function in part (b) to predict the amount of milk produced in 2008.
(e) The actual amount of milk produced in 2008 was 190 billion pounds. How does your prediction in part (d) compare to this?

6. Social Networking The data in the table below represent the percent of U.S. citizens aged 12 and older who have a profile on at least one social network.

Year	Percent on a Social Networking Site
2008 $(x = 8)$	24
2009 $(x = 9)$	34
2010 $(x = 10)$	48
2011 $(x = 11)$	52
2012 $(x = 12)$	53
2013 $(x = 13)$	62
2014 $(x = 14)$	67

Source: *Edison Research, 2014*

(a) Using a graphing utility, draw a scatter diagram of the data using 8 for 2008, 9 for 2009, and so on, as the independent variable, and percent on social networking site as the dependent variable.
(b) Using a graphing utility, build a logarithmic model from the data.
(c) Graph the logarithmic function found in part (b) on the scatter diagram.
(d) Use the model to predict the percent of U.S. citizens on social networking sites in 2015.
(e) Use the model to predict the year in which 90% of U.S. citizens will be on social networking sites.

7. Population Model The following data represent the population of the United States. An ecologist is interested in building a model that describes the population of the United States.

Year	Population
1900	76,212,168
1910	92,228,496
1920	106,021,537
1930	123,202,624
1940	132,164,569
1950	151,325,798
1960	179,323,175
1970	203,302,031
1980	226,542,203
1990	248,709,873
2000	281,421,906
2010	308,745,538

Source: *U.S. Census Bureau*

(a) Using a graphing utility, draw a scatter diagram of the data using years since 1900 as the independent variable and population as the dependent variable.
(b) Using a graphing utility, build a logistic model from the data.
(c) Using a graphing utility, draw the function found in part (b) on the scatter diagram.
(d) Based on the function found in part (b), what is the carrying capacity of the United States?
(e) Use the function found in part (b) to predict the population of the United States in 2012.
(f) When will the United States population be 350,000,000?
(g) Compare actual U.S. Census figures to the predictions found in parts (e) and (f). Discuss any differences.

8. Population Model The data on the right represent the world population. An ecologist is interested in building a model that describes the world population.

(a) Using a graphing utility, draw a scatter diagram of the data using years since 2000 as the independent variable and population as the dependent variable.
(b) Using a graphing utility, build a logistic model from the data.
(c) Using a graphing utility, draw the function found in part (b) on the scatter diagram.
(d) Based on the function found in part (b), what is the carrying capacity of the world?
(e) Use the function found in part (b) to predict the population of the world in 2021.
(f) When will world population be 10 billion?

Year	Population (billions)	Year	Population (billions)
2001	6.17	2008	6.71
2002	6.24	2009	6.79
2003	6.32	2010	6.87
2004	6.40	2011	6.94
2005	6.47	2012	7.02
2006	6.55	2013	7.10
2007	6.63	2014	7.18

Source: *U.S. Census Bureau*

9. Cell Phone Towers The following data represent the number of cell sites in service in the United States from 1985 to 2013 at the end of each year.

Year	Cell Sites (thousands)	Year	Cell Sites (thousands)	Year	Cell Sites (thousands)
1985 $(x = 1)$	0.9	1995 $(x = 11)$	22.7	2005 $(x = 21)$	183.7
1986 $(x = 2)$	1.5	1996 $(x = 12)$	30.0	2006 $(x = 22)$	195.6
1987 $(x = 3)$	2.3	1997 $(x = 13)$	51.6	2007 $(x = 23)$	213.3
1988 $(x = 4)$	3.2	1998 $(x = 14)$	65.9	2008 $(x = 24)$	242.1
1989 $(x = 5)$	4.2	1999 $(x = 15)$	81.7	2009 $(x = 25)$	247.1
1990 $(x = 6)$	5.6	2000 $(x = 16)$	104.3	2010 $(x = 26)$	253.1
1991 $(x = 7)$	7.8	2001 $(x = 17)$	127.5	2011 $(x = 27)$	283.4
1992 $(x = 8)$	10.3	2002 $(x = 18)$	139.3	2012 $(x = 28)$	301.8
1993 $(x = 9)$	12.8	2003 $(x = 19)$	163.0	2013 $(x = 29)$	304.4
1994 $(x = 10)$	17.9	2004 $(x = 20)$	175.7		

Source: *©2014 CTIA-The Wireless Association®. All Rights Reserved.*

(a) Using a graphing utility, draw a scatter diagram of the data using 1 for 1985, 2 for 1986, and so on as the independent variable, and number of cell sites as the dependent variable.
(b) Using a graphing utility, build a logistic model from the data.
(c) Graph the logistic function found in part (b) on the scatter diagram.
(d) What is the predicted carrying capacity for cell sites in the United States?
(e) Use the model to predict the number of cell sites in the United States at the end of 2019.

10. Cable Rates The data on the right represent the average monthly rate charged for expanded basic cable television in the United States from 1995 to 2014. A market researcher believes that external factors, such as the growth of satellite television and internet programming, have affected the cost of basic cable. She is interested in building a model that will describe the average monthly cost of basic cable.

(a) Using a graphing utility, draw a scatter diagram of the data using 0 for 1995, 1 for 1996, and so on, as the independent variable and average monthly rate as the dependent variable.
(b) Using a graphing utility, build a logistic model from the data.
(c) Graph the logistic function found in part (b) on the scatter diagram.
(d) Based on the model found in part (b), what is the maximum possible average monthly rate for basic cable?
(e) Use the model to predict the average rate for basic cable in 2018.

Year	Average Monthly Rate (dollars)	Year	Average Monthly Rate (dollars)
1995 $(x = 0)$	22.35	2005 $(x = 10)$	43.04
1996 $(x = 1)$	24.28	2006 $(x = 11)$	45.26
1997 $(x = 2)$	26.31	2007 $(x = 12)$	47.27
1998 $(x = 3)$	27.88	2008 $(x = 13)$	49.65
1999 $(x = 4)$	28.94	2009 $(x = 14)$	52.37
2000 $(x = 5)$	31.22	2010 $(x = 15)$	54.44
2001 $(x = 6)$	33.75	2011 $(x = 16)$	57.46
2002 $(x = 7)$	36.47	2012 $(x = 17)$	61.63
2003 $(x = 8)$	38.95	2013 $(x = 18)$	64.41
2004 $(x = 9)$	41.04	2014 $(x = 19)$	66.61

Source: *Federal Communications Commission, 2014*

Mixed Practice

11. **Online Advertising Revenue** The data in the table below represent the U.S. online advertising revenues for the years 2005–2014.

Year	U.S. Online Advertising Revenue ($ billions)
2005 $(x = 0)$	12.5
2006 $(x = 1)$	16.9
2007 $(x = 2)$	21.2
2008 $(x = 3)$	23.4
2009 $(x = 4)$	22.7
2010 $(x = 5)$	26.0
2011 $(x = 6)$	31.7
2012 $(x = 7)$	36.6
2013 $(x = 8)$	42.8
2014 $(x = 9)$	49.5

Source: *marketingcharts.com*

(a) Using a graphing utility, draw a scatter diagram of the data using 0 for 2005, 1 for 2006, and so on as the independent variable, and online advertising revenue as the dependent variable.
(b) Based on the scatter diagram drawn in part (a), decide what model (linear, quadratic, cubic, exponential, logarithmic, or logistic) that you think best describes the relation between year and revenue.
(c) Using a graphing utitlity, find the model of best fit.
(d) Using a graphing utility, draw the model of best fit on the scatter diagram drawn in part (a).
(e) Use your model to predict the online advertising revenue in 2016.

12. **Age versus Total Cholesterol** The following data represent the age and average total cholesterol for adult males at various ages.

Age	Total Cholesterol
27	189
40	205
50	215
60	210
70	210
80	194

(a) Using a graphing utility, draw a scatter diagram of the data using age, x, as the independent variable and total cholesterol, y, as the dependent variable.
(b) Based on the scatter diagram drawn in part (a), decide on a model (linear, quadratic, cubic, exponential, logarithmic, or logistic) that you think best describes the relation between age and total cholesterol. Be sure to justify your choice of model.
(c) Using a graphing utility, find the model of best fit.
(d) Using a graphing utility, draw the model of best fit on the scatter diagram drawn in part (a).
(e) Use your model to predict the total cholesterol of a 35-year-old male.

13. **Golfing** The data below represent the expected percentage of putts that will be made by professional golfers on the PGA Tour depending on distance. For example, it is expected that 99.3% of 2-foot putts will be made.

Distance (feet)	Expected Percentage	Distance (feet)	Expected Percentage
2	99.3	14	25.0
3	94.8	15	22.0
4	85.8	16	20.0
5	74.7	17	19.0
6	64.7	18	17.0
7	55.6	19	16.0
8	48.5	20	14.0
9	43.4	21	13.0
10	38.3	22	12.0
11	34.2	23	11.0
12	30.1	24	11.0
13	27.0	25	10.0

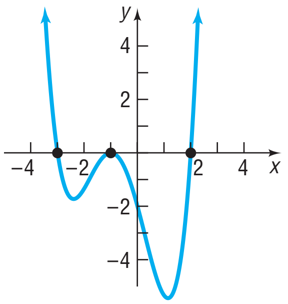
Source: *TheSandTrap.com*

(a) Using a graphing utility, draw a scatter diagram of the data with distance as the independent variable.
(b) Based on the scatter diagram drawn in part (a), decide on a model (linear, quadratic, cubic, exponential, logarithmic, or logistic) that you think best describes the relation between distance and expected percentage. Be sure to justify your choice of model.
(c) Using a graphing utility, find the model of best fit.
(d) Graph the function found in part (c) on the scatter diagram.
(e) Use the function found in part (c) to predict what percentage of 30-foot putts will be made.

Retain Your Knowledge

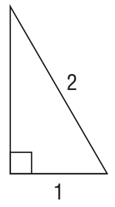
Problems 14–17 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

14. Construct a polynomial function that might have the graph shown. (More than one answer is possible.)



15. Rationalize the denominator of $\dfrac{3}{\sqrt{2}}$.

16. Use the Pythagorean Theorem to find the exact length of the unlabeled side in the given right triangle.



17. Graph the equation $(x-3)^2 + y^2 = 25$.

Chapter Review

Things to Know

Composite function (p. 408)	$(f \circ g)(x) = f(g(x))$ The domain of $f \circ g$ is the set of all numbers x in the domain of g for which $g(x)$ is in the domain of f.
One-to-one function (p. 416)	A function for which any two different inputs in the domain correspond to two different outputs in the range For any choice of elements x_1, x_2 in the domain of f, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.
Horizontal-line test (p. 417)	If every horizontal line intersects the graph of a function f in at most one point, f is one-to-one.
Inverse function f^{-1} of f (pp. 418–420, 421)	Domain of f = range of f^{-1}; range of f = domain of f^{-1} $f^{-1}(f(x)) = x$ for all x in the domain of f $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1} The graphs of f and f^{-1} are symmetric with respect to the line $y = x$.
Properties of the exponential function (pp. 430, 433–434, 435)	$f(x) = Ca^x, \quad a > 1, C > 0$ — Domain: the interval $(-\infty, \infty)$ Range: the interval $(0, \infty)$ x-intercepts: none; y-intercept: C Horizontal asymptote: x-axis $(y = 0)$ as $x \to -\infty$ Increasing; one-to-one; smooth; continuous See Figure 24 for a typical graph. $f(x) = Ca^x, \quad 0 < a < 1, C > 0$ — Domain: the interval $(-\infty, \infty)$ Range: the interval $(0, \infty)$ x-intercepts: none; y-intercept: C Horizontal asymptote: x-axis $(y = 0)$ as $x \to \infty$ Decreasing; one-to-one; smooth; continuous See Figure 29 for a typical graph.
Number e (p. 436)	Value approached by the expression $\left(1 + \dfrac{1}{n}\right)^n$ as $n \to \infty$; that is, $\lim_{n\to\infty}\left(1 + \dfrac{1}{n}\right)^n = e$.
Property of exponents (p. 437)	If $a^u = a^v$, then $u = v$.
Properties of the logarithmic function (pp. 445–446, 447, 448)	$f(x) = \log_a x, \quad a > 1$ $(y = \log_a x$ means $x = a^y)$ — Domain: the interval $(0, \infty)$ Range: the interval $(-\infty, \infty)$ x-intercept: 1; y-intercept: none Vertical asymptote: $x = 0$ (y-axis) Increasing; one-to-one; smooth; continuous See Figure 45(a) for a typical graph.

	$f(x) = \log_a x, \quad 0 < a < 1$ $(y = \log_a x \text{ means } x = a^y)$	Domain: the interval $(0, \infty)$ Range: the interval $(-\infty, \infty)$ x-intercept: 1; y-intercept: none Vertical asymptote: $x = 0$ (y-axis) Decreasing; one-to-one; smooth; continuous See Figure 45(b) for a typical graph.

Natural logarithm (p. 449) $\quad y = \ln x$ means $x = e^y$.

Properties of logarithms (pp. 459–460, 462)

$\log_a 1 = 0 \qquad \log_a a = 1 \qquad a^{\log_a M} = M \qquad \log_a a^r = r$

$\log_a (MN) = \log_a M + \log_a N \qquad \log_a \left(\dfrac{M}{N}\right) = \log_a M - \log_a N$

$\log_a M^r = r \log_a M \qquad a^r = e^{r \ln a}$

If $M = N$, then $\log_a M = \log_a N$.

If $\log_a M = \log_a N$, then $M = N$.

Formulas

Change-of-Base Formula (p. 463) $\quad \log_a M = \dfrac{\log_b M}{\log_b a}$

Compound Interest Formula (p. 476) $\quad A = P \cdot \left(1 + \dfrac{r}{n}\right)^{nt}$

Continuous compounding (p. 478) $\quad A = Pe^{rt}$

Effective rate of interest (p. 478) $\quad$ Compounding n times per year: $r_e = \left(1 + \dfrac{r}{n}\right)^n - 1$

Continuous compounding: $r_e = e^r - 1$

Present Value Formulas (p. 479) $\quad P = A \cdot \left(1 + \dfrac{r}{n}\right)^{-nt}$ or $P = Ae^{-rt}$

Growth and decay (p. 485, 487) $\quad A(t) = A_0 e^{kt}$

Newton's Law of Cooling (p. 488) $\quad u(t) = T + (u_0 - T)e^{kt} \quad k < 0$

Logistic model (p. 489) $\quad P(t) = \dfrac{c}{1 + ae^{-bt}}$

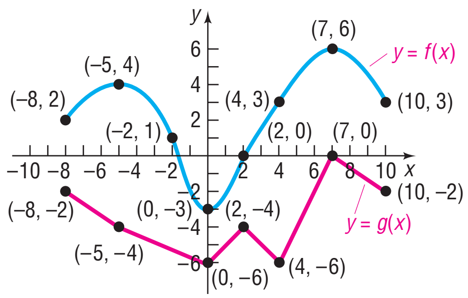
Objectives

Section		You should be able to ...	Examples	Review Exercises
6.1	1	Form a composite function (p. 408)	1, 2, 4, 5	1–7
	2	Find the domain of a composite function (p. 409)	2–4	5–7
6.2	1	Determine whether a function is one-to-one (p. 416)	1, 2	8(a), 9
	2	Determine the inverse of a function defined by a map or a set of ordered pairs (p. 418)	3, 4	8(b)
	3	Obtain the graph of the inverse function from the graph of the function (p. 421)	7	9
	4	Find the inverse of a function defined by an equation (p. 422)	8, 9, 10	10–13
6.3	1	Evaluate exponential functions (p. 428)	1	14(a), (c), 47(a)
	2	Graph exponential functions (p. 432)	3–6	31–33
	3	Define the number e (p. 436)	pg. 436	33
	4	Solve exponential equations (p. 437)	7, 8	35, 36, 39, 41
6.4	1	Change exponential statements to logarithmic statements and logarithmic statements to exponential statements (p. 446)	2, 3	15, 16
	2	Evaluate logarithmic expressions (p. 446)	4	14(b), (d), 19, 46(b), 48(a), 49
	3	Determine the domain of a logarithmic function (p. 447)	5	17, 18, 34(a)
	4	Graph logarithmic functions (p. 448)	6, 7	34(b), 46(a)
	5	Solve logarithmic equations (p. 452)	8, 9	37, 40, 46(c), 48(b)

Section	You should be able to ...	Examples	Review Exercises
6.5	1 Work with the properties of logarithms (p. 458)	1, 2	20, 21
	2 Write a logarithmic expression as a sum or difference of logarithms (p. 460)	3–5	22–25
	3 Write a logarithmic expression as a single logarithm (p. 461)	6	26–28
	4 Evaluate a logarithm whose base is neither 10 nor e (p. 462)	7, 8	29
	5 Graph a logarithmic function whose base is neither 10 nor e (p. 464)	9	30
6.6	1 Solve logarithmic equations (p. 467)	1–3	37, 43
	2 Solve exponential equations (p. 469)	4–7	38, 42, 44, 45
	3 Solve logarithmic and exponential equations using a graphing utility (p. 471)	8	35–45
6.7	1 Determine the future value of a lump sum of money (p. 475)	1–3	50
	2 Calculate effective rates of return (p. 478)	4	50
	3 Determine the present value of a lump sum of money (p. 479)	5	51
	4 Determine the rate of interest or the time required to double a lump sum of money (p. 480)	6, 7	50
6.8	1 Find equations of populations that obey the law of uninhibited growth (p. 484)	1, 2	54
	2 Find equations of populations that obey the law of decay (p. 487)	3	52, 55
	3 Use Newton's Law of Cooling (p. 488)	4	53
	4 Use logistic models (p. 489)	5, 6	56
6.9	1 Build an exponential model from data (p. 496)	1	57
	2 Build a logarithmic model from data (p. 498)	2	58
	3 Build a logistic model from data (p. 498)	3	59

Review Exercises

1. Evaluate each expression using the graphs of $y = f(x)$ and $y = g(x)$ shown in the figure.



(a) $(g \circ f)(-8)$
(b) $(f \circ g)(-8)$
(c) $(g \circ g)(7)$
(d) $(g \circ f)(-5)$

In Problems 2–4, for the given functions f and g find:
(a) $(f \circ g)(2)$
(b) $(g \circ f)(-2)$
(c) $(f \circ f)(4)$
(d) $(g \circ g)(-1)$

2. $f(x) = 3x - 5; \quad g(x) = 1 - 2x^2$

3. $f(x) = \sqrt{x + 2}: \quad g(x) = 2x^2 + 1$

4. $f(x) = e^x; \quad g(x) = 3x - 2$

In Problems 5–7, find $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$ for each pair of functions. State the domain of each composite function.

5. $f(x) = 2 - x; \quad g(x) = 3x + 1$

6. $f(x) = \sqrt{3x}; \quad g(x) = 1 + x + x^2$

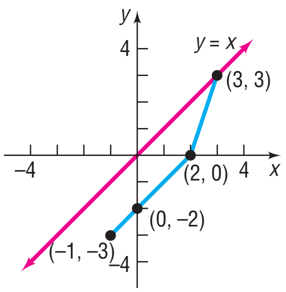
7. $f(x) = \dfrac{x + 1}{x - 1}; \quad g(x) = \dfrac{1}{x}$

In Problem 8, (a) verify that the function is one-to-one, and (b) find the inverse of the given function.

8. $\{(1, 2), (3, 5), (5, 8), (6, 10)\}$

In Problem 9, state why the graph of the function is one-to-one. Then draw the graph of the inverse function f^{-1}. For convenience (and as a hint), the graph of $y = x$ is also given.

9.



In Problems 10–13, the function f is one-to-one. Find the inverse of each function and check your answer. State the domain and the range of f and f^{-1}.

10. $f(x) = \dfrac{2x+3}{5x-2}$

11. $f(x) = \dfrac{1}{x-1}$

12. $f(x) = \sqrt{x-2}$

13. $f(x) = x^{1/3} + 1$

In Problem 14, $f(x) = 3^x$ and $g(x) = \log_3 x$.

14. Evaluate: (a) $f(4)$ (b) $g(9)$ (c) $f(-2)$ (d) $g\left(\dfrac{1}{27}\right)$

15. Convert $5^2 = z$ to an equivalent statement involving a logarithm.

16. Convert $\log_5 u = 13$ to an equivalent statement involving an exponent.

In Problems 17 and 18, find the domain of each logarithmic function.

17. $f(x) = \log(3x-2)$

18. $H(x) = \log_2(x^2 - 3x + 2)$

In Problems 19–21, evaluate each expression. Do not use a calculator.

19. $\log_2\left(\dfrac{1}{8}\right)$

20. $\ln e^{\sqrt{2}}$

21. $2^{\log_2 0.4}$

In Problems 22–25, write each expression as the sum and/or difference of logarithms. Express powers as factors.

22. $\log_3\left(\dfrac{uv^2}{w}\right), \quad u > 0, v > 0, w > 0$

23. $\log_2(a^2\sqrt{b})^4, \quad a > 0, b > 0$

24. $\log(x^2\sqrt{x^3+1}), \quad x > 0$

25. $\ln\left(\dfrac{2x+3}{x^2-3x+2}\right)^2, \quad x > 2$

In Problems 26–28, write each expression as a single logarithm.

26. $3\log_4 x^2 + \dfrac{1}{2}\log_4\sqrt{x}$

27. $\ln\left(\dfrac{x-1}{x}\right) + \ln\left(\dfrac{x}{x+1}\right) - \ln(x^2-1)$

28. $\dfrac{1}{2}\ln(x^2+1) - 4\ln\dfrac{1}{2} - \dfrac{1}{2}[\ln(x-4) + \ln x]$

29. Use the Change-of-Base Formula and a calculator to evaluate $\log_4 19$. Round your answer to three decimal places.

30. Graph $y = \log_3 x$ using a graphing utility and the Change-of-Base Formula.

In Problems 31–34, use the given function f to:

(a) Find the domain of f. (b) Graph f. (c) From the graph, determine the range and any asymptotes of f.
(d) Find f^{-1}, the inverse of f. (e) Find the domain and the range of f^{-1}. (f) Graph f^{-1}.

31. $f(x) = 2^{x-3}$

32. $f(x) = 1 + 3^{-x}$

33. $f(x) = 3e^{x-2}$

34. $f(x) = \dfrac{1}{2}\ln(x+3)$

In Problems 35–45, solve each equation. Express irrational solutions in exact form and as a decimal rounded to 3 decimal places. Verify your results using a graphing utility.

35. $8^{6+3x} = 4$

36. $3^{x^2+x} = \sqrt{3}$

37. $\log_x 64 = -3$

38. $5^x = 3^{x+2}$

39. $25^{2x} = 5^{x^2-12}$

40. $\log_3\sqrt{x-2} = 2$

41. $8 = 4^{x^2}\cdot 2^{5x}$

42. $2^x\cdot 5 = 10^x$

43. $\log_6(x+3) + \log_6(x+4) = 1$

44. $e^{1-x} = 5$

45. $9^x + 4\cdot 3^x - 3 = 0$

---

46. Suppose that $f(x) = \log_2(x-2) + 1$.

(a) Graph f.

(b) What is $f(6)$? What point is on the graph of f?

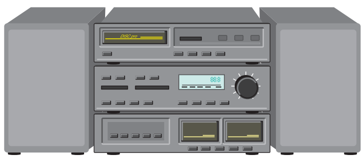
(c) Solve $f(x) = 4$. What point is on the graph of f?

(d) Based on the graph drawn in part (a), solve $f(x) > 0$.

(e) Find $f^{-1}(x)$. Graph f^{-1} on the same Cartesian plane as f.

47. Amplifying Sound An amplifier's power output P (in watts) is related to its decibel voltage gain d by the formula

$$P = 25e^{0.1d}$$



(a) Find the power output for a decibel voltage gain of 4 decibels.

(b) For a power output of 50 watts, what is the decibel voltage gain?

48. Limiting Magnitude of a Telescope A telescope is limited in its usefulness by the brightness of the star that it is aimed at and by the diameter of its lens. One measure of a star's brightness is its *magnitude;* the dimmer the star, the larger its magnitude. A formula for the limiting magnitude L of a telescope, that is, the magnitude of the dimmest star that it can be used to view, is given by

$$L = 9 + 5.1 \log d$$

where d is the diameter (in inches) of the lens.
(a) What is the limiting magnitude of a 3.5-inch telescope?
(b) What diameter is required to view a star of magnitude 14?

49. Salvage Value The number of years n for a piece of machinery to depreciate to a known salvage value can be found using the formula

$$n = \frac{\log s - \log i}{\log(1 - d)}$$

where s is the salvage value of the machinery, i is its initial value, and d is the annual rate of depreciation.
(a) How many years will it take for a piece of machinery to decline in value from \$90,000 to \$10,000 if the annual rate of depreciation is 0.20 (20%)?
(b) How many years will it take for a piece of machinery to lose half of its value if the annual rate of depreciation is 15%?

50. Funding a College Education A child's grandparents purchase a \$10,000 bond fund that matures in 18 years to be used for her college education. The bond fund pays 4% interest compounded semiannually. How much will the bond fund be worth at maturity? What is the effective rate of interest? How long will it take the bond to double in value under these terms?

51. Funding a College Education A child's grandparents wish to purchase a bond that matures in 18 years to be used for her college education. The bond pays 4% interest compounded semiannually. How much should they pay so that the bond will be worth \$85,000 at maturity?

52. Estimating the Date That a Prehistoric Man Died The bones of a prehistoric man found in the desert of New Mexico contain approximately 5% of the original amount of carbon 14. If the half-life of carbon 14 is 5730 years, approximately how long ago did the man die?

53. Temperature of a Skillet A skillet is removed from an oven whose temperature is 450°F and placed in a room whose temperature is 70°F. After 5 minutes, the temperature of the skillet is 400°F. How long will it be until its temperature is 150°F?

54. World Population The annual growth rate of the world's population in 2015 was $k = 1.08\% = 0.0108$. The population of the world in 2015 was 7,214,958,996. Letting $t = 0$ represent 2015, use the uninhibited growth model to predict the world's population in the year 2020.
Source: *U.S. Census Bureau*

55. Radioactive Decay The half-life of cobalt is 5.27 years. If 100 grams of radioactive cobalt is present now, how much will be present in 20 years? In 40 years?

56. Logistic Growth The logistic growth model

$$P(t) = \frac{0.8}{1 + 1.67e^{-0.16t}}$$

represents the proportion of new cars with a global positioning system (GPS). Let $t = 0$ represent 2006, $t = 1$ represent 2007, and so on.
(a) What proportion of new cars in 2006 had a GPS?
(b) Determine the maximum proportion of new cars that have a GPS.
(c) Using a graphing utility, graph $P = P(t)$.
(d) When will 75% of new cars have a GPS?

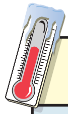
57. Rising Tuition The following data represent the average in-state tuition and fees (in 2013 dollars) at public four-year colleges and universities in the United States from the academic year 1983–84 to the academic year 2013–14.

Academic Year	Tuition and Fees (2013 dollars)
1983–84 ($x = 0$)	2684
1988–89 ($x = 5$)	3111
1993–94 ($x = 10$)	4101
1998–99 ($x = 15$)	4648
2003–04 ($x = 20$)	5900
2008–09 ($x = 25$)	7008
2013–14 ($x = 30$)	8893

Source: *The College Board*

(a) Using a graphing utility, draw a scatter diagram with academic year as the independent variable.
(b) Using a graphing utility, build an exponential model from the data.
(c) Express the function found in part (b) in the form $A(t) = A_0e^{kt}$.
(d) Graph the exponential function found in part (b) or (c) on the scatter diagram.
(e) Predict the academic year when the average tuition will reach \$12,000.

58. Wind Chill Factor The following data represent the wind speed (mph) and wind chill factor at an air temperature of 15°F.



Wind Speed (mph)	Wind Chill Factor (°F)
5	7
10	3
15	0
20	−2
25	−4
30	−5
35	−7

Source: *U.S. National Weather Service*

(a) Using a graphing utility, draw a scatter diagram with wind speed as the independent variable.
(b) Using a graphing utility, build a logarithmic model from the data.
(c) Using a graphing utility, draw the logarithmic function found in part (b) on the scatter diagram.
(d) Use the function found in part (b) to predict the wind chill factor if the air temperature is 15°F and the wind speed is 23 mph.

59. Spreading of a Disease Jack and Diane live in a small town of 50 people. Unfortunately, both Jack and Diane have a cold. Those who come in contact with someone who has this cold will themselves catch the cold. The following data represent the number of people in the small town who have caught the cold after t days.

Days, t	Number of People with Cold, C
0	2
1	4
2	8
3	14
4	22
5	30
6	37
7	42
8	44

(a) Using a graphing utility, draw a scatter diagram of the data. Comment on the type of relation that appears to exist between the days and number of people with a cold.
(b) Using a graphing utility, build a logistic model from the data.
(c) Graph the function found in part (b) on the scatter diagram.
(d) According to the function found in part (b), what is the maximum number of people who will catch the cold? In reality, what is the maximum number of people who could catch the cold?
(e) Sometime between the second and third day, 10 people in the town had a cold. According to the model found in part (b), when did 10 people have a cold?
(f) How long will it take for 46 people to catch the cold?

Chapter Test

CHAPTER **Test Prep** VIDEOS

The Chapter Test Prep Videos are step-by-step solutions available in MyMathLab®, or on this text's YouTube™ Channel. Flip back to the Resources for Success page for a link to this text's YouTube channel.

1. Given $f(x) = \dfrac{x+2}{x-2}$ and $g(x) = 2x + 5$, find:
(a) $f \circ g$ and state its domain (b) $(g \circ f)(-2)$
(c) $(f \circ g)(-2)$

2. Determine whether the function is one-to-one.
(a) $y = 4x^2 + 3$
(b) $y = \sqrt{x+3} - 5$

3. Find the inverse of $f(x) = \dfrac{2}{3x-5}$ and check your answer. State the domain and the range of f and f^{-1}.

4. If the point $(3, -5)$ is on the graph of a one-to-one function f, what point must be on the graph of f^{-1}?

In Problems 5–7, solve each equation.

5. $3^x = 243$ **6.** $\log_b 16 = 2$ **7.** $\log_5 x = 4$

In Problems 8–11, use a calculator to evaluate each expression. Round your answer to three decimal places.

8. $e^3 + 2$ **9.** $\log 20$
10. $\log_3 21$ **11.** $\ln 133$

In Problems 12 and 13, use the given function f to:

(a) Find the domain of f.
(b) Graph f.
(c) From the graph, determine the range and any asymptotes of f.
(d) Find f^{-1}, the inverse of f.
(e) Find the domain and the range of f^{-1}.
(f) Graph f^{-1}.

12. $f(x) = 4^{x+1} - 2$ **13.** $f(x) = 1 - \log_5(x-2)$

In Problems 14–19, solve each equation.

14. $5^{x+2} = 125$ **15.** $\log(x+9) = 2$
16. $8 - 2e^{-x} = 4$ **17.** $\log(x^2 + 3) = \log(x + 6)$
18. $7^{x+3} = e^x$ **19.** $\log_2(x-4) + \log_2(x+4) = 3$

20. Write $\log_2\left(\dfrac{4x^3}{x^2 - 3x - 18}\right)$ as the sum and/or difference of logarithms. Express powers as factors.

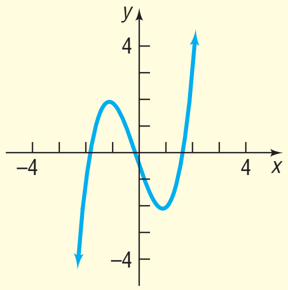
21. A 50-mg sample of a radioactive substance decays to 34 mg after 30 days. How long will it take for there to be 2 mg remaining?

22. (a) If \$1000 is invested at 5% compounded monthly, how much is there after 8 months?
(b) If you want to have \$1000 in 9 months, how much do you need to place in a savings account now that pays 5% compounded quarterly?
(c) How long does it take to double your money if you can invest it at 6% compounded annually?

23. The decibel level, D, of sound is given by the equation $D = 10 \log\left(\dfrac{I}{I_0}\right)$, where I is the intensity of the sound and $I_0 = 10^{-12}$ watt per square meter.
(a) If the shout of a single person measures 80 decibels, how loud will the sound be if two people shout at the same time? That is, how loud would the sound be if the intensity doubled?
(b) The pain threshold for sound is 125 decibels. If the Athens Olympic Stadium 2004 (Olympiako Stadio Athinas 'Spyros Louis') can seat 74,400 people, how many people in the crowd need to shout at the same time for the resulting sound level to meet or exceed the pain threshold? (Ignore any possible sound dampening.)

Cumulative Review

1. (a) Is the following graph the graph of a function? If it is, is the function one-to-one?



(b) Assuming the graph is a function, what type of function might it be (polynomial, exponential, and so on)? Why?

2. For the function $f(x) = 2x^2 - 3x + 1$, find the following:
(a) $f(3)$ (b) $f(-x)$ (c) $f(x+h)$

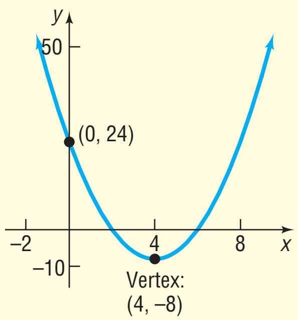
3. Determine which of the following points are on the graph of $x^2 + y^2 = 1$.
(a) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

4. Solve the equation $3(x-2) = 4(x+5)$.

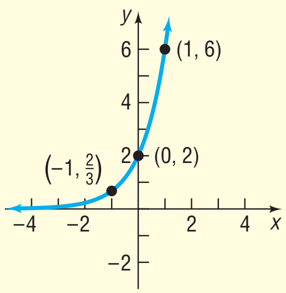
5. Graph the line $2x - 4y = 16$.

6. (a) Graph the quadratic function $f(x) = -x^2 + 2x - 3$ by determining whether its graph opens up or down and by finding its vertex, axis of symmetry, y-intercept, and x-intercept(s), if any.
(b) Solve $f(x) \le 0$.

7. Determine the quadratic function whose graph is given in the figure.



8. Is the graph of the following function a polynomial, exponential, or logarithmic function? Determine the function whose graph is given. Exponential;



9. Graph $f(x) = 3(x+1)^3 - 2$ using transformations.

10. (a) Given that $f(x) = x^2 + 2$ and $g(x) = \dfrac{2}{x-3}$, find $f(g(x))$ and state its domain. What is $f(g(5))$?
(b) If $f(x) = x + 2$ and $g(x) = \log_2 x$, find $(f(g(x))$ and state its domain. What is $f(g(14))$?

11. For the polynomial function $f(x) = 4x^3 + 9x^2 - 30x - 8$:
(a) Find the real zeros of f.
(b) Determine the intercepts of the graph of f.
(c) Use a graphing utility to approximate the local maxima and local minima.
(d) Draw a complete graph of f. Be sure to label the intercepts and turning points.

12. For the function $g(x) = 3^x + 2$:
(a) Graph g using transformations. State the domain, range, and horizontal asymptote of g.
(b) Determine the inverse of g. State the domain, range, and vertical asymptote of g^{-1}.
(c) On the same graph as g, graph g^{-1}.

13. Solve the equation $4^{x-3} = 8^{2x}$.

14. Solve the equation: $\log_3(x+1) + \log_3(2x-3) = \log_9 9$

15. Suppose that $f(x) = \log_3(x+2)$. Solve:
(a) $f(x) = 0$ (b) $f(x) > 0$
(c) $f(x) = 3$

16. Data Analysis The following data represent the percent of all drivers by age that have been stopped by the police for any reason within the past year. The median age represents the midpoint of the upper and lower limit for the age range.

Age Range	Median Age, x	Percentage Stopped, y
16–19	17.5	18.2
20–29	24.5	16.8
30–39	34.5	11.3
40–49	44.5	9.4
50–59	54.5	7.7
≥60	69.5	3.8

(a) Using your graphing utility, draw a scatter diagram of the data treating median age, x, as the independent variable.
(b) Determine a model that you feel best describes the relation between median age and percentage stopped. You may choose from among linear, quadratic, cubic, exponential, logarithmic, or logistic models.
(c) Provide a justification for the model that you selected in part (b).

Chapter Projects

I. Depreciation of Cars Kelley Blue Book is a guide that provides the current retail price of cars. You can access the Kelley Blue Book at your library or online at *www.kbb.com*.

1. Identify three cars that you are considering purchasing, and find the Kelley Blue Book value of the cars for 0 (brand new), 1, 2, 3, 4, and 5 years of age. Online, the value of the car can be found by selecting What should I pay for a used car? Enter the year, make, and model of the car you are selecting. To be consistent, we will assume the cars will be driven 12,000 miles per year, so a 1-year-old car will have 12,000 miles, a 2-year-old car will have 24,000 miles, and so on. Choose the same options for each year, and finally determine the suggested retail price for cars that are in Excellent, Good, and Fair shape. You should have a total of 16 observations (1 for a brand new car, 3 for a 1-year-old car, 3 for a 2-year-old car, and so on).
2. Draw a scatter diagram of the data with age as the independent variable and value as the dependent variable using Excel, a TI-graphing calculator, or some other spreadsheet. The Chapter 4 project describes how to draw a scatter diagram in Excel.
3. Determine the exponential function of best fit. Graph the exponential function of best fit on the scatter diagram. To do this in Excel, click on any data point in the scatter diagram. Now select the Chart Element icon (+). Check the box for Trendline, select the arrow to the right, and choose More Options. Select the Exponential radio button and select Display Equation on Chart. See Figure 71. Move the Trendline Options window off to the side, and you will see the exponential function of best fit displayed on the scatter diagram. Do you think the function accurately describes the relation between age of the car and suggested retail price?

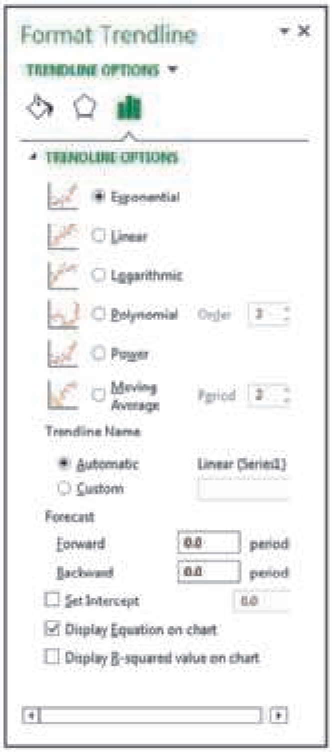


Figure 71

4. The exponential function of best fit is of the form $y = Ce^{rx}$, where y is the suggested retail value of the car and x is the age of the car (in years). What does the value of C represent? What does the value of r represent? What is the depreciation rate for each car that you are considering?
5. Write a report detailing which car you would purchase based on the depreciation rate you found for each car.

The following projects are available on the Instructor's Resource Center (IRC):

II. Hot Coffee A fast-food restaurant wants a special container to hold coffee. The restaurant wishes the container to quickly cool the coffee from 200° to 130°F and keep the liquid between 110° and 130°F as long as possible. The restaurant has three containers to select from. Which one should be purchased?

III. Project at Motorola ***Thermal Fatigue of Solder Connections*** Product reliability is a major concern of a manufacturer. Here a logarithmic transformation is used to simplify the analysis of a cell phone's ability to withstand temperature change.

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7 Analytic Geometry

The Orbit of the Hale-Bopp Comet

The orbits of the Hale-Bopp Comet and Earth can be modeled using *ellipses*, the subject of Section 7.3. The Internet-based Project at the end of this chapter explores the possibility of the Hale-Bopp Comet colliding with Earth.

—See the Internet-based Chapter Project I—

••• A Look Back

In Chapter 1, we introduced rectangular coordinates and showed how geometry problems can be solved algebraically. In Section 2.3, we defined a circle geometrically and then used the distance formula and rectangular coordinates to obtain an equation for a circle.

A Look Ahead •••

In this chapter, geometric definitions are given for the conics, and the distance formula and rectangular coordinates are used to obtain their equations.

Historically, Apollonius (200 BC) was among the first to study *conics* and discover some of their interesting properties. Today, conics are still studied because of their many uses. *Paraboloids of revolution* (parabolas rotated about their axes of symmetry) are used as signal collectors (the satellite dishes used with radar and dish TV, for example), as solar energy collectors, and as reflectors (telescopes, light projection, and so on). The planets circle the Sun in approximately *elliptical* orbits. Elliptical surfaces can be used to reflect signals such as light and sound from one place to another. A third conic, the *hyperbola,* can be used to determine the location of lightning strikes.

The Greeks used Euclidean geometry to study conics. However, we shall use the more powerful methods of analytic geometry, which employs both algebra and geometry, for our study of conics.

Outline

OBJECTIVE 1 Know the Names of the Conics (p. 514)

1 Know the Names of the Conics

The word *conic* derives from the word *cone,* which is a geometric figure that can be constructed in the following way: Let a and g be two distinct lines that intersect at a point V. Keep the line a fixed. Now rotate the line g about a, while maintaining the same angle between a and g. The collection of points swept out (generated) by the line g is called a **right circular cone**. See Figure 1. The fixed line a is called the **axis** of the cone; the point V is its **vertex**; the lines that pass through V and make the same angle with a as g are **generators** of the cone. Each generator is a line that lies entirely on the cone. The cone consists of two parts, called **nappes**, that intersect at the vertex.

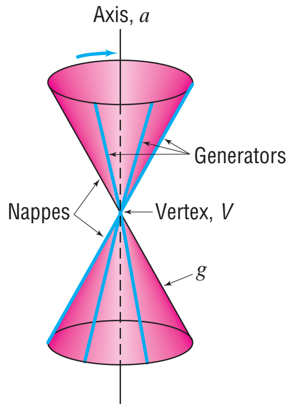


Figure 1 Right circular cone

Conics, an abbreviation for **conic sections**, are curves that result from the intersection of a right circular cone and a plane. The conics we shall study arise when the plane does not contain the vertex, as shown in Figure 2. These conics are **circles** when the plane is perpendicular to the axis of the cone and intersects each generator; **ellipses** when the plane is tilted slightly so that it intersects each generator, but intersects only one nappe of the cone; **parabolas** when the plane is tilted farther so that it is parallel to one (and only one) generator and intersects only one nappe of the cone; and **hyperbolas** when the plane intersects both nappes.

If the plane contains the vertex, the intersection of the plane and the cone is a point, a line, or a pair of intersecting lines. These are usually called **degenerate conics**.

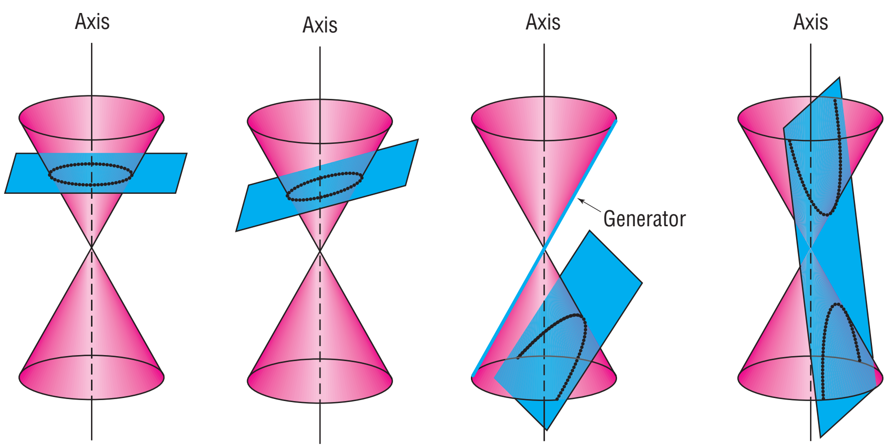


Figure 2 **(a)** Circle **(b)** Ellipse **(c)** Parabola **(d)** Hyperbola

Conic sections are used in modeling many different applications. For example, parabolas are used in describing searchlights and telescopes (see Figures 16 and 17 on page 521). Ellipses are used to model the orbits of planets and whispering chambers (see pages 532–533). And hyperbolas are used to locate lightning strikes and model nuclear cooling towers (see Problems 76 and 77 in Section 7.4).

7.2 The Parabola

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Distance Formula (Section 1.1, p. 85)
- Symmetry (Section 2.1, pp. 166–168)
- Square Root Method (Section 1.3, p. 112)
- Completing the Square (Chapter R, Review, Section R.5, p. 57)
- Graphing Techniques: Transformations (Section 3.5, pp. 256–264)

Now Work the **'Are You Prepared?'** problems on page 522.

OBJECTIVES
1 Analyze Parabolas with Vertex at the Origin (p. 515)
2 Analyze Parabolas with Vertex at (h, k) (p. 519)
3 Solve Applied Problems Involving Parabolas (p. 520)

In Section 4.3, we learned that the graph of a quadratic function is a parabola. In this section, we give a geometric definition of a parabola and use it to obtain an equation.

DEFINITION

A **parabola** is the collection of all points P in a plane that are the same distance d from a fixed point F as they are from a fixed line D. The point F is called the **focus** of the parabola, and the line D is its **directrix**. As a result, a parabola is the set of points P for which

$$d(F, P) = d(P, D) \tag{1}$$

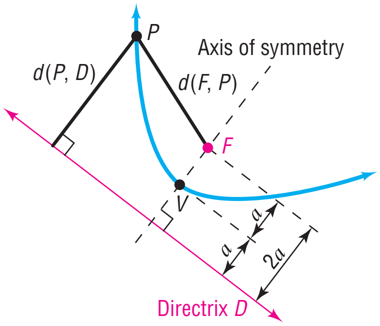


Figure 3 Parabola

Figure 3 shows a parabola (in blue). The line through the focus F and perpendicular to the directrix D is called the **axis of symmetry** of the parabola. The point of intersection of the parabola with its axis of symmetry is called the **vertex** V.

Because the vertex V lies on the parabola, it must satisfy equation (1): $d(F, V) = d(V, D)$. The vertex is midway between the focus and the directrix. We shall let a equal the distance $d(F, V)$ from F to V. Now we are ready to derive an equation for a parabola. To do this, we use a rectangular system of coordinates positioned so that the vertex V, focus F, and directrix D of the parabola are conveniently located.

1 Analyze Parabolas with Vertex at the Origin

If we choose to locate the vertex V at the origin $(0, 0)$, we can conveniently position the focus F on either the x-axis or the y-axis. First, consider the case where the focus F is on the positive x-axis, as shown in Figure 4. Because the distance from F to V is a, the coordinates of F will be $(a, 0)$ with $a > 0$. Similarly, because the distance from V to the directrix D is also a, and because D must be perpendicular to the x-axis (since the x-axis is the axis of symmetry), the equation of the directrix D must be $x = -a$.

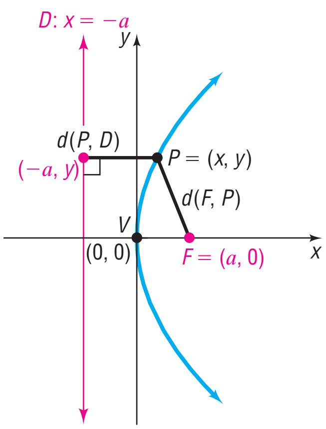


Figure 4

Now, if $P = (x, y)$ is any point on the parabola, P must satisfy equation (1):

$$d(F, P) = d(P, D)$$

So we have

$$\sqrt{(x-a)^2+(y-0)^2} = \sqrt{(x-(-a))^2+(y-y)^2} \quad \text{Use the Distance Formula.}$$
$$(x-a)^2+y^2 = (x+a)^2 \quad \text{Square both sides.}$$
$$x^2-2ax+a^2+y^2 = x^2+2ax+a^2 \quad \text{Multiply out.}$$
$$y^2 = 4ax \quad \text{Simplify.}$$

THEOREM

Equation of a Parabola: Vertex at (0, 0), Focus at (a, 0), $a > 0$

The equation of a parabola with vertex at $(0, 0)$, focus at $(a, 0)$, and directrix $x = -a$, $a > 0$, is

$$y^2 = 4ax \tag{2}$$

Recall that a is the distance from the vertex to the focus of a parabola. When graphing the parabola $y^2 = 4ax$ it is helpful to determine the "opening" by finding the points that lie directly above or below the focus $(a, 0)$. This is done by letting $x = a$ in $y^2 = 4ax$, so $y^2 = 4a(a) = 4a^2$, or $y = \pm 2a$. The line segment joining the two points, $(a, 2a)$ and $(a, -2a)$, is called the **latus rectum**; its length is $4a$.

EXAMPLE 1 Finding the Equation of a Parabola and Graphing It

Find an equation of the parabola with vertex at $(0, 0)$ and focus at $(3, 0)$. Graph the equation.

Solution The distance from the vertex $(0, 0)$ to the focus $(3, 0)$ is $a = 3$. Based on equation (2), the equation of this parabola is

$$y^2 = 4ax$$
$$y^2 = 12x \quad a = 3$$

To graph this parabola, find the two points that determine the latus rectum by letting $x = 3$. Then

$$y^2 = 12x = 12(3) = 36$$
$$y = \pm 6 \quad \text{Solve for } y.$$

The points $(3, 6)$ and $(3, -6)$ determine the latus rectum. These points help in graphing the parabola because they determine the "opening." See Figure 5.

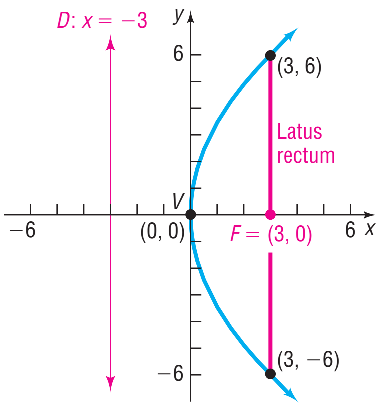


Figure 5 $y^2 = 12x$

Now Work PROBLEM 21

EXAMPLE 2 Graphing a Parabola Using a Graphing Utility

Graph the parabola $y^2 = 12x$.

Solution To graph the parabola $y^2 = 12x$, we need to graph the two functions $Y_1 = \sqrt{12x}$ and $Y_2 = -\sqrt{12x}$ on a square screen. Figure 6 shows the graph of $y^2 = 12x$. Notice that the graph fails the vertical line test, so $y^2 = 12x$ is not a function.

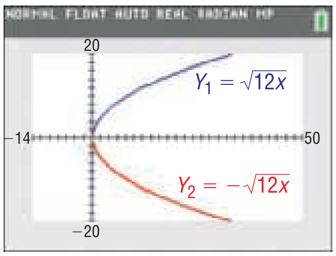


Figure 6 $Y_1 = \sqrt{12x}$; $Y_2 = -\sqrt{12x}$

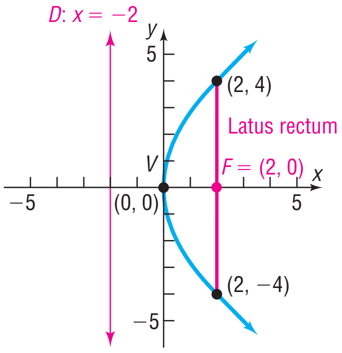
By reversing the steps used to obtain equation (2), it follows that the graph of an equation of the form of equation (2), $y^2 = 4ax$, is a parabola; its vertex is at $(0, 0)$, its focus is at $(a, 0)$, its directrix is the line $x = -a$, and its axis of symmetry is the x-axis.

For the remainder of this section, the direction **"Analyze the equation"** will mean to find the vertex, focus, and directrix of the parabola and graph it.

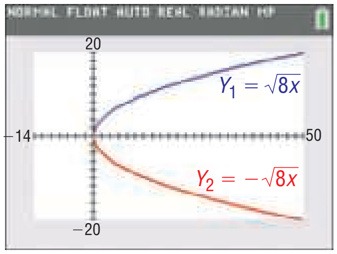
EXAMPLE 3 **Analyzing the Equation of a Parabola**

Analyze the equation: $y^2 = 8x$

Solution The equation $y^2 = 8x$ is of the form $y^2 = 4ax$, where $4a = 8$, so $a = 2$. Consequently, the graph of the equation is a parabola with vertex at $(0, 0)$ and focus on the positive x-axis at $(a, 0) = (2, 0)$. The directrix is the vertical line $x = -2$. The two points that determine the latus rectum are obtained by letting $x = 2$. Then $y^2 = 16$, so $y = \pm 4$. The points $(2, -4)$ and $(2, 4)$ determine the latus rectum. See Figure 7(a) for the graph drawn by hand. Figure 7(b) shows the graph obtained using a graphing utility.



(a)



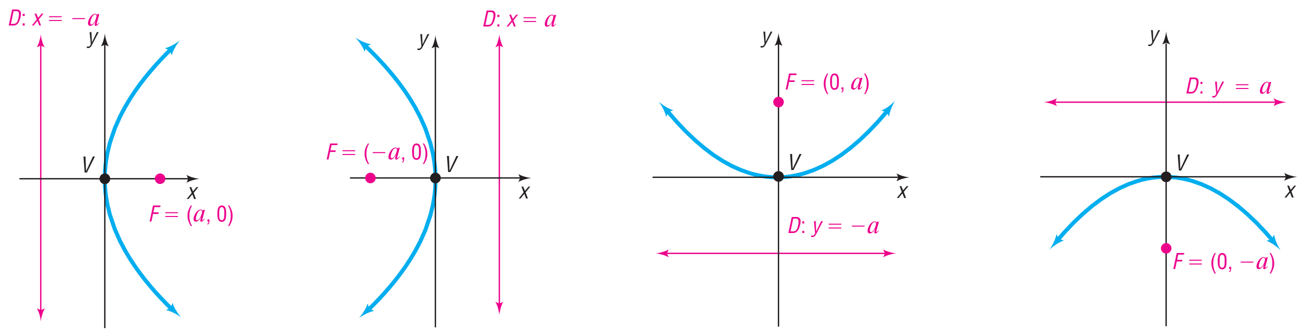
(b)

Figure 7 $y^2 = 8x$

Recall that we obtained equation (2) after placing the focus on the positive x-axis. If the focus is placed on the negative x-axis, positive y-axis, or negative y-axis, a different form of the equation for the parabola results. The four forms of the equation of a parabola with vertex at $(0, 0)$ and focus on a coordinate axis a distance a from $(0, 0)$ are given in Table 1, and their graphs are given in Figure 8. Notice that each graph is symmetric with respect to its axis of symmetry.

Table 1

Equations of a Parabola Vertex at (0, 0); Focus on an Axis; $a > 0$				
Vertex	**Focus**	**Directrix**	**Equation**	**Description**
$(0, 0)$	$(a, 0)$	$x = -a$	$y^2 = 4ax$	Parabola, axis of symmetry is the x-axis, opens right
$(0, 0)$	$(-a, 0)$	$x = a$	$y^2 = -4ax$	Parabola, axis of symmetry is the x-axis, opens left
$(0, 0)$	$(0, a)$	$y = -a$	$x^2 = 4ay$	Parabola, axis of symmetry is the y-axis, opens up
$(0, 0)$	$(0, -a)$	$y = a$	$x^2 = -4ay$	Parabola, axis of symmetry is the y-axis, opens down



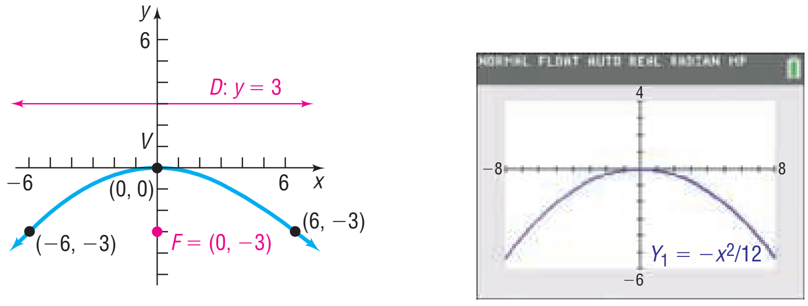
(a) $y^2 = 4ax$ (b) $y^2 = -4ax$ (c) $x^2 = 4ay$ (d) $x^2 = -4ay$

Figure 8

EXAMPLE 4 Analyzing the Equation of a Parabola

Analyze the equation: $x^2 = -12y$

Solution The equation $x^2 = -12y$ is of the form $x^2 = -4ay$, with $a = 3$. Consequently, the graph of the equation is a parabola with vertex at $(0, 0)$, focus at $(0, -3)$, and directrix the line $y = 3$. The parabola opens down, and its axis of symmetry is the y-axis. To obtain the points defining the latus rectum, let $y = -3$. Then $x^2 = 36$, so $x = \pm 6$. The points $(-6, -3)$ and $(6, -3)$ determine the latus rectum. See Figure 9(a) for the graph drawn by hand. Figure 9(b) shows the graph using a graphing utility.



(a) (b)

Figure 9 $x^2 = -12y$

Now Work PROBLEM 41

EXAMPLE 5 Finding the Equation of a Parabola

Find the equation of the parabola with focus at $(0, 4)$ and directrix the line $y = -4$. Graph the equation.

Solution A parabola whose focus is at $(0, 4)$ and whose directrix is the horizontal line $y = -4$ will have its vertex at $(0, 0)$. (Do you see why? The vertex is midway between the focus and the directrix.) Since the focus is on the positive y-axis at $(0, 4)$, the equation of this parabola is of the form $x^2 = 4ay$, with $a = 4$; that is,

$$x^2 = 4ay = \underset{\substack{\uparrow \\ a\,=\,4}}{4}(4)y = 16y$$

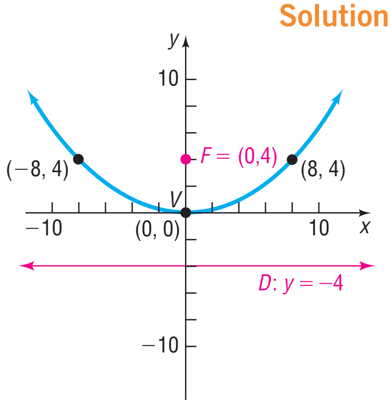


Figure 10 $x^2 = 16y$

Letting $y = 4$, we find $x^2 = 64$, so $x = \pm 8$. The points $(8, 4)$ and $(-8, 4)$ determine the latus rectum. Figure 10 shows the graph of $x^2 = 16y$.

✓**Check:** Verify the graph drawn in Figure 10 by graphing $Y_1 = \dfrac{x^2}{16}$ using a graphing utility.

EXAMPLE 6 Finding the Equation of a Parabola

Find the equation of a parabola with vertex at $(0, 0)$ if its axis of symmetry is the x-axis and its graph contains the point $\left(-\frac{1}{2}, 2\right)$. Find its focus and directrix, and graph the equation.

Solution The vertex is at the origin, the axis of symmetry is the x-axis, and the graph contains a point in the second quadrant, so the parabola opens to the left. From Table 1, note that the form of the equation is

$$y^2 = -4ax$$

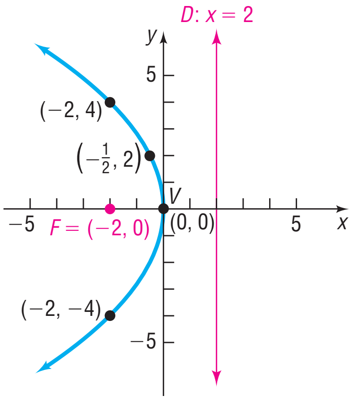


Figure 11 $y^2 = -8x$

Because the point $\left(-\frac{1}{2}, 2\right)$ is on the parabola, the coordinates $x = -\frac{1}{2}$, $y = 2$ must satisfy the equation $y^2 = -4ax$. Substituting $x = -\frac{1}{2}$ and $y = 2$ into the equation leads to

$$2^2 = -4a\left(-\frac{1}{2}\right) \qquad y^2 = -4ax;\, x = -\frac{1}{2},\, y = 2$$

$$a = 2$$

The equation of the parabola is

$$y^2 = -4(2)x = -8x$$

The focus is at $(-2, 0)$ and the directrix is the line $x = 2$. Letting $x = -2$ gives $y^2 = 16$, so $y = \pm 4$. The points $(-2, 4)$ and $(-2, -4)$ determine the latus rectum. See Figure 11.

Now Work PROBLEM 29

2 Analyze Parabolas with Vertex at (h, k)

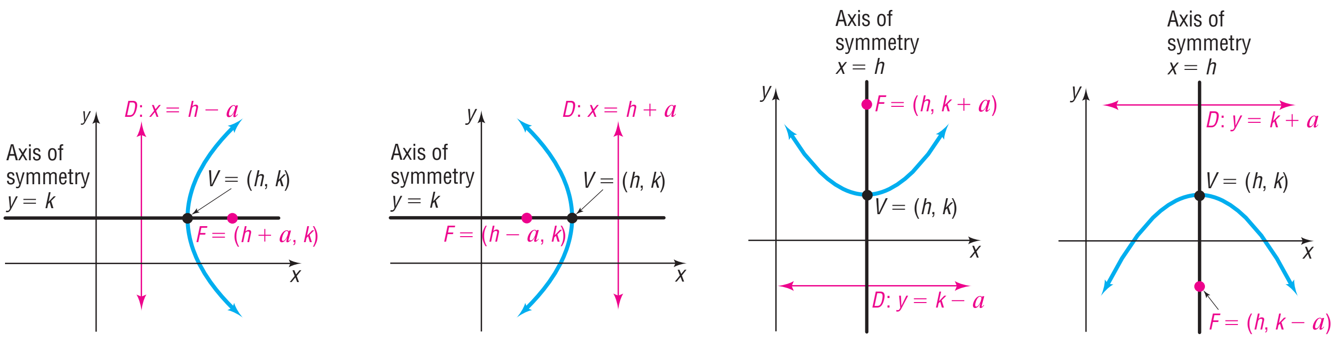
If a parabola with vertex at the origin and axis of symmetry along a coordinate axis is shifted horizontally h units and then vertically k units, the result is a parabola with vertex at (h, k) and axis of symmetry parallel to a coordinate axis. The equations of such parabolas have the same forms as those in Table 1 but x is replaced by $x - h$ (the horizontal shift) and y is replaced by $y - k$ (the vertical shift). Table 2 gives the forms of the equations of such parabolas. Figures 12(a)–(d) illustrate the graphs for $h > 0$, $k > 0$.

Note: It is not recommended that Table 2 be memorized. Rather, use the ideas of transformations (shift horizontally h units, vertically k units), along with the fact that a represents the distance from the vertex to the focus, to determine the various components of a parabola. It is also helpful to remember that parabolas of the form "$x^2 =$ " open up or down, while parabolas of the form "$y^2 =$ " open left or right.

Table 2

Equations of a Parabola: Vertex at (h, k); Axis of Symmetry Parallel to a Coordinate Axis; $a > 0$

Vertex	Focus	Directrix	Equation	Description
(h, k)	$(h + a, k)$	$x = h - a$	$(y - k)^2 = 4a(x - h)$	Axis of symmetry is parallel to the x-axis, opens right
(h, k)	$(h - a, k)$	$x = h + a$	$(y - k)^2 = -4a(x - h)$	Axis of symmetry is parallel to the x-axis, opens left
(h, k)	$(h, k + a)$	$y = k - a$	$(x - h)^2 = 4a(y - k)$	Axis of symmetry is parallel to the y-axis, opens up
(h, k)	$(h, k - a)$	$y = k + a$	$(x - h)^2 = -4a(y - k)$	Axis of symmetry is parallel to the y-axis, opens down



(a) $(y - k)^2 = 4a(x - h)$ **(b)** $(y - k)^2 = -4a(x - h)$ **(c)** $(x - h)^2 = 4a(y - k)$ **(d)** $(x - h)^2 = -4a(y - k)$

Figure 12

EXAMPLE 7 **Finding the Equation of a Parabola, Vertex Not at the Origin**

Find an equation of the parabola with vertex at $(-2, 3)$ and focus at $(0, 3)$. Graph the equation.

Solution The vertex $(-2, 3)$ and focus $(0, 3)$ both lie on the horizontal line $y = 3$ (the axis of symmetry). The distance a from the vertex $(-2, 3)$ to the focus $(0, 3)$ is $a = 2$. Also, because the focus lies to the right of the vertex, the parabola opens to the right. Consequently, the form of the equation is

$$(y - k)^2 = 4a(x - h)$$

where $(h, k) = (-2, 3)$ and $a = 2$. Therefore, the equation is

$$(y - 3)^2 = 4 \cdot 2[x - (-2)]$$
$$(y - 3)^2 = 8(x + 2)$$

To find the points that define the latus rectum, let $x = 0$, so that $(y - 3)^2 = 16$. Then $y - 3 = \pm 4$, so $y = -1$ or $y = 7$. The points $(0, -1)$ and $(0, 7)$ determine the latus rectum; the line $x = -4$ is the directrix. See Figure 13. ■

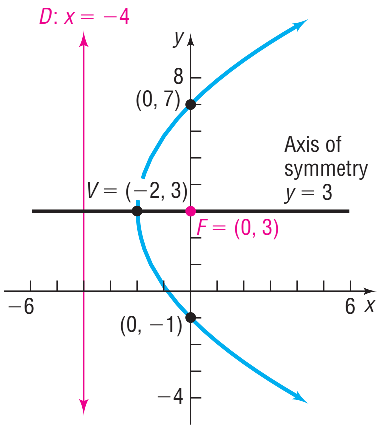


Figure 13 $(y - 3)^2 = 8(x + 2)$

Now Work PROBLEM 31

EXAMPLE 8 **Using a Graphing Utility to Graph a Parabola, Vertex Not at Origin**

Using a graphing utility, graph the equation $(y - 3)^2 = 8(x + 2)$.

Solution First, solve the equation for y.

$$(y - 3)^2 = 8(x + 2)$$
$$y - 3 = \pm\sqrt{8(x + 2)} \quad \text{Use the Square Root Method.}$$
$$y = 3 \pm \sqrt{8(x + 2)} \quad \text{Add 3 to both sides.}$$

Figure 14 shows the graphs of the equations $Y_1 = 3 + \sqrt{8(x + 2)}$ and $Y_2 = 3 - \sqrt{8(x + 2)}$. ■

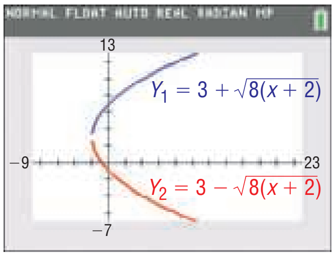


Figure 14

Polynomial equations define parabolas whenever they involve two variables that are quadratic in one variable and linear in the other.

EXAMPLE 9 **Analyzing the Equation of a Parabola**

Analyze the equation: $x^2 + 4x - 4y = 0$

Solution To analyze the equation $x^2 + 4x - 4y = 0$, complete the square involving the variable x.

$$x^2 + 4x - 4y = 0$$
$$x^2 + 4x = 4y \quad \text{Isolate the terms involving } x \text{ on the left side.}$$
$$x^2 + 4x + 4 = 4y + 4 \quad \text{Complete the square on the left side.}$$
$$(x + 2)^2 = 4(y + 1) \quad \text{Factor.}$$

This equation is of the form $(x - h)^2 = 4a(y - k)$, with $h = -2$, $k = -1$, and $a = 1$. The graph is a parabola with vertex at $(h, k) = (-2, -1)$ that opens up. The focus is $a = 1$ unit above the vertex at $(-2, 0)$, and the directrix is the line $y = -2$. See Figure 15. ■

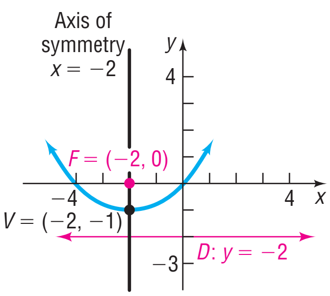


Figure 15 $x^2 + 4x - 4y = 0$

Now Work PROBLEM 49

3 Solve Applied Problems Involving Parabolas

Parabolas find their way into many applications. For example, as discussed in Section 4.4, suspension bridges have cables in the shape of a parabola. Another property of parabolas that is used in applications is their reflecting property.

Suppose that a mirror is shaped like a **paraboloid of revolution**, a surface formed by rotating a parabola about its axis of symmetry. If a light (or any other emitting source) is placed at the focus of the parabola, all the rays emanating from the light reflect off the mirror in lines parallel to the axis of symmetry. This principle is used in the design of searchlights, flashlights, certain automobile headlights, and other such devices. See Figure 16.

Conversely, suppose that rays of light (or other signals) emanate from a distant source so that they are essentially parallel. When these rays strike the surface of a parabolic mirror whose axis of symmetry is parallel to these rays, they are reflected to a single point at the focus. This principle is used in the design of some solar energy devices, satellite dishes, and the mirrors used in some types of telescopes. See Figure 17.

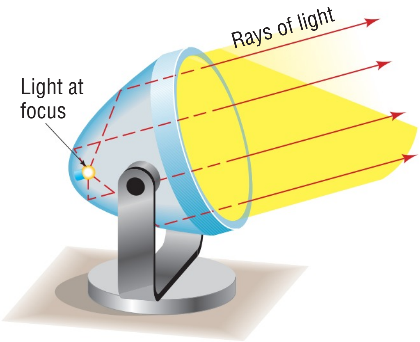


Figure 16 Searchlight

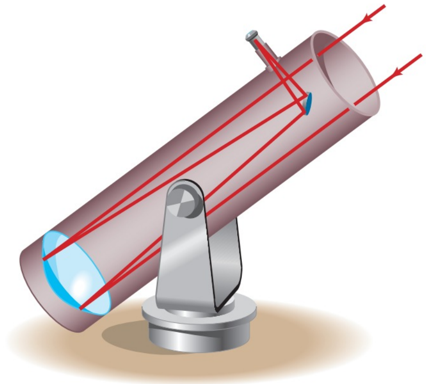


Figure 17 Telescope

EXAMPLE 10 Satellite Dish

A satellite dish is shaped like a paraboloid of revolution. The signals that emanate from a satellite strike the surface of the dish and are reflected to a single point, where the receiver is located. If the dish is 8 feet across at its opening and 3 feet deep at its center, at what position should the receiver be placed? That is, where is the focus?

Solution Figure 18(a) shows the satellite dish. On a rectangular coordinate system, draw the parabola used to form the dish so that the vertex of the parabola is at the origin and its focus is on the positive y-axis. See Figure 18(b).

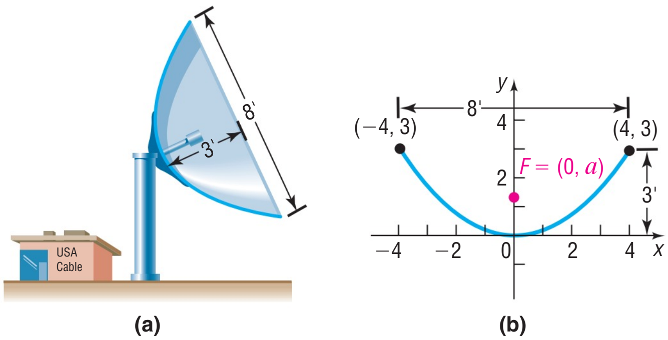


Figure 18

The form of the equation of the parabola is

$$x^2 = 4ay$$

and its focus is at $(0, a)$. Since $(4, 3)$ is a point on the graph, this gives

$$4^2 = 4a(3) \qquad x^2 = 4ay;\ x = 4,\ y = 3$$

$$a = \frac{4}{3} \qquad \text{Solve for } a.$$

The receiver should be located $1\frac{1}{3}$ feet (1 foot, 4 inches) from the base of the dish, along its axis of symmetry. ■

Now Work PROBLEM 65

7.2 Assess Your Understanding

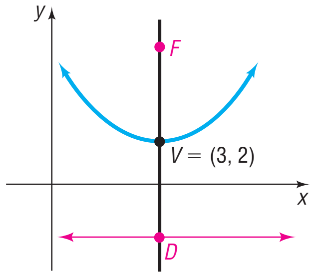
'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. The formula for the distance d from $P_1 = (x_1, y_1)$ to $P_2 = (x_2, y_2)$ is $d =$ ____________________. (p. 85)
2. To complete the square of $x^2 - 4x$, add ____. (p. 57)
3. Use the Square Root Method to find the real solutions of $(x + 4)^2 = 9$. (p. 112)
4. The point that is symmetric with respect to the x-axis to the point $(-2, 5)$ is ________. (pp. 166–168)
5. To graph $y = (x - 3)^2 + 1$, shift the graph of $y = x^2$ to the right ____ units and then _____ 1 unit. (pp. 256–264)

Concepts and Vocabulary

6. A(n) ____________ is the collection of all points in a plane that are the same distance from a fixed point as they are from a fixed line.
7. The line through the focus and perpendicular to the directrix is called the ________________ of the parabola.
8. For the parabola $y^2 = 4ax$, the line segment joining the two points $(a, 2a)$ and $(a, -2a)$ is called the ____________.

Answer Problems 9–12 using the figure.



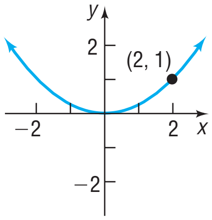
9. If $a > 0$, the equation of the parabola is of the form
   (a) $(y - k)^2 = 4a(x - h)$ (b) $(y - k)^2 = -4a(x - h)$
   (c) $(x - h)^2 = 4a(y - k)$ (d) $(x - h)^2 = -4a(y - k)$
10. The coordinates of the vertex are __________.
11. If $a = 4$, then the coordinates of the focus are ________.
   (a) $(-1, 2)$ (b) $(3, -2)$ (c) $(7, 2)$ (d) $(3, 6)$
12. If $a = 4$, then the equation of the directrix is ________.
   (a) $x = -3$ (b) $x = 3$ (c) $y = -2$ (d) $y = 2$

Skill Building

In Problems 13–20, the graph of a parabola is given. Match each graph to its equation.

(A) $y^2 = 4x$ (C) $y^2 = -4x$ (E) $(y - 1)^2 = 4(x - 1)$ (G) $(y - 1)^2 = -4(x - 1)$

(B) $x^2 = 4y$ (D) $x^2 = -4y$ (F) $(x + 1)^2 = 4(y + 1)$ (H) $(x + 1)^2 = -4(y + 1)$

13.


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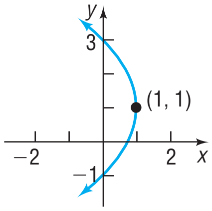

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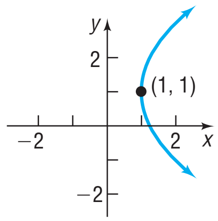

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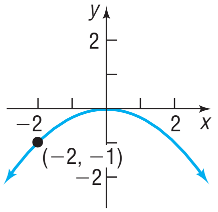

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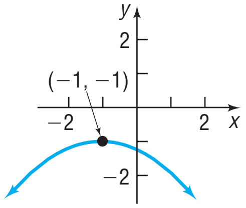

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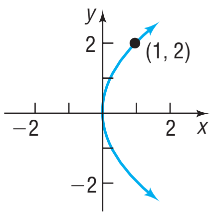

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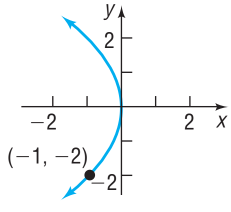

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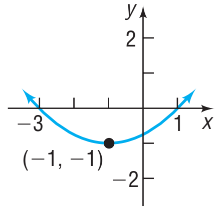


In Problems 21–38, find the equation of the parabola described. Find the two points that define the latus rectum, and graph the equation by hand.

21. Focus at $(4, 0)$; vertex at $(0, 0)$
22. Focus at $(0, 2)$; vertex at $(0, 0)$
23. Focus at $(0, -3)$; vertex at $(0, 0)$
24. Focus at $(-4, 0)$; vertex at $(0, 0)$
25. Focus at $(-2, 0)$; directrix the line $x = 2$
26. Focus at $(0, -1)$; directrix the line $y = 1$
27. Directrix the line $y = -\frac{1}{2}$; vertex at $(0, 0)$
28. Directrix the line $x = -\frac{1}{2}$; vertex at $(0, 0)$
29. Vertex at $(0, 0)$; axis of symmetry the y-axis; containing the point $(2, 3)$
30. Vertex at $(0, 0)$; axis of symmetry the x-axis; containing the point $(2, 3)$

31. Vertex at $(2, -3)$; focus at $(2, -5)$

32. Vertex at $(4, -2)$; focus at $(6, -2)$

33. Vertex at $(-1, -2)$; focus at $(0, -2)$

34. Vertex at $(3, 0)$; focus at $(3, -2)$

35. Focus at $(-3, 4)$; directrix the line $y = 2$

36. Focus at $(2, 4)$; directrix the line $x = -4$

37. Focus at $(-3, -2)$; directrix the line $x = 1$

38. Focus at $(-4, 4)$; directrix the line $y = -2$

In Problems 39–56, find the vertex, focus, and directrix of each parabola. Graph the equation by hand. Verify your graph using a graphing utility.

39. $x^2 = 4y$ **40.** $y^2 = 8x$ **41.** $y^2 = -16x$ **42.** $x^2 = -4y$

43. $(y - 2)^2 = 8(x + 1)$ **44.** $(x + 4)^2 = 16(y + 2)$ **45.** $(x - 3)^2 = -(y + 1)$ **46.** $(y + 1)^2 = -4(x - 2)$

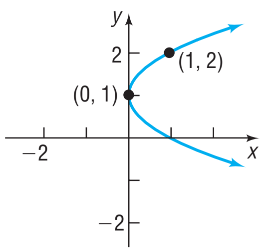
47. $(y + 3)^2 = 8(x - 2)$ **48.** $(x - 2)^2 = 4(y - 3)$ **49.** $y^2 - 4y + 4x + 4 = 0$ **50.** $x^2 + 6x - 4y + 1 = 0$

51. $x^2 + 8x = 4y - 8$ **52.** $y^2 - 2y = 8x - 1$ **53.** $y^2 + 2y - x = 0$ **54.** $x^2 - 4x = 2y$

55. $x^2 - 4x = y + 4$ **56.** $y^2 + 12y = -x + 1$

In Problems 57–64, write an equation for each parabola.

57.



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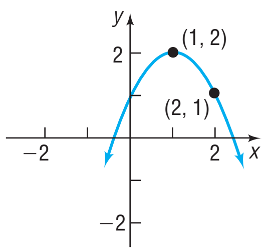

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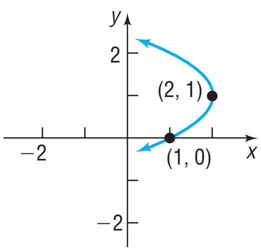

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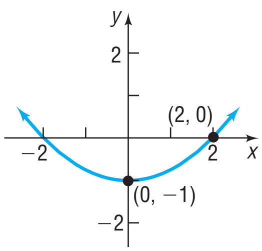

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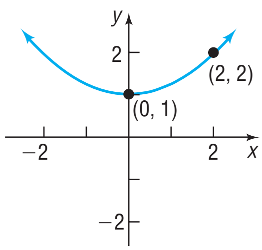

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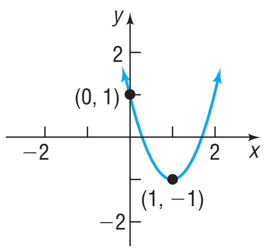

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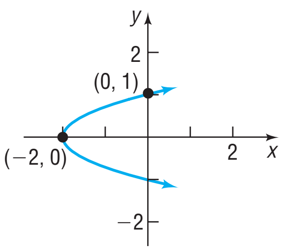

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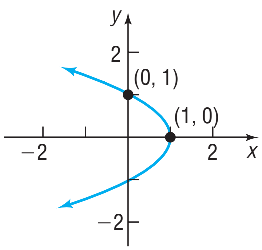


Applications and Extensions

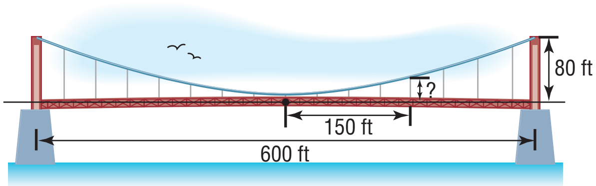
65. Satellite Dish A satellite dish is shaped like a paraboloid of revolution. The signals that emanate from a satellite strike the surface of the dish and are reflected to a single point, where the receiver is located. If the dish is 10 feet across at its opening and 4 feet deep at its center, at what position should the receiver be placed?

66. Constructing a TV Dish A cable TV receiving dish is in the shape of a paraboloid of revolution. Find the location of the receiver, which is placed at the focus, if the dish is 6 feet across at its opening and 2 feet deep.

67. Constructing a Flashlight The reflector of a flashlight is in the shape of a paraboloid of revolution. Its diameter is 4 inches and its depth is 1 inch. How far from the vertex should the light bulb be placed so that the rays will be reflected parallel to the axis?

68. Constructing a Headlight A sealed-beam headlight is in the shape of a paraboloid of revolution. The bulb, which is placed at the focus, is 1 inch from the vertex. If the depth is to be 2 inches, what is the diameter of the headlight at its opening?

69. Suspension Bridge The cables of a suspension bridge are in the shape of a parabola, as shown in the figure. The towers supporting the cable are 600 feet apart and 80 feet high. If the cables touch the road surface midway between the towers, what is the height of the cable from the road at a point 150 feet from the center of the bridge?

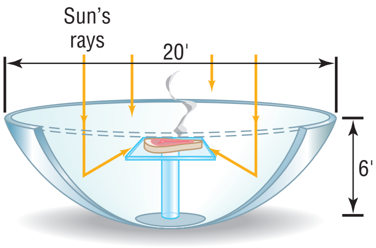


70. Suspension Bridge The cables of a suspension bridge are in the shape of a parabola. The towers supporting the cable are 400 feet apart and 100 feet high. If the cables are at a height of 10 feet midway between the towers, what is the height of the cable at a point 50 feet from the center of the bridge?

71. Searchlight A searchlight is shaped like a paraboloid of revolution. If the light source is located 2 feet from the base along the axis of symmetry and the opening is 5 feet across, how deep should the searchlight be?

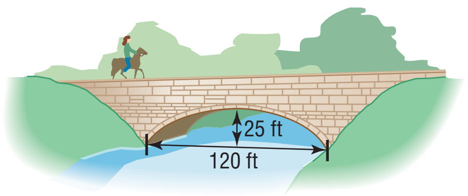
72. Searchlight A searchlight is shaped like a paraboloid of revolution. If the light source is located 2 feet from the base along the axis of symmetry and the depth of the searchlight is 4 feet, what should the width of the opening be?

73. **Solar Heat** A mirror is shaped like a paraboloid of revolution and will be used to concentrate the rays of the sun at its focus, creating a heat source. See the figure. If the mirror is 20 feet across at its opening and is 6 feet deep, where will the heat source be concentrated?



74. **Reflecting Telescope** A reflecting telescope contains a mirror shaped like a paraboloid of revolution. If the mirror is 4 inches across at its opening and is 3 inches deep, where will the collected light be concentrated?

75. **Parabolic Arch Bridge** A bridge is built in the shape of a parabolic arch. The bridge has a span of 120 feet and a maximum height of 25 feet. See the illustration. Choose a suitable rectangular coordinate system and find the height of the arch at distances of 10, 30, and 50 feet from the center.



76. **Parabolic Arch Bridge** A bridge is to be built in the shape of a parabolic arch and is to have a span of 100 feet. The height of the arch a distance of 40 feet from the center is to be 10 feet. Find the height of the arch at its center.

77. **Gateway Arch** The Gateway Arch in St. Louis is often mistaken to be parabolic in shape. In fact, it is a *catenary*, which has a more complicated formula than a parabola. The Arch is 630 feet high and 630 feet wide at its base.

(a) Find the equation of a parabola with the same dimensions. Let x equal the horizontal distance from the center of the arch.

(b) The table below gives the height of the Arch at various widths; find the corresponding heights for the parabola found in (a).

Width (ft)	Height (ft)
567	100
478	312.5
308	525

(c) Do the data support the notion that the Arch is in the shape of a parabola?

Source: *gatewayarch.com*

78. Show that an equation of the form

$$Ax^2 + Ey = 0 \qquad A \neq 0, E \neq 0$$

is the equation of a parabola with vertex at $(0, 0)$ and axis of symmetry the y-axis. Find its focus and directrix.

79. Show that an equation of the form

$$Cy^2 + Dx = 0 \qquad C \neq 0, D \neq 0$$

is the equation of a parabola with vertex at $(0, 0)$ and axis of symmetry the x-axis. Find its focus and directrix.

80. Show that the graph of an equation of the form

$$Ax^2 + Dx + Ey + F = 0 \qquad A \neq 0$$

(a) Is a parabola if $E \neq 0$.
(b) Is a vertical line if $E = 0$ and $D^2 - 4AF = 0$.
(c) Is two vertical lines if $E = 0$ and $D^2 - 4AF > 0$.
(d) Contains no points if $E = 0$ and $D^2 - 4AF < 0$.

81. Show that the graph of an equation of the form

$$Cy^2 + Dx + Ey + F = 0 \qquad C \neq 0$$

(a) Is a parabola if $D \neq 0$.
(b) Is a horizontal line if $D = 0$ and $E^2 - 4CF = 0$.
(c) Is two horizontal lines if $D = 0$ and $E^2 - 4CF > 0$.
(d) Contains no points if $D = 0$ and $E^2 - 4CF < 0$.

Retain Your Knowledge

Problems 82–85 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

82. For $x = 9y^2 - 36$, list the intercepts and test for symmetry.

83. Solve: $4^{x+1} = 8^{x-1}$

84. Find the vertex of the graph of $f(x) = -2x^2 + 8x - 5$.

85. If $f(x) = x^2 + 2x - 3$, find $\dfrac{f(x+h) - f(x)}{h}$.

'Are You Prepared?' Answers

1. $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ 2. 4 3. $\{-7, -1\}$ 4. $(-2, -5)$ 5. 3; up

7.3 The Ellipse

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Distance Formula (Section 1.1, p. 85)
- Completing the Square (Chapter R, Review, Section R.5, p. 57)
- Intercepts (Section 2.1, pp. 165–166)
- Symmetry (Section 2.1, pp. 166–168)
- Circles (Section 2.3, pp. 189–193)
- Graphing Techniques: Transformations (Section 3.5, pp. 256–264)

Now Work the **'Are You Prepared?'** problems on page 533.

OBJECTIVES
1. Analyze Ellipses with Center at the Origin (p. 525)
2. Analyze Ellipses with Center at (h, k) (p. 530)
3. Solve Applied Problems Involving Ellipses (p. 532)

DEFINITION An **ellipse** is the collection of all points in a plane, the sum of whose distances from two fixed points, called the **foci**, is a constant.

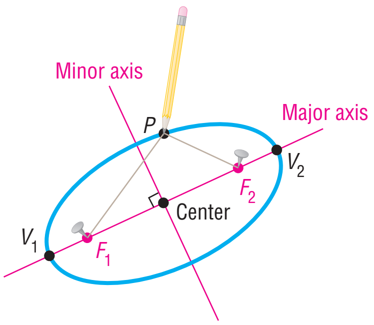


Figure 19 Ellipse

The definition contains within it a physical means for drawing an ellipse. Find a piece of string (the length of this string is the constant referred to in the definition). Then take two thumbtacks (the foci) and stick them into a piece of cardboard so that the distance between them is less than the length of the string. Now attach the ends of the string to the thumbtacks and, using the point of a pencil, pull the string taut. See Figure 19. Keeping the string taut, rotate the pencil around the two thumbtacks. The pencil traces out an ellipse, as shown in Figure 19.

In Figure 19, the foci are labeled F_1 and F_2. The line containing the foci is called the **major axis**. The midpoint of the line segment joining the foci is the **center** of the ellipse. The line through the center and perpendicular to the major axis is the **minor axis**.

The two points of intersection of the ellipse and the major axis are the **vertices**, V_1 and V_2, of the ellipse. The distance from one vertex to the other is the **length of the major axis**. The ellipse is symmetric with respect to its major axis, with respect to its minor axis, and with respect to its center.

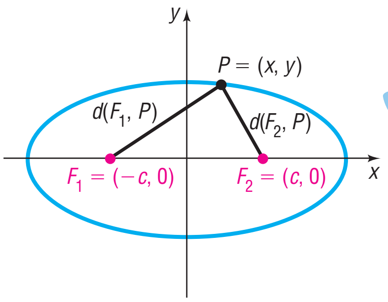


Figure 20

1 Analyze Ellipses with Center at the Origin

With these ideas in mind, we are ready to find the equation of an ellipse in a rectangular coordinate system. First, place the center of the ellipse at the origin. Second, position the ellipse so that its major axis coincides with a coordinate axis, say the x-axis, as shown in Figure 20. If c is the distance from the center to a focus, one focus is at $F_1 = (-c, 0)$ and the other at $F_2 = (c, 0)$. As we shall see, it is convenient to let $2a$ denote the constant distance referred to in the definition. Then, if $P = (x, y)$ is any point on the ellipse,

$$d(F_1, P) + d(F_2, P) = 2a$$ **Sum of the distances from P to the foci equals a constant, $2a$.**

$$\sqrt{(x - (-c))^2 + (y - 0)^2} + \sqrt{(x - c)^2 + (y - 0)^2} = 2a$$ **Use the Distance Formula.**

$$\sqrt{(x + c)^2 + y^2} = 2a - \sqrt{(x - c)^2 + y^2}$$ **Isolate one radical.**

$$(x + c)^2 + y^2 = 4a^2 - 4a\sqrt{(x - c)^2 + y^2} + (x - c)^2 + y^2$$ **Square both sides.**

$$x^2 + 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x - c)^2 + y^2} + x^2 - 2cx + c^2 + y^2$$ **Multiply out.**

$$4cx - 4a^2 = -4a\sqrt{(x - c)^2 + y^2}$$ **Simplify; isolate the radical.**

$$cx - a^2 = -a\sqrt{(x - c)^2 + y^2}$$ **Divide each side by 4.**

$$(cx - a^2)^2 = a^2[(x - c)^2 + y^2]$$ **Square both sides again.**

$$c^2x^2 - 2a^2cx + a^4 = a^2(x^2 - 2cx + c^2 + y^2)$$ **Multiply out.**

$$(c^2 - a^2)x^2 - a^2y^2 = a^2c^2 - a^4$$ **Rearrange the terms.**

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2) \quad (1)$$ **Multiply each side by -1; factor out a^2 on the right side.**

To obtain points on the ellipse off the x-axis, it must be that $a > c$. To see why, look again at Figure 20. Then

$$d(F_1, P) + d(F_2, P) > d(F_1, F_2)$$ **The sum of the lengths of two sides of a triangle is greater than the length of the third side.**

$$2a > 2c$$ **$d(F_1, P) + d(F_2, P) = 2a$; $d(F_1, F_2) = 2c$**

$$a > c$$

Because $a > c > 0$, this means $a^2 > c^2$, so $a^2 - c^2 > 0$. Let $b^2 = a^2 - c^2$, $b > 0$. Then $a > b$ and equation (1) can be written as

$$b^2x^2 + a^2y^2 = a^2b^2$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$ **Divide each side by a^2b^2.**

As you can verify, the graph of this equation has symmetry with respect to the x-axis, y-axis, and origin.

Because the major axis is the x-axis, find the vertices of this ellipse by letting $y = 0$. The vertices satisfy the equation $\frac{x^2}{a^2} = 1$, the solutions of which are $x = \pm a$. Consequently, the vertices of this ellipse are $V_1 = (-a, 0)$ and $V_2 = (a, 0)$. The y-intercepts of the ellipse, found by letting $x = 0$, have coordinates $(0, -b)$ and $(0, b)$. These four intercepts, $(a, 0)$, $(-a, 0)$, $(0, b)$, and $(0, -b)$, are used to graph the ellipse.

THEOREM

Equation of an Ellipse: Center at (0, 0); Major Axis along the x-Axis

An equation of the ellipse with center at $(0, 0)$, foci at $(-c, 0)$ and $(c, 0)$, and vertices at $(-a, 0)$ and $(a, 0)$ is

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{where } a > b > 0 \text{ and } b^2 = a^2 - c^2 \qquad (2)}$$

The major axis is the x-axis. See Figure 21.

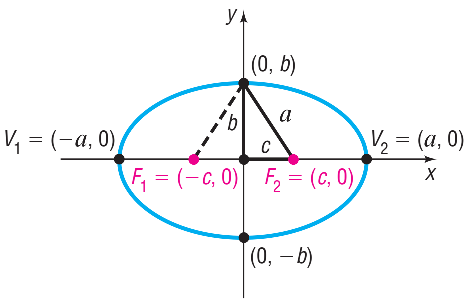


Figure 21

Notice in Figure 21 the right triangle formed by the points $(0, 0)$, $(c, 0)$, and $(0, b)$. Because $b^2 = a^2 - c^2$ (or $b^2 + c^2 = a^2$), the distance from the focus at $(c, 0)$ to the point $(0, b)$ is a.

This can be seen another way. Look at the two right triangles in Figure 21. They are congruent. Do you see why? Because the sum of the distances from the foci to a point on the ellipse is $2a$, it follows that the distance from $(c, 0)$ to $(0, b)$ is a.

EXAMPLE 1 Finding an Equation of an Ellipse

Find an equation of the ellipse with center at the origin, one focus at $(3, 0)$, and a vertex at $(-4, 0)$. Graph the equation.

Solution The ellipse has its center at the origin and, since the given focus and vertex lie on the x-axis, the major axis is the x-axis. The distance from the center, $(0, 0)$, to one of the foci, $(3, 0)$, is $c = 3$. The distance from the center, $(0, 0)$, to one of the vertices, $(-4, 0)$, is $a = 4$. From equation (2), it follows that

$$b^2 = a^2 - c^2 = 16 - 9 = 7$$

so an equation of the ellipse is

$$\frac{x^2}{16} + \frac{y^2}{7} = 1$$

Figure 22 shows the graph. ■

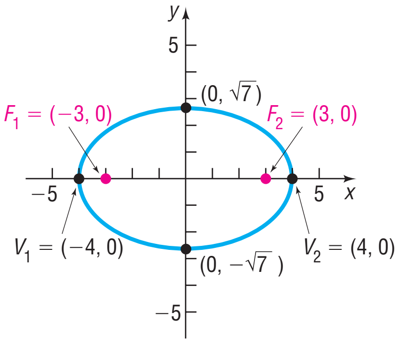


Figure 22 $\frac{x^2}{16} + \frac{y^2}{7} = 1$

In Figure 22, the intercepts of the equation are used to graph the ellipse. Following this practice will make it easier for you to obtain an accurate graph of an ellipse when graphing by hand. The intercepts also tell you how to set the initial viewing window when using a graphing utility.

Now Work PROBLEM 27

EXAMPLE 2 Graphing an Ellipse Using a Graphing Utility

Use a graphing utility to graph the ellipse $\frac{x^2}{16} + \frac{y^2}{7} = 1$.

Solution First, solve $\frac{x^2}{16} + \frac{y^2}{7} = 1$ for y.

$$\frac{y^2}{7} = 1 - \frac{x^2}{16} \qquad \text{Subtract } \frac{x^2}{16} \text{ from each side.}$$

$$y^2 = 7\left(1 - \frac{x^2}{16}\right) \qquad \text{Multiply both sides by 7.}$$

$$y = \pm\sqrt{7\left(1 - \frac{x^2}{16}\right)} \qquad \text{Apply the Square Root Method.}$$

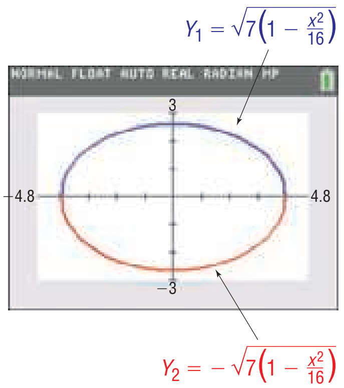


Figure 23

Figure 23* shows the graphs of $Y_1 = \sqrt{7\left(1 - \frac{x^2}{16}\right)}$ and $Y_2 = -\sqrt{7\left(1 - \frac{x^2}{16}\right)}$. ■

In Figure 23 a square screen is used. As with circles and parabolas, this is done to avoid a distorted view of the graph.

An equation of the form of equation (2), with $a > b$, is the equation of an ellipse with center at the origin, foci on the x-axis at $(-c, 0)$ and $(c, 0)$, where $c^2 = a^2 - b^2$, and major axis along the x-axis.

For the remainder of this section, the direction **"Analyze the equation"** will mean to find the center, major axis, foci, and vertices of the ellipse and graph it.

*The initial viewing window selected was $X\text{min} = -4$, $X\text{max} = 4$, $Y\text{min} = -3$, $Y\text{max} = 3$. Then we used the ZOOM-SQUARE option to obtain the window shown.

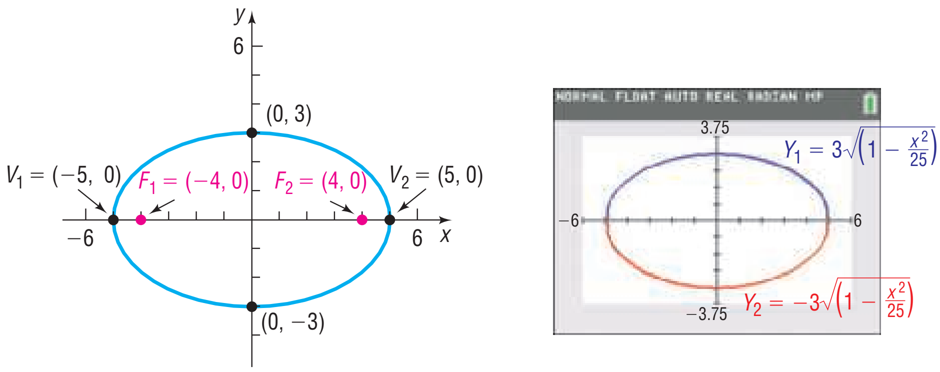
EXAMPLE 3 Analyzing the Equation of an Ellipse

Analyze the equation: $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

Solution The given equation is of the form of equation (2), with $a^2 = 25$ and $b^2 = 9$. The equation is that of an ellipse with center $(0, 0)$ and major axis along the x-axis. The vertices are at $(\pm a, 0) = (\pm 5, 0)$. Because $b^2 = a^2 - c^2$, this means

$$c^2 = a^2 - b^2 = 25 - 9 = 16$$

The foci are at $(\pm c, 0) = (\pm 4, 0)$. The y-intercepts are $(0, \pm b) = (0, \pm 3)$. Figure 24(a) shows the graph drawn by hand. Figure 24(b) shows the graph obtained using a graphing utility.



(a) **(b)**

Figure 24 $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

Now Work PROBLEM 17

If the major axis of an ellipse with center at $(0, 0)$ lies on the y-axis, then the foci are at $(0, -c)$ and $(0, c)$. Using the same steps as before, the definition of an ellipse leads to the following result:

THEOREM

Equation of an Ellipse; Center at $(0, 0)$; Major Axis along the y-Axis

An equation of the ellipse with center at $(0, 0)$, foci at $(0, -c)$ and $(0, c)$, and vertices at $(0, -a)$ and $(0, a)$ is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad \text{where } a > b > 0 \text{ and } b^2 = a^2 - c^2 \qquad \textbf{(3)}$$

The major axis is the y-axis.

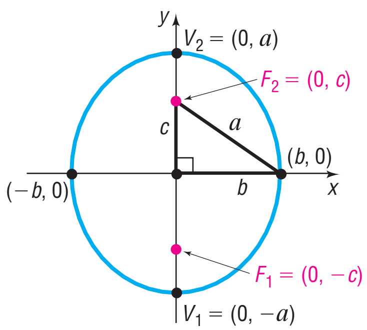


Figure 25 $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1, a > b > 0$

Figure 25 illustrates the graph of such an ellipse. Again, notice the right triangle formed by the points at $(0, 0)$, $(b, 0)$, and $(0, c)$, so that $a^2 = b^2 + c^2$ (or $b^2 = a^2 - c^2$).

Look closely at equations (2) and (3). Although they may look alike, there is a difference! In equation (2), the larger number, a^2, is in the denominator of the x^2-term, so the major axis of the ellipse is along the x-axis. In equation (3), the larger number, a^2, is in the denominator of the y^2-term, so the major axis is along the y-axis.

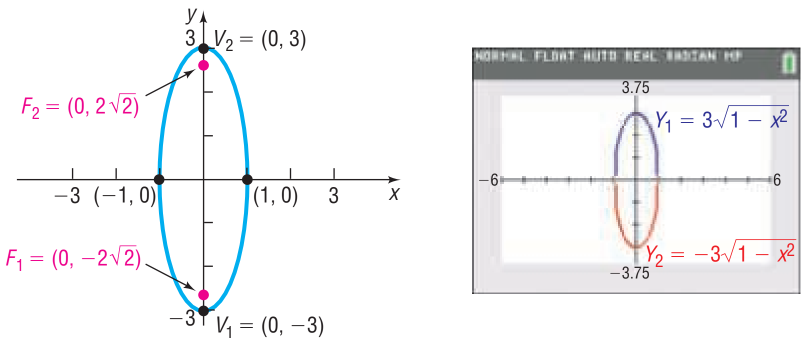
EXAMPLE 4 Analyzing the Equation of an Ellipse

Analyze the equation: $9x^2 + y^2 = 9$

Solution To put the equation in proper form, divide each side by 9.

$$x^2 + \frac{y^2}{9} = 1$$

The larger denominator, 9, is in the y^2-term so, based on equation (3), this is the equation of an ellipse with center at the origin and major axis along the y-axis. Also, $a^2 = 9$, $b^2 = 1$, and $c^2 = a^2 - b^2 = 9 - 1 = 8$. The vertices are at $(0, \pm a) = (0, \pm 3)$, and the foci are at $(0, \pm c) = (0, \pm 2\sqrt{2})$. The x-intercepts are at $(\pm b, 0) = (\pm 1, 0)$. Figure 26(a) shows the graph drawn by hand. Figure 26(b) shows the graph obtained using a graphing utility.



(a) (b)

Figure 26 $9x^2 + y^2 = 9$

Now Work PROBLEM 21

EXAMPLE 5 Finding an Equation of an Ellipse

Find an equation of the ellipse having one focus at $(0, 2)$ and vertices at $(0, -3)$ and $(0, 3)$. Graph the equation.

Solution By plotting the given focus and vertices, we find that the major axis is the y-axis. Because the vertices are at $(0, -3)$ and $(0, 3)$, the center of this ellipse is at their midpoint, the origin. The distance from the center, $(0, 0)$, to the given focus, $(0, 2)$, is $c = 2$. The distance from the center, $(0, 0)$, to one of the vertices, $(0, 3)$, is $a = 3$. So $b^2 = a^2 - c^2 = 9 - 4 = 5$. The form of the equation of this ellipse is given by equation (3).

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

$$\frac{x^2}{5} + \frac{y^2}{9} = 1$$

Figure 27 shows the graph.

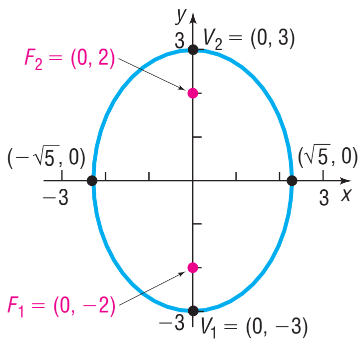


Figure 27 $\frac{x^2}{5} + \frac{y^2}{9} = 1$

Now Work PROBLEM 29

The circle may be considered a special kind of ellipse. To see why, let $a = b$ in equation (2) or (3). Then

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$$

$$x^2 + y^2 = a^2$$

This is the equation of a circle with center at the origin and radius a. The value of c is

$$c^2 = a^2 - b^2 = 0$$
$$\uparrow$$
$$a = b$$

This indicates that the closer the two foci of an ellipse are to the center, the more the ellipse will look like a circle.

2 Analyze Ellipses with Center at (h, k)

If an ellipse with center at the origin and major axis coinciding with a coordinate axis is shifted horizontally h units and then vertically k units, the result is an ellipse with center at (h, k) and major axis parallel to a coordinate axis. The equations of such ellipses have the same forms as those given in equations (2) and (3), except that x is replaced by $x - h$ (the horizontal shift) and y is replaced by $y - k$ (the vertical shift). Table 3 gives the forms of the equations of such ellipses, and Figure 28 shows their graphs.

Note: It is not recommended that Table 3 be memorized. Rather, use transformations (shift horizontally h units, vertically k units), along with the fact that a represents the distance from the center to the vertices, c represents the distance from the center to the foci, and $b^2 = a^2 - c^2$ (or $c^2 = a^2 - b^2$).

Table 3

Equations of an Ellipse: Center at (h, k); Major Axis Parallel to a Coordinate Axis

Center	Major Axis	Foci	Vertices	Equation
(h, k)	Parallel to the x-axis	$(h + c, k)$	$(h + a, k)$	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1,$
		$(h - c, k)$	$(h - a, k)$	$a > b > 0$ and $b^2 = a^2 - c^2$
(h, k)	Parallel to the y-axis	$(h, k + c)$	$(h, k + a)$	$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1,$
		$(h, k - c)$	$(h, k - a)$	$a > b > 0$ and $b^2 = a^2 - c^2$

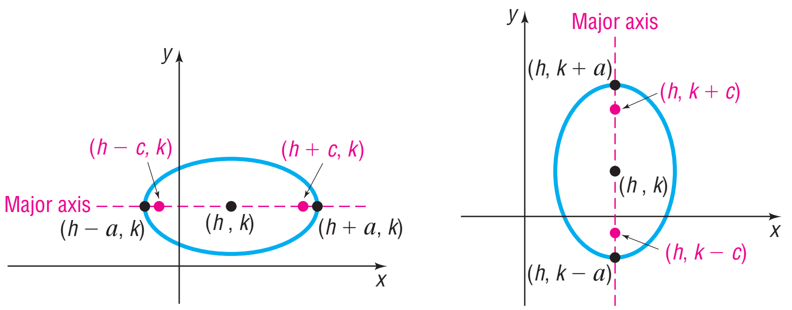


Figure 28 (a) $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ (b) $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$

EXAMPLE 6 **Finding an Equation of an Ellipse, Center Not at the Origin**

Find an equation for the ellipse with center at $(2, -3)$, one focus at $(3, -3)$, and one vertex at $(5, -3)$. Graph the equation.

Solution The center is at $(h, k) = (2, -3)$, so $h = 2$ and $k = -3$. Plot the center, focus, and vertex, and note that the points all lie on the line $y = -3$. Therefore, the major axis is parallel to the x-axis. The distance from the center $(2, -3)$ to a focus $(3, -3)$ is $c = 1$; the distance from the center $(2, -3)$ to a vertex $(5, -3)$ is $a = 3$. Then $b^2 = a^2 - c^2 = 9 - 1 = 8$. The form of the equation is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \qquad h = 2, k = -3, a = 3, b = 2\sqrt{2}$$

$$\frac{(x-2)^2}{9} + \frac{(y+3)^2}{8} = 1$$

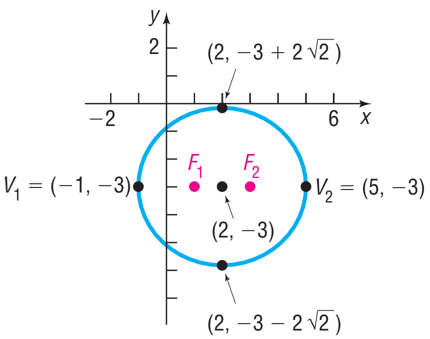


Figure 29 $\dfrac{(x-2)^2}{9} + \dfrac{(y+3)^2}{8} = 1$

To graph the equation, use the center $(h, k) = (2, -3)$ to locate the vertices. The major axis is parallel to the x-axis, so the vertices are $a = 3$ units left and right of the center $(2, -3)$. Therefore, the vertices are

$$V_1 = (2 - 3, -3) = (-1, -3) \quad \text{and} \quad V_2 = (2 + 3, -3) = (5, -3)$$

Since $c = 1$ and the major axis is parallel to the x-axis, the foci are 1 unit left and right of the center. Therefore, the foci are

$$F_1 = (2 - 1, -3) = (1, -3) \quad \text{and} \quad F_2 = (2 + 1, -3) = (3, -3)$$

Finally, use the value of $b = 2\sqrt{2}$ to find the two points above and below the center.

$$(2, -3 - 2\sqrt{2}) \quad \text{and} \quad (2, -3 + 2\sqrt{2})$$

Figure 29 shows the graph. ■

Now Work PROBLEM 55

EXAMPLE 7 Using a Graphing Utility to Graph an Ellipse, Center Not at the Origin

Using a graphing utility, graph the ellipse: $\dfrac{(x-2)^2}{9} + \dfrac{(y+3)^2}{8} = 1$

Solution First, solve the equation $\dfrac{(x-2)^2}{9} + \dfrac{(y+3)^2}{8} = 1$ for y.

$$\frac{(y+3)^2}{8} = 1 - \frac{(x-2)^2}{9} \qquad \text{Subtract } \frac{(x-2)^2}{9} \text{ from each side.}$$

$$(y+3)^2 = 8\left[1 - \frac{(x-2)^2}{9}\right] \qquad \text{Multiply each side by 8.}$$

$$y + 3 = \pm\sqrt{8\left[1 - \frac{(x-2)^2}{9}\right]} \qquad \text{Apply the Square Root Method.}$$

$$y = -3 \pm \sqrt{8\left[1 - \frac{(x-2)^2}{9}\right]} \qquad \text{Subtract 3 from each side.}$$

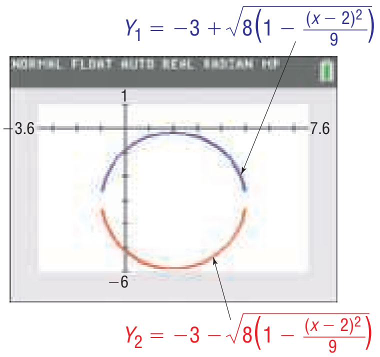


Figure 30

Figure 30 shows the graphs of $Y_1 = -3 + \sqrt{8\left[1 - \dfrac{(x-2)^2}{9}\right]}$ and $Y_2 = -3 - \sqrt{8\left[1 - \dfrac{(x-2)^2}{9}\right]}$. ■

EXAMPLE 8 Analyzing the Equation of an Ellipse

Analyze the equation: $4x^2 + y^2 - 8x + 4y + 4 = 0$

Solution Complete the squares in x and in y.

$$4x^2 + y^2 - 8x + 4y + 4 = 0$$

$$4x^2 - 8x + y^2 + 4y = -4 \qquad \text{Group like variables; place the constant on the right side.}$$

$$4(x^2 - 2x) + (y^2 + 4y) = -4 \qquad \text{Factor out 4 from the first two terms.}$$

$$4(x^2 - 2x + 1) + (y^2 + 4y + 4) = -4 + 4 + 4 \qquad \text{Complete each square.}$$

$$4(x-1)^2 + (y+2)^2 = 4 \qquad \text{Factor.}$$

$$(x-1)^2 + \frac{(y+2)^2}{4} = 1 \qquad \text{Divide each side by 4.}$$

This is the equation of an ellipse with center at $(1, -2)$ and major axis parallel to the y-axis. Since $a^2 = 4$ and $b^2 = 1$, we have $c^2 = a^2 - b^2 = 4 - 1 = 3$. The vertices are at $(h, k \pm a) = (1, -2 \pm 2)$ or $(1, -4)$ and $(1, 0)$. The foci are at $(h, k \pm c) = (1, -2 \pm \sqrt{3})$ or $(1, -2 - \sqrt{3})$ and $(1, -2 + \sqrt{3})$. Figure 31(a) shows the graph drawn by hand. Figure 31(b) shows the graph obtained using a graphing utility.

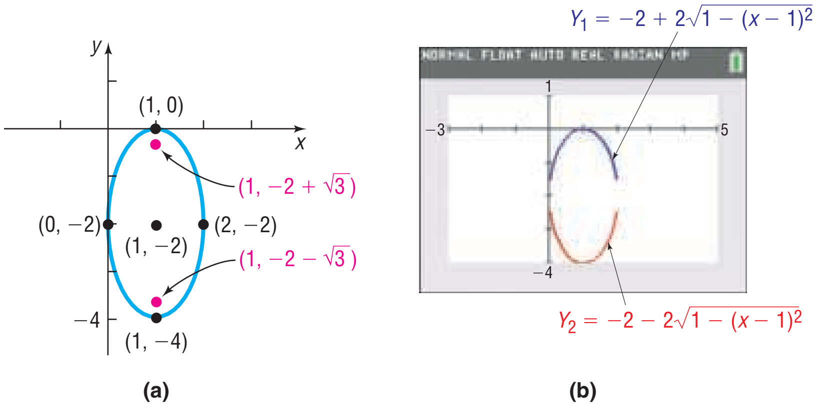


Figure 31 $4x^2 + y^2 - 8x + 4y + 4 = 0$

Now Work PROBLEM 47

3 Solve Applied Problems Involving Ellipses

Ellipses are found in many applications in science and engineering. For example, the orbits of the planets around the Sun are elliptical, with the Sun's position at a focus. See Figure 32.

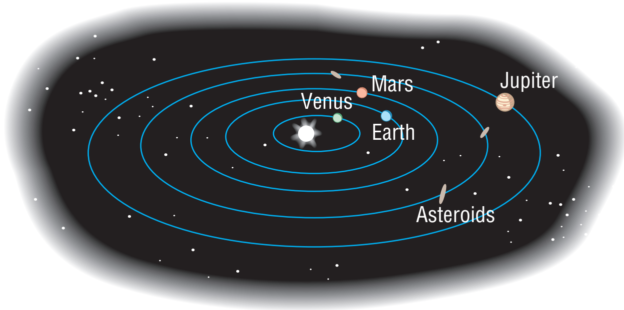


Figure 32 Planet orbits

Stone and concrete bridges are often shaped as semielliptical arches. Elliptical gears are used in machinery when a variable rate of motion is required.

Ellipses also have an interesting reflection property. If a source of light (or sound) is placed at one focus, the waves transmitted by the source reflect off the ellipse and concentrate at the other focus. This is the principle behind *whispering galleries*, which are rooms designed with elliptical ceilings. A person standing at one focus of the ellipse can whisper and be heard by a person standing at the other focus, because all the sound waves that reach the ceiling are reflected to the other person.

EXAMPLE 9 **A Whispering Gallery**

The whispering gallery in the Museum of Science and Industry in Chicago is 47.3 feet long. The distance from the center of the room to the foci is 20.3 feet. Find an equation that describes the shape of the room. How high is the room at its center?

Source: *Chicago Museum of Science and Industry website; www.msichicago.org*

Solution Set up a rectangular coordinate system so that the center of the ellipse is at the origin and the major axis is along the x-axis. The equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Since the length of the room is 47.3 feet, the distance from the center of the room to each vertex (the end of the room) is $\frac{47.3}{2} = 23.65$ feet, so $a = 23.65$ feet. The distance from the center of the room to each focus is $c = 20.3$ feet. See Figure 33.

Since $b^2 = a^2 - c^2$, this means that $b^2 = 23.65^2 - 20.3^2 = 147.2325$. An equation that describes the shape of the room is given by

$$\frac{x^2}{23.65^2} + \frac{y^2}{147.2325} = 1$$

The height of the room at its center is $b = \sqrt{147.2325} \approx 12.1$ feet. ■

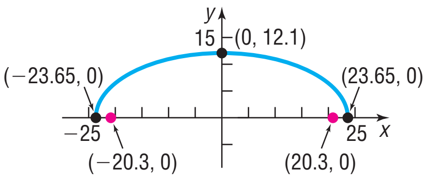


Figure 33

Now Work PROBLEM 71

7.3 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. The distance d from $P_1 = (2, -5)$ to $P_2 = (4, -2)$ is $d =$ ________. (p. 85)

2. To complete the square of $x^2 - 3x$, add ______. (p. 57)

3. Find the intercepts of the equation $y^2 = 16 - 4x^2$. (pp. 165–166)

4. The point that is symmetric with respect to the y-axis to the point $(-2, 5)$ is ________. (pp. 166–168)

5. To graph $y = (x + 1)^2 - 4$, shift the graph of $y = x^2$ to the (left/right) ______ unit(s) and then (up/down) ________ unit(s). (pp. 256–264)

6. The standard equation of a circle with center at $(2, -3)$ and radius 1 is ________. (pp. 189–193)

Concepts and Vocabulary

7. A(n) ________ is the collection of all points in a plane the sum of whose distances from two fixed points is a constant.

8. For an ellipse, the foci lie on a line called the ________.
(a) minor axis (b) major axis
(c) directrix (d) latus rectum

9. For the ellipse $\frac{x^2}{4} + \frac{y^2}{25} = 1$, the vertices are the points ____________ and ________.

10. For the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$, the value of a is _____, the value of b is _____, and the major axis is the _____-axis.

11. If the center of an ellipse is $(2, -3)$, the major axis is parallel to the x-axis, and the distance from the center of the ellipse to its vertices is $a = 4$ units, then the coordinates of the vertices are ____________ and ____________.

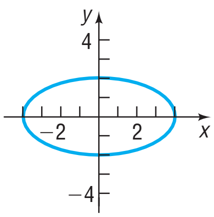
12. If the foci of an ellipse are $(-4, 4)$ and $(6, 4)$, then the coordinates of the center of the ellipse are ________.
(a) $(1, 4)$ (b) $(4, 1)$
(c) $(1, 0)$ (d) $(5, 4)$

Skill Building

In Problems 13–16, the graph of an ellipse is given. Match each graph to its equation.

(A) $\frac{x^2}{4} + y^2 = 1$ (B) $x^2 + \frac{y^2}{4} = 1$ (C) $\frac{x^2}{16} + \frac{y^2}{4} = 1$ (D) $\frac{x^2}{4} + \frac{y^2}{16} = 1$

13.



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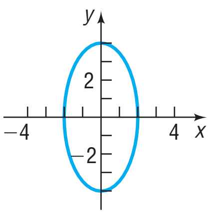

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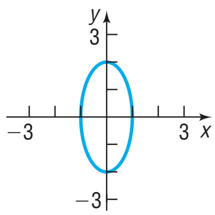

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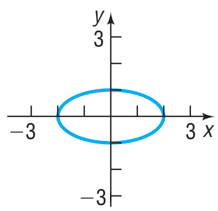


In Problems 17–26, find the vertices and foci of each ellipse. Graph each equation by hand. Verify your graph using a graphing utility.

17. $\dfrac{x^2}{25} + \dfrac{y^2}{4} = 1$ **18.** $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ **19.** $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$ **20.** $x^2 + \dfrac{y^2}{16} = 1$

21. $4x^2 + y^2 = 16$ **22.** $x^2 + 9y^2 = 18$ **23.** $4y^2 + x^2 = 8$ **24.** $4y^2 + 9x^2 = 36$

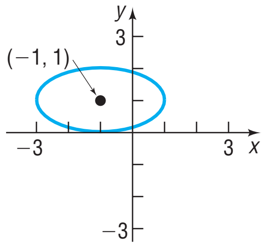
25. $x^2 + y^2 = 16$ **26.** $x^2 + y^2 = 4$

In Problems 27–38, find an equation for each ellipse. Graph the equation by hand.

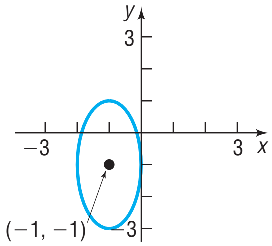
27. Center at $(0, 0)$; focus at $(3, 0)$; vertex at $(5, 0)$
28. Center at $(0, 0)$; focus at $(-1, 0)$; vertex at $(3, 0)$
29. Center at $(0, 0)$; focus at $(0, -4)$; vertex at $(0, 5)$
30. Center at $(0, 0)$; focus at $(0, 1)$; vertex at $(0, -2)$
31. Foci at $(\pm 2, 0)$; length of the major axis is 6
32. Foci at $(0, \pm 2)$; length of the major axis is 8
33. Focus at $(-4, 0)$; vertices at $(\pm 5, 0)$
34. Focus at $(0, -4)$; vertices at $(0, \pm 8)$
35. Foci at $(0, \pm 3)$; x-intercepts are ± 2
36. Vertices at $(\pm 4, 0)$; y-intercepts are ± 1
37. Center at $(0, 0)$; vertex at $(0, 4)$; $b = 1$
38. Vertices at $(\pm 5, 0)$; $c = 2$

In Problems 39–42, write an equation for each ellipse.

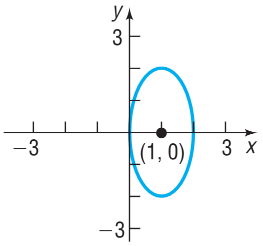
39.



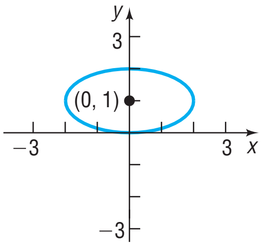
40.



41.



42.



In Problems 43–54, analyze each equation; that is, find the center, foci, and vertices of each ellipse. Graph each equation by hand. Verify your graph using a graphing utility.

43. $\dfrac{(x-3)^2}{4} + \dfrac{(y+1)^2}{9} = 1$ **44.** $\dfrac{(x+4)^2}{9} + \dfrac{(y+2)^2}{4} = 1$ **45.** $(x+5)^2 + 4(y-4)^2 = 16$

46. $9(x-3)^2 + (y+2)^2 = 18$ **47.** $x^2 + 4x + 4y^2 - 8y + 4 = 0$ **48.** $x^2 + 3y^2 - 12y + 9 = 0$

49. $2x^2 + 3y^2 - 8x + 6y + 5 = 0$ **50.** $4x^2 + 3y^2 + 8x - 6y = 5$ **51.** $9x^2 + 4y^2 - 18x + 16y - 11 = 0$

52. $x^2 + 9y^2 + 6x - 18y + 9 = 0$ **53.** $4x^2 + y^2 + 4y = 0$ **54.** $9x^2 + y^2 - 18x = 0$

In Problems 55–64, find an equation for each ellipse. Graph the equation by hand.

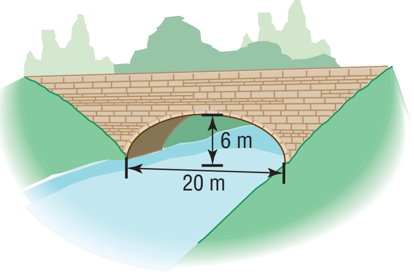
55. Center at $(2, -2)$; vertex at $(7, -2)$; focus at $(4, -2)$
56. Center at $(-3, 1)$; vertex at $(-3, 3)$; focus at $(-3, 0)$
57. Vertices at $(4, 3)$ and $(4, 9)$; focus at $(4, 8)$
58. Foci at $(1, 2)$ and $(-3, 2)$; vertex at $(-4, 2)$
59. Foci at $(5, 1)$ and $(-1, 1)$; length of the major axis is 8
60. Vertices at $(2, 5)$ and $(2, -1)$; $c = 2$
61. Center at $(1, 2)$; focus at $(4, 2)$; contains the point $(1, 3)$
62. Center at $(1, 2)$; focus at $(1, 4)$; contains the point $(2, 2)$
63. Center at $(1, 2)$; vertex at $(4, 2)$; contains the point $(1, 5)$
64. Center at $(1, 2)$; vertex at $(1, 4)$; contains the point $(1 + \sqrt{3}, 3)$

In Problems 65–68, graph each function. Be sure to label all the intercepts. [**Hint:** *Notice that each function is half an ellipse.*]

65. $f(x) = \sqrt{16 - 4x^2}$ **66.** $f(x) = \sqrt{9 - 9x^2}$ **67.** $f(x) = -\sqrt{64 - 16x^2}$ **68.** $f(x) = -\sqrt{4 - 4x^2}$

Applications and Extensions

69. Semielliptical Arch Bridge An arch in the shape of the upper half of an ellipse is used to support a bridge that is to span a river 20 meters wide. The center of the arch is 6 meters above the center of the river. See the figure. Write an equation for the ellipse in which the x-axis coincides with the water level and the y-axis passes through the center of the arch.



70. Semielliptical Arch Bridge The arch of a bridge is a semiellipse with a horizontal major axis. The span is 30 feet, and the top of the arch is 10 feet above the major axis. The roadway is horizontal and is 2 feet above the top of the arch. Find the vertical distance from the roadway to the arch at 5-foot intervals along the roadway.

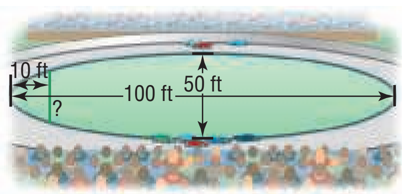
71. Whispering Gallery A hall 100 feet in length is to be designed as a whispering gallery. If the foci are located 25 feet from the center, how high will the ceiling be at the center?

72. Whispering Gallery Jim, standing at one focus of a whispering gallery, is 6 feet from the nearest wall. His friend is standing at the other focus, 100 feet away. What is the length of this whispering gallery? How high is its elliptical ceiling at the center?

73. Semielliptical Arch Bridge A bridge is built in the shape of a semielliptical arch. The bridge has a span of 120 feet and a maximum height of 25 feet. Choose a suitable rectangular coordinate system and find the height of the arch at distances of 10, 30, and 50 feet from the center.

74. Semielliptical Arch Bridge A bridge is to be built in the shape of a semielliptical arch and is to have a span of 100 feet. The height of the arch, at a distance of 40 feet from the center, is to be 10 feet. Find the height of the arch at its center.

75. Racetrack Design Consult the figure. A racetrack is in the shape of an ellipse, 100 feet long and 50 feet wide. What is the width 10 feet from a vertex?



76. Semielliptical Arch Bridge An arch for a bridge over a highway is in the form of half an ellipse. The top of the arch is 20 feet above the ground level (the major axis). The highway has four lanes, each 12 feet wide; a center safety strip 8 feet wide; and two side strips, each 4 feet wide. What should the span of the bridge be (the length of its major axis) if the height 28 feet from the center is to be 13 feet?

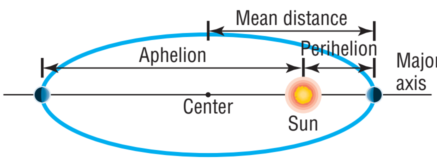
77. Installing a Vent Pipe A homeowner is putting in a fireplace that has a 4-inch-radius vent pipe. He needs to cut an elliptical hole in his roof to accommodate the pipe. If the pitch of his roof is $\frac{5}{4}$, (a rise of 5, run of 4) what are the dimensions of the hole?

Source: *www.doe.virginia.gov*

78. Volume of a Football A football is in the shape of a **prolate spheroid,** which is simply a solid obtained by rotating an ellipse $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right)$ about its major axis. An inflated NFL football averages 11.125 inches in length and 28.25 inches in center circumference. If the volume of a prolate spheroid is $\frac{4}{3}\pi ab^2$, how much air does the football contain? (Neglect material thickness).

Source: *www.nfl.com*

*In Problems 79–83, use the fact that the orbit of a planet about the Sun is an ellipse, with the Sun at one focus. The **aphelion** of a planet is its greatest distance from the Sun, and the **perihelion** is its shortest distance. The **mean distance** of a planet from the Sun is the length of the semimajor axis of the elliptical orbit. See the illustration.*



79. Earth The mean distance of Earth from the Sun is 93 million miles. If the aphelion of Earth is 94.5 million miles, what is the perihelion? Write an equation for the orbit of Earth around the Sun.

80. Mars The mean distance of Mars from the Sun is 142 million miles. If the perihelion of Mars is 128.5 million miles, what is the aphelion? Write an equation for the orbit of Mars about the Sun.

81. Jupiter The aphelion of Jupiter is 507 million miles. If the distance from the center of its elliptical orbit to the Sun is 23.2 million miles, what is the perihelion? What is the mean distance? Write an equation for the orbit of Jupiter around the Sun.

82. Pluto The perihelion of Pluto is 4551 million miles, and the distance from the center of its elliptical orbit to the Sun is 897.5 million miles. Find the aphelion of Pluto. What is the mean distance of Pluto from the Sun? Write an equation for the orbit of Pluto about the Sun.

83. Elliptical Orbit A planet orbits a star in an elliptical orbit with the star located at one focus. The perihelion of the planet is 5 million miles. The **eccentricity** e of a conic section is $e=\frac{c}{a}$. If the eccentricity of the orbit is 0.75, find the aphelion of the planet.†

84. A rectangle is inscribed in an ellipse with major axis of length 14 meters and minor axis of length 4 meters. Find the maximum area of a rectangle inscribed in the ellipse. Round your answer to two decimal places.†

85. Let D be the line given by the equation $x+5=0$. Let E be the conic section given by the equation $x^2+5y^2=20$. Let the point C be the vertex of E with the smaller x-coordinate, and let B be the endpoint of the minor axis of E with the larger y-coordinate. Determine the exact y-coordinate of the point M on D that is equidistant from points B and C.†

86. Show that an equation of the form

$$Ax^2+Cy^2+F=0, \qquad A\neq 0, C\neq 0, F\neq 0$$

where A and C are of the same sign and F is of opposite sign,
(a) is the equation of an ellipse with center at $(0,0)$ if $A\neq C$.
(b) is the equation of a circle with center $(0,0)$ if $A=C$.

87. Show that the graph of an equation of the form

$$Ax^2+Cy^2+Dx+Ey+F=0, \qquad A\neq 0, C\neq 0$$

where A and C are of the same sign,
(a) is an ellipse if $\frac{D^2}{4A}+\frac{E^2}{4C}-F$ is the same sign as A.
(b) is a point if $\frac{D^2}{4A}+\frac{E^2}{4C}-F=0$.
(c) contains no points if $\frac{D^2}{4A}+\frac{E^2}{4C}-F$ is of opposite sign to A.

†Courtesy of the Joliet Junior College Mathematics Department

Explaining Concepts: Discussion and Writing

88. The **eccentricity** e of an ellipse is defined as the number $\frac{c}{a}$, where a is the distance of a vertex from the center and c is the distance of a focus from the center. Because $a > c$, it follows that $e < 1$. Write a brief paragraph about the general shape of each of the following ellipses. Be sure to justify your conclusions.

(a) Eccentricity close to 0 (b) Eccentricity $= 0.5$ (c) Eccentricity close to 1

Retain Your Knowledge

Problems 89–92 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

89. Find the zeros of the quadratic function $f(x) = (x - 5)^2 - 12$. What are the x-intercepts, if any, of the graph of the function?

90. Find the domain of the rational function $f(x) = \dfrac{2x - 3}{x - 5}$. Find any horizontal, vertical, or oblique asymptotes.

91. Solve: $|4 - 5x| - 8 \le 3$

92. Determine the point(s) of intersection of the graphs of $f(x) = 2x^2 + 7x - 4$ and $g(x) = 6x + 11$ by solving $f(x) = g(x)$.

'Are You Prepared?' Answers

1. $\sqrt{13}$ **2.** $\frac{9}{4}$ **3.** $(-2, 0), (2, 0), (0, -4), (0, 4)$ **4.** $(2, 5)$ **5.** left 1; down 4 **6.** $(x - 2)^2 + (y + 3)^2 = 1$

7.4 The Hyperbola

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Distance Formula (Section 1.1, p. 85)
- Completing the Square (Chapter R, Review, Section R.5, p. 57)
- Intercepts (Section 2.1, pp. 165–166)
- Symmetry (Section 2.1, pp. 166–168)
- Asymptotes (Section 5.4, pp. 374–379)
- Graphing Techniques: Transformations (Section 3.5, pp. 256–264)
- Square Root Method (Section 1.3, p. 112)

Now Work the **'Are You Prepared?'** problems on page 546.

OBJECTIVES
1. Analyze Hyperbolas with Center at the Origin (p. 537)
2. Find the Asymptotes of a Hyperbola (p. 541)
3. Analyze Hyperbolas with Center at (h, k) (p. 543)
4. Solve Applied Problems Involving Hyperbolas (p. 545)

DEFINITION

A **hyperbola** is the collection of all points in a plane, the difference of whose distances from two fixed points, called the **foci**, is a constant.

Figure 34 illustrates a hyperbola with foci F_1 and F_2. The line containing the foci is called the **transverse axis**. The midpoint of the line segment joining the foci is the **center** of the hyperbola. The line through the center and perpendicular to the transverse axis is the **conjugate axis**. The hyperbola consists of two separate curves, called **branches**, that are symmetric with respect to the transverse axis, conjugate axis, and center. The two points of intersection of the hyperbola and the transverse axis are the **vertices**, V_1 and V_2, of the hyperbola.

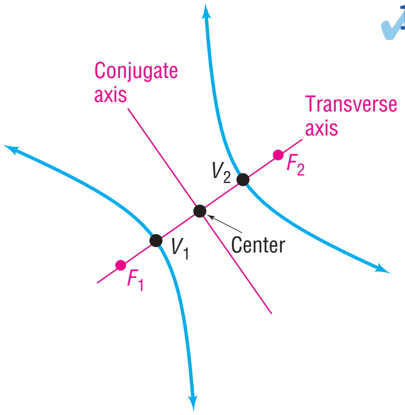


Figure 34 Hyperbola

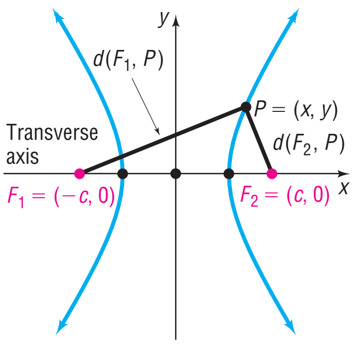


Figure 35
$d(F_1, P) - d(F_2, P) = \pm 2a$

1 Analyze Hyperbolas with Center at the Origin

With these ideas in mind, we are now ready to find the equation of a hyperbola in the rectangular coordinate system. First, place the center at the origin. Next, position the hyperbola so that its transverse axis coincides with a coordinate axis. Suppose that the transverse axis coincides with the x-axis, as shown in Figure 35.

If c is the distance from the center to a focus, one focus is at $F_1 = (-c, 0)$ and the other at $F_2 = (c, 0)$. Now we let the constant difference of the distances from any point $P = (x, y)$ on the hyperbola to the foci F_1 and F_2 be denoted by $\pm 2a$. (If P is on the right branch, the $+$ sign is used; if P is on the left branch, the $-$ sign is used.) The coordinates of P must satisfy the equation

$$d(F_1, P) - d(F_2, P) = \pm 2a \quad \textbf{Difference of the distances from } P \textbf{ to the foci equals } \pm 2a.$$

$$\sqrt{(x - (-c))^2 + y^2} - \sqrt{(x - c)^2 + y^2} = \pm 2a \quad \textbf{Use the Distance Formula.}$$

$$\sqrt{(x + c)^2 + y^2} = \pm 2a + \sqrt{(x - c)^2 + y^2} \quad \textbf{Isolate one radical.}$$

$$(x + c)^2 + y^2 = 4a^2 \pm 4a\sqrt{(x - c)^2 + y^2} + (x - c)^2 + y^2 \quad \textbf{Square both sides.}$$

Next multiply out.

$$x^2 + 2cx + c^2 + y^2 = 4a^2 \pm 4a\sqrt{(x - c)^2 + y^2} + x^2 - 2cx + c^2 + y^2$$

$$4cx - 4a^2 = \pm 4a\sqrt{(x - c)^2 + y^2} \quad \textbf{Simplify; isolate the radical.}$$

$$cx - a^2 = \pm a\sqrt{(x - c)^2 + y^2} \quad \textbf{Divide each side by 4.}$$

$$(cx - a^2)^2 = a^2[(x - c)^2 + y^2] \quad \textbf{Square both sides.}$$

$$c^2x^2 - 2ca^2x + a^4 = a^2(x^2 - 2cx + c^2 + y^2) \quad \textbf{Multiply out.}$$

$$c^2x^2 + a^4 = a^2x^2 + a^2c^2 + a^2y^2 \quad \textbf{Distribute and simplify.}$$

$$(c^2 - a^2)x^2 - a^2y^2 = a^2c^2 - a^4 \quad \textbf{Rearrange terms.}$$

$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2) \quad \textbf{Factor out } a^2 \textbf{ on the right side.} \qquad (1)$$

To obtain points on the hyperbola off the x-axis, it must be that $a < c$. To see why, look again at Figure 35.

$$d(F_1, P) < d(F_2, P) + d(F_1, F_2) \quad \textbf{Use triangle } F_1PF_2.$$

$$d(F_1, P) - d(F_2, P) < d(F_1, F_2)$$

$$2a < 2c \quad P \textbf{ is on the right branch, so } d(F_1, P) - d(F_2, P) = 2a;\ d(F_1, F_2) = 2c.$$

$$a < c \quad \textbf{Divide each side by 2.}$$

Since $a < c$, we also have $a^2 < c^2$, so $c^2 - a^2 > 0$. Let $b^2 = c^2 - a^2$, $b > 0$. Then equation (1) can be written as

$$b^2x^2 - a^2y^2 = a^2b^2$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \textbf{Divide each side by } a^2b^2.$$

To find the vertices of the hyperbola defined by this equation, let $y = 0$. The vertices satisfy the equation $\dfrac{x^2}{a^2} = 1$, the solutions of which are $x = \pm a$. Consequently, the vertices of the hyperbola are $V_1 = (-a, 0)$ and $V_2 = (a, 0)$. Notice that the distance from the center $(0, 0)$ to either vertex is a.

THEOREM

Equation of a Hyperbola: Center at (0, 0); Transverse Axis along the *x*-Axis

An equation of the hyperbola with center at $(0, 0)$, foci at $(-c, 0)$ and $(c, 0)$, and vertices at $(-a, 0)$ and $(a, 0)$ is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{where } b^2 = c^2 - a^2 \qquad \textbf{(2)}$$

The transverse axis is the x-axis.

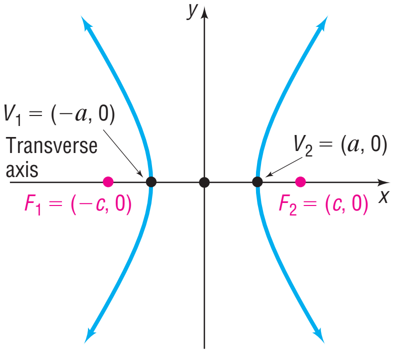


Figure 36 $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1, \quad b^2 = c^2 - a^2$

See Figure 36. As you can verify, the hyperbola defined by equation (2) is symmetric with respect to the x-axis, y-axis, and origin. To find the y-intercepts, if any, let $x = 0$ in equation (2). This results in the equation $\frac{y^2}{b^2} = -1$, which has no real solution, so the hyperbola defined by equation (2) has no y-intercepts. In fact, since $\frac{x^2}{a^2} - 1 = \frac{y^2}{b^2} \geq 0$, it follows that $\frac{x^2}{a^2} \geq 1$. There are no points on the graph for $-a < x < a$.

EXAMPLE 1 **Finding and Graphing an Equation of a Hyperbola**

Find an equation of the hyperbola with center at the origin, one focus at $(3, 0)$, and one vertex at $(-2, 0)$. Graph the equation.

Solution The hyperbola has its center at the origin. Plot the center, focus, and vertex. Since they all lie on the x-axis, the transverse axis coincides with the x-axis. One focus is at $(c, 0) = (3, 0)$, so $c = 3$. One vertex is at $(-a, 0) = (-2, 0)$, so $a = 2$. From equation (2), it follows that $b^2 = c^2 - a^2 = 9 - 4 = 5$, so an equation of the hyperbola is

$$\frac{x^2}{4} - \frac{y^2}{5} = 1$$

To graph a hyperbola, it is helpful to locate and plot other points on the graph. For example, to find the points above and below the foci, we let $x = \pm 3$. Then

$$\frac{x^2}{4} - \frac{y^2}{5} = 1$$

$$\frac{(\pm 3)^2}{4} - \frac{y^2}{5} = 1 \qquad x = \pm 3$$

$$\frac{9}{4} - \frac{y^2}{5} = 1$$

$$\frac{y^2}{5} = \frac{5}{4}$$

$$y^2 = \frac{25}{4}$$

$$y = \pm\frac{5}{2}$$

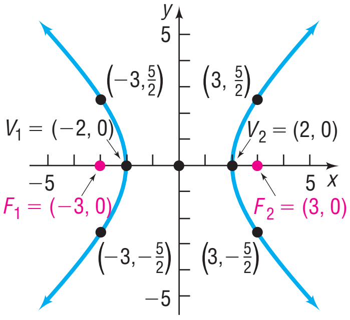


Figure 37 $\dfrac{x^2}{4} - \dfrac{y^2}{5} = 1$

The points above and below the foci are $\left(\pm 3, \frac{5}{2}\right)$ and $\left(\pm 3, -\frac{5}{2}\right)$. These points determine the "opening" of the hyperbola. See Figure 37. ■

EXAMPLE 2 Using a Graphing Utility to Graph a Hyperbola

Using a graphing utility, graph the hyperbola: $\dfrac{x^2}{4} - \dfrac{y^2}{5} = 1$

Solution To graph the hyperbola $\dfrac{x^2}{4} - \dfrac{y^2}{5} = 1$, we need to graph the two functions $Y_1 = \sqrt{5}\sqrt{\dfrac{x^2}{4} - 1}$ and $Y_2 = -\sqrt{5}\sqrt{\dfrac{x^2}{4} - 1}$. As with graphing circles, parabolas, and ellipses on a graphing utility, use a square screen setting so that the graph is not distorted. Figure 38 shows the graph of the hyperbola. ■

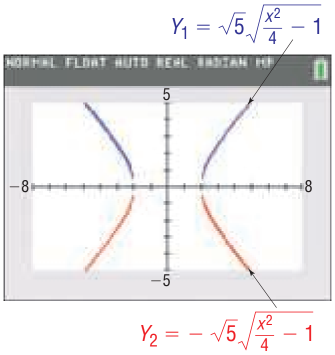


Figure 38

Now Work PROBLEM 19

An equation of the form of equation (2) is the equation of a hyperbola with center at the origin, foci on the x-axis at $(-c, 0)$ and $(c, 0)$, where $c^2 = a^2 + b^2$, and transverse axis along the x-axis.

For the next two examples of this section, the direction **"Analyze the equation"** will mean to find the center, transverse axis, vertices, and foci of the hyperbola and graph it.

EXAMPLE 3 Analyzing the Equation of a Hyperbola

Analyze the equation: $\dfrac{x^2}{16} - \dfrac{y^2}{4} = 1$

Solution The given equation is of the form of equation (2), with $a^2 = 16$ and $b^2 = 4$. The graph of the equation is a hyperbola with center at $(0, 0)$ and transverse axis along the x-axis. Also, $c^2 = a^2 + b^2 = 16 + 4 = 20$. The vertices are at $(\pm a, 0) = (\pm 4, 0)$, and the foci are at $(\pm c, 0) = (\pm 2\sqrt{5}, 0)$.

To locate the points on the graph above and below the foci, let $x = \pm 2\sqrt{5}$. Then

$$\begin{aligned}
\frac{x^2}{16} - \frac{y^2}{4} &= 1\\
\frac{(\pm 2\sqrt{5})^2}{16} - \frac{y^2}{4} &= 1\\
\frac{20}{16} - \frac{y^2}{4} &= 1\\
\frac{5}{4} - \frac{y^2}{4} &= 1\\
\frac{y^2}{4} &= \frac{1}{4}\\
y &= \pm 1
\end{aligned}$$

The points above and below the foci are $(\pm 2\sqrt{5}, 1)$ and $(\pm 2\sqrt{5}, -1)$. See Figure 39(a) for the graph drawn by hand. Figure 39(b) shows the graph obtained using a graphing utility, where $Y_1 = \sqrt{4\left(\dfrac{x^2}{16} - 1\right)}$ and $Y_2 = -\sqrt{4\left(\dfrac{x^2}{16} - 1\right)}$.

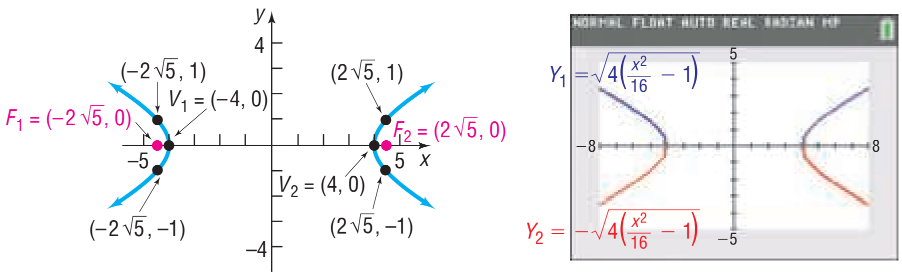


Figure 39 $\dfrac{x^2}{16} - \dfrac{y^2}{4} = 1$ (a) (b) ■

THEOREM

Equation of a Hyperbola; Center at (0, 0) Transverse Axis along the *y*-Axis

An equation of the hyperbola with center at $(0, 0)$, foci at $(0, -c)$ and $(0, c)$, and vertices at $(0, -a)$ and $(0, a)$ is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \qquad \text{where } b^2 = c^2 - a^2 \tag{3}$$

The transverse axis is the y-axis.

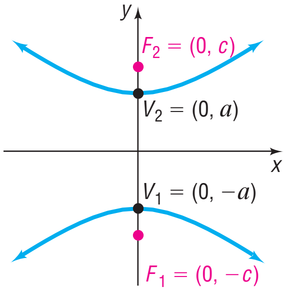


Figure 40
$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1,\ b^2 = c^2 - a^2$

Figure 40 shows the graph of a typical hyperbola defined by equation (3).

An equation of the form of equation (2), $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, is the equation of a hyperbola with center at the origin, foci on the x-axis at $(-c, 0)$ and $(c, 0)$, where $c^2 = a^2 + b^2$, and transverse axis along the x-axis.

An equation of the form of equation (3), $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, is the equation of a hyperbola with center at the origin, foci on the y-axis at $(0, -c)$ and $(0, c)$, where $c^2 = a^2 + b^2$, and transverse axis along the y-axis.

Notice the difference in the forms of equations (2) and (3). When the y^2-term is subtracted from the x^2-term, the transverse axis is along the x-axis. When the x^2-term is subtracted from the y^2-term, the transverse axis is along the y-axis.

EXAMPLE 4 **Analyzing the Equation of a Hyperbola**

Analyze the equation: $4y^2 - x^2 = 4$

Solution To put the equation in proper form, divide each side by 4:

$$y^2 - \frac{x^2}{4} = 1$$

Since the x^2-term is subtracted from the y^2-term, the equation is that of a hyperbola with center at the origin and transverse axis along the y-axis. Comparing the above equation to equation (3), note that $a^2 = 1$, $b^2 = 4$, and $c^2 = a^2 + b^2 = 5$. The vertices are at $(0, \pm a) = (0, \pm 1)$, and the foci are at $(0, \pm c) = (0, \pm\sqrt{5})$.

To locate points on the graph to the left and right of the foci, let $y = \pm\sqrt{5}$. Then

$$\begin{aligned} 4y^2 - x^2 &= 4 \\ 4(\pm\sqrt{5})^2 - x^2 &= 4 \qquad y = \pm\sqrt{5} \\ 20 - x^2 &= 4 \\ x^2 &= 16 \\ x &= \pm 4 \end{aligned}$$

Four other points on the graph are $(\pm 4, \sqrt{5})$ and $(\pm 4, -\sqrt{5})$. See Figure 41(a) for the graph drawn by hand. Figure 41(b) shows the graph obtained using a graphing utility.

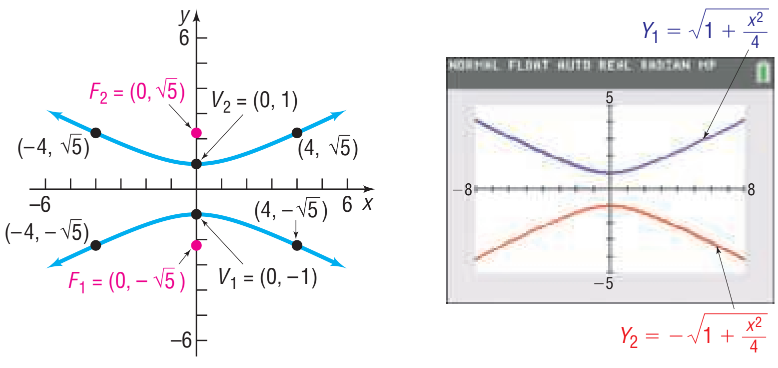


Figure 41 $4y^2 - x^2 = 4$ **(a)** **(b)**

EXAMPLE 5 **Finding an Equation of a Hyperbola**

Find an equation of the hyperbola that has one vertex at $(0, 2)$ and foci at $(0, -3)$ and $(0, 3)$. Graph the equation.

Solution Since the foci are at $(0, -3)$ and $(0, 3)$, the center of the hyperbola, which is at their midpoint, is the origin. Also, the transverse axis is along the y-axis. The given information also reveals that $c = 3$, $a = 2$, and $b^2 = c^2 - a^2 = 9 - 4 = 5$. The form of the equation of the hyperbola is given by equation (3):

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

$$\frac{y^2}{4} - \frac{x^2}{5} = 1$$

Let $y = \pm 3$ to obtain points on the graph on either side of the foci. See Figure 42. ■

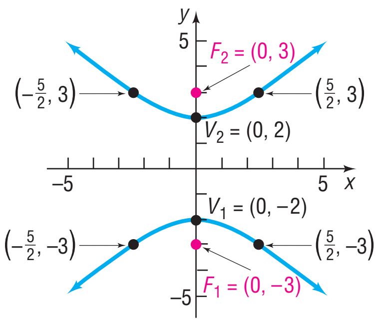


Figure 42 $\frac{y^2}{4} - \frac{x^2}{5} = 1$

Now Work PROBLEM 21

Look at the equations of the hyperbolas in Examples 3 and 5. For the hyperbola in Example 3, $a^2 = 16$ and $b^2 = 4$, so $a > b$; for the hyperbola in Example 5, $a^2 = 4$ and $b^2 = 5$, so $a < b$. We conclude that, for hyperbolas, there are no requirements involving the relative sizes of a and b. Contrast this situation to the case of an ellipse, in which the relative sizes of a and b dictate which axis is the major axis. Hyperbolas have another feature to distinguish them from ellipses and parabolas: Hyperbolas have asymptotes.

2 Find the Asymptotes of a Hyperbola

Recall from Section 5.4 that a horizontal or oblique asymptote of a graph is a line with the property that the distance from the line to points on the graph approaches 0 as $x \to -\infty$ or as $x \to \infty$. Asymptotes provide information about the end behavior of the graph of a hyperbola.

THEOREM

Asymptotes of a Hyperbola

The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has the two oblique asymptotes

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x \tag{4}$$

■

Proof Begin by solving for y in the equation of the hyperbola.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$$

$$y^2 = b^2\left(\frac{x^2}{a^2} - 1\right)$$

Since $x \neq 0$, the right side can be rearranged in the form

$$y^2 = \frac{b^2x^2}{a^2}\left(1 - \frac{a^2}{x^2}\right)$$

$$y = \pm\frac{bx}{a}\sqrt{1 - \frac{a^2}{x^2}}$$

Now, as $x \to -\infty$ or as $x \to \infty$, the term $\frac{a^2}{x^2}$ approaches 0, so the expression under the radical approaches 1. So, as $x \to -\infty$ or as $x \to \infty$, the value of y approaches $\pm\frac{bx}{a}$; that is, the graph of the hyperbola approaches the lines

$$y = -\frac{b}{a}x \quad \text{and} \quad y = \frac{b}{a}x$$

These lines are oblique asymptotes of the hyperbola. ■

The asymptotes of a hyperbola are not part of the hyperbola, but they do serve as a guide for graphing the hyperbola. For example, suppose that we want to graph the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

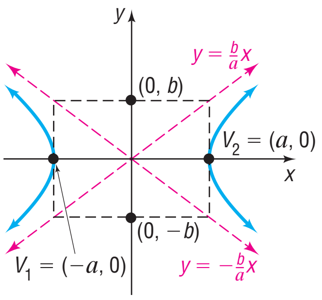


Figure 43 $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Begin by plotting the vertices $(-a, 0)$ and $(a, 0)$. Then plot the points $(0, -b)$ and $(0, b)$ and use these four points to construct a rectangle, as shown in Figure 43. The diagonals of this rectangle have slopes $\frac{b}{a}$ and $-\frac{b}{a}$, and their extensions are the asymptotes $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ of the hyperbola. If we graph the asymptotes, we can use them to establish the "opening" of the hyperbola and avoid plotting other points.

THEOREM

Asymptotes of a Hyperbola

The hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ has the two oblique asymptotes:

$$y = \frac{a}{b}x \quad \text{and} \quad y = -\frac{a}{b}x \tag{5}$$

You are asked to prove this result in Problem 84.

For the remainder of this section, the direction **"Analyze the equation"** will mean to find the center, transverse axis, vertices, foci, and asymptotes of the hyperbola and graph it.

EXAMPLE 6 Analyzing the Equation of a Hyperbola

Analyze the equation: $\frac{y^2}{4} - x^2 = 1$

Solution Since the x^2-term is subtracted from the y^2-term, the equation is of the form of equation (3) and is a hyperbola with center at the origin and transverse axis along the y-axis. Comparing this equation to equation (3), note that $a^2 = 4$, $b^2 = 1$, and $c^2 = a^2 + b^2 = 5$. The vertices are at $(0, \pm a) = (0, \pm 2)$, and the foci are at $(0, \pm c) = (0, \pm\sqrt{5})$. Using equation (5) with $a = 2$ and $b = 1$, the asymptotes are the lines $y = \frac{a}{b}x = 2x$ and $y = -\frac{a}{b}x = -2x$. Form the rectangle containing the points $(0, \pm a) = (0, \pm 2)$ and $(\pm b, 0) = (\pm 1, 0)$. The extensions of the diagonals of this rectangle are the asymptotes. Now graph the asymptotes and the hyperbola. See Figure 44. ■

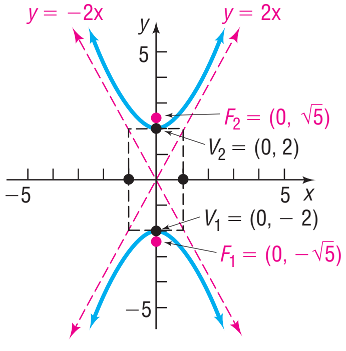


Figure 44 $\frac{y^2}{4} - x^2 = 1$

EXAMPLE 7 Analyzing the Equation of a Hyperbola

Analyze the equation: $9x^2 - 4y^2 = 36$

Solution Divide each side of the equation by 36 to put the equation in proper form.

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

The center of the hyperbola is the origin. Since the x^2-term is first in the equation, the transverse axis is along the x-axis and the vertices and foci lie on the x-axis. Using equation (2), note that $a^2 = 4$, $b^2 = 9$, and $c^2 = a^2 + b^2 = 13$. The vertices are $a = 2$ units left and right of the center at $(\pm a, 0) = (\pm 2, 0)$, the foci are $c = \sqrt{13}$ units left and right of the center at $(\pm c, 0) = (\pm\sqrt{13}, 0)$, and the asymptotes have the equations

$$y = \frac{b}{a}x = \frac{3}{2}x \quad \text{and} \quad y = -\frac{b}{a}x = -\frac{3}{2}x$$

To graph the hyperbola by hand, form the rectangle containing the points $(\pm a, 0)$ and $(0, \pm b)$, that is, $(-2, 0)$, $(2, 0)$, $(0, -3)$, and $(0, 3)$. The extensions of the diagonals of this rectangle are the asymptotes. See Figure 45(a) for the graph drawn by hand. Figure 45(b) shows the graph obtained using a graphing utility.

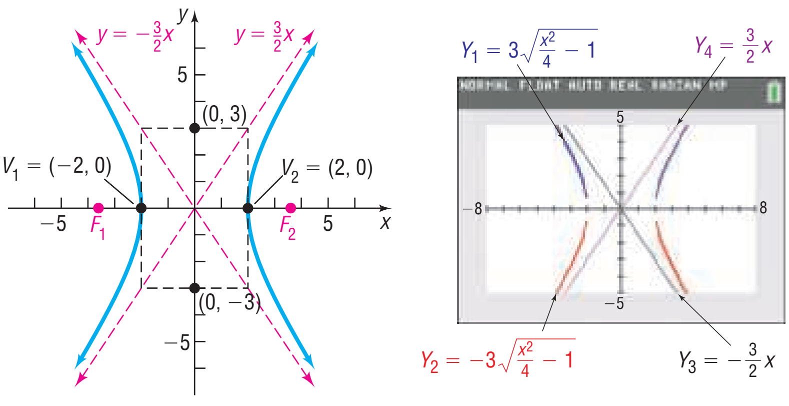


Figure 45 $9x^2 - 4y^2 = 36$

Seeing the Concept

Refer to Figure 45(b). Create a TABLE using Y_1 and Y_4 with $x = 10$, 100, 1000, and 10,000. Compare the values of Y_1 and Y_4. Repeat for Y_2 and Y_3. Now, create a TABLE using Y_1 and Y_3 with $x = -10, -100, -1000$, and $-10{,}000$. Repeat for Y_2 and Y_4.

Now Work PROBLEM 31

3 Analyze Hyperbolas with Center at (h, k)

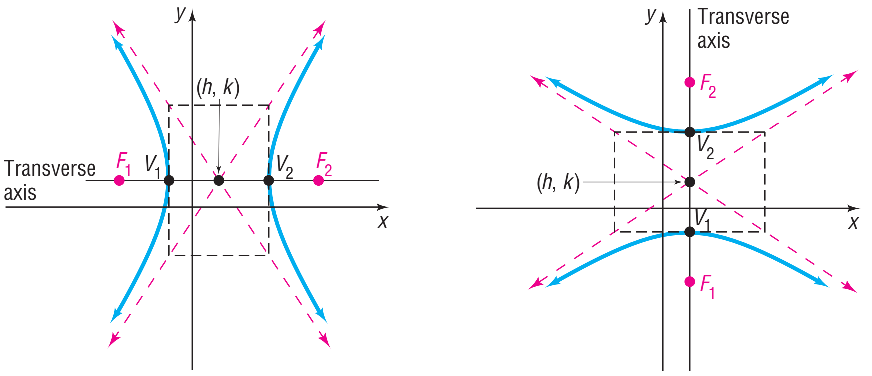
If a hyperbola with center at the origin and transverse axis coinciding with a coordinate axis is shifted horizontally h units and then vertically k units, the result is a hyperbola with center at (h, k) and transverse axis parallel to a coordinate axis. The equations of such hyperbolas have the same forms as those given in equations (2) and (3), except that x is replaced by $x - h$ (the horizontal shift) and y is replaced by $y - k$ (the vertical shift). Table 4 gives the forms of the equations of such hyperbolas. See Figure 46 on the next page for typical graphs.

Table 4

Hyperbolas with Center at (h, k) and Transverse Axis Parallel to a Coordinate Axis

Center	Transverse Axis	Foci	Vertices	Equation	Asymptotes
(h, k)	Parallel to the x-axis	$(h \pm c, k)$	$(h \pm a, k)$	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, $b^2 = c^2 - a^2$	$y - k = \pm\frac{b}{a}(x - h)$
(h, k)	Parallel to the y-axis	$(h, k \pm c)$	$(h, k \pm a)$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$, $b^2 = c^2 - a^2$	$y - k = \pm\frac{a}{b}(x - h)$

Note: It is not recommended that Table 4 be memorized. Rather, use transformations (shift horizontally h units, vertically k units), along with the fact that a represents the distance from the center to the vertices, c represents the distance from the center to the foci, and $b^2 = c^2 - a^2$ (or $c^2 = a^2 + b^2$). ■



(a) $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$ (b) $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$

Figure 46

EXAMPLE 8 Finding an Equation of a Hyperbola, Center Not at the Origin

Find an equation for the hyperbola with center at $(1, -2)$, one focus at $(4, -2)$, and one vertex at $(3, -2)$. Graph the equation by hand.

Solution The center is at $(h, k) = (1, -2)$, so $h = 1$ and $k = -2$. Since the center, focus, and vertex all lie on the line $y = -2$, the transverse axis is parallel to the x-axis. The distance from the center $(1, -2)$ to the focus $(4, -2)$ is $c = 3$; the distance from the center $(1, -2)$ to the vertex $(3, -2)$ is $a = 2$. Thus, $b^2 = c^2 - a^2 = 9 - 4 = 5$. The equation is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-1)^2}{4} - \frac{(y+2)^2}{5} = 1$$

See Figure 47. ■

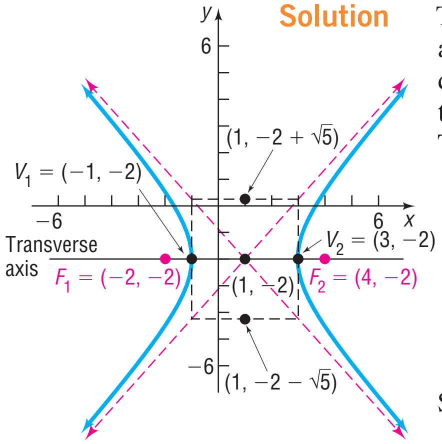


Figure 47 $\dfrac{(x-1)^2}{4} - \dfrac{(y+2)^2}{5} = 1$

Now Work PROBLEM 41

EXAMPLE 9 Analyzing the Equation of a Hyperbola

Analyze the equation: $-x^2 + 4y^2 - 2x - 16y + 11 = 0$

Solution Complete the squares in x and in y.

$$-x^2 + 4y^2 - 2x - 16y + 11 = 0$$

$$-(x^2 + 2x) + 4(y^2 - 4y) = -11 \quad \text{Group terms.}$$

$$-(x^2 + 2x + 1) + 4(y^2 - 4y + 4) = -11 - 1 + 16 \quad \text{Complete each square.}$$

$$-(x+1)^2 + 4(y-2)^2 = 4 \quad \text{Factor.}$$

$$(y-2)^2 - \frac{(x+1)^2}{4} = 1 \quad \text{Divide each side by 4.}$$

This is the equation of a hyperbola with center at $(-1, 2)$ and transverse axis parallel to the y-axis. Also, $a^2 = 1$ and $b^2 = 4$, so $c^2 = a^2 + b^2 = 5$. Since the transverse axis is parallel to the y-axis, the vertices and foci are located a and c units above and below the center, respectively. The vertices are at $(h, k \pm a) = (-1, 2 \pm 1)$, or $(-1, 1)$ and $(-1, 3)$. The foci are at $(h, k \pm c) = (-1, 2 \pm \sqrt{5})$. The asymptotes are

$y - 2 = \frac{1}{2}(x + 1)$ and $y - 2 = -\frac{1}{2}(x + 1)$. Figure 48(a) shows the graph drawn by hand. Figure 48(b) shows the graph obtained using a graphing utility.

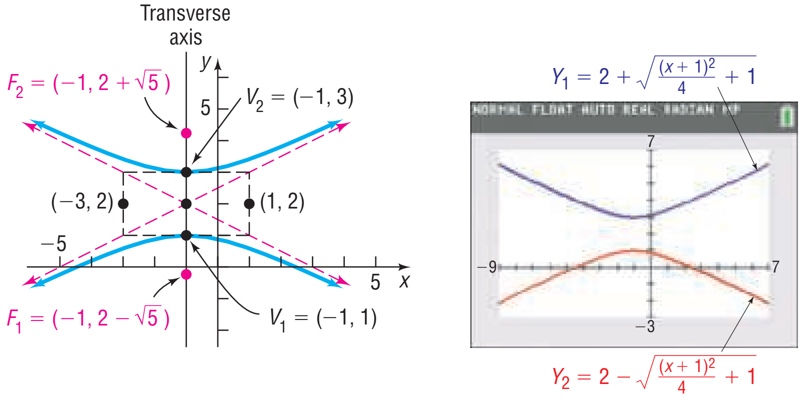


Figure 48 $-x^2 + 4y^2 - 2x - 16y + 11 = 0$

Now Work PROBLEM 55

4 Solve Applied Problems Involving Hyperbolas

Look at Figure 49. Suppose that three microphones are located at points O_1, O_2, and O_3 (the foci of the two hyperbolas). In addition, suppose that a gun is fired at S and the microphone at O_1 records the gunshot 1 second after the microphone at O_2. Because sound travels at about 1100 feet per second, we conclude that the microphone at O_1 is 1100 feet farther from the gunshot than O_2. We can model this situation by saying that S lies on a branch of a hyperbola with foci at O_1 and O_2. (Do you see why? The difference of the distances from S to O_1 and from S to O_2 is the constant 1100.) If the third microphone at O_3 records the gunshot 2 seconds after O_1, then S lies on a branch of a second hyperbola with foci at O_1 and O_3. In this case, the constant difference is 2200. The intersection of the two hyperbolas identifies the location of S.

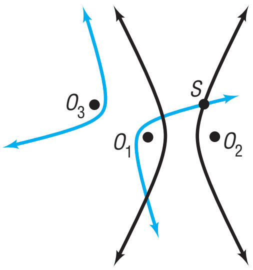


Figure 49

EXAMPLE 10 **Lightning Strikes**

Suppose that two people standing 1 mile apart both see a flash of lightning. After a period of time, the person standing at point A hears the thunder. One second later, the person standing at point B hears the thunder. If the person at B is due west of the person at A and the lightning strike is known to occur due north of the person standing at point A, where did the lightning strike occur?

Solution See Figure 50 in which the ordered pair (x, y) represents the location of the lightning strike. We know that sound travels at 1100 feet per second, so the person at point A is 1100 feet closer to the lightning strike than the person at point B. Since the difference of the distance from (x, y) to B and the distance from (x, y) to A is the constant 1100, the point (x, y) lies on a hyperbola whose foci are at A and B.

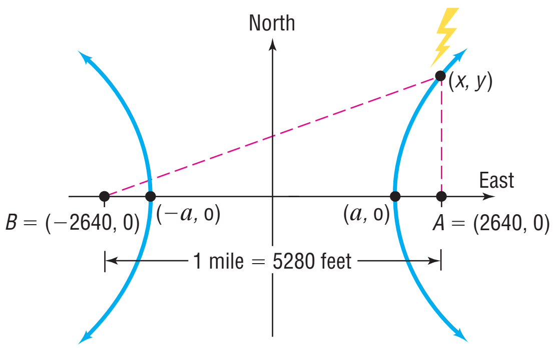


Figure 50

An equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $2a = 1100$, so $a = 550$.

Because the distance between the two people is 1 mile (5280 feet) and each person is at a focus of the hyperbola, this means

$$2c = 5280$$
$$c = \frac{5280}{2} = 2640$$

Since $b^2 = c^2 - a^2 = 2640^2 - 550^2 = 6{,}667{,}100$, the equation of the hyperbola that describes the location of the lightning strike is

$$\frac{x^2}{550^2} - \frac{y^2}{6{,}667{,}100} = 1$$

Refer to Figure 50. Since the lightning strike occurred due north of the individual at the point $A = (2640, 0)$, let $x = 2640$ and solve the resulting equation.

$$\frac{2640^2}{550^2} - \frac{y^2}{6{,}667{,}100} = 1 \qquad x = 2640$$

$$-\frac{y^2}{6{,}667{,}100} = -22.04 \qquad \text{Subtract } \frac{2640^2}{550^2} \text{ from both sides.}$$

$$y^2 = 146{,}942{,}884 \qquad \text{Multiply both sides by } -6{,}667{,}100.$$

$$y = 12{,}122 \qquad y > 0 \text{ since the lightning strike occurred in quadrant I.}$$

The lightning strike occurred 12,122 feet north of the person standing at point A.

Check: The difference between the distance from $(2640, 12{,}122)$ to the person at the point $B = (-2640, 0)$ and the distance from $(2640, 12{,}122)$ to the person at the point $A = (2640, 0)$ should be 1100. Using the distance formula, the difference in the distances is

$$\sqrt{[2640 - (-2640)]^2 + (12{,}122 - 0)^2} - \sqrt{(2640 - 2640)^2 + (12{,}122 - 0)^2} = 1100$$

as required. ■

Now Work PROBLEM 75

7.4 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. The distance d from $P_1 = (3, -4)$ to $P_2 = (-2, 1)$ is $d =$ _____. (p. 85)
2. To complete the square of $x^2 + 5x$, add ___. (p. 57)
3. Find the intercepts of the equation $y^2 = 9 + 4x^2$. (pp. 165–166)
4. ***True or False*** The equation $y^2 = 9 + x^2$ is symmetric with respect to the x-axis, the y-axis, and the origin. (pp. 166–168)
5. To graph $y = (x - 5)^3 - 4$, shift the graph of $y = x^3$ to the (left/right) ________ unit(s) and then (up/down) ________ unit(s). (pp. 256–264)
6. Find the vertical asymptotes, if any, and the horizontal or oblique asymptote, if any, of $y = \dfrac{x^2 - 9}{x^2 - 4}$. (pp. 374–379)

Concepts and Vocabulary

7. A(n) __________ is the collection of points in a plane the difference of whose distances from two fixed points is a constant.

8. For a hyperbola, the foci lie on a line called the __________ _____.

Answer Problems 9–11 using the figure to the right.

9. The equation of the hyperbola is of the form

(a) $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$

(b) $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$

(c) $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

(d) $\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1$

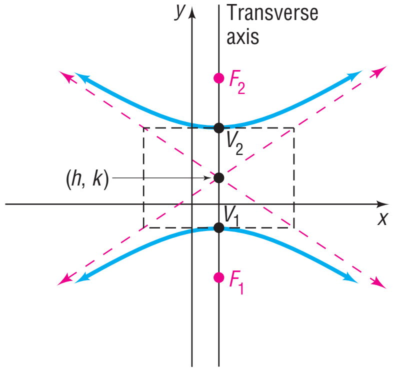
10. If the center of the hyperbola is $(2, 1)$ and $a = 3$, then the coordinates of the vertices are ______ and ________ .

11. If the center of the hyperbola is $(2, 1)$ and $c = 5$, then the coordinates of the foci are ______ and ________ .

12. In a hyperbola, if $a = 3$ and $c = 5$, then $b =$ ________.
(a) 1 (b) 2 (c) 4 (d) 8

13. For the hyperbola $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$, the value of a is ____ , the value of b is ____ , and the transverse axis is the ____ -axis.

14. For the hyperbola $\dfrac{y^2}{16} - \dfrac{x^2}{81} = 1$, the asymptotes are __________ and ________ .

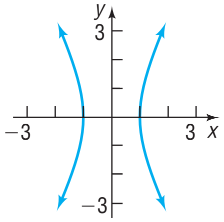


Skill Building

In Problems 15–18, the graph of a hyperbola is given. Match each graph to its equation.

(A) $\dfrac{x^2}{4} - y^2 = 1$ (B) $x^2 - \dfrac{y^2}{4} = 1$ (C) $\dfrac{y^2}{4} - x^2 = 1$ (D) $y^2 - \dfrac{x^2}{4} = 1$

15.



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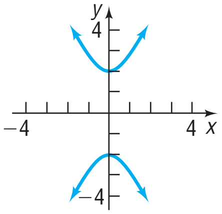

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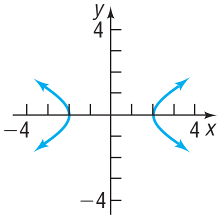

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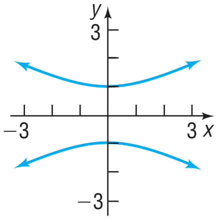


In Problems 19–28, find an equation for the hyperbola described. Graph the equation by hand.

19. Center at $(0, 0)$; focus at $(3, 0)$; vertex at $(1, 0)$

20. Center at $(0, 0)$; focus at $(0, 5)$; vertex at $(0, 3)$

21. Center at $(0, 0)$; focus at $(0, -6)$; vertex at $(0, 4)$

22. Center at $(0, 0)$; focus at $(-3, 0)$; vertex at $(2, 0)$

23. Foci at $(-5, 0)$ and $(5, 0)$; vertex at $(3, 0)$

24. Focus at $(0, 6)$; vertices at $(0, -2)$ and $(0, 2)$

25. Vertices at $(0, -6)$ and $(0, 6)$; asymptote the line $y = 2x$

26. Vertices at $(-4, 0)$ and $(4, 0)$; asymptote the line $y = 2x$

27. Foci at $(-4, 0)$ and $(4, 0)$; asymptote the line $y = -x$

28. Foci at $(0, -2)$ and $(0, 2)$; asymptote the line $y = -x$

In Problems 29–36, find the center, transverse axis, vertices, foci, and asymptotes. Graph each equation by hand. Verify your graph using a graphing utility.

29. $\dfrac{x^2}{25} - \dfrac{y^2}{9} = 1$

30. $\dfrac{y^2}{16} - \dfrac{x^2}{4} = 1$

31. $4x^2 - y^2 = 16$

32. $4y^2 - x^2 = 16$

33. $y^2 - 9x^2 = 9$

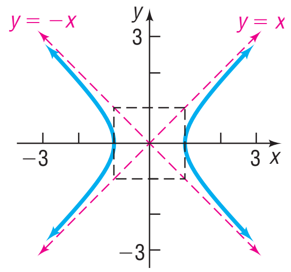
34. $x^2 - y^2 = 4$

35. $y^2 - x^2 = 25$

36. $2x^2 - y^2 = 4$

In Problems 37–40, write an equation for each hyperbola.

37.



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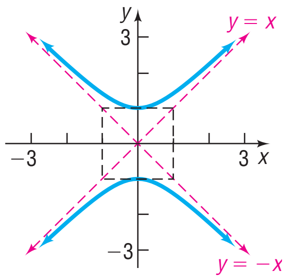

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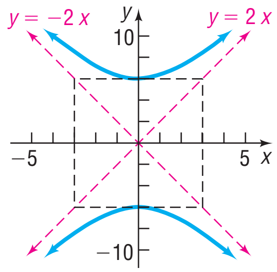

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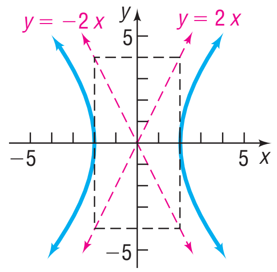


In Problems 41–48, find an equation for the hyperbola described. Graph the equation by hand.

41. Center at $(4, -1)$; focus at $(7, -1)$; vertex at $(6, -1)$

42. Center at $(-3, 1)$; focus at $(-3, 6)$; vertex at $(-3, 4)$

43. Center at $(-3, -4)$; focus at $(-3, -8)$; vertex at $(-3, -2)$

44. Center at $(1, 4)$; focus at $(-2, 4)$; vertex at $(0, 4)$

45. Foci at $(3, 7)$ and $(7, 7)$; vertex at $(6, 7)$

46. Focus at $(-4, 0)$ vertices at $(-4, 4)$ and $(-4, 2)$

47. Vertices at $(-1, -1)$ and $(3, -1)$; asymptote the line $y + 1 = \frac{3}{2}(x - 1)$

48. Vertices at $(1, -3)$ and $(1, 1)$; asymptote the line $y + 1 = \frac{3}{2}(x - 1)$

In Problems 49–62, find the center, transverse axis, vertices, foci, and asymptotes. Graph each equation by hand. Verify your graph using a graphing utility.

49. $\frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} = 1$

50. $\frac{(y+3)^2}{4} - \frac{(x-2)^2}{9} = 1$

51. $(y-2)^2 - 4(x+2)^2 = 4$

52. $(x+4)^2 - 9(y-3)^2 = 9$

53. $(x+1)^2 - (y+2)^2 = 4$

54. $(y-3)^2 - (x+2)^2 = 4$

55. $x^2 - y^2 - 2x - 2y - 1 = 0$

56. $y^2 - x^2 - 4y + 4x - 1 = 0$

57. $y^2 - 4x^2 - 4y - 8x - 4 = 0$

58. $2x^2 - y^2 + 4x + 4y - 4 = 0$

59. $4x^2 - y^2 - 24x - 4y + 16 = 0$

60. $2y^2 - x^2 + 2x + 8y + 3 = 0$

61. $y^2 - 4x^2 - 16x - 2y - 19 = 0$

62. $x^2 - 3y^2 + 8x - 6y + 4 = 0$

In Problems 63–66, graph each function by hand. Be sure to label any intercepts. [**Hint:** *Notice that each function is half a hyperbola.*]

63. $f(x) = \sqrt{16 + 4x^2}$

64. $f(x) = -\sqrt{9 + 9x^2}$

65. $f(x) = -\sqrt{-25 + x^2}$

66. $f(x) = \sqrt{-1 + x^2}$

Mixed Practice

In Problems 67–74, analyze each equation.

67. $\frac{(x-3)^2}{4} - \frac{y^2}{25} = 1$

68. $\frac{(y+2)^2}{16} - \frac{(x-2)^2}{4} = 1$

69. $x^2 = 16(y - 3)$

70. $y^2 = -12(x + 1)$

71. $25x^2 + 9y^2 - 250x + 400 = 0$

72. $x^2 + 36y^2 - 2x + 288y + 541 = 0$

73. $x^2 - 6x - 8y - 31 = 0$

74. $9x^2 - y^2 - 18x - 8y - 88 = 0$

Applications and Extensions

75. Fireworks Display Suppose that two people standing 2 miles apart both see the burst from a fireworks display. After a period of time the first person, standing at point A, hears the burst. One second later the second person, standing at point B, hears the burst. If the person at point B is due west of the person at point A, and if the display is known to occur due north of the person at point A, where did the fireworks display occur?

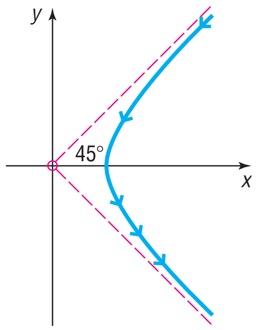
76. Lightning Strikes Suppose that two people standing 1 mile apart both see a flash of lightning. After a period of time the first person, standing at point A, hears the thunder. Two seconds later the second person, standing at point B, hears the thunder. If the person at point B is due west of the person at point A, and if the lightning strike is known to occur due north of the person standing at point A, where did the lightning strike occur?

77. Nuclear Power Plant Some nuclear power plants utilize "natural draft" cooling towers in the shape of a **hyperboloid**, a solid obtained by rotating a hyperbola about its conjugate axis. Suppose that such a cooling tower has a base diameter of 400 feet and the diameter at its narrowest point, 360 feet above the ground, is 200 feet. If the diameter at the top of the tower is 300 feet, how tall is the tower?

Source: *Bay Area Air Quality Management District*

78. An Explosion Two recording devices are set 2400 feet apart, with the device at point A to the west of the device at point B. At a point between the devices 300 feet from point B, a small amount of explosive is detonated. The recording devices record the time until the sound reaches each. How far directly north of point B should a second explosion be done so that the measured time difference recorded by the devices is the same as that for the first detonation?

79. Rutherford's Experiment In May 1911, Ernest Rutherford published a paper in *Philosophical Magazine*. In this article, he described the motion of alpha particles as they are shot at a piece of gold foil 0.00004 cm thick. Before conducting this experiment, Rutherford expected that the alpha particles would shoot through the foil just as a bullet would shoot through snow. Instead, a small fraction of the alpha particles bounced off the foil. This led to the conclusion that the nucleus of an atom is dense, while the remainder of the atom is sparse. Only the density of the nucleus could cause the alpha particles to deviate from their path. The figure shows a diagram from Rutherford's paper that indicates that the deflected alpha particles follow the path of one branch of a hyperbola.



(a) Find an equation of the asymptotes under this scenario.

(b) If the vertex of the path of the alpha particles is 10 cm from the center of the hyperbola, find a model that describes the path of the particle.

80. Hyperbolic Mirrors Hyperbolas have interesting reflective properties that make them useful for lenses and mirrors. For example, if a ray of light strikes a convex hyperbolic mirror on a line that would (theoretically) pass through its rear focus, it is reflected through the front focus. This property, and that of the parabola, were used to develop the *Cassegrain* telescope in 1672. The focus of the parabolic mirror and the rear focus of the hyperbolic mirror are the same point. The rays are collected by the parabolic mirror, then are reflected toward the (common) focus, and thus are reflected by the hyperbolic mirror through the opening to its front focus, where the eyepiece is located. If the equation of the hyperbola is $\frac{y^2}{9} - \frac{x^2}{16} = 1$ and the focal length (distance from the vertex to the focus) of the parabola is 6, find the equation of the parabola.

Source: *www.enchantedlearning.com*

81. The **eccentricity** e of a hyperbola is defined as the number $\frac{c}{a}$, where a is the distance of a vertex from the center and c is the distance of a focus from the center. Because $c > a$, it follows that $e > 1$. Describe the general shape of a hyperbola whose eccentricity is close to 1. What is the shape if e is very large?

82. A hyperbola for which $a = b$ is called an **equilateral hyperbola**. Find the eccentricity e of an equilateral hyperbola.

[**Note:** The eccentricity of a hyperbola is defined in Problem 81.]

83. Two hyperbolas that have the same set of asymptotes are called **conjugate**. Show that the hyperbolas

$$\frac{x^2}{4} - y^2 = 1 \quad \text{and} \quad y^2 - \frac{x^2}{4} = 1$$

are conjugate. Graph each hyperbola on the same set of coordinate axes.

84. Prove that the hyperbola

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

has the two oblique asymptotes

$$y = \frac{a}{b}x \quad \text{and} \quad y = -\frac{a}{b}x$$

85. Show that the graph of an equation of the form

$$Ax^2 + Cy^2 + F = 0 \qquad A \neq 0, C \neq 0, F \neq 0$$

where A and C are opposite in sign, is a hyperbola with center at $(0, 0)$.

86. Show that the graph of an equation of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \qquad A \neq 0, C \neq 0$$

where A and C are opposite in sign,

(a) is a hyperbola if $\frac{D^2}{4A} + \frac{E^2}{4C} - F \neq 0$.

(b) is two intersecting lines if $\frac{D^2}{4A} + \frac{E^2}{4C} - F = 0$.

Retain Your Knowledge

Problems 87–90 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

87. Solve: $8x^3 - 12x^2 - 50x + 75 = 0$

88. The function $f(x) = \frac{x+5}{x-6}$, $x \neq 6$, is one-to-one. Find its inverse function.

89. Solve the inequality: $\frac{x^2 - 16}{x^2 - 25} \leq 0$

90. Solve: $\log_7(x - 5) + \log_7(x + 1) = 1$

'Are You Prepared?' Answers

1. $5\sqrt{2}$ **2.** $\frac{25}{4}$ **3.** $(0, -3), (0, 3)$ **4.** True **5.** right 5; down 4 **6.** Vertical: $x = -2, x = 2$; horizontal: $y = 1$

Chapter Review

Things to Know

Equations

Parabola (pp. 515–521)	See Tables 1 and 2 (pp. 517 and 519).
Ellipse (pp. 525–533)	See Table 3 (p. 530).
Hyperbola (pp. 536–546)	See Table 4 (p. 543).

Definitions

Parabola (p. 515)	Set of points P in a plane for which $d(F, P) = d(P, D)$, where F is the focus and D is the directrix
Ellipse (p. 525)	Set of points P in a plane, the sum of whose distances from two fixed points (the foci) is a constant
Hyperbola (p. 536)	Set of points P in a plane, the difference of whose distances from two fixed points (the foci) is a constant

Objectives

Section		You should be able to . . .	Examples	Review Exercises
7.1	1	Know the names of the conics (p. 514)		1–10
7.2	1	Analyze parabolas with vertex at the origin (p. 515)	1–6	1, 11
	2	Analyze parabolas with vertex at (h, k) (p. 519)	7–9	4, 6, 9, 14
	3	Solve applied problems involving parabolas (p. 520)	10	21
7.3	1	Analyze ellipses with center at the origin (p. 525)	1–5	3, 13
	2	Analyze ellipses with center at (h, k) (p. 530)	6–8	8, 10, 16, 20
	3	Solve applied problems involving ellipses (p. 532)	9	22
7.4	1	Analyze hyperbolas with center at the origin (p. 537)	1–5	2, 5, 12, 19
	2	Find the asymptotes of a hyperbola (p. 541)	6,7	2, 5, 7
	3	Analyze hyperbolas with center at (h, k) (p. 543)	8, 9	7, 15, 17, 18
	4	Solve applied problems involving hyperbolas (p. 545)	10	23

Review Exercises

In Problems 1–10, identify each equation. If it is a parabola, give its vertex, focus, and directrix; if it is an ellipse, give its center, vertices, and foci; if it is a hyperbola, give its center, vertices, foci, and asymptotes.

1. $y^2 = -16x$

2. $\dfrac{x^2}{25} - y^2 = 1$

3. $\dfrac{y^2}{25} + \dfrac{x^2}{16} = 1$

4. $x^2 + 4y = 4$

5. $4x^2 - y^2 = 8$

6. $x^2 - 4x = 2y$

7. $y^2 - 4y - 4x^2 + 8x = 4$

8. $4x^2 + 9y^2 - 16x - 18y = 11$

9. $4x^2 - 16x + 16y + 32 = 0$

10. $9x^2 + 4y^2 - 18x + 8y = 23$

In Problems 11–18, find an equation of the conic described. Graph the equation.

11. Parabola; focus at $(-2, 0)$; directrix the line $x = 2$

12. Hyperbola; center at $(0, 0)$; focus at $(0, 4)$; vertex at $(0, -2)$

13. Ellipse; foci at $(-3, 0)$ and $(3, 0)$; vertex at $(4, 0)$

14. Parabola; vertex at $(2, -3)$; focus at $(2, -4)$

15. Hyperbola; center at $(-2, -3)$; focus at $(-4, -3)$; vertex at $(-3, -3)$

16. Ellipse; foci at $(-4, 2)$ and $(-4, 8)$; vertex at $(-4, 10)$

17. Center at $(-1, 2)$; $a = 3$; $c = 4$; transverse axis parallel to the x-axis

18. Vertices at $(0, 1)$ and $(6, 1)$; asymptote the line $3y + 2x = 9$

19. Find an equation of the hyperbola whose foci are the vertices of the ellipse $4x^2 + 9y^2 = 36$ and whose vertices are the foci of this ellipse.

20. Describe the collection of points in a plane so that the distance from each point to the point $(3, 0)$ is three-fourths of its distance from the line $x = \frac{16}{3}$.

21. Searchlight A searchlight is shaped like a paraboloid of revolution. If a light source is located 1 foot from the vertex along the axis of symmetry and the opening is 2 feet across, how deep should the mirror be in order to reflect the light rays parallel to the axis of symmetry?

22. Semielliptical Arch Bridge A bridge is built in the shape of a semielliptical arch. The bridge has a span of 60 feet and a maximum height of 20 feet. Find the height of the arch at distances of 5, 10, and 20 feet from the center.

23. Calibrating Instruments In a test of their recording devices, a team of seismologists positioned two of the devices 2000 feet apart, with the device at point A to the west of the device at point B. At a point between the devices and 200 feet from point B, a small amount of explosive was detonated and a note made of the time at which the sound reached each device. A second explosion is to be carried out at a point directly north of point B. How far north should the site of the second explosion be chosen so that the measured time difference recorded by the devices for the second detonation is the same as that recorded for the first detonation?

Chapter Test

CHAPTER **Test Prep** VIDEOS

The Chapter Test Prep Videos are step-by-step solutions available in MyMathLab®, or on this text's YouTube Channel. Flip back to the Resources for Success page for a link to this text's YouTube channel.

In Problems 1–3, identify each equation. If it is a parabola, give its vertex, focus, and directrix; if an ellipse, give its center, vertices, and foci; if a hyperbola, give its center, vertices, foci, and asymptotes.

1. $\dfrac{(x+1)^2}{4} - \dfrac{y^2}{9} = 1$ **2.** $8y = (x-1)^2 - 4$ **3.** $2x^2 + 3y^2 + 4x - 6y = 13$

In Problems 4–6, find an equation of the conic described; graph the equation.

4. Parabola: focus $(-1, 4.5)$, vertex $(-1, 3)$

5. Ellipse: center $(0, 0)$, vertex $(0, -4)$, focus $(0, 3)$

6. Hyperbola: center $(2, 2)$, vertex $(2, 4)$, contains the point $\left(2 + \sqrt{10}, 5\right)$

7. A parabolic reflector (paraboloid of revolution) is used by TV crews at football games to pick up the referee's announcements, quarterback signals, and so on. A microphone is placed at the focus of the parabola. If a certain reflector is 4 feet wide and 1.5 feet deep, where should the microphone be placed?

Cumulative Review

1. For $f(x) = -3x^2 + 5x - 2$, find

$$\frac{f(x+h) - f(x)}{h} \qquad h \neq 0$$

2. In the complex number system, solve the equation

$$9x^4 + 33x^3 - 71x^2 - 57x - 10 = 0$$

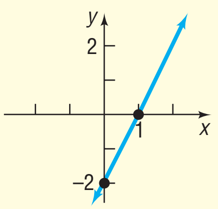
3. For what numbers x is $6 - x \geq x^2$?

4. (a) Find the domain and range of $y = 3^x + 2$.
(b) Find the inverse of $y = 3^x + 2$ and state its domain and range.

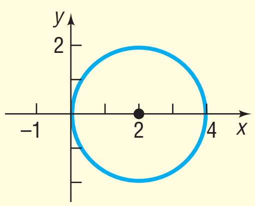
5. $f(x) = \log_4 (x - 2)$
(a) Solve $f(x) = 2$.
(b) Solve $f(x) \leq 2$.

6. Find an equation for each of the following graphs.

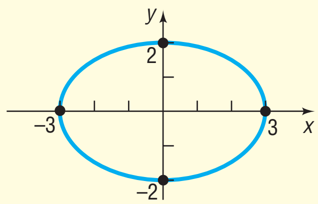
(a) Line:



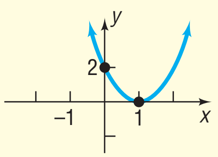
(b) Circle:



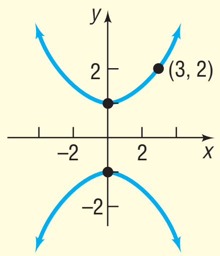
(c) Ellipse:



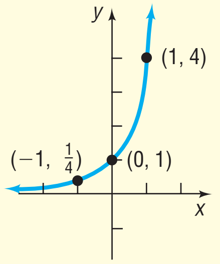
(d) Parabola:



(e) Hyperbola:



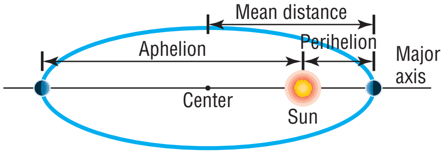
(f) Exponential:



Chapter Projects

I. Comet Hale-Bopp The orbits of planets and some comets about the Sun are ellipses, with the Sun at one focus. The **aphelion** of a planet is its greatest distance from the Sun, and the **perihelion** is its shortest distance. The **mean distance** of a planet from the Sun is the length of the semimajor axis of the elliptical orbit. See the illustration.

1. Research the history of Comet Hale-Bopp on the Internet. In particular, determine the aphelion and perihelion. Often these values are given in terms of astronomical units. What is an astronomical unit? What is it equivalent to in miles? In kilometers? What is the orbital period of Comet Hale-Bopp? When will it next be visible from Earth? How close does it come to Earth?
2. Find a model for the orbit of Comet Hale-Bopp around the Sun. Use the x-axis as the major axis.
3. Comet Hale-Bopp has an orbit that is roughly perpendicular to that of Earth. Find a model for the orbit of Earth using the y-axis as the major axis.
4. Use a graphing utility or some other graphing technology to graph the paths of the orbits. Based on the graphs, do the paths of the orbits intersect? Does this mean that Comet Hale-Bopp will collide with Earth?



The following projects can be found at the Instructor's Resource Center (IRC):

II. The Orbits of Neptune and Pluto The astronomical body known as Pluto and the planet Neptune travel around the Sun in elliptical orbits. Pluto, at times, comes closer to the Sun than Neptune, the outermost planet. This project examines and analyzes the two orbits.

III. Project at Motorola ***Distorted Deployable Space Reflector Antennas*** An engineer designs an antenna that will deploy in space to collect sunlight.

IV. Constructing a Bridge over the East River The size of ships using a river and fluctuations in water height due to tides or flooding must be considered when designing a bridge that will cross a major waterway.

8 Systems of Equations and Inequalities

Economic Outcomes

Annual Earnings of Young Adults

For both males and females, earnings increase with education: full-time workers with at least a bachelor's degree have higher median earnings than those with less education. For example, in 2014, male college graduates earned 84% more than male high school completers. Females with a bachelor's or higher degree earned 81% more than female high school completers. Males and females who dropped out of high school earned 31% and 29% less, respectively, than male and female high school completers.

The median earnings of young adults who had at least a bachelor's degree declined in the 1970s relative to their counterparts who were high school completers, before increasing between 1980 and 2014. Males with a bachelor's degree or higher had earnings 19% higher than male high school completers in 1980 and had earnings 84% higher in 2014. Among females, those with at least a bachelor's degree had earnings 34% higher than female high school completers in 1980, compared with earnings 81% higher in 2014.

—See Chapter Project I—

••• A Look Back

In Chapters 1, 4, 5, and 6 we solved various kinds of equations and inequalities involving a single variable.

A Look Ahead •••

In this chapter we take up the problem of solving equations and inequalities containing two or more variables. There are various ways to solve such problems.

The *method of substitution* for solving equations in several variables dates back to ancient times.

The *method of elimination*, although it had existed for centuries, was put into systematic order by Karl Friedrich Gauss (1777–1855) and by Camille Jordan (1838–1922).

The theory of *matrices* was developed in 1857 by Arthur Cayley (1821–1895), although only later were matrices used as we use them in this chapter. Matrices have become a very flexible instrument, useful in almost all areas of mathematics.

The method of *determinants* was invented by Takakazu Seki Kôwa (1642–1708) in 1683 in Japan and by Gottfried Wilhelm von Leibniz (1646–1716) in 1693 in Germany. *Cramer's Rule* is named after Gabriel Cramer (1704–1752) of Switzerland, who popularized the use of determinants for solving linear systems.

Section 8.5, on *partial fraction decomposition*, provides an application of systems of equations. This particular application is one that is used in integral calculus.

Section 8.8 introduces *linear programming*, a modern application of linear inequalities. This topic is particularly useful for students interested in operations research.

Outline

8.1 Systems of Linear Equations: Substitution and Elimination

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Linear Equations (Section 1.2, pp. 102–103)
- Lines (Section 2.2, pp. 173–184)

Now Work the **'Are You Prepared?'** problems on page 564.

OBJECTIVES
1. Solve Systems of Equations by Substitution (p. 556)
2. Solve Systems of Equations by Elimination (p. 557)
3. Identify Inconsistent Systems of Equations Containing Two Variables (p. 559)
4. Express the Solution of a System of Dependent Equations Containing Two Variables (p. 559)
5. Solve Systems of Three Equations Containing Three Variables (p. 560)
6. Identify Inconsistent Systems of Equations Containing Three Variables (p. 562)
7. Express the Solution of a System of Dependent Equations Containing Three Variables (p. 563)

EXAMPLE 1 **Movie Theater Ticket Sales**

A movie theater sells tickets for \$10.00 each, with seniors receiving a discount of \$2.00. One evening the theater took in \$4630 in revenue. If x represents the number of tickets sold at \$10.00 and y the number of tickets sold at the discounted price of \$8.00, write an equation that relates these variables.

Solution Each nondiscounted ticket brings in \$10.00, so x tickets will bring in $10x$ dollars. Similarly, y discounted tickets bring in $8y$ dollars. Because the total brought in is \$4630, this means

$$10x + 8y = 4630$$

■

In Example 1, suppose it is also known that 525 tickets were sold that evening. Then a second equation relating the variables x and y is

$$x + y = 525$$

The two equations

$$\begin{cases} 10x + 8y = 4630 \\ x + y = 525 \end{cases}$$

form a *system* of equations.

In general, a **system of equations** is a collection of two or more equations, each containing one or more variables. Example 2 gives some illustrations of systems of equations.

EXAMPLE 2 **Examples of Systems of Equations**

(a) $\begin{cases} 2x + y = 5 & (1) \\ -4x + 6y = -2 & (2) \end{cases}$ **Two equations containing two variables, x and y**

(b) $\begin{cases} x + y^2 = 5 & (1) \\ 2x + y = 4 & (2) \end{cases}$ **Two equations containing two variables, x and y**

(c) $\begin{cases} x + y + z = 6 & (1) \\ 3x - 2y + 4z = 9 & (2) \\ x - y - z = 0 & (3) \end{cases}$ **Three equations containing three variables, x, y, and z**

(d) $\begin{cases} x + y + z = 5 & (1) \\ x - y \quad = 2 & (2) \end{cases}$ **Two equations containing three variables, x, y, and z**

(e) $\begin{cases} x + y + z = 6 & (1) \\ 2x \quad + 2z = 4 & (2) \\ \quad y + z = 2 & (3) \\ x \quad = 4 & (4) \end{cases}$ **Four equations containing three variables, x, y, and z**

We use a brace to remind us that we are dealing with a system of equations, and we number each equation in the system for convenient reference.

A **solution** of a system of equations consists of values for the variables that are solutions of each equation of the system. To **solve** a system of equations means to find all solutions of the system.

For example, $x = 2, y = 1$ is a solution of the system in Example 2(a), because

$$\begin{cases} 2x + y = 5 & (1) \\ -4x + 6y = -2 & (2) \end{cases} \qquad \begin{cases} 2(2) + 1 = 4 + 1 = 5 \\ -4(2) + 6(1) = -8 + 6 = -2 \end{cases}$$

This solution may also be written as the ordered pair $(2, 1)$.

A solution of the system in Example 2(b) is $x = 1, y = 2$, because

$$\begin{cases} x + y^2 = 5 & (1) \\ 2x + y = 4 & (2) \end{cases} \qquad \begin{cases} 1 + 2^2 = 1 + 4 = 5 \\ 2(1) + 2 = 2 + 2 = 4 \end{cases}$$

Another solution of the system in Example 2(b) is $x = \frac{11}{4}, y = -\frac{3}{2}$, which you can check for yourself.

A solution of the system in Example 2(c) is $x = 3, y = 2, z = 1$, because

$$\begin{cases} x + y + z = 6 & (1) \\ 3x - 2y + 4z = 9 & (2) \\ x - y - z = 0 & (3) \end{cases} \qquad \begin{cases} 3 + 2 + 1 = 6 \\ 3(3) - 2(2) + 4(1) = 9 - 4 + 4 = 9 \\ 3 - 2 - 1 = 0 \end{cases}$$

This solution may also be written as the ordered triplet $(3, 2, 1)$.

Note that $x = 3, y = 3, z = 0$ is not a solution of the system in Example 2(c).

$$\begin{cases} x + y + z = 6 & (1) \\ 3x - 2y + 4z = 9 & (2) \\ x - y - z = 0 & (3) \end{cases} \qquad \begin{cases} 3 + 3 + 0 = 6 \\ 3(3) - 2(3) + 4(0) = 3 \neq 9 \\ 3 - 3 - 0 = 0 \end{cases}$$

Although $x = 3, y = 3$, and $z = 0$ satisfy equations (1) and (3), they do not satisfy equation (2). Any solution of the system must satisfy *each* equation of the system.

Now Work PROBLEM 11

When a system of equations has at least one solution, it is said to be **consistent**. When a system of equations has no solution, it is called **inconsistent**.

An equation in n variables is said to be **linear** if it is equivalent to an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $x_1, x_2, \ldots, x_n$ are n distinct variables, $a_1, a_2, \ldots, a_n, b$ are constants, and at least one of the a's is not 0.

Some examples of linear equations are

$$2x + 3y = 2 \qquad 5x - 2y + 3z = 10 \qquad 8x + 8y - 2z + 5w = 0$$

If each equation in a system of equations is linear, the result is a **system of linear equations**. The systems in Examples 2(a), (c), (d), and (e) are linear, whereas the system in Example 2(b) is nonlinear. In this chapter we shall solve linear systems in Sections 8.1 to 8.3. Nonlinear systems are discussed in Section 8.6.

We begin by discussing a system of two linear equations containing two variables. The problem of solving such a system can be viewed as a geometry problem. The graph of each equation in such a system is a line. So a system of two linear equations containing two variables represents a pair of lines. The lines either (1) intersect, or (2) are parallel, or (3) are **coincident** (that is, identical).

1. If the lines intersect, the system of equations has one solution, given by the point of intersection. The system is **consistent** and the equations are **independent**. See Figure 1(a).
2. If the lines are parallel, the system of equations has no solution, because the lines never intersect. The system is **inconsistent**. See Figure 1(b).
3. If the lines are coincident (the lines lie on top of each other), the system of equations has infinitely many solutions, represented by the totality of points on the line. The system is **consistent** and the equations are **dependent**. See Figure 1(c).

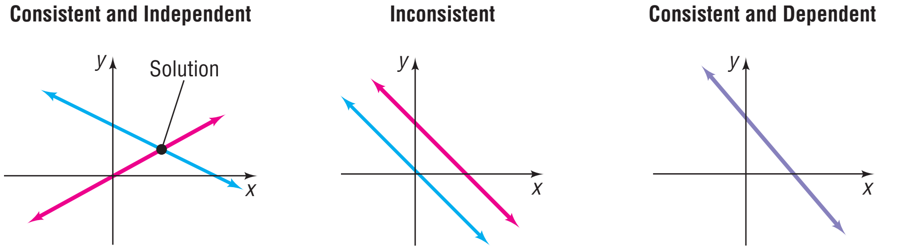


Figure 1 **(a)** Intersecting lines; system has one solution **(b)** Parallel lines; system has no solution **(c)** Coincident lines; system has infinitely many solutions

EXAMPLE 3 Solving a System of Linear Equations Using a Graphing Utility

Solve: $\begin{cases} 2x + y = -1 & \textbf{(1)} \\ -4x + 6y = 42 & \textbf{(2)} \end{cases}$

Solution First, solve each equation for y. This is equivalent to writing each equation in slope–intercept form. Equation (1) in slope–intercept form is $Y_1 = -2x - 1$. Equation (2) in slope–intercept form is $Y_2 = \dfrac{2}{3}x + 7$. Figure 2 shows the graphs using a TI-84 Plus C. The lines intersect, so the system is consistent and the equations are independent. Using INTERSECT gives the solution $(-3, 5)$. ■

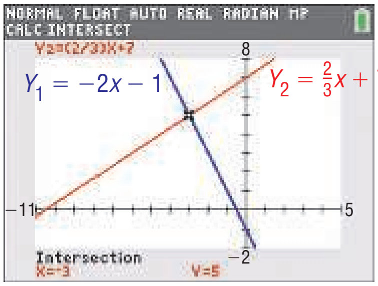


Figure 2

1 Solve Systems of Equations by Substitution

Most of the time algebraic methods must be used to obtain exact solutions. A number of methods are available for solving systems of linear equations algebraically. In this section, we introduce two methods: *substitution* and *elimination*. We illustrate the **method of substitution** by solving the system given in Example 3.

EXAMPLE 4 How to Solve a System of Linear Equations by Substitution

Solve: $\begin{cases} 2x + y = -1 & \textbf{(1)} \\ -4x + 6y = 42 & \textbf{(2)} \end{cases}$

Step-by-Step Solution

Step 1: Pick one of the equations, and solve for one of the variables in terms of the remaining variable(s).

Solve equation (1) for y.

$$2x + y = -1 \quad \textbf{Equation (1)}$$
$$y = -2x - 1 \quad \textbf{Subtract } 2x \textbf{ from each side of (1).}$$

Step 2: Substitute the result into the remaining equation(s).

Substitute $-2x - 1$ for y in equation (2). The result is an equation containing just the variable x, which we can solve.

$$-4x + 6y = 42 \quad \textbf{Equation (2)}$$
$$-4x + 6(-2x - 1) = 42 \quad \textbf{Substitute } -2x - 1 \textbf{ for } y \textbf{ in (2).}$$

Step 3: If one equation in one variable results, solve this equation. Otherwise, repeat Steps 1 and 2 until a single equation with one variable remains.

$$-4x - 12x - 6 = 42 \quad \textbf{Distribute.}$$
$$-16x - 6 = 42 \quad \textbf{Combine like terms.}$$
$$-16x = 48 \quad \textbf{Add 6 to both sides.}$$
$$x = \frac{48}{-16} \quad \textbf{Divide both sides by } -16.$$
$$x = -3 \quad \textbf{Simplify.}$$

Step 4: Find the values of the remaining variables by back-substitution.

Because we know that $x = -3$, we can find the value of y by **back-substitution**, that is, by substituting -3 for x in one of the original equations. Equation (1) seems easier to work with, so we will back-substitute into equation (1).

$$2x + y = -1 \quad \textbf{Equation (1)}$$
$$2(-3) + y = -1 \quad \textbf{Substitute } x = -3 \textbf{ into equation (1).}$$
$$-6 + y = -1 \quad \textbf{Simplify.}$$
$$y = -1 + 6 \quad \textbf{Add 6 to both sides.}$$
$$y = 5$$

Step 5: Check the solution found.

We have $x = -3$ and $y = 5$. Verify that both equations are satisfied (true) for these values.

$$\begin{cases} 2x + y = -1; & 2(-3) + 5 = -6 + 5 = -1 \\ -4x + 6y = 42; & -4(-3) + 6(5) = 12 + 30 = 42 \end{cases}$$

The solution of the system is $x = -3$ and $y = 5$. The solution can also be written as the ordered pair $(-3, 5)$. ■

Now Use Substitution to Work PROBLEM 21

2 Solve Systems of Equations by Elimination

A second method for solving a system of linear equations is the *method of elimination*. This method is usually preferred over substitution if substitution leads to fractions or if the system contains more than two variables. Elimination also provides the necessary motivation for solving systems using matrices (the subject of Section 8.2).

In Words

When using elimination, get the coefficients of one of the variables to be opposites of each other.

The idea behind the **method of elimination** is to replace the original system of equations by an equivalent system so that adding two of the equations eliminates a variable. The rules for obtaining equivalent equations are the same as those studied earlier. However, we may also interchange any two equations of the system and/or replace any equation in the system by the sum (or difference) of that equation and a nonzero multiple of any other equation in the system.

Rules for Obtaining an Equivalent System of Equations

1. Interchange any two equations of the system.
2. Multiply (or divide) each side of an equation by the same nonzero constant.
3. Replace any equation in the system by the sum (or difference) of that equation and a nonzero multiple of any other equation in the system.

An example will give you the idea. As you work through the example, pay particular attention to the pattern being followed.

EXAMPLE 5 How to Solve a System of Linear Equations by Elimination

Solve: $\begin{cases} 2x + 3y = 1 & \textbf{(1)} \\ -x + y = -3 & \textbf{(2)} \end{cases}$

Step-by-Step Solution

Step 1: Multiply both sides of one or both equations by a nonzero constant so that the coefficients of one of the variables are additive inverses.

Multiply both sides of equation (2) by 2 so that the coefficients of x in the two equations are additive inverses.

$$\begin{cases} 2x + 3y = 1 & \textbf{(1)} \\ -x + y = -3 & \textbf{(2)} \end{cases}$$

$$\begin{cases} 2x + 3y = 1 & \textbf{(1)} \\ 2(-x + y) = 2(-3) & \textbf{(2)} \quad \textbf{Multiply by 2.} \end{cases}$$

$$\begin{cases} 2x + 3y = 1 & \textbf{(1)} \\ -2x + 2y = -6 & \textbf{(2)} \end{cases}$$

Step 2: Add the equations to eliminate the variable. Solve the resulting equation for the remaining unknown.

$$\begin{cases} 2x + 3y = 1 & \textbf{(1)} \\ -2x + 2y = -6 & \textbf{(2)} \end{cases}$$

$$\begin{aligned} 5y &= -5 && \textbf{Add equations (1) and (2).} \\ y &= -1 && \textbf{Divide both sides by 5.} \end{aligned}$$

Step 3: Back-substitute the value of the variable found in Step 2 into one of the *original* equations to find the value of the remaining variable.

Back-substitute $y = -1$ into equation (1) and solve for x.

$$\begin{aligned} 2x + 3y &= 1 && \textbf{Equation (1)} \\ 2x + 3(-1) &= 1 && \textbf{Substitute } y = -1 \textbf{ into equation (1).} \\ 2x - 3 &= 1 && \textbf{Simplify.} \\ 2x &= 4 && \textbf{Add 3 to both sides.} \\ x &= 2 && \textbf{Divide both sides by 2.} \end{aligned}$$

Step 4: Check the solution found.

The check is left to you.

The solution of the system is $x = 2$ and $y = -1$. The solution also can be written as the ordered pair $(2, -1)$. ■

Now Use Elimination to Work PROBLEM 21

EXAMPLE 6 Movie Theater Ticket Sales

A movie theater sells tickets for \$10.00 each, with seniors receiving a discount of \$2.00. One evening the theater sold 525 tickets and took in \$4630 in revenue. How many of each type of ticket were sold?

Solution If x represents the number of tickets sold at \$10.00 and y the number of tickets sold at the discounted price of \$8.00, then the given information results in the system of equations

$$\begin{cases} 10x + 8y = 4630 & \textbf{(1)} \\ \quad x + \quad y = \quad 525 & \textbf{(2)} \end{cases}$$

Using the method of elimination, first multiply the second equation by -8, and then add the equations.

$$\begin{aligned} \begin{cases} 10x + 8y &= \quad 4630 \\ -8x - 8y &= -4200 \end{cases} \\ \hline 2x &= 430 \qquad \textbf{Add the equations.} \\ x &= 215 \end{aligned}$$

Since $x + y = 525$, then $y = 525 - x = 525 - 215 = 310$. So 215 nondiscounted tickets and 310 senior discount tickets were sold. ■

3 Identify Inconsistent Systems of Equations Containing Two Variables

The previous examples dealt with consistent systems of equations that had a single solution. The next two examples deal with two other possibilities that may occur, the first being a system that has no solution.

EXAMPLE 7 An Inconsistent System of Linear Equations

Solve: $\begin{cases} 2x + \quad y = 5 & \textbf{(1)} \\ 4x + 2y = 8 & \textbf{(2)} \end{cases}$

Solution We choose to use the method of substitution and solve equation (1) for y.

$$\begin{aligned} 2x + y &= 5 && \textbf{(1)} \\ y &= -2x + 5 && \textbf{Subtract } 2x \textbf{ from each side.} \end{aligned}$$

Now substitute $-2x + 5$ for y in equation (2) and solve for x.

$$\begin{aligned} 4x + 2y &= 8 && \textbf{(2)} \\ 4x + 2(-2x + 5) &= 8 && \textbf{Substitute } y = -2x + 5 \textbf{ in (2).} \\ 4x - 4x + 10 &= 8 && \textbf{Multiply out.} \\ 0 &= -2 && \textbf{Subtract 10 from both sides.} \end{aligned}$$

This statement is false. Conclude that the system has no solution and is therefore inconsistent. ■

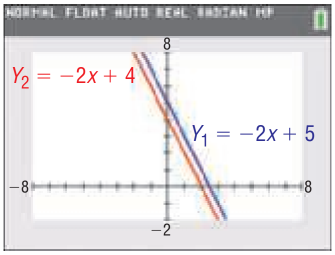


Figure 3

Figure 3 illustrates the pair of lines whose equations form the system in Example 7. Notice that the graphs of the two equations are lines, each with slope -2; one has a y-intercept of 5, the other a y-intercept of 4. The lines are parallel and have no point of intersection. This geometric statement is equivalent to the algebraic statement that the system has no solution.

4 Express the Solution of a System of Dependent Equations Containing Two Variables

EXAMPLE 8 Solving a System of Dependent Equations

Solve: $\begin{cases} \quad 2x + \quad y = \quad 4 & \textbf{(1)} \\ -6x - 3y = -12 & \textbf{(2)} \end{cases}$

Solution We choose to use the method of elimination.

$$\begin{cases} 2x + y = 4 & (1) \\ -6x - 3y = -12 & (2) \end{cases}$$

$$\begin{cases} 6x + 3y = 12 & (1) \quad \text{Multiply each side of equation (1) by 3.} \\ -6x - 3y = -12 & (2) \end{cases}$$
$$0 = 0 \qquad \text{Add equations (1) and (2).}$$

The statement $0 = 0$ is true. This means the equation $6x + 3y = 12$ is equivalent to $-6x - 3y = -12$. Therefore, the original system is equivalent to a system containing one equation, so the equations are dependent. This means that any values of x and y that satisfy $6x + 3y = 12$ or, equivalently, $2x + y = 4$ are solutions. For example, $x = 2$, $y = 0$; $x = 0$, $y = 4$; $x = -2$, $y = 8$; $x = 4$, $y = -4$; and so on, are solutions. There are, in fact, infinitely many values of x and y for which $2x + y = 4$, so the original system has infinitely many solutions. We will write the solution of the original system either as

$$y = -2x + 4, \quad \text{where } x \text{ can be any real number}$$

or as

$$x = -\frac{1}{2}y + 2, \quad \text{where } y \text{ can be any real number.}$$

The solution can also be expressed as $\{(x, y) \mid y = -2x + 4, x \text{ is any real number}\}$ or as $\left\{(x, y) \,\middle|\, x = -\frac{1}{2}y + 2, y \text{ is any real number}\right\}$. ■

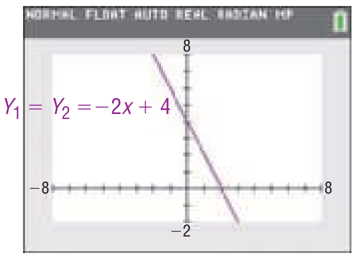


Figure 4 $y = -2x + 4$

Figure 4 illustrates the situation presented in Example 8. Notice that the graphs of the two equations are lines, each with slope -2 and each with y-intercept 4. The lines are coincident. Notice also that equation (2) in the original system is -3 times equation (1), indicating that the two equations are dependent.

For the system in Example 8, some of the infinite number of solutions can be written down by assigning values to x and then finding $y = -2x + 4$.

If $x = -1$, then $y = -2(-1) + 4 = 6$.
If $x = 0$, then $y = 4$.
If $x = 2$, then $y = 0$.

The ordered pairs $(-1, 6)$, $(0, 4)$, and $(2, 0)$ are three of the points on the line in Figure 4.

Now Work PROBLEMS 27 AND 31

5 Solve Systems of Three Equations Containing Three Variables

Just like a system of two linear equations containing two variables, a system of three linear equations containing three variables has (1) exactly one solution (a consistent system with independent equations), or (2) no solution (an inconsistent system), or (3) infinitely many solutions (a consistent system with dependent equations).

The problem of solving a system of three linear equations containing three variables can be viewed as a geometry problem. The graph of each equation in such a system is a plane in space. A system of three linear equations containing three variables represents three planes in space. Figure 5 illustrates some of the possibilities.

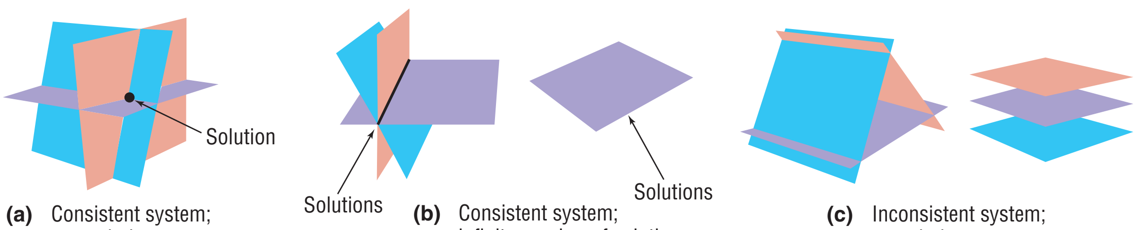


Figure 5 **(a)** Consistent system; one solution **(b)** Consistent system; infinite number of solutions **(c)** Inconsistent system; no solution

Recall that a **solution** to a system of equations consists of values for the variables that are solutions of each equation of the system. For example, $x = 3, y = -1, z = -5$ or, using an ordered triplet, $(3, -1, -5)$ is a solution to the system of equations

$$\begin{cases} x + y + z = -3 & (1) \\ 2x - 3y + 6z = -21 & (2) \\ -3x + 5y = -14 & (3) \end{cases} \qquad \begin{aligned} &3 + (-1) + (-5) = -3 \\ &2(3) - 3(-1) + 6(-5) = 6 + 3 - 30 = -21 \\ &-3(3) + 5(-1) = -9 - 5 = -14 \end{aligned}$$

because these values of the variables are solutions of each equation.

Typically, when solving a system of three linear equations containing three variables, we use the method of elimination. Recall that the idea behind the method of elimination is to form equivalent equations so that adding two of the equations eliminates a variable.

EXAMPLE 9 Solving a System of Three Linear Equations with Three Variables

Use the method of elimination to solve the system of equations.

$$\begin{cases} x + y - z = -1 & (1) \\ 4x - 3y + 2z = 16 & (2) \\ 2x - 2y - 3z = 5 & (3) \end{cases}$$

Solution For a system of three equations, attempt to eliminate one variable at a time, using pairs of equations, until an equation with a single variable remains. Our strategy for solving this system is to use equation (1) to eliminate the variable x from equations (2) and (3). We can then treat the new equations (2) and (3) as a system with two unknowns. Alternatively, we could use equation (1) to eliminate either y or z from equations (2) and (3). Try one of these approaches for yourself.

Begin by multiplying each side of equation (1) by -4 and adding the result to equation (2). (Do you see why? The coefficients of x are now negatives of one another.) Also multiply equation (1) by -2 and add the result to equation (3). Notice that these two procedures result in the elimination of the variable x from equations (2) and (3).

$$\begin{aligned} x + y - z &= -1 \quad (1) \quad \textbf{Multiply by } -4. & -4x - 4y + 4z &= 4 \quad (1) \\ 4x - 3y + 2z &= 16 \quad (2) & 4x - 3y + 2z &= 16 \quad (2) \\ & & \hline -7y + 6z &= 20 \quad \textbf{Add} \end{aligned}$$

$$\begin{aligned} x + y - z &= -1 \quad (1) \quad \textbf{Multiply by } -2. & -2x - 2y + 2z &= 2 \quad (1) \\ 2x - 2y - 3z &= 5 \quad (3) & 2x - 2y - 3z &= 5 \quad (3) \\ & & \hline -4y - z &= 7 \quad \textbf{Add} \end{aligned}$$

$$\begin{cases} x + y - z = -1 & (1) \\ -7y + 6z = 20 & (2) \\ -4y - z = 7 & (3) \end{cases}$$

Now concentrate on the new equations (2) and (3), treating them as a system of two equations containing two variables. It is easier to eliminate z. Multiply each side of equation (3) by 6, and add equations (2) and (3). The result is the new equation (3).

$$\begin{aligned} -7y + 6z &= 20 \quad (2) & -7y + 6z &= 20 \quad (2) \\ -4y - z &= 7 \quad (3) \quad \textbf{Multiply by 6.} & -24y - 6z &= 42 \quad (3) \\ & & \hline -31y &= 62 \quad \textbf{Add} \end{aligned} \qquad \begin{cases} x + y - z = -1 & (1) \\ -7y + 6z = 20 & (2) \\ -31y = 62 & (3) \end{cases}$$

Now solve equation (3) for y by dividing both sides of the equation by -31.

$$\begin{cases} x + y - z = -1 & (1) \\ -7y + 6z = 20 & (2) \\ y = -2 & (3) \end{cases}$$

Back-substitute $y = -2$ in equation (2) and solve for z.

$$\begin{aligned} -7y + 6z &= 20 && \textbf{(2)} \\ -7(-2) + 6z &= 20 && \textbf{Substitute } y = -2 \textbf{ in (2).} \\ 6z &= 6 && \textbf{Subtract 14 from both sides of the equation.} \\ z &= 1 && \textbf{Divide both sides of the equation by 6.} \end{aligned}$$

Finally, back-substitute $y = -2$ and $z = 1$ in equation (1) and solve for x.

$$\begin{aligned} x + y - z &= -1 && \textbf{(1)} \\ x + (-2) - 1 &= -1 && \textbf{Substitute } y = -2 \textbf{ and } z = 1 \textbf{ in (1).} \\ x - 3 &= -1 && \textbf{Simplify.} \\ x &= 2 && \textbf{Add 3 to both sides.} \end{aligned}$$

The solution of the original system is $x = 2, y = -2, z = 1$, or, using an ordered triplet, $(2, -2, 1)$. You should check this solution. ■

Look back over the solution given in Example 9. Note the pattern of removing one of the variables from two of the equations, followed by solving this system of two equations and two unknowns. Although which variables to remove is your choice, the methodology remains the same for all systems.

Now Work PROBLEM 45

6 Identify Inconsistent Systems of Equations Containing Three Variables

EXAMPLE 10 **Identify an Inconsistent System of Linear Equations**

Solve: $\begin{cases} 2x + y - z = -2 & \textbf{(1)} \\ x + 2y - z = -9 & \textbf{(2)} \\ x - 4y + z = 1 & \textbf{(3)} \end{cases}$

Solution Our strategy is the same as in Example 9. However, in this system, it seems easiest to eliminate the variable z first. Do you see why?

Multiply each side of equation (1) by -1, and add the result to equation (2). Also, add equations (2) and (3).

$$\begin{aligned} 2x + y - z &= -2 \quad \textbf{(1)} \quad \textbf{Multiply by } -1. \\ x + 2y - z &= -9 \quad \textbf{(2)} \end{aligned} \qquad \begin{aligned} -2x - y + z &= 2 \quad \textbf{(1)} \\ x + 2y - z &= -9 \quad \textbf{(2)} \\ \hline -x + y &= -7 \quad \textbf{Add} \end{aligned}$$

$$\begin{aligned} x + 2y - z &= -9 \quad \textbf{(2)} \\ x - 4y + z &= 1 \quad \textbf{(3)} \\ \hline 2x - 2y &= -8 \quad \textbf{Add} \end{aligned} \qquad \begin{cases} 2x + y - z = -2 & \textbf{(1)} \\ -x + y = -7 & \textbf{(2)} \\ 2x - 2y = -8 & \textbf{(3)} \end{cases}$$

Now concentrate on the new equations (2) and (3), treating them as a system of two equations containing two variables. Multiply each side of equation (2) by 2, and add the result to equation (3).

$$\begin{aligned} -x + y &= -7 \quad \textbf{(2)} \quad \textbf{Multiply by 2.} \\ 2x - 2y &= -8 \quad \textbf{(3)} \end{aligned} \qquad \begin{aligned} -2x + 2y &= -14 \quad \textbf{(2)} \\ 2x - 2y &= -8 \quad \textbf{(3)} \\ \hline 0 &= -22 \quad \textbf{Add} \end{aligned} \qquad \begin{cases} 2x + y - z = -2 & \textbf{(1)} \\ -x + y = -7 & \textbf{(2)} \\ 0 = -22 & \textbf{(3)} \end{cases}$$

Equation (3) has no solution, so the system is inconsistent. ■

7 Express the Solution of a System of Dependent Equations Containing Three Variables

EXAMPLE 11 Solving a System of Dependent Equations

Solve: $\begin{cases} x - 2y - z = 8 & \textbf{(1)} \\ 2x - 3y + z = 23 & \textbf{(2)} \\ 4x - 5y + 5z = 53 & \textbf{(3)} \end{cases}$

Solution Our plan is to eliminate x from equations (2) and (3). Multiply each side of equation (1) by -2, and add the result to equation (2). Also, multiply each side of equation (1) by -4, and add the result to equation (3).

$$\begin{array}{lll} x - 2y - z = 8 \quad \textbf{(1)} & \textbf{Multiply by } -2. & -2x + 4y + 2z = -16 \quad \textbf{(1)} \\ 2x - 3y + z = 23 \quad \textbf{(2)} & & \underline{2x - 3y + z = 23} \quad \textbf{(2)} \\ & & y + 3z = 7 \quad \textbf{Add} \\ x - 2y - z = 8 \quad \textbf{(1)} & \textbf{Multiply by } -4. & -4x + 8y + 4z = -32 \quad \textbf{(1)} \\ 4x - 5y + 5z = 53 \quad \textbf{(3)} & & \underline{4x - 5y + 5z = 53} \quad \textbf{(3)} \\ & & 3y + 9z = 21 \quad \textbf{Add} \end{array}$$

$$\begin{cases} x - 2y - z = 8 & \textbf{(1)} \\ y + 3z = 7 & \textbf{(2)} \\ 3y + 9z = 21 & \textbf{(3)} \end{cases}$$

Treat equations (2) and (3) as a system of two equations containing two variables, and eliminate the variable y by multiplying both sides of equation (2) by -3 and adding the result to equation (3).

$$\begin{array}{lll} y + 3z = 7 \quad \textbf{(2)} & \textbf{Multiply by } -3. & -3y - 9z = -21 \\ 3y + 9z = 21 \quad \textbf{(3)} & & \underline{3y + 9z = 21} \quad \textbf{Add} \\ & & 0 = 0 \end{array} \qquad \begin{cases} x - 2y - z = 8 & \textbf{(1)} \\ y + 3z = 7 & \textbf{(2)} \\ 0 = 0 & \textbf{(3)} \end{cases}$$

The original system is equivalent to a system containing two equations, so the equations are dependent and the system has infinitely many solutions. If we solve equation (2) for y, we can express y in terms of z as $y = -3z + 7$. Substitute this expression into equation (1) to determine x in terms of z.

$$\begin{aligned} x - 2y - z &= 8 && \textbf{(1)} \\ x - 2(-3z + 7) - z &= 8 && \textbf{Substitute } y = -3z + 7 \textbf{ in (1).} \\ x + 6z - 14 - z &= 8 && \textbf{Multiply out.} \\ x + 5z &= 22 && \textbf{Combine like terms.} \\ x &= -5z + 22 && \textbf{Solve for } x. \end{aligned}$$

We write the solution to the system as

$$\begin{cases} x = -5z + 22 \\ y = -3z + 7 \end{cases} \quad \text{where } z \text{ can be any real number.}$$

This way of writing the solution makes it easier to find specific solutions of the system. To find specific solutions, choose any value of z and use the equations $x = -5z + 22$ and $y = -3z + 7$ to determine x and y. For example, if $z = 0$, then $x = 22$ and $y = 7$, and if $z = 1$, then $x = 17$ and $y = 4$.

Using ordered triplets, the solution is

$$\{(x, y, z) \mid x = -5z + 22,\ y = -3z + 7,\ z \text{ is any real number}\}$$ ■

Now Work PROBLEM 47

Two distinct points in the Cartesian plane determine a unique line. Given three noncollinear points, we can find the (unique) quadratic function whose graph contains these three points.

EXAMPLE 12 **Curve Fitting**

Find real numbers *a, b,* and *c* so that the graph of the quadratic function $y = ax^2 + bx + c$ contains the points $(-1, -4)$, $(1, 6)$, and $(3, 0)$.

Solution The three points must satisfy the equation $y = ax^2 + bx + c$.

For the point $(-1, -4)$ we have: $-4 = a(-1)^2 + b(-1) + c$ $\quad -4 = a - b + c$

For the point $(1, 6)$ we have: $6 = a(1)^2 + b(1) + c$ $\quad 6 = a + b + c$

For the point $(3, 0)$ we have: $0 = a(3)^2 + b(3) + c$ $\quad 0 = 9a + 3b + c$

Determine a, b, and c so that each equation is satisfied. That is, solve the following system of three equations containing three variables:

$$\begin{cases} a - b + c = -4 & \textbf{(1)} \\ a + b + c = 6 & \textbf{(2)} \\ 9a + 3b + c = 0 & \textbf{(3)} \end{cases}$$

Solving this system of equations, we obtain $a = -2$, $b = 5$, and $c = 3$. So the quadratic function whose graph contains the points $(-1, -4)$, $(1, 6)$, and $(3, 0)$ is

$$y = -2x^2 + 5x + 3 \qquad y = ax^2 + bx + c,\quad a = -2, b = 5, c = 3$$

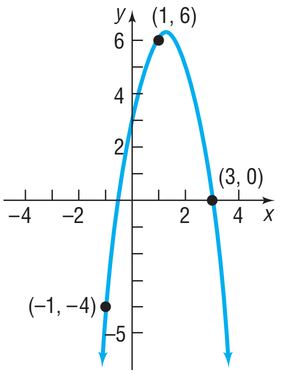


Figure 6 $y = -2x^2 + 5x + 3$

Figure 6 shows the graph of the function along with the three points. ■

Now Work PROBLEM 73

8.1 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Solve the equation: $3x + 4 = 8 - x$. (pp. 102–103)

2. (a) Graph the line: $3x + 4y = 12$.
   (b) What is the slope of a line parallel to this line? (pp. 173–184)

Concepts and Vocabulary

3. If a system of equations has no solution, it is said to be ____________.

4. If a system of equations has one solution, the system is ____________ and the equations are ____________.

5. If the solution to a system of two linear equations containing two unknowns is $x = 3, y = -2$, then the lines intersect at the point __________.

6. If the lines that make up a system of two linear equations are coincident, then the system is __________ and the equations are __________.

7. If a system of two linear equations in two variables is inconsistent, which of the following best describes the graphs of the lines in the system?
   (a) intersecting (b) parallel
   (c) coincident (d) perpendicular

8. If a system of dependent equations containing three variables has the general solution
   $\{(x, y, z) \mid x = -z + 4, y = -2z + 5, z \text{ is any real number}\}$
   which of the following is one of the infinite number of solutions of the system?
   (a) $(1, -1, 3)$ (b) $(0, 4, 5)$ (c) $(4, -3, 0)$ (d) $(-1, 5, 7)$

Skill Building

In Problems 9–18, verify that the values of the variables listed are solutions of the system of equations.

9. $\begin{cases} 2x - y = 5 \\ 5x + 2y = 8 \end{cases}$ $x = 2, y = -1; (2, -1)$

10. $\begin{cases} 3x + 2y = 2 \\ x - 7y = -30 \end{cases}$ $x = -2, y = 4; (-2, 4)$

11. $\begin{cases} 3x - 4y = 4 \\ \frac{1}{2}x - 3y = -\frac{1}{2} \end{cases}$ $x = 2, y = \frac{1}{2}; \left(2, \frac{1}{2}\right)$

12. $\begin{cases} 2x + \frac{1}{2}y = 0 \\ 3x - 4y = -\frac{19}{2} \end{cases}$ $x = -\frac{1}{2}, y = 2; \left(-\frac{1}{2}, 2\right)$

13. $\begin{cases} x - y = 3 \\ \frac{1}{2}x + y = 3 \end{cases}$ $x = 4, y = 1; (4, 1)$

14. $\begin{cases} x - y = 3 \\ -3x + y = 1 \end{cases}$ $x = -2, y = -5; (-2, -5)$

15. $\begin{cases} 3x + 3y + 2z = 4 \\ x - y - z = 0 \\ 2y - 3z = -8 \end{cases}$ $x = 1, y = -1, z = 2; (1, -1, 2)$

16. $\begin{cases} 4x - z = 7 \\ 8x + 5y - z = 0 \\ -x - y + 5z = 6 \end{cases}$ $x = 2, y = -3, z = 1; (2, -3, 1)$

17. $\begin{cases} 3x + 3y + 2z = 4 \\ x - 3y + z = 10 \\ 5x - 2y - 3z = 8 \end{cases}$ $x = 2, y = -2, z = 2; (2, -2, 2)$

18. $\begin{cases} 4x - 5z = 6 \\ 5y - z = -17 \\ -x - 6y + 5z = 24 \end{cases}$ $x = 4, y = -3, z = 2; (4, -3, 2)$

In Problems 19–56, solve each system of equations. If the system has no solution, say that it is inconsistent. For Problems 19–30, graph the lines of the system.

19. $\begin{cases} x + y = 8 \\ x - y = 4 \end{cases}$

20. $\begin{cases} x + 2y = -7 \\ x + y = -3 \end{cases}$

21. $\begin{cases} 5x - y = 21 \\ 2x + 3y = -12 \end{cases}$

22. $\begin{cases} x + 3y = 5 \\ 2x - 3y = -8 \end{cases}$

23. $\begin{cases} 3x = 24 \\ x + 2y = 0 \end{cases}$

24. $\begin{cases} 4x + 5y = -3 \\ -2y = -8 \end{cases}$

25. $\begin{cases} 3x - 6y = 2 \\ 5x + 4y = 1 \end{cases}$

26. $\begin{cases} 2x + 4y = \frac{2}{3} \\ 3x - 5y = -10 \end{cases}$

27. $\begin{cases} 2x + y = 1 \\ 4x + 2y = 3 \end{cases}$

28. $\begin{cases} x - y = 5 \\ -3x + 3y = 2 \end{cases}$

29. $\begin{cases} 2x - y = 0 \\ 4x + 2y = 12 \end{cases}$

30. $\begin{cases} 3x + 3y = -1 \\ 4x + y = \frac{8}{3} \end{cases}$

31. $\begin{cases} x + 2y = 4 \\ 2x + 4y = 8 \end{cases}$

32. $\begin{cases} 3x - y = 7 \\ 9x - 3y = 21 \end{cases}$

33. $\begin{cases} 2x - 3y = -1 \\ 10x + y = 11 \end{cases}$

34. $\begin{cases} 3x - 2y = 0 \\ 5x + 10y = 4 \end{cases}$

35. $\begin{cases} 2x + 3y = 6 \\ x - y = \frac{1}{2} \end{cases}$

36. $\begin{cases} \frac{1}{2}x + y = -2 \\ x - 2y = 8 \end{cases}$

37. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = 3 \\ \frac{1}{4}x - \frac{2}{3}y = -1 \end{cases}$

38. $\begin{cases} \frac{1}{3}x - \frac{3}{2}y = -5 \\ \frac{3}{4}x + \frac{1}{3}y = 11 \end{cases}$

39. $\begin{cases} 3x - 5y = 3 \\ 15x + 5y = 21 \end{cases}$

40. $\begin{cases} 2x - y = -1 \\ x + \frac{1}{2}y = \frac{3}{2} \end{cases}$

41. $\begin{cases} \frac{1}{x} + \frac{1}{y} = 8 \\ \frac{3}{x} - \frac{5}{y} = 0 \end{cases}$

42. $\begin{cases} \frac{4}{x} - \frac{3}{y} = 0 \\ \frac{6}{x} + \frac{3}{2y} = 2 \end{cases}$

[**Hint:** Let $u = \frac{1}{x}$ and $v = \frac{1}{y}$, and solve for u and v. Then $x = \frac{1}{u}$ and $y = \frac{1}{v}$.]

43. $\begin{cases} x - y = 6 \\ 2x - 3z = 16 \\ 2y + z = 4 \end{cases}$

44. $\begin{cases} 2x + y = -4 \\ -2y + 4z = 0 \\ 3x - 2z = -11 \end{cases}$

45. $\begin{cases} x - 2y + 3z = 7 \\ 2x + y + z = 4 \\ -3x + 2y - 2z = -10 \end{cases}$

46. $\begin{cases} 2x + y - 3z = 0 \\ -2x + 2y + z = -7 \\ 3x - 4y - 3z = 7 \end{cases}$

47. $\begin{cases} x - y - z = 1 \\ 2x + 3y + z = 2 \\ 3x + 2y = 0 \end{cases}$

48. $\begin{cases} 2x - 3y - z = 0 \\ -x + 2y + z = 5 \\ 3x - 4y - z = 1 \end{cases}$

49. $\begin{cases} x - y - z = 1 \\ -x + 2y - 3z = -4 \\ 3x - 2y - 7z = 0 \end{cases}$

50. $\begin{cases} 2x - 3y - z = 0 \\ 3x + 2y + 2z = 2 \\ x + 5y + 3z = 2 \end{cases}$

51. $\begin{cases} 2x - 2y + 3z = 6 \\ 4x - 3y + 2z = 0 \\ -2x + 3y - 7z = 1 \end{cases}$

52. $\begin{cases} 3x - 2y + 2z = 6 \\ 7x - 3y + 2z = -1 \\ 2x - 3y + 4z = 0 \end{cases}$

53. $\begin{cases} x + y - z = 6 \\ 3x - 2y + z = -5 \\ x + 3y - 2z = 14 \end{cases}$

54. $\begin{cases} x - y + z = -4 \\ 2x - 3y + 4z = -15 \\ 5x + y - 2z = 12 \end{cases}$

55. $\begin{cases} x + 2y - z = -3 \\ 2x - 4y + z = -7 \\ -2x + 2y - 3z = 4 \end{cases}$

56. $\begin{cases} x + 4y - 3z = -8 \\ 3x - y + 3z = 12 \\ x + y + 6z = 1 \end{cases}$

Applications and Extensions

57. The perimeter of a rectangular floor is 90 feet. Find the dimensions of the floor if the length is twice the width.

58. The length of fence required to enclose a rectangular field is 3000 meters. What are the dimensions of the field if it is known that the difference between its length and width is 50 meters?

59. Orbital Launches In 2014 there was a total of 92 commercial and noncommercial orbital launches worldwide. In addition, the number of noncommercial orbital launches was three times the number of commercial orbital launches. Determine the number of commercial and noncommercial orbital launches in 2014.
Source: *Federal Aviation Administration*

60. Movie Theater Tickets A movie theater charges \$9.00 for adults and \$7.00 for senior citizens. On a day when 325 people paid for admission, the total receipts were \$2495. How many who paid were adults? How many were seniors?

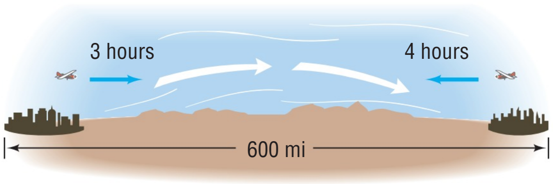
61. Mixing Nuts A store sells cashews for \$5.00 per pound and peanuts for \$1.50 per pound. The manager decides to mix 30 pounds of peanuts with some cashews and sell the mixture for \$3.00 per pound. How many pounds of cashews should be mixed with the peanuts so that the mixture will produce the same revenue as selling the nuts separately?

62. Mixing a Solution A chemist wants to make 14 liters of a 40% acid solution. She has solutions that are 30% acid and 65% acid. How much of each must she mix?

63. Presale Order A wireless store owner takes presale orders for a new smartphone and tablet. He gets 340 preorders for the smartphone and 250 preorders for the tablet. The combined value of the preorders is \$270,500. If the price of a smartphone and tablet together is \$965, how much does each device cost?

64. Financial Planning A recently retired couple needs \$12,000 per year to supplement their Social Security. They have \$150,000 to invest to obtain this income. They have decided on two investment options: AA bonds yielding 10% per annum and a Bank Certificate yielding 5%.
(a) How much should be invested in each to realize exactly \$12,000?
(b) If, after 2 years, the couple requires \$14,000 per year in income, how should they reallocate their investment to achieve the new amount?

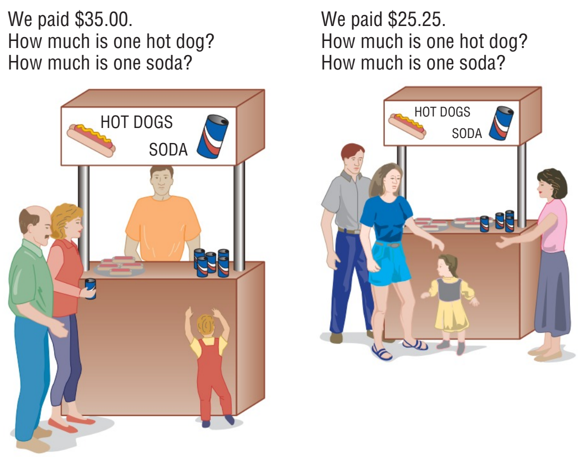
65. Computing Wind Speed With a tail wind, a small Piper aircraft can fly 600 miles in 3 hours. Against this same wind, the Piper can fly the same distance in 4 hours. Find the average wind speed and the average airspeed of the Piper.



66. Computing Wind Speed The average airspeed of a single-engine aircraft is 150 miles per hour. If the aircraft flew the same distance in 2 hours with the wind as it flew in 3 hours against the wind, what was the wind speed?

67. Restaurant Management A restaurant manager wants to purchase 200 sets of dishes. One design costs \$25 per set, and another costs \$45 per set. If she has only \$7400 to spend, how many sets of each design should she order?

68. Cost of Fast Food One group of people purchased 10 hot dogs and 5 soft drinks at a cost of \$35.00. A second group bought 7 hot dogs and 4 soft drinks at a cost of \$25.25. What is the cost of a single hot dog? A single soft drink?



69. Computing a Refund The grocery store we use does not mark prices on its goods. My wife went to this store, bought three 1-pound packages of bacon and two cartons of eggs, and paid a total of \$13.45. Not knowing that she went to the store, I also went to the same store, purchased two 1-pound packages of bacon and three cartons of eggs, and paid a total of \$11.45. Now we want to return two 1-pound packages of bacon and two cartons of eggs. How much will be refunded?

70. Finding the Current of a Stream Pamela requires 3 hours to swim 15 miles downstream on the Illinois River. The return trip upstream takes 5 hours. Find Pamela's average speed in still water. How fast is the current? (Assume that Pamela's speed is the same in each direction.)

71. Pharmacy A doctor's prescription calls for a daily intake containing 40 milligrams (mg) of vitamin C and 30 mg of vitamin D. Your pharmacy stocks two liquids that can be used: One contains 20% vitamin C and 30% vitamin D, the other 40% vitamin C and 20% vitamin D. How many milligrams of each compound should be mixed to fill the prescription?

72. Pharmacy A doctor's prescription calls for the creation of pills that contain 12 units of vitamin B_{12} and 12 units of vitamin E. Your pharmacy stocks two powders that can be used to make these pills: One contains 20% vitamin B_{12} and 30% vitamin E, the other 40% vitamin B_{12} and 20% vitamin E. How many units of each powder should be mixed in each pill?

73. Curve Fitting Find real numbers a, b, and c so that the graph of the function $y = ax^2 + bx + c$ contains the points $(-1, 4)$, $(2, 3)$, and $(0, 1)$.

74. Curve Fitting Find real numbers a, b, and c so that the graph of the function $y = ax^2 + bx + c$ contains the points $(-1, -2)$, $(1, -4)$, and $(2, 4)$.

75. IS–LM Model in Economics In economics, the IS curve is a linear equation that represents all combinations of income Y and interest rates r that maintain an equilibrium in the market for goods in the economy. The LM curve is a linear equation that represents all combinations of income Y and interest rates r that maintain an equilibrium in the market for money in the economy. In an economy, suppose that the equilibrium level of income (in millions of dollars) and interest rates satisfy the system of equations

$$\begin{cases} 0.06Y - 5000r = 240 \\ 0.06Y + 6000r = 900 \end{cases}$$

Find the equilibrium level of income and interest rates.

76. IS–LM Model in Economics In economics, the IS curve is a linear equation that represents all combinations of income Y and interest rates r that maintain an equilibrium in the market for goods in the economy. The LM curve is a linear equation that represents all combinations of income Y and interest rates r that maintain an equilibrium in the market for money in the economy. In an economy, suppose that the equilibrium level of income (in millions of dollars) and interest rates satisfy the system of equations

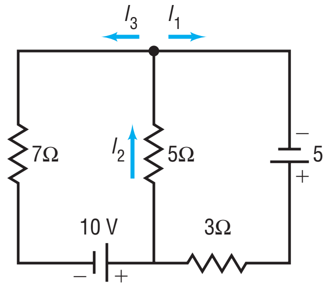
$$\begin{cases} 0.05Y - 1000r = 10 \\ 0.05Y + 800r = 100 \end{cases}$$

Find the equilibrium level of income and interest rates.

77. Electricity: Kirchhoff's Rules An application of Kirchhoff's Rules to the circuit shown results in the following system of equations:

$$\begin{cases} I_2 = I_1 + I_3 \\ 5 - 3I_1 - 5I_2 = 0 \\ 10 - 5I_2 - 7I_3 = 0 \end{cases}$$

Find the currents I_1, I_2, and I_3.

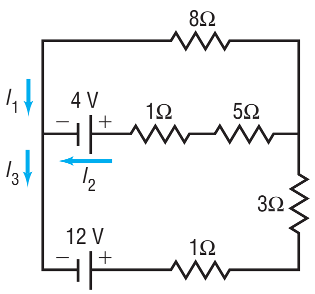


Source: *Physics for Scientists & Engineers,* 9th ed., by Serway. © 2013 Cengage Learning.

78. Electricity: Kirchhoff's Rules An application of Kirchhoff's Rules to the circuit shown below results in the following system of equations:

$$\begin{cases} I_3 = I_1 + I_2 \\ 8 = 4I_3 + 6I_2 \\ 8I_1 = 4 + 6I_2 \end{cases}$$

Find the currents I_1, I_2, and I_3.



Source: *Physics for Scientists & Engineers,* 9th ed., by Serway. © 2013 Cengage Learning.

79. Theater Revenues A Broadway theater has 500 seats, divided into orchestra, main, and balcony seating. Orchestra seats sell for \$150, main seats for \$135, and balcony seats for \$110. If all the seats are sold, the gross revenue to the theater is \$64,250. If all the main and balcony seats are sold, but only half the orchestra seats are sold, the gross revenue is \$56,750. How many of each kind of seat are there?

80. Theater Revenues A movie theater charges \$11.00 for adults, \$6.50 for children, and \$9.00 for senior citizens. One day the theater sold 405 tickets and collected \$3315 in receipts. Twice as many children's tickets were sold as adult tickets. How many adults, children, and senior citizens went to the theater that day?

81. Nutrition A dietitian wishes a patient to have a meal that has 66 grams (g) of protein, 94.5 g of carbohydrates, and 910 milligrams (mg) of calcium. The hospital food service tells the dietitian that the dinner for today is chicken, corn, and 2% milk. Each serving of chicken has 30 g of protein, 35 g of carbohydrates, and 200 mg of calcium. Each serving of corn has 3 g of protein, 16 g of carbohydrates, and 10 mg of calcium. Each glass of 2% milk has 9 g of protein, 13 g of carbohydrates, and 300 mg of calcium. How many servings of each food should the dietitian provide for the patient?

82. Investments Kelly has \$20,000 to invest. As her financial planner, you recommend that she diversify into three investments: Treasury bills that yield 5% simple interest, Treasury bonds that yield 7% simple interest, and corporate bonds that yield 10% simple interest. Kelly wishes to earn \$1390 per year in income. Also, Kelly wants her investment in Treasury bills to be \$3000 more than her investment in corporate bonds. How much money should Kelly place in each investment?

83. Prices of Fast Food One group of customers bought 8 deluxe hamburgers, 6 orders of large fries, and 6 large colas for \$52.20. A second group ordered 10 deluxe hamburgers, 6 large fries, and 8 large colas and paid \$63.20. Is there sufficient information to determine the price of each food item? If not, construct a table showing the various possibilities. Assume that the hamburgers cost between \$3.50 and \$4.50, the fries between \$1.50 and \$2.00, and the colas between \$1.20 and \$1.80.

84. Prices of Fast Food Use the information given in Problem 83. Suppose that a third group purchased 3 deluxe hamburgers, 2 large fries, and 4 large colas for \$21.90. Now is there sufficient information to determine the price of each food item? If so, determine each price.

85. Painting a House Three painters (Beth, Bill, and Edie), working together, can paint the exterior of a home in 10 hours (h). Bill and Edie together have painted a similar house in 15 h. One day, all three worked on this same kind of house for 4 h, after which Edie left. Beth and Bill required 8 more hours to finish. Assuming no gain or loss in efficiency, how long should it take each person to complete such a job alone?

Explaining Concepts: Discussion and Writing

86. Make up a system of three linear equations containing three variables that has:
(a) No solution
(b) Exactly one solution
(c) Infinitely many solutions

Give the three systems to a friend to solve and critique.

87. Write a brief paragraph outlining your strategy for solving a system of two linear equations containing two variables.

88. Do you prefer the method of substitution or the method of elimination for solving a system of two linear equations containing two variables? Give your reasons.

Retain Your Knowledge

Problems 89–92 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

89. Graph $f(x) = -3^{1-x} + 2$.

90. Factor each of the following:
(a) $4(2x-3)^3 \cdot 2 \cdot (x^3+5)^2 + 2(x^3+5) \cdot 3x^2 \cdot (2x-3)^4$
(b) $\frac{1}{2}(3x-5)^{-\frac{1}{2}} \cdot 3 \cdot (x+3)^{-\frac{1}{2}} - \frac{1}{2}(x+3)^{-\frac{3}{2}}(3x-5)^{\frac{1}{2}}$

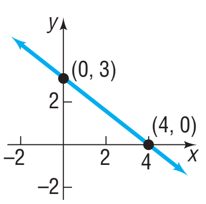
91. Solve: $|3x-2| + 5 \le 9$

92. The exponential function $f(x) = 1 + 2^x$ is one-to-one. Find f^{-1}.

'Are You Prepared?' Answers

1. $\{1\}$

2. (a)



(b) $-\frac{3}{4}$

8.2 Systems of Linear Equations: Matrices

OBJECTIVES

The systematic approach of the method of elimination for solving a system of linear equations provides another method of solution that involves a simplified notation.

Consider the following system of linear equations:

$$\begin{cases} x + 4y = 14 \\ 3x - 2y = 0 \end{cases}$$

If we choose not to write the symbols used for the variables, we can represent this system as

$$\left[\begin{array}{cc|c} 1 & 4 & 14 \\ 3 & -2 & 0 \end{array}\right]$$

where it is understood that the first column represents the coefficients of the variable x, the second column the coefficients of y, and the third column the constants on the right side of the equal signs. The vertical line serves as a reminder of the equal signs. The large square brackets are used to denote a *matrix* in algebra.

DEFINITION

A **matrix** is a rectangular array of numbers:

$$\begin{array}{cc} & \begin{matrix} \textbf{Column 1} & \textbf{Column 2} & & \textbf{Column } j & & \textbf{Column } n \end{matrix} \\ \begin{matrix} \textbf{Row 1} \\ \textbf{Row 2} \\ \vdots \\ \textbf{Row } i \\ \vdots \\ \textbf{Row } m \end{matrix} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \end{array} \qquad \textbf{(1)}$$

Each number a_{ij} of the matrix has two indexes: the **row index** i and the **column index** j. The matrix shown in display (1) has m rows and n columns. The numbers a_{ij} are usually referred to as the **entries** of the matrix. For example, a_{23} refers to the entry in the second row, third column.

In Words
To augment means to increase or expand. An augmented matrix broadens the idea of matrices to systems of linear equations.

1 Write the Augmented Matrix of a System of Linear Equations

Now we will use matrix notation to represent a system of linear equations. The matrix used to represent a system of linear equations is called an **augmented matrix**. In writing the augmented matrix of a system, the variables of each equation must be on the left side of the equal sign and the constants on the right side. A variable that does not appear in an equation has a coefficient of 0.

EXAMPLE 1 Writing the Augmented Matrix of a System of Linear Equations

Write the augmented matrix of each system of equations.

(a) $\begin{cases} 3x - 4y = -6 & \textbf{(1)} \\ 2x - 3y = -5 & \textbf{(2)} \end{cases}$ (b) $\begin{cases} 2x - y + z = 0 & \textbf{(1)} \\ x + z - 1 = 0 & \textbf{(2)} \\ x + 2y - 8 = 0 & \textbf{(3)} \end{cases}$

Solution (a) The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -4 & -6 \\ 2 & -3 & -5 \end{array}\right]$$

(b) Care must be taken that the system be written so that the coefficients of all variables are present (if any variable is missing, its coefficient is 0). Also, all constants must be to the right of the equal sign. We need to rearrange the given system as follows:

$$\begin{cases} 2x - y + z = 0 & \textbf{(1)} \\ x + z - 1 = 0 & \textbf{(2)} \\ x + 2y - 8 = 0 & \textbf{(3)} \end{cases}$$

$$\begin{cases} 2x - y + z = 0 & \textbf{(1)} \\ x + 0\cdot y + z = 1 & \textbf{(2)} \\ x + 2y + 0\cdot z = 8 & \textbf{(3)} \end{cases}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 0 & 8 \end{array}\right]$$

■

Caution Be sure variables and constants are lined up correctly before writing the augmented matrix. ■

If we do not include the constants to the right of the equal sign (that is, to the right of the vertical bar in the augmented matrix of a system of equations), the resulting matrix is called the **coefficient matrix** of the system. For the systems discussed in Example 1, the coefficient matrices are

$$\begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

Now Work PROBLEM 9

2 Write the System of Equations from the Augmented Matrix

EXAMPLE 2 **Writing the System of Linear Equations from the Augmented Matrix**

Write the system of linear equations that corresponds to each augmented matrix.

(a) $\left[\begin{array}{cc|c} 5 & 2 & 13 \\ -3 & 1 & -10 \end{array}\right]$ (b) $\left[\begin{array}{ccc|c} 3 & -1 & -1 & 7 \\ 2 & 0 & 2 & 8 \\ 0 & 1 & 1 & 0 \end{array}\right]$

Solution (a) The matrix has two rows and so represents a system of two equations. The two columns to the left of the vertical bar indicate that the system has two variables. If x and y are used to denote these variables, the system of equations is

$$\begin{cases} 5x + 2y = 13 & \textbf{(1)} \\ -3x + y = -10 & \textbf{(2)} \end{cases}$$

(b) Since the augmented matrix has three rows, it represents a system of three equations. Since there are three columns to the left of the vertical bar, the system contains three variables. If x, y, and z are the three variables, the system of equations is

$$\begin{cases} 3x - y - z = 7 & \textbf{(1)} \\ 2x \qquad + 2z = 8 & \textbf{(2)} \\ \qquad y + z = 0 & \textbf{(3)} \end{cases}$$

■

3 Perform Row Operations on a Matrix

Row operations on a matrix are used to solve systems of equations when the system is written as an augmented matrix. There are three basic row operations.

Row Operations

1. Interchange any two rows.
2. Replace a row by a nonzero multiple of that row.
3. Replace a row by the sum of that row and a constant nonzero multiple of some other row.

These three row operations correspond to the three rules given earlier for obtaining an equivalent system of equations. When a row operation is performed on a matrix, the resulting matrix represents a system of equations equivalent to the system represented by the original matrix.

For example, consider the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & -1 & 2 \end{array}\right]$$

Suppose that we want to apply a row operation to this matrix that results in a matrix whose entry in row 2, column 1 is a 0. The row operation to use is

Multiply each entry in row 1 by -4, and add the result to the corresponding entries in row 2. **(2)**

If we use R_2 to represent the new entries in row 2 and r_1 and r_2 to represent the original entries in rows 1 and 2, respectively, we can represent the row operation in statement (2) by

$$R_2 = -4r_1 + r_2$$

Then

$$\left[\begin{array}{rr|r} 1 & 2 & 3 \\ 4 & -1 & 2 \end{array}\right] \xrightarrow[R_2 = -4r_1 + r_2]{} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ -4(1)+4 & -4(2)+(-1) & -4(3)+2 \end{array}\right] = \left[\begin{array}{rr|r} 1 & 2 & 3 \\ 0 & -9 & -10 \end{array}\right]$$

As desired, we now have the entry 0 in row 2, column 1.

EXAMPLE 3 Applying a Row Operation to an Augmented Matrix

Apply the row operation $R_2 = -3r_1 + r_2$ to the augmented matrix

$$\left[\begin{array}{rr|r} 1 & -2 & 2 \\ 3 & -5 & 9 \end{array}\right]$$

Solution The row operation $R_2 = -3r_1 + r_2$ says that the entries in row 2 are to be replaced by the entries obtained after multiplying each entry in row 1 by -3 and adding the result to the corresponding entries in row 2.

$$\left[\begin{array}{rr|r} 1 & -2 & 2 \\ 3 & -5 & 9 \end{array}\right] \xrightarrow[R_2 = -3r_1 + r_2]{} \left[\begin{array}{cc|c} 1 & -2 & 2 \\ -3(1)+3 & (-3)(-2)+(-5) & -3(2)+9 \end{array}\right] = \left[\begin{array}{rr|r} 1 & -2 & 2 \\ 0 & 1 & 3 \end{array}\right]$$

■

Now Work PROBLEM 19

EXAMPLE 4 Finding a Particular Row Operation

Find a row operation that results in the augmented matrix

$$\left[\begin{array}{rr|r} 1 & -2 & 2 \\ 0 & 1 & 3 \end{array}\right]$$

having a 0 in row 1, column 2.

Solution We want a 0 in row 1, column 2. Because there is a 1 in row 2, column 2, this result can be accomplished by multiplying row 2 by 2 and adding the result to row 1. That is, apply the row operation $R_1 = 2r_2 + r_1$.

$$\left[\begin{array}{rr|r} 1 & -2 & 2 \\ 0 & 1 & 3 \end{array}\right] \xrightarrow[R_1 = 2r_2 + r_1]{} \left[\begin{array}{cc|c} 2(0)+1 & 2(1)+(-2) & 2(3)+2 \\ 0 & 1 & 3 \end{array}\right] = \left[\begin{array}{rr|r} 1 & 0 & 8 \\ 0 & 1 & 3 \end{array}\right]$$

■

A word about the notation introduced here. A row operation such as $R_1 = 2r_2 + r_1$ changes the entries in row 1. Note also that for this type of row operation, we change the entries in a given row by multiplying the entries in some other row by an appropriate nonzero number and adding the results to the original entries of the row to be changed.

4 Solve a System of Linear Equations Using Matrices

To solve a system of linear equations using matrices, use row operations on the augmented matrix of the system to obtain a matrix that is in *row echelon form*.

DEFINITION

A matrix is in **row echelon form** when the following conditions are met:

1. The entry in row 1, column 1 is a 1, and only 0's appear below it.
2. The first nonzero entry in each row after the first row is a 1, only 0's appear below it, and the 1 appears to the right of the first nonzero entry in any row above.
3. Any rows that contain all 0's to the left of the vertical bar appear at the bottom.

For example, for a system of three equations containing three variables, x, y, and z, with a unique solution, the augmented matrix is in row echelon form if it is of the form

$$\left[\begin{array}{ccc|c} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 1 & f \end{array}\right]$$

where a, b, c, d, e, and f are real numbers. The last row of this augmented matrix states that $z = f$. We can then determine the value of y using back-substitution with $z = f$, since row 2 represents the equation $y + cz = e$. Finally, x is determined using back-substitution again.

Two advantages of solving a system of equations by writing the augmented matrix in row echelon form are the following:

1. The process is algorithmic; that is, it consists of repetitive steps that can be programmed on a computer.
2. The process works on any system of linear equations, no matter how many equations or variables are present.

The next example shows how to solve a system of linear equations by writing its augmented matrix in row echelon form.

EXAMPLE 5 How to Solve a System of Linear Equations Using Matrices

$$\text{Solve: } \begin{cases} 2x + 2y \phantom{{}+z} = 6 & (1) \\ x + y + z = 1 & (2) \\ 3x + 4y - z = 13 & (3) \end{cases}$$

Step-by-Step Solution

Step 1: Write the augmented matrix that represents the system.

Write the augmented matrix of the system.

$$\left[\begin{array}{ccc|c} 2 & 2 & 0 & 6 \\ 1 & 1 & 1 & 1 \\ 3 & 4 & -1 & 13 \end{array}\right]$$

Step 2: Perform row operations that result in the entry in row 1, column 1 becoming 1.

To get a 1 in row 1, column 1, interchange rows 1 and 2. [Note that this is equivalent to interchanging equations (1) and (2) of the system.]

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 0 & 6 \\ 3 & 4 & -1 & 13 \end{array}\right]$$

Step 3: Perform row operations that leave the entry in row 1, column 1 a 1, while causing the entries in column 1 below row 1 to become 0's.

Next, we want a 0 in row 2, column 1 and a 0 in row 3, column 1. Use the row operations $R_2 = -2r_1 + r_2$ and $R_3 = -3r_1 + r_3$ to accomplish this. Note that row 1 is unchanged using these row operations. Also, do you see that performing these row operations simultaneously is the same as doing one followed by the other?

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 0 & 6 \\ 3 & 4 & -1 & 13 \end{array}\right] \underset{\substack{R_2 = -2r_1 + r_2 \\ R_3 = -3r_1 + r_3}}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 4 \\ 0 & 1 & -4 & 10 \end{array}\right]$$

Step 4: Perform row operations that result in the entry in row 2, column 2 becoming 1 with 0's below it.

We want the entry in row 2, column 2 to be 1. We also want to have a 0 below the 1 in row 2, column 2. Interchanging rows 2 and 3 accomplishes both goals.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 4 \\ 0 & 1 & -4 & 10 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -4 & 10 \\ 0 & 0 & -2 & 4 \end{array}\right]$$

Step 5: Repeat Step 4, placing a 1 in row 3, column 3.

To obtain a 1 in row 3, column 3, use the row operation $R_3 = -\frac{1}{2}r_3$. The result is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -4 & 10 \\ 0 & 0 & -2 & 4 \end{array}\right] \underset{R_3 = -\frac{1}{2}r_3}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -4 & 10 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Step 6: The matrix on the right in Step 5 is the row echelon form of the augmented matrix. Use back-substitution to solve the original system.

The third row of the augmented matrix represents the equation $z = -2$. Using $z = -2$, back-substitute into the equation $y - 4z = 10$ (from the second row) and obtain

$$\begin{aligned} y - 4z &= 10 \\ y - 4(-2) &= 10 \qquad \mathbf{z = -2} \\ y &= 2 \qquad \textbf{Solve for } \boldsymbol{y}. \end{aligned}$$

Finally, back-substitute $y = 2$ and $z = -2$ into the equation $x + y + z = 1$ (from the first row) and obtain

$$\begin{aligned} x + y + z &= 1 \\ x + 2 + (-2) &= 1 \qquad \mathbf{y = 2, z = -2} \\ x &= 1 \qquad \textbf{Solve for } \boldsymbol{x}. \end{aligned}$$

The solution of the system is $x = 1, y = 2, z = -2$ or, using an ordered triplet, $(1, 2, -2)$. ■

In Words

To obtain an augmented matrix in row echelon form:

- Add rows, exchange rows, or multiply a row by a nonzero constant.
- Work from top to bottom and left to right.
- Get 1's in the main diagonal with 0's below the 1's.
- Once the entry in row 1, column 1 is 1 with 0's below it, do not use row 1 in your row operations.
- Once the entries in row 1, column 1 and row 2, column 2 are 1 with 0's below, do not use row 1 or 2 in your row operations (and so on).

Matrix Method for Solving a System of Linear Equations (Row Echelon Form)

STEP 1: Write the augmented matrix that represents the system.

STEP 2: Perform row operations that place the entry 1 in row 1, column 1.

STEP 3: Perform row operations that leave the entry 1 in row 1, column 1 unchanged, while causing 0's to appear below it in column 1.

STEP 4: Perform row operations that place the entry 1 in row 2, column 2, but leave the entries in columns to the left unchanged. If it is impossible to place a 1 in row 2, column 2, proceed to place a 1 in row 2, column 3. Once a 1 is in place, perform row operations to place 0's below it. (Place any rows that contain only 0's on the left side of the vertical bar, at the bottom of the matrix.)

STEP 5: Now repeat Step 4, placing a 1 in the next row, but one column to the right. Continue until the bottom row or the vertical bar is reached.

STEP 6: The matrix that results is the row echelon form of the augmented matrix. Analyze the system of equations corresponding to it to solve the original system.

EXAMPLE 6 Solving a System of Linear Equations Using Matrices (Row Echelon Form)

$$\text{Solve: } \begin{cases} x - y + z = 8 & \textbf{(1)} \\ 2x + 3y - z = -2 & \textbf{(2)} \\ 3x - 2y - 9z = 9 & \textbf{(3)} \end{cases}$$

Algebraic Solution

STEP 1: The augmented matrix of the system is

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 8 \\ 2 & 3 & -1 & -2 \\ 3 & -2 & -9 & 9 \end{array}\right]$$

STEP 2: Because the entry 1 is already present in row 1, column 1, go to Step 3.

STEP 3: Perform the row operations $R_2 = -2r_1 + r_2$ and $R_3 = -3r_1 + r_3$. Each of these leaves the entry 1 in row 1, column 1 unchanged, while causing 0's to appear under it.

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 8 \\ 2 & 3 & -1 & -2 \\ 3 & -2 & -9 & 9 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & -1 & 1 & 8 \\ 0 & 5 & -3 & -18 \\ 0 & 1 & -12 & -15 \end{array}\right]$$

$$R_2 = -2r_1 + r_2$$
$$R_3 = -3r_1 + r_3$$

STEP 4: The easiest way to obtain the entry 1 in row 2, column 2 without altering column 1 is to interchange rows 2 and 3 (another way would be to multiply row 2 by $\frac{1}{5}$, but this introduces fractions).

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 5 & -3 & -18 \end{array}\right]$$

To get a 0 under the 1 in row 2, column 2, perform the row operation $R_3 = -5r_2 + r_3$.

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 5 & -3 & -18 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 57 & 57 \end{array}\right]$$

$$R_3 = -5r_2 + r_3$$

STEP 5: Continuing, obtain a 1 in row 3, column 3 by using $R_3 = \frac{1}{57}r_3$.

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 57 & 57 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$R_3 = \frac{1}{57}r_3$$

STEP 6: The matrix on the right is the row echelon form of the augmented matrix. The system of equations represented by the matrix in row echelon form is

$$\begin{cases} x - y + z = 8 & \textbf{(1)} \\ y - 12z = -15 & \textbf{(2)} \\ z = 1 & \textbf{(3)} \end{cases}$$

Using $z = 1$, back-substitute to get

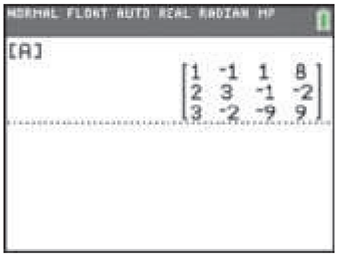
$$\begin{cases} x - y + 1 = 8 & \textbf{(1)} \\ y - 12(1) = -15 & \textbf{(2)} \end{cases} \xrightarrow[\text{Simplify.}]{} \begin{cases} x - y = 7 & \textbf{(1)} \\ y = -3 & \textbf{(2)} \end{cases}$$

Graphing Solution

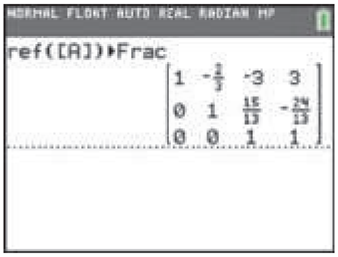
The augmented matrix of the system is

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 8 \\ 2 & 3 & -1 & -2 \\ 3 & -2 & -9 & 9 \end{array}\right]$$

Enter this matrix into a graphing utility and name it A. See Figure 7(a). Using the REF (Row Echelon Form) command on matrix A gives the results shown in Figure 7(b).



(a)



(b)

Figure 7 Row echelon form

The system of equations represented by the matrix in row echelon form is

$$\begin{cases} x - \frac{2}{3}y - 3z = 3 & \textbf{(1)} \\ y + \frac{15}{13}z = -\frac{24}{13} & \textbf{(2)} \\ z = 1 & \textbf{(3)} \end{cases}$$

Using $z = 1$, back-substitute to get

$$\begin{cases} x - \frac{2}{3}y - 3(1) = 3 & \textbf{(1)} \\ y + \frac{15}{13}(1) = -\frac{24}{13} & \textbf{(2)} \end{cases}$$

$$\begin{cases} x - \frac{2}{3}y = 6 & \textbf{(1)} \\ y = -\frac{39}{13} = -3 & \textbf{(2)} \end{cases}$$

Using $y = -3$ from the second equation, back-substitute into $x - \frac{2}{3}y = 6$ to get $x = 4$. The solution of the system is $x = 4, y = -3, z = 1$ or, using an ordered triplet, $(4, -3, 1)$. ■

Using $y = -3$, back-substitute into $x - y = 7$ to get $x = 4$. The solution of the system is $x = 4, y = -3, z = 1$ or, using an ordered triplet, $(4, -3, 1)$.

Notice that the row echelon form of the augmented matrix in the graphing solution differs from the row echelon form in the algebraic solution, yet both matrices provide the same solution! This is because the two solutions used different row operations to obtain the row echelon form. In all likelihood, the two solutions parted ways in Step 4 of the algebraic solution, where we avoided introducing fractions by interchanging rows 2 and 3.

Sometimes it is advantageous to write a matrix in **reduced row echelon form**. In this form, row operations are used to obtain entries that are 0 above (as well as below) the leading 1 in a row. For example, the row echelon form obtained in the algebraic solution to Example 6 is

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

To write this matrix in reduced row echelon form, proceed as follows:

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 1 & 1 \end{array}\right] \underset{R_1 = r_2 + r_1}{\rightarrow} \left[\begin{array}{rrr|r} 1 & 0 & -11 & -7 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 1 & 1 \end{array}\right] \underset{\substack{R_1 = 11r_3 + r_1 \\ R_2 = 12r_3 + r_2}}{\rightarrow} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

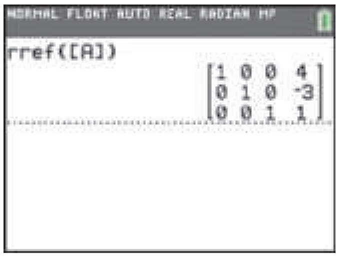


Figure 8 Reduced row echelon form

The matrix is now written in reduced row echelon form. The advantage of writing the matrix in this form is that the solution to the system, $x = 4$, $y = -3$, $z = 1$, is readily found, without the need to back-substitute. Another advantage will be seen in Section 8.4, where the inverse of a matrix is discussed. The method used to write a matrix in reduced row echelon form is called **Gauss–Jordan elimination**.

Most graphing utilities also have the ability to put a matrix in reduced row echelon form. Figure 8 shows the reduced row echelon form of the augmented matrix from Example 6 using the RREF command on a TI-84 Plus C graphing calculator.

Now Work PROBLEMS 39 AND 49

The matrix method for solving a system of linear equations also identifies systems that have infinitely many solutions and systems that are inconsistent.

EXAMPLE 7 Solving a Dependent System of Linear Equations Using Matrices

Solve: $\begin{cases} 6x - y - z = 4 & \textbf{(1)} \\ -12x + 2y + 2z = -8 & \textbf{(2)} \\ 5x + y - z = 3 & \textbf{(3)} \end{cases}$

Solution Start with the augmented matrix of the system and proceed to obtain a 1 in row 1, column 1 with 0's below.

$$\left[\begin{array}{rrr|r} 6 & -1 & -1 & 4 \\ -12 & 2 & 2 & -8 \\ 5 & 1 & -1 & 3 \end{array}\right] \underset{R_1 = -1r_3 + r_1}{\rightarrow} \left[\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ -12 & 2 & 2 & -8 \\ 5 & 1 & -1 & 3 \end{array}\right] \underset{\substack{R_2 = 12r_1 + r_2 \\ R_3 = -5r_1 + r_3}}{\rightarrow} \left[\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ 0 & -22 & 2 & 4 \\ 0 & 11 & -1 & -2 \end{array}\right]$$

Obtaining a 1 in row 2, column 2 without altering column 1 can be accomplished by $R_2 = -\frac{1}{22}r_2$, by $R_3 = \frac{1}{11}r_3$ and interchanging rows 2 and 3, or by $R_2 = \frac{23}{11}r_3 + r_2$. We shall use the first of these.

$$\left[\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ 0 & -22 & 2 & 4 \\ 0 & 11 & -1 & -2 \end{array}\right] \underset{R_2 = -\frac{1}{22}r_2}{\longrightarrow} \left[\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ 0 & 1 & -\frac{1}{11} & -\frac{2}{11} \\ 0 & 11 & -1 & -2 \end{array}\right] \underset{R_3 = -11r_2 + r_3}{\longrightarrow} \left[\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ 0 & 1 & -\frac{1}{11} & -\frac{2}{11} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This matrix is in row echelon form. Because the bottom row consists entirely of 0's, the system actually consists of only two equations.

$$\begin{cases} x - 2y = 1 & \textbf{(1)} \\ y - \frac{1}{11}z = -\frac{2}{11} & \textbf{(2)} \end{cases}$$

To make it easier to write down some of the solutions, we express both x and y in terms of z.

From the second equation, $y = \frac{1}{11}z - \frac{2}{11}$. Now back-substitute this solution for y into the first equation to get

$$x = 2y + 1 = 2\left(\frac{1}{11}z - \frac{2}{11}\right) + 1 = \frac{2}{11}z + \frac{7}{11}$$

The original system is equivalent to the system

$$\begin{cases} x = \frac{2}{11}z + \frac{7}{11} & \textbf{(1)} \\ y = \frac{1}{11}z - \frac{2}{11} & \textbf{(2)} \end{cases} \quad \text{where } z \text{ can be any real number.}$$

Let's look at the situation. The original system of three equations is equivalent to a system containing two equations. This means that any values of x, y, z that satisfy both

$$x = \frac{2}{11}z + \frac{7}{11} \quad \text{and} \quad y = \frac{1}{11}z - \frac{2}{11}$$

are solutions. For example, $z = 0$, $x = \frac{7}{11}$, $y = -\frac{2}{11}$; $z = 1$, $x = \frac{9}{11}$, $y = -\frac{1}{11}$; and $z = -1$, $x = \frac{5}{11}$, $y = -\frac{3}{11}$ are some of the solutions of the original system. There are, in fact, infinitely many values of x, y, and z for which the two equations are satisfied. That is, the original system has infinitely many solutions. We write the solution of the original system as

$$\begin{cases} x = \frac{2}{11}z + \frac{7}{11} \\ y = \frac{1}{11}z - \frac{2}{11} \end{cases} \quad \text{where } z \text{ can be any real number}$$

or, using ordered triplets, as

$$\left\{ (x, y, z) \,\middle|\, x = \frac{2}{11}z + \frac{7}{11},\ y = \frac{1}{11}z - \frac{2}{11},\ z \text{ any real number} \right\}$$

■

The solution can also be found by writing the augmented matrix in reduced row echelon form. Begin with the row echelon form.

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & -\frac{1}{11} & -\frac{2}{11} \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 = 2r_2 + r_1} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{2}{11} & \frac{7}{11} \\ 0 & 1 & -\frac{1}{11} & -\frac{2}{11} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The matrix on the right is in reduced row echelon form. The corresponding system of equations is

$$\begin{cases} x - \frac{2}{11}z = \frac{7}{11} & \textbf{(1)} \\ y - \frac{1}{11}z = -\frac{2}{11} & \textbf{(2)} \end{cases} \quad \text{where } z \text{ can be any real number}$$

or, equivalently,

$$\begin{cases} x = \frac{2}{11}z + \frac{7}{11} \\ y = \frac{1}{11}z - \frac{2}{11} \end{cases} \quad \text{where } z \text{ can be any real number}$$

Now Work PROBLEM 55

EXAMPLE 8 Solving an Inconsistent System of Linear Equations Using Matrices

Solve: $\begin{cases} x + y + z = 6 \\ 2x - y - z = 3 \\ x + 2y + 2z = 0 \end{cases}$

Solution Begin with the augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & -1 & 3 \\ 1 & 2 & 2 & 0 \end{array}\right] \xrightarrow[R_3 = -1r_1 + r_3]{R_2 = -2r_1 + r_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & -3 & -9 \\ 0 & 1 & 1 & -6 \end{array}\right] \xrightarrow{\text{Interchange rows 2 and 3.}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & -6 \\ 0 & -3 & -3 & -9 \end{array}\right] \xrightarrow{R_3 = 3r_2 + r_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & -6 \\ 0 & 0 & 0 & -27 \end{array}\right]$$

This matrix is in row echelon form. The bottom row is equivalent to the equation

$$0x + 0y + 0z = -27$$

which has no solution. The original system is inconsistent. ■

Now Work PROBLEM 29

The matrix method is especially effective for systems of equations for which the number of equations and the number of variables are unequal. Here, too, such a system is either inconsistent or consistent. If it is consistent, it has either exactly one solution or infinitely many solutions.

EXAMPLE 9 Solving a System of Linear Equations Using Matrices

$$\text{Solve: } \begin{cases} x - 2y + z = 0 & (1) \\ 2x + 2y - 3z = -3 & (2) \\ y - z = -1 & (3) \\ -x + 4y + 2z = 13 & (4) \end{cases}$$

Solution Begin with the augmented matrix.

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 2 & 2 & -3 & -3 \\ 0 & 1 & -1 & -1 \\ -1 & 4 & 2 & 13 \end{array}\right] \underset{\substack{R_2 = -2r_1 + r_2 \\ R_4 = r_1 + r_4}}{\rightarrow} \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 6 & -5 & -3 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & 3 & 13 \end{array}\right] \underset{\text{Interchange rows 2 and 3.}}{\rightarrow} \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 6 & -5 & -3 \\ 0 & 2 & 3 & 13 \end{array}\right]$$

$$\underset{\substack{R_3 = -6r_2 + r_3 \\ R_4 = -2r_2 + r_4}}{\rightarrow} \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 5 & 15 \end{array}\right] \underset{R_4 = -5r_3 + r_4}{\rightarrow} \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We could stop here, since the matrix is in row echelon form, and back-substitute $z = 3$ to find x and y. Or we can continue and obtain the reduced row echelon form.

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{R_1 = 2r_2 + r_1}{\rightarrow} \left[\begin{array}{rrr|r} 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{\substack{R_1 = r_3 + r_1 \\ R_2 = r_3 + r_2}}{\rightarrow} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The matrix is now in reduced row echelon form, and we can see that the solution is $x = 1$, $y = 2$, $z = 3$ or, using an ordered triplet, $(1, 2, 3)$. ■

Now Work PROBLEM 71

EXAMPLE 10 Financial Planning

Adam and Michelle require an additional \$25,000 in annual income (beyond their pension benefits). They are rather risk averse and have narrowed their investment choices down to Treasury notes that yield 3%, Treasury bonds that yield 5%, and corporate bonds that yield 6%. They have \$600,000 to invest and want the amount invested in Treasury notes to equal the total amount invested in Treasury bonds and corporate bonds. How much should they place in each investment?

Solution Let n, b, and c represent the amounts invested in Treasury notes, Treasury bonds, and corporate bonds, respectively. There is a total of \$600,000 to invest, which means that the sum of the amounts invested in Treasury notes, Treasury bonds, and corporate bonds should equal \$600,000. The first equation is

$$n + b + c = 600{,}000 \qquad (1)$$

If \$100,000 were invested in Treasury notes, the income would be $0.03(\$100,000) = \$3000$. In general, if n dollars were invested in Treasury notes, the income would be $0.03n$. Since the total income is to be \$25,000, the second equation is

$$0.03n + 0.05b + 0.06c = 25{,}000 \quad \textbf{(2)}$$

The amount invested in Treasury notes should equal the amount invested in Treasury bonds and corporate bonds, so the third equation is

$$n = b + c \quad \text{or} \quad n - b - c = 0 \quad \textbf{(3)}$$

We have the following system of equations:

$$\begin{cases} n + b + c = 600{,}000 & \textbf{(1)} \\ 0.03n + 0.05b + 0.06c = 25{,}000 & \textbf{(2)} \\ n - b - c = 0 & \textbf{(3)} \end{cases}$$

Begin with the augmented matrix and proceed as follows:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 600{,}000 \\ 0.03 & 0.05 & 0.06 & 25{,}000 \\ 1 & -1 & -1 & 0 \end{array}\right] \underset{\substack{R_2 = -0.03r_1 + r_2 \\ R_3 = -r_1 + r_3}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 600{,}000 \\ 0 & 0.02 & 0.03 & 7000 \\ 0 & -2 & -2 & -600{,}000 \end{array}\right]$$

$$\underset{R_2 = \frac{1}{0.02}r_2}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 600{,}000 \\ 0 & 1 & 1.5 & 350{,}000 \\ 0 & -2 & -2 & -600{,}000 \end{array}\right] \underset{R_3 = 2r_2 + r_3}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 600{,}000 \\ 0 & 1 & 1.5 & 350{,}000 \\ 0 & 0 & 1 & 100{,}000 \end{array}\right]$$

The matrix is now in row echelon form. The final matrix represents the system

$$\begin{cases} n + b + c = 600{,}000 & \textbf{(1)} \\ b + 1.5c = 350{,}000 & \textbf{(2)} \\ c = 100{,}000 & \textbf{(3)} \end{cases}$$

From equation (3), note that Adam and Michelle should invest \$100,000 in corporate bonds. Back-substitute \$100,000 into equation (2) to find that $b = 200{,}000$, so Adam and Michelle should invest \$200,000 in Treasury bonds. Back-substitute these values into equation (1) and find that $n = 300{,}000$, so \$300,000 should be invested in Treasury notes. ■

8.2 Assess Your Understanding

Concepts and Vocabulary

1. An m by n rectangular array of numbers is called a(n) ________.

2. The matrix used to represent a system of linear equations is called a(n) ____________ matrix.

3. The notation a_{35} refers to the entry in the ________ row and ________ column of a matrix.

4. ***True or False*** The matrix $\left[\begin{array}{cc|c} 1 & 3 & -2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right]$ is in row echelon form.

5. Which of the following matrices is in reduced row echelon form?

(a) $\left[\begin{array}{cc|c} 1 & 2 & 9 \\ 3 & -1 & -1 \end{array}\right]$ (b) $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 4 \end{array}\right]$

(c) $\left[\begin{array}{cc|c} 1 & 2 & 9 \\ 0 & 0 & 28 \end{array}\right]$ (d) $\left[\begin{array}{cc|c} 1 & 2 & 9 \\ 0 & 1 & 4 \end{array}\right]$

6. Which of the following statements accurately describes the system represented by the matrix $\left[\begin{array}{ccc|c} 1 & 5 & -2 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 5 \end{array}\right]$?

(a) The system has one solution.
(b) The system has infinitely many solutions.
(c) The system has no solution.
(d) The number of solutions cannot be determined.

Skill Building

In Problems 7–18, write the augmented matrix of the given system of equations.

7. $\begin{cases} x - 5y = 5 \\ 4x + 3y = 6 \end{cases}$

8. $\begin{cases} 3x + 4y = 7 \\ 4x - 2y = 5 \end{cases}$

9. $\begin{cases} 2x + 3y - 6 = 0 \\ 4x - 6y + 2 = 0 \end{cases}$

10. $\begin{cases} 9x - y = 0 \\ 3x - y - 4 = 0 \end{cases}$

11. $\begin{cases} 0.01x - 0.03y = 0.06 \\ 0.13x + 0.10y = 0.20 \end{cases}$

12. $\begin{cases} \frac{4}{3}x - \frac{3}{2}y = \frac{3}{4} \\ -\frac{1}{4}x + \frac{1}{3}y = \frac{2}{3} \end{cases}$

13. $\begin{cases} x - y + z = 10 \\ 3x + 3y = 5 \\ x + y + 2z = 2 \end{cases}$

14. $\begin{cases} 5x - y - z = 0 \\ x + y = 5 \\ 2x - 3z = 2 \end{cases}$

15. $\begin{cases} x + y - z = 2 \\ 3x - 2y = 2 \\ 5x + 3y - z = 1 \end{cases}$

16. $\begin{cases} 2x + 3y - 4z = 0 \\ x - 5z + 2 = 0 \\ x + 2y - 3z = -2 \end{cases}$

17. $\begin{cases} x - y - z = 10 \\ 2x + y + 2z = -1 \\ -3x + 4y = 5 \\ 4x - 5y + z = 0 \end{cases}$

18. $\begin{cases} x - y + 2z - w = 5 \\ x + 3y - 4z + 2w = 2 \\ 3x - y - 5z - w = -1 \end{cases}$

In Problems 19–26, write the system of equations corresponding to each augmented matrix. Then perform the indicated row operation(s) on the given augmented matrix.

19. $\left[\begin{array}{cc|c} 1 & -3 & -2 \\ 2 & -5 & 5 \end{array}\right]$ $R_2 = -2r_1 + r_2$

20. $\left[\begin{array}{cc|c} 1 & -3 & -3 \\ 2 & -5 & -4 \end{array}\right]$ $R_2 = -2r_1 + r_2$

21. $\left[\begin{array}{ccc|c} 1 & -3 & 4 & 3 \\ 3 & -5 & 6 & 6 \\ -5 & 3 & 4 & 6 \end{array}\right]$ $R_2 = -3r_1 + r_2$, $R_3 = 5r_1 + r_3$

22. $\left[\begin{array}{ccc|c} 1 & -3 & 3 & -5 \\ -4 & -5 & -3 & -5 \\ -3 & -2 & 4 & 6 \end{array}\right]$ $R_2 = 4r_1 + r_2$, $R_3 = 3r_1 + r_3$

23. $\left[\begin{array}{ccc|c} 1 & -3 & 2 & -6 \\ 2 & -5 & 3 & -4 \\ -3 & -6 & 4 & 6 \end{array}\right]$ $R_2 = -2r_1 + r_2$, $R_3 = 3r_1 + r_3$

24. $\left[\begin{array}{ccc|c} 1 & -3 & -4 & -6 \\ 6 & -5 & 6 & -6 \\ -1 & 1 & 4 & 6 \end{array}\right]$ $R_2 = -6r_1 + r_2$, $R_3 = r_1 + r_3$

25. $\left[\begin{array}{ccc|c} 5 & -3 & 1 & -2 \\ 2 & -5 & 6 & -2 \\ -4 & 1 & 4 & 6 \end{array}\right]$ $R_1 = -2r_2 + r_1$, $R_3 = 2r_2 + r_3$

26. $\left[\begin{array}{ccc|c} 4 & -3 & -1 & 2 \\ 3 & -5 & 2 & 6 \\ -3 & -6 & 4 & 6 \end{array}\right]$ $R_1 = -r_2 + r_1$, $R_3 = r_2 + r_3$

In Problems 27–38, the reduced row echelon form of a system of linear equations is given. Write the system of equations corresponding to the given matrix. Use x, y; or x, y, z; or x_1, x_2, x_3, x_4 as variables. Determine whether the system is consistent or inconsistent. If it is consistent, give the solution.

27. $\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -1 \end{array}\right]$

28. $\left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 0 \end{array}\right]$

29. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 3 \end{array}\right]$

30. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right]$

31. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & -4 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

32. $\left[\begin{array}{ccc|c} 1 & 0 & 4 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

33. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right]$

34. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 3 & 0 \end{array}\right]$

35. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 2 \\ 0 & 1 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

36. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right]$

37. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

38. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$

In Problems 39–74, solve each system of equations using matrices (row operations). If the system has no solution, say that it is inconsistent.

39. $\begin{cases} x + y = 8 \\ x - y = 4 \end{cases}$

40. $\begin{cases} x + 2y = 5 \\ x + y = 3 \end{cases}$

41. $\begin{cases} 2x - 4y = -2 \\ 3x + 2y = 3 \end{cases}$

42. $\begin{cases} 3x + 3y = 3 \\ 4x + 2y = \dfrac{8}{3} \end{cases}$

43. $\begin{cases} x + 2y = 4 \\ 2x + 4y = 8 \end{cases}$

44. $\begin{cases} 3x - y = 7 \\ 9x - 3y = 21 \end{cases}$

45. $\begin{cases} 2x + 3y = 6 \\ x - y = \dfrac{1}{2} \end{cases}$

46. $\begin{cases} \dfrac{1}{2}x + y = -2 \\ x - 2y = 8 \end{cases}$

47. $\begin{cases} 3x - 5y = 3 \\ 15x + 5y = 21 \end{cases}$

48. $\begin{cases} 2x - y = -1 \\ x + \dfrac{1}{2}y = \dfrac{3}{2} \end{cases}$

49. $\begin{cases} x - y = 6 \\ 2x - 3z = 16 \\ 2y + z = 4 \end{cases}$

50. $\begin{cases} 2x + y = -4 \\ -2y + 4z = 0 \\ 3x - 2z = -11 \end{cases}$

51. $\begin{cases} x - 2y + 3z = 7 \\ 2x + y + z = 4 \\ -3x + 2y - 2z = -10 \end{cases}$

52. $\begin{cases} 2x + y - 3z = 0 \\ -2x + 2y + z = -7 \\ 3x - 4y - 3z = 7 \end{cases}$

53. $\begin{cases} 2x - 2y - 2z = 2 \\ 2x + 3y + z = 2 \\ 3x + 2y = 0 \end{cases}$

54. $\begin{cases} 2x - 3y - z = 0 \\ -x + 2y + z = 5 \\ 3x - 4y - z = 1 \end{cases}$

55. $\begin{cases} -x + y + z = -1 \\ -x + 2y - 3z = -4 \\ 3x - 2y - 7z = 0 \end{cases}$

56. $\begin{cases} 2x - 3y - z = 0 \\ 3x + 2y + 2z = 2 \\ x + 5y + 3z = 2 \end{cases}$

57. $\begin{cases} 2x - 2y + 3z = 6 \\ 4x - 3y + 2z = 0 \\ -2x + 3y - 7z = 1 \end{cases}$

58. $\begin{cases} 3x - 2y + 2z = 6 \\ 7x - 3y + 2z = -1 \\ 2x - 3y + 4z = 0 \end{cases}$

59. $\begin{cases} x + y - z = 6 \\ 3x - 2y + z = -5 \\ x + 3y - 2z = 14 \end{cases}$

60. $\begin{cases} x - y + z = -4 \\ 2x - 3y + 4z = -15 \\ 5x + y - 2z = 12 \end{cases}$

61. $\begin{cases} x + 2y - z = -3 \\ 2x - 4y + z = -7 \\ -2x + 2y - 3z = 4 \end{cases}$

62. $\begin{cases} x + 4y - 3z = -8 \\ 3x - y + 3z = 12 \\ x + y + 6z = 1 \end{cases}$

63. $\begin{cases} 3x + y - z = \dfrac{2}{3} \\ 2x - y + z = 1 \\ 4x + 2y = \dfrac{8}{3} \end{cases}$

64. $\begin{cases} x + y = 1 \\ 2x - y + z = 1 \\ x + 2y + z = \dfrac{8}{3} \end{cases}$

65. $\begin{cases} x + y + z + w = 4 \\ 2x - y + z = 0 \\ 3x + 2y + z - w = 6 \\ x - 2y - 2z + 2w = -1 \end{cases}$

66. $\begin{cases} x + y + z + w = 4 \\ -x + 2y + z = 0 \\ 2x + 3y + z - w = 6 \\ -2x + y - 2z + 2w = -1 \end{cases}$

67. $\begin{cases} x + 2y + z = 1 \\ 2x - y + 2z = 2 \\ 3x + y + 3z = 3 \end{cases}$

68. $\begin{cases} x + 2y - z = 3 \\ 2x - y + 2z = 6 \\ x - 3y + 3z = 4 \end{cases}$

69. $\begin{cases} x - y + z = 5 \\ 3x + 2y - 2z = 0 \end{cases}$

70. $\begin{cases} 2x + y - z = 4 \\ -x + y + 3z = 1 \end{cases}$

71. $\begin{cases} 2x + 3y - z = 3 \\ x - y - z = 0 \\ -x + y + z = 0 \\ x + y + 3z = 5 \end{cases}$

72. $\begin{cases} x - 3y + z = 1 \\ 2x - y - 4z = 0 \\ x - 3y + 2z = 1 \\ x - 2y = 5 \end{cases}$

73. $\begin{cases} 4x + y + z - w = 4 \\ x - y + 2z + 3w = 3 \end{cases}$

74. $\begin{cases} -4x + y = 5 \\ 2x - y + z - w = 5 \\ z + w = 4 \end{cases}$

Applications and Extensions

75. Curve Fitting Find the function $y = ax^2 + bx + c$ whose graph contains the points $(1, 2)$, $(-2, -7)$, and $(2, -3)$.

76. Curve Fitting Find the function $y = ax^2 + bx + c$ whose graph contains the points $(1, -1)$, $(3, -1)$, and $(-2, 14)$.

77. Curve Fitting Find the function $f(x) = ax^3 + bx^2 + cx + d$ for which $f(-3) = -112$, $f(-1) = -2$, $f(1) = 4$, and $f(2) = 13$.

78. Curve Fitting Find the function $f(x) = ax^3 + bx^2 + cx + d$ for which $f(-2) = -10$, $f(-1) = 3$, $f(1) = 5$, and $f(3) = 15$.

79. Nutrition A dietitian at Palos Community Hospital wants a patient to have a meal that has 78 grams (g) of protein, 59 g of carbohydrates, and 75 milligrams (mg) of vitamin A. The hospital food service tells the dietitian that the dinner for today is salmon steak, baked eggs, and acorn squash. Each serving of salmon steak has 30 g of protein, 20 g of carbohydrates, and 2 mg of vitamin A. Each serving of baked eggs contains 15 g of protein, 2 g of carbohydrates, and 20 mg of vitamin A. Each serving of acorn squash contains 3 g of protein, 25 g of carbohydrates, and 32 mg of vitamin A. How many servings of each food should the dietitian provide for the patient?

80. Nutrition A dietitian at General Hospital wants a patient to have a meal that has 47 grams (g) of protein, 58 g of carbohydrates, and 630 milligrams (mg) of calcium. The hospital food service tells the dietitian that the dinner for today is pork chops, corn on the cob, and 2% milk. Each serving of pork chops has 23 g of protein, 0 g of carbohydrates, and 10 mg of calcium. Each serving of corn on the cob contains 3 g of protein, 16 g of carbohydrates, and 10 mg of calcium. Each glass of 2% milk contains 9 g of protein, 13 g of carbohydrates, and 300 mg of calcium. How many servings of each food should the dietitian provide for the patient?

81. Financial Planning Carletta has \$10,000 to invest. As her financial consultant, you recommend that she invest in Treasury bills that yield 6%, Treasury bonds that yield 7%, and corporate bonds that yield 8%. Carletta wants to have an annual income of \$680, and the amount invested in corporate bonds must be half that invested in Treasury bills. Find the amount in each investment.

82. Landscaping A landscape company is hired to plant trees in three new subdivisions. The company charges the developer for each tree planted, an hourly rate to plant the trees, and a fixed delivery charge. In one subdivision it took 166 labor hours to plant 250 trees for a cost of \$7520. In a second subdivision it took 124 labor hours to plant 200 trees for a cost of \$5945. In the final subdivision it took 200 labor hours to plant 300 trees for a cost of \$8985. Determine the cost for each tree, the hourly labor charge, and the fixed delivery charge.
Source: *www.bx.org*

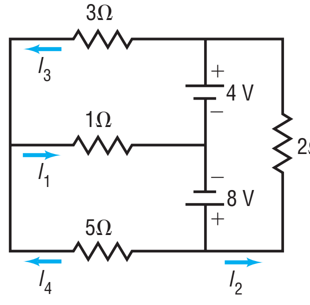
83. Production To manufacture an automobile requires painting, drying, and polishing. Epsilon Motor Company produces three types of cars: the Delta, the Beta, and the Sigma. Each Delta requires 10 hours (hr) for painting, 3 hr for drying, and 2 hr for polishing. A Beta requires 16 hr for painting, 5 hr for drying, and 3 hr for polishing, and a Sigma requires 8 hr for painting, 2 hr for drying, and 1 hr for polishing. If the company has 240 hr for painting, 69 hr for drying, and 41 hr for polishing per month, how many of each type of car are produced?

84. Production A Florida juice company completes the preparation of its products by sterilizing, filling, and labeling bottles. Each case of orange juice requires 9 minutes (min) for sterilizing, 6 min for filling, and 1 min for labeling. Each case of grapefruit juice requires 10 min for sterilizing, 4 min for filling, and 2 min for labeling. Each case of tomato juice requires 12 min for sterilizing, 4 min for filling, and 1 min for labeling. If the company runs the sterilizing machine for 398 min, the filling machine for 164 min, and the labeling machine for 58 min, how many cases of each type of juice are prepared?

85. Electricity: Kirchhoff's Rules An application of Kirchhoff's Rules to the circuit shown results in the following system of equations:

$$\begin{cases} -4 + 8 - 2I_2 = 0 \\ 8 = 5I_4 + I_1 \\ 4 = 3I_3 + I_1 \\ I_3 + I_4 = I_1 \end{cases}$$

Find the currents I_1, I_2, I_3, and I_4.

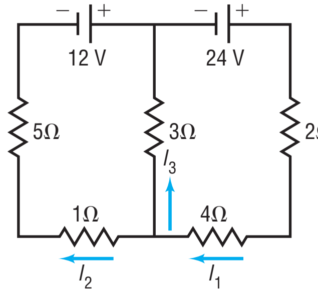


Source: *Based on Raymond Serway.* Physics, *3rd ed. (Philadelphia: Saunders, 1990), Prob. 34, p. 790.*

86. Electricity: Kirchhoff's Rules An application of Kirchhoff's Rules to the circuit shown results in the following system of equations:

$$\begin{cases} I_1 = I_3 + I_2 \\ 24 - 6I_1 - 3I_3 = 0 \\ 12 + 24 - 6I_1 - 6I_2 = 0 \end{cases}$$

Find the currents I_1, I_2, and I_3.



Source: *Ibid., Prob. 38, p. 791.*

87. Financial Planning Three retired couples each require an additional annual income of \$2000 per year. As their financial consultant, you recommend that they invest some money in Treasury bills that yield 7%, some money in corporate bonds that yield 9%, and some money in "junk bonds" that yield 11%. Prepare a table for each couple showing the various ways that their goals can be achieved:
(a) If the first couple has \$20,000 to invest.
(b) If the second couple has \$25,000 to invest.
(c) If the third couple has \$30,000 to invest.
(d) What advice would you give each couple regarding the amount to invest and the choices available?

[**Hint:** Higher yields generally carry more risk.]

88. Financial Planning A young couple has \$25,000 to invest. As their financial consultant, you recommend that they invest some money in Treasury bills that yield 7%, some money in corporate bonds that yield 9%, and some money in junk bonds that yield 11%. Prepare a table showing the various ways that this couple can achieve the following goals:
(a) \$1500 per year in income
(b) \$2000 per year in income
(c) \$2500 per year in income
(d) What advice would you give this couple regarding the income that they require and the choices available?

[**Hint:** Higher yields generally carry more risk.]

89. Pharmacy A doctor's prescription calls for a daily intake of a supplement containing 40 milligrams (mg) of vitamin C and 30 mg of vitamin D. Your pharmacy stocks three supplements that can be used: one contains 20% vitamin C and 30% vitamin D; a second, 40% vitamin C and 20% vitamin D; and a third, 30% vitamin C and 50% vitamin D. Create a table showing the possible combinations that could be used to fill the prescription.

90. Pharmacy A doctor's prescription calls for the creation of pills that contain 12 units of vitamin B_{12} and 12 units of vitamin E. Your pharmacy stocks three powders that can be used to make these pills: one contains 20% vitamin B_{12} and 30% vitamin E; a second, 40% vitamin B_{12} and 20% vitamin E; and a third, 30% vitamin B_{12} and 40% vitamin E. Create a table showing the possible combinations of these powders that could be mixed in each pill. Hint: 10 units of the first powder contains $10(0.2) = 2$ units of vitamin B_{12}.

Explaining Concepts: Discussion and Writing

91. Write a brief paragraph or two outlining your strategy for solving a system of linear equations using matrices.

92. When solving a system of linear equations using matrices, do you prefer to place the augmented matrix in row echelon form or in reduced row echelon form? Give reasons for your choice.

93. Make up a system of three linear equations containing three variables that has:
(a) No solution
(b) Exactly one solution
(c) Infinitely many solutions

Give the three systems to a friend to solve and critique.

Retain Your Knowledge

Problems 94–97 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

94. Solve: $x^2 - 3x < 6 + 2x$

95. Graph: $f(x) = \dfrac{2x^2 - x - 1}{x^2 + 2x + 1}$

96. State the domain of $f(x) = -e^{x+5} - 6$.

97. Find the complex zeros of $f(x) = x^4 + 21x^2 - 100$.

8.3 Systems of Linear Equations: Determinants

OBJECTIVES
1. Evaluate 2 by 2 Determinants (p. 585)
2. Use Cramer's Rule to Solve a System of Two Equations Containing Two Variables (p. 586)
3. Evaluate 3 by 3 Determinants (p. 588)
4. Use Cramer's Rule to Solve a System of Three Equations Containing Three Variables (p. 590)
5. Know Properties of Determinants (p. 591)

The preceding section described a method of using matrices to solve a system of linear equations. This section deals with yet another method for solving systems of linear equations; however, it can be used only when the number of equations equals the number of variables. Although the method works for any system (provided that the number of equations equals the number of variables), it is most often used for systems of two equations containing two variables or three equations containing three variables. This method, called *Cramer's Rule,* is based on the concept of a *determinant*.

1 Evaluate 2 by 2 Determinants

DEFINITION

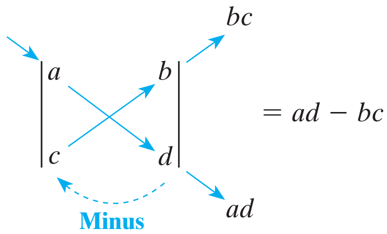
If a, b, c, and d are four real numbers, the symbol

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

is called a **2 by 2 determinant**. Its value is the number $ad - bc$; that is,

$$\boxed{D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad \textbf{(1)}}$$

The following device may be helpful for remembering the value of a 2 by 2 determinant:



EXAMPLE 1 **Evaluating a 2 by 2 Determinant**

Evaluate: $\begin{vmatrix} 3 & -2 \\ 6 & 1 \end{vmatrix}$

Algebraic Solution

$$\begin{vmatrix} 3 & -2 \\ 6 & 1 \end{vmatrix} = (3)(1) - (-2)(6)$$
$$= 3 - (-12)$$
$$= 15$$

Graphing Solution

First, enter the matrix whose entries are those of the determinant into the graphing utility and name it A. Using the determinant command, obtain the result shown in Figure 9.

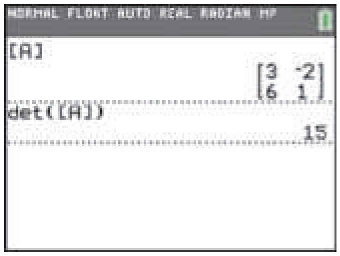


Figure 9

Now Work PROBLEM 7

2 Use Cramer's Rule to Solve a System of Two Equations Containing Two Variables

Let's see the role that a 2 by 2 determinant plays in the solution of a system of two equations containing two variables. Consider the system

$$\begin{cases} ax + by = s & \textbf{(1)} \\ cx + dy = t & \textbf{(2)} \end{cases} \qquad \textbf{(2)}$$

We use the method of elimination to solve this system.

Provided that $d \neq 0$ and $b \neq 0$, this system is equivalent to the system

$$\begin{cases} adx + bdy = sd & \textbf{(1)} \quad \textbf{Multiply by } \boldsymbol{d}\textbf{.} \\ bcx + bdy = tb & \textbf{(2)} \quad \textbf{Multiply by } \boldsymbol{b}\textbf{.} \end{cases}$$

Subtract the second equation from the first equation and obtain

$$\begin{cases} (ad - bc)x + 0 \cdot y = sd - tb & \textbf{(1)} \\ bcx + bdy = tb & \textbf{(2)} \end{cases}$$

Now the first equation can be rewritten using determinant notation.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} x = \begin{vmatrix} s & b \\ t & d \end{vmatrix}$$

If $D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$, solve for x to get

$$x = \frac{\begin{vmatrix} s & b \\ t & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{\begin{vmatrix} s & b \\ t & d \end{vmatrix}}{D} \qquad \textbf{(3)}$$

Return now to the original system (2). Provided that $a \neq 0$ and $c \neq 0$, the system is equivalent to

$$\begin{cases} acx + bcy = cs & \textbf{(1)} \quad \textbf{Multiply by } \boldsymbol{c}\textbf{.} \\ acx + ady = at & \textbf{(2)} \quad \textbf{Multiply by } \boldsymbol{a}\textbf{.} \end{cases}$$

Subtract the first equation from the second equation and obtain

$$\begin{cases} acx + \quad bcy = cs & \textbf{(1)} \\ 0 \cdot x + (ad - bc)y = at - cs & \textbf{(2)} \end{cases}$$

The second equation can now be rewritten using determinant notation.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} y = \begin{vmatrix} a & s \\ c & t \end{vmatrix}$$

If $D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$, solve for y to get

$$y = \frac{\begin{vmatrix} a & s \\ c & t \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{\begin{vmatrix} a & s \\ c & t \end{vmatrix}}{D} \qquad \textbf{(4)}$$

Equations (3) and (4) lead to the following result, called **Cramer's Rule**.

THEOREM

Cramer's Rule for Two Equations Containing Two Variables

The solution to the system of equations

$$\begin{cases} ax + by = s & (1) \\ cx + dy = t & (2) \end{cases} \qquad (5)$$

is given by

$$x = \frac{\begin{vmatrix} s & b \\ t & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & s \\ c & t \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad (6)$$

provided that

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$$

In the derivation given for Cramer's Rule, we assumed that none of the numbers a, b, c, and d was 0. In Problem 65 you will be asked to complete the proof under the less stringent condition that $D = ad - bc \neq 0$.

Now look carefully at the pattern in Cramer's Rule. The denominator in the solution (6) is the determinant of the coefficients of the variables.

$$\begin{cases} ax + by = s \\ cx + dy = t \end{cases} \qquad D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

In the solution for x, the numerator is the determinant, denoted by D_x, formed by replacing the entries in the first column (the coefficients of x) of D by the constants on the right side of the equal sign.

$$D_x = \begin{vmatrix} s & b \\ t & d \end{vmatrix}$$

In the solution for y, the numerator is the determinant, denoted by D_y, formed by replacing the entries in the second column (the coefficients of y) of D by the constants on the right side of the equal sign.

$$D_y = \begin{vmatrix} a & s \\ c & t \end{vmatrix}$$

Cramer's Rule then states that if $D \neq 0$,

$$x = \frac{D_x}{D} \qquad y = \frac{D_y}{D} \qquad (7)$$

EXAMPLE 2 **Solving a System of Linear Equations Using Determinants**

Use Cramer's Rule, if applicable, to solve the system

$$\begin{cases} 3x - 2y = 4 & (1) \\ 6x + y = 13 & (2) \end{cases}$$

Algebraic Solution

The determinant D of the coefficients of the variables is

$$D = \begin{vmatrix} 3 & -2 \\ 6 & 1 \end{vmatrix} = (3)(1) - (-2)(6) = 15$$

Graphing Solution

Enter the coefficient matrix into the graphing utility. Call it A and evaluate $\det[A]$. Since $\det[A] \neq 0$, Cramer's Rule can be used. Enter the matrices D_x and D_y into the graphing utility and call them B and C, respectively. Finally, find x by

Because $D \neq 0$, Cramer's Rule (7) can be used.

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 4 & -2 \\ 13 & 1 \end{vmatrix}}{15} = \frac{(4)(1) - (-2)(13)}{15} = \frac{30}{15} = 2 \qquad y = \frac{D_y}{D} = \frac{\begin{vmatrix} 3 & 4 \\ 6 & 13 \end{vmatrix}}{15} = \frac{(3)(13) - (4)(6)}{15} = \frac{15}{15} = 1$$

calculating $\dfrac{\det[B]}{\det[A]}$, and find y by calculating $\dfrac{\det[C]}{\det[A]}$. The results are shown in Figure 10.

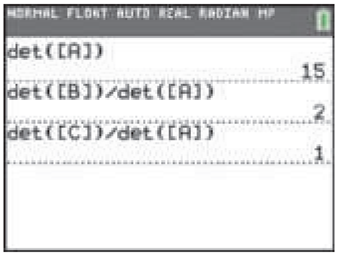


Figure 10

The solution is $x = 2, y = 1$ or, using an ordered pair, $(2, 1)$. ■

In attempting to use Cramer's Rule, if the determinant D of the coefficients of the variables is found to equal 0 (so that Cramer's Rule is not applicable), then the system either is consistent with dependent equations or is inconsistent. To determine whether the system has no solution or infinitely many solutions, solve the system using the methods of Section 8.1 or 8.2.

Now Work PROBLEM 15

3 Evaluate 3 by 3 Determinants

To use Cramer's Rule to solve a system of three equations containing three variables, we need to define a 3 by 3 determinant.

A **3 by 3 determinant** is symbolized by

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \tag{8}$$

in which $a_{11}, a_{12}, \ldots$, are real numbers.

As with matrices, we use a double subscript to identify an entry by indicating its row and column numbers. For example, the entry a_{23} is in row 2, column 3.

The value of a 3 by 3 determinant may be defined in terms of 2 by 2 determinants by the following formula:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}_{\substack{\text{2 by 2 determinant left after} \\ \text{removing the row and column containing } a_{11}}} \overset{\text{Minus}}{-} a_{12}\underbrace{\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}}_{\substack{\text{2 by 2 determinant left after} \\ \text{removing the row and column containing } a_{12}}} \overset{\text{Plus}}{+} a_{13}\underbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}_{\substack{\text{2 by 2 determinant left after} \\ \text{removing the row and column containing } a_{13}}} \tag{9}$$

In Words
The graphic below should help you visualize the minor of a_{11} in a 3 by 3 determinant:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The 2 by 2 determinants shown in formula (9) are called **minors** of the 3 by 3 determinant. For an n by n determinant, the **minor** M_{ij} of entry a_{ij} is the determinant that results from removing the ith row and the jth column.

EXAMPLE 3 **Finding Minors of a 3 by 3 Determinant**

For the determinant $A = \begin{vmatrix} 2 & -1 & 3 \\ -2 & 5 & 1 \\ 0 & 6 & -9 \end{vmatrix}$, find: (a) M_{12} (b) M_{23}

Solution (a) M_{12} is the determinant that results from removing the first row and the second column from A.

$$A = \begin{vmatrix} 2 & -1 & 3 \\ -2 & 5 & 1 \\ 0 & 6 & -9 \end{vmatrix} \qquad M_{12} = \begin{vmatrix} -2 & 1 \\ 0 & -9 \end{vmatrix} = (-2)(-9) - (1)(0) = 18$$

(b) M_{23} is the determinant that results from removing the second row and the third column from A.

$$A = \begin{vmatrix} 2 & -1 & 3 \\ -2 & 5 & 1 \\ 0 & 6 & -9 \end{vmatrix} \qquad M_{23} = \begin{vmatrix} 2 & -1 \\ 0 & 6 \end{vmatrix} = (2)(6) - (-1)(0) = 12$$

Referring to formula (9), note that each element a_{ij} in the first row of the determinant is multiplied by its minor, but this term is sometimes added and other times subtracted. To determine whether to add or subtract a term, consider the *cofactor*.

DEFINITION

For an n by n determinant A, the **cofactor** of entry a_{ij}, denoted by A_{ij}, is given by

$$\boxed{A_{ij} = (-1)^{i+j} M_{ij}}$$

where M_{ij} is the minor of entry a_{ij}.

The exponent of $(-1)^{i+j}$ is the sum of the row and column of the entry a_{ij}, so if $i + j$ is even, $(-1)^{i+j}$ equals 1, and if $i + j$ is odd, $(-1)^{i+j}$ equals -1.

To find the value of a determinant, multiply each entry in any row or column by its cofactor and sum the results. This process is referred to as **expanding across a row or column**. For example, the value of the 3 by 3 determinant in formula (9) was found by expanding across row 1.

Expanding down column 2 gives

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{2+2} a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{3+2} a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

Expand down column 2.

Expanding across row 3 gives

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{3+1} a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + (-1)^{3+2} a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + (-1)^{3+3} a_{33} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Expand across row 3.

It can be shown that the value of a determinant does not depend on the choice of the row or column used in the expansion. However, expanding across a row or column that has an entry equal to 0 reduces the amount of work needed to compute the value of the determinant.

EXAMPLE 4 Evaluating a 3 by 3 Determinant

Find the value of the 3 by 3 determinant: $\begin{vmatrix} 3 & 0 & -1 \\ 4 & 6 & 2 \\ 8 & -2 & 3 \end{vmatrix}$

Solution Because of the 0 in row 1, column 2, it is easiest to expand across row 1 or down column 2. We choose to expand across row 1.

$$\begin{vmatrix} 3 & 0 & -1 \\ 4 & 6 & 2 \\ 8 & -2 & 3 \end{vmatrix} = (-1)^{1+1} \cdot 3 \cdot \begin{vmatrix} 6 & 2 \\ -2 & 3 \end{vmatrix} + (-1)^{1+2} \cdot 0 \cdot \begin{vmatrix} 4 & 2 \\ 8 & 3 \end{vmatrix} + (-1)^{1+3} \cdot (-1) \cdot \begin{vmatrix} 4 & 6 \\ 8 & -2 \end{vmatrix}$$
$$= 3(18 - (-4)) - 0 + (-1)(-8 - 48)$$
$$= 3(22) + (-1)(-56)$$
$$= 66 + 56 = 122$$

Now Work PROBLEM 11

4 Use Cramer's Rule to Solve a System of Three Equations Containing Three Variables

Consider the following system of three equations containing three variables.

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = c_1 \\ a_{21}x + a_{22}y + a_{23}z = c_2 \\ a_{31}x + a_{32}y + a_{33}z = c_3 \end{cases} \qquad \textbf{(10)}$$

It can be shown that if the determinant D of the coefficients of the variables is not 0, that is, if

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$$

then the unique solution of system (10) is given by

THEOREM

Cramer's Rule for Three Equations Containing Three Variables

$$\boxed{x = \frac{D_x}{D} \qquad y = \frac{D_y}{D} \qquad z = \frac{D_z}{D}}$$

where

$$D_x = \begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix} \qquad D_y = \begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix} \qquad D_z = \begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix}$$

Do you see the similarity between this pattern and the pattern observed earlier for a system of two equations containing two variables?

EXAMPLE 5 Using Cramer's Rule

Use Cramer's Rule, if applicable, to solve the following system:

$$\begin{cases} 2x + y - z = 3 & \textbf{(1)} \\ -x + 2y + 4z = -3 & \textbf{(2)} \\ x - 2y - 3z = 4 & \textbf{(3)} \end{cases}$$

Solution The value of the determinant D of the coefficients of the variables is

$$D = \begin{vmatrix} 2 & 1 & -1 \\ -1 & 2 & 4 \\ 1 & -2 & -3 \end{vmatrix} = (-1)^{1+1} \cdot 2 \cdot \begin{vmatrix} 2 & 4 \\ -2 & -3 \end{vmatrix} + (-1)^{1+2} \cdot 1 \cdot \begin{vmatrix} -1 & 4 \\ 1 & -3 \end{vmatrix} + (-1)^{1+3}(-1)\begin{vmatrix} -1 & 2 \\ 1 & -2 \end{vmatrix}$$
$$= 2(2) - 1(-1) + (-1)(0)$$
$$= 4 + 1 = 5$$

Because $D \neq 0$, proceed to find the values of D_x, D_y, and D_z. To find D_x, replace the coefficients of x in D with the constants and then evaluate the determinant.

$$D_x = \begin{vmatrix} 3 & 1 & -1 \\ -3 & 2 & 4 \\ 4 & -2 & -3 \end{vmatrix} = (-1)^{1+1} \cdot 3 \cdot \begin{vmatrix} 2 & 4 \\ -2 & -3 \end{vmatrix} + (-1)^{1+2} \cdot 1 \cdot \begin{vmatrix} -3 & 4 \\ 4 & -3 \end{vmatrix} + (-1)^{1+3}(-1)\begin{vmatrix} -3 & 2 \\ 4 & -2 \end{vmatrix}$$

$$= 3(2) - 1(-7) + (-1)(-2) = 15$$

$$D_y = \begin{vmatrix} 2 & 3 & -1 \\ -1 & -3 & 4 \\ 1 & 4 & -3 \end{vmatrix} = (-1)^{1+1} \cdot 2 \cdot \begin{vmatrix} -3 & 4 \\ 4 & -3 \end{vmatrix} + (-1)^{1+2} \cdot 3 \cdot \begin{vmatrix} -1 & 4 \\ 1 & -3 \end{vmatrix} + (-1)^{1+3}(-1)\begin{vmatrix} -1 & -3 \\ 1 & 4 \end{vmatrix}$$

$$= 2(-7) - 3(-1) + (-1)(-1) = -10$$

$$D_z = \begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & -3 \\ 1 & -2 & 4 \end{vmatrix} = (-1)^{1+1} \cdot 2 \cdot \begin{vmatrix} 2 & -3 \\ -2 & 4 \end{vmatrix} + (-1)^{1+2} \cdot 1 \cdot \begin{vmatrix} -1 & -3 \\ 1 & 4 \end{vmatrix} + (-1)^{1+3} \cdot 3 \cdot \begin{vmatrix} -1 & 2 \\ 1 & -2 \end{vmatrix}$$

$$= 2(2) - 1(-1) + 3(0) = 5$$

As a result,

$$x = \frac{D_x}{D} = \frac{15}{5} = 3 \qquad y = \frac{D_y}{D} = \frac{-10}{5} = -2 \qquad z = \frac{D_z}{D} = \frac{5}{5} = 1$$

The solution is $x = 3$, $y = -2$, $z = 1$ or, using an ordered triplet, $(3, -2, 1)$. ■

Cramer's Rule cannot be used when the determinant of the coefficients on the variables, D, is 0. But can anything be learned about the system other than it is not a consistent and independent system if $D = 0$? The answer is yes!

Cramer's Rule with Inconsistent or Dependent Systems

- If $D = 0$ and at least one of the determinants D_x, D_y, or D_z is different from 0, then the system is inconsistent and the solution set is $\varnothing$, or $\{\ \}$.
- If $D = 0$ and all the determinants D_x, D_y, and D_z equal 0, then the system is consistent and dependent, so there are infinitely many solutions. The system must be solved using row reduction techniques.

Now Work PROBLEM 33

5 Know Properties of Determinants

Determinants have several properties that are sometimes helpful for obtaining their value. We list some of them here.

THEOREM

The value of a determinant changes sign if any two rows (or any two columns) are interchanged. **(11)**

Proof for 2 by 2 Determinants

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad \text{and} \quad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad = -(ad - bc)$$ ■

EXAMPLE 6 **Demonstrating Theorem (11)**

$$\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = 2 \qquad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$$

THEOREM

If all the entries in any row (or any column) equal 0, the value of the determinant is 0. **(12)**

Proof Expand across the row (or down the column) containing the 0's.

THEOREM

If any two rows (or any two columns) of a determinant have corresponding entries that are equal, the value of the determinant is 0. **(13)**

In Problem 68, you are asked to prove this result for a 3 by 3 determinant in which the entries in column 1 equal the entries in column 3.

EXAMPLE 7 **Demonstrating Theorem (13)**

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} + (-1)^{1+2} \cdot 2 \cdot \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} + (-1)^{1+3} \cdot 3 \cdot \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

$$= 1(-3) - 2(-6) + 3(-3) = -3 + 12 - 9 = 0$$

THEOREM

If any row (or any column) of a determinant is multiplied by a nonzero number k, the value of the determinant is also changed by a factor of k. **(14)**

In Problem 67, you are asked to prove this result for a 3 by 3 determinant using row 2.

EXAMPLE 8 **Demonstrating Theorem (14)**

$$\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} = 6 - 8 = -2$$

$$\begin{vmatrix} k & 2k \\ 4 & 6 \end{vmatrix} = 6k - 8k = -2k = k(-2) = k\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix}$$

THEOREM

If the entries of any row (or any column) of a determinant are multiplied by a nonzero number k and the result is added to the corresponding entries of another row (or column), the value of the determinant remains unchanged. **(15)**

In Problem 69, you are asked to prove this result for a 3 by 3 determinant using rows 1 and 2.

EXAMPLE 9 **Demonstrating Theorem (15)**

$$\begin{vmatrix} 3 & 4 \\ 5 & 2 \end{vmatrix} = -14 \qquad \begin{vmatrix} 3 & 4 \\ 5 & 2 \end{vmatrix} \underset{\uparrow}{\rightarrow} \begin{vmatrix} -7 & 0 \\ 5 & 2 \end{vmatrix} = -14$$

Multiply row 2 by -2 and add to row 1.

Now Work PROBLEM 45

8.3 Assess Your Understanding

Concepts and Vocabulary

1. $D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = $ ________.

2. Using Cramer's Rule, the value of x that satisfies the system of equations $\begin{cases} 2x + 3y = 5 \\ x - 4y = -3 \end{cases}$ is $x = \dfrac{\_\_\_\_\_\_\_\_}{\begin{vmatrix} 2 & 3 \\ 1 & -4 \end{vmatrix}}$.

3. ***True or False*** A determinant can never equal 0.

4. ***True or False*** When using Cramer's Rule, if $D = 0$, then the system of linear equations is inconsistent.

5. ***True or False*** If any row (or any column) of a determinant is multiplied by a nonzero number k, the value of the determinant remains unchanged.

6. If any two rows of a determinant are interchanged, its value is best described by which of the following?
(a) changes sign (b) becomes zero (c) remains the same
(d) no longer relates to the original value

Skill Building

In Problems 7–14, find the value of each determinant.

7. $\begin{vmatrix} 6 & 4 \\ -1 & 3 \end{vmatrix}$

8. $\begin{vmatrix} 8 & -3 \\ 4 & 2 \end{vmatrix}$

9. $\begin{vmatrix} -3 & -1 \\ 4 & 2 \end{vmatrix}$

10. $\begin{vmatrix} -4 & 2 \\ -5 & 3 \end{vmatrix}$

11. $\begin{vmatrix} 3 & 4 & 2 \\ 1 & -1 & 5 \\ 1 & 2 & -2 \end{vmatrix}$

12. $\begin{vmatrix} 1 & 3 & -2 \\ 6 & 1 & -5 \\ 8 & 2 & 3 \end{vmatrix}$

13. $\begin{vmatrix} 4 & -1 & 2 \\ 6 & -1 & 0 \\ 1 & -3 & 4 \end{vmatrix}$

14. $\begin{vmatrix} 3 & -9 & 4 \\ 1 & 4 & 0 \\ 8 & -3 & 1 \end{vmatrix}$

In Problems 15–42, solve each system of equations using Cramer's Rule if it is applicable. If Cramer's Rule is not applicable, say so.

15. $\begin{cases} x + y = 8 \\ x - y = 4 \end{cases}$

16. $\begin{cases} x + 2y = 5 \\ x - y = 3 \end{cases}$

17. $\begin{cases} 5x - y = 13 \\ 2x + 3y = 12 \end{cases}$

18. $\begin{cases} x + 3y = 5 \\ 2x - 3y = -8 \end{cases}$

19. $\begin{cases} 3x = 24 \\ x + 2y = 0 \end{cases}$

20. $\begin{cases} 4x + 5y = -3 \\ -2y = -4 \end{cases}$

21. $\begin{cases} 3x - 6y = 24 \\ 5x + 4y = 12 \end{cases}$

22. $\begin{cases} 2x + 4y = 16 \\ 3x - 5y = -9 \end{cases}$

23. $\begin{cases} 3x - 2y = 4 \\ 6x - 4y = 0 \end{cases}$

24. $\begin{cases} -x + 2y = 5 \\ 4x - 8y = 6 \end{cases}$

25. $\begin{cases} 2x - 4y = -2 \\ 3x + 2y = 3 \end{cases}$

26. $\begin{cases} 3x + 3y = 3 \\ 4x + 2y = \dfrac{8}{3} \end{cases}$

27. $\begin{cases} 2x - 3y = -1 \\ 10x + 10y = 5 \end{cases}$

28. $\begin{cases} 3x - 2y = 0 \\ 5x + 10y = 4 \end{cases}$

29. $\begin{cases} 2x + 3y = 6 \\ x - y = \dfrac{1}{2} \end{cases}$

30. $\begin{cases} \dfrac{1}{2}x + y = -2 \\ x - 2y = 8 \end{cases}$

31. $\begin{cases} 3x - 5y = 3 \\ 15x + 5y = 21 \end{cases}$

32. $\begin{cases} 2x - y = -1 \\ x + \dfrac{1}{2}y = \dfrac{3}{2} \end{cases}$

33. $\begin{cases} x + y - z = 6 \\ 3x - 2y + z = -5 \\ x + 3y - 2z = 14 \end{cases}$

34. $\begin{cases} x - y + z = -4 \\ 2x - 3y + 4z = -15 \\ 5x + y - 2z = 12 \end{cases}$

35. $\begin{cases} x + 2y - z = -3 \\ 2x - 4y + z = -7 \\ -2x + 2y - 3z = 4 \end{cases}$

36. $\begin{cases} x + 4y - 3z = -8 \\ 3x - y + 3z = 12 \\ x + y + 6z = 1 \end{cases}$

37. $\begin{cases} x - 2y + 3z = 1 \\ 3x + y - 2z = 0 \\ 2x - 4y + 6z = 2 \end{cases}$

38. $\begin{cases} x - y + 2z = 5 \\ 3x + 2y = 4 \\ -2x + 2y - 4z = -10 \end{cases}$

39. $\begin{cases} x + 2y - z = 0 \\ 2x - 4y + z = 0 \\ -2x + 2y - 3z = 0 \end{cases}$

40. $\begin{cases} x + 4y - 3z = 0 \\ 3x - y + 3z = 0 \\ x + y + 6z = 0 \end{cases}$

41. $\begin{cases} x - 2y + 3z = 0 \\ 3x + y - 2z = 0 \\ 2x - 4y + 6z = 0 \end{cases}$

42. $\begin{cases} x - y + 2z = 0 \\ 3x + 2y = 0 \\ -2x + 2y - 4z = 0 \end{cases}$

In Problems 43–50, use properties of determinants to find the value of each determinant if it is known that

$$\begin{vmatrix} x & y & z \\ u & v & w \\ 1 & 2 & 3 \end{vmatrix} = 4$$

43. $\begin{vmatrix} 1 & 2 & 3 \\ u & v & w \\ x & y & z \end{vmatrix}$

44. $\begin{vmatrix} x & y & z \\ u & v & w \\ 2 & 4 & 6 \end{vmatrix}$

45. $\begin{vmatrix} x & y & z \\ -3 & -6 & -9 \\ u & v & w \end{vmatrix}$

46. $\begin{vmatrix} 1 & 2 & 3 \\ x - u & y - v & z - w \\ u & v & w \end{vmatrix}$

47. $\begin{vmatrix} 1 & 2 & 3 \\ x-3 & y-6 & z-9 \\ 2u & 2v & 2w \end{vmatrix}$ **48.** $\begin{vmatrix} x & y & z-x \\ u & v & w-u \\ 1 & 2 & 2 \end{vmatrix}$ **49.** $\begin{vmatrix} 1 & 2 & 3 \\ 2x & 2y & 2z \\ u-1 & v-2 & w-3 \end{vmatrix}$ **50.** $\begin{vmatrix} x+3 & y+6 & z+9 \\ 3u-1 & 3v-2 & 3w-3 \\ 1 & 2 & 3 \end{vmatrix}$

Mixed Practice

In Problems 51–56, solve for x.

51. $\begin{vmatrix} x & x \\ 4 & 3 \end{vmatrix} = 5$ **52.** $\begin{vmatrix} x & 1 \\ 3 & x \end{vmatrix} = -2$ **53.** $\begin{vmatrix} x & 1 & 1 \\ 4 & 3 & 2 \\ -1 & 2 & 5 \end{vmatrix} = 2$

54. $\begin{vmatrix} 3 & 2 & 4 \\ 1 & x & 5 \\ 0 & 1 & -2 \end{vmatrix} = 0$ **55.** $\begin{vmatrix} x & 2 & 3 \\ 1 & x & 0 \\ 6 & 1 & -2 \end{vmatrix} = 7$ **56.** $\begin{vmatrix} x & 1 & 2 \\ 1 & x & 3 \\ 0 & 1 & 2 \end{vmatrix} = -4x$

Applications and Extensions

57. Geometry: Equation of a Line An equation of the line containing the two points (x_1, y_1) and (x_2, y_2) may be expressed as the determinant

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Prove this result by expanding the determinant and comparing the result to the two-point form of the equation of a line.

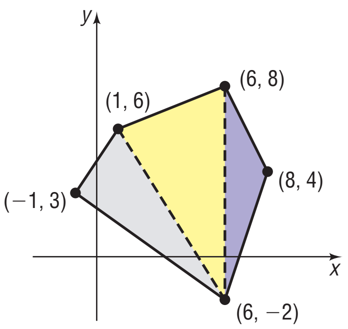
58. Geometry: Collinear Points Using the result obtained in Problem 57, show that three distinct points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear (lie on the same line) if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

59. Geometry: Area of a Triangle A triangle has vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3). The area of the triangle is given by the absolute value of D, where $D = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}$.

Use this formula to find the area of a triangle with vertices $(2, 3)$, $(5, 2)$, and $(6, 5)$.

60. Geometry: Area of a Polygon The formula from Problem 59 can be used to find the area of a polygon. To do so, divide the polygon into non-overlapping triangular regions and find the sum of the areas. Use this approach to find the area of the given polygon.



61. Geometry: Area of a Polygon Another approach for finding the area of a polygon by using determinants is to use the formula

$$A = \frac{1}{2}\left(\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix} + \cdots + \begin{vmatrix} x_n & y_n \\ x_1 & y_1 \end{vmatrix}\right)$$

where $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ are the n corner points in counterclockwise order. Use this formula to compute the area of the polygon from Problem 60 again. Which method do you prefer?

62. Show that the formula in Problem 61 yields the same result as the formula used in Problem 59.

63. Geometry: Equation of a Circle An equation of the circle containing the distinct points (x_1, y_1), (x_2, y_2), and (x_3, y_3) can be found using the following equation.

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ x & x_1 & x_2 & x_3 \\ y & y_1 & y_2 & y_3 \\ x^2+y^2 & x_1^2+y_1^2 & x_2^2+y_2^2 & x_3^2+y_3^2 \end{vmatrix} = 0$$

Find the equation of the circle containing the points $(7, -5)$, $(3, 3)$, and $(6, 2)$. Write the equation in standard form.

64. Show that $\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} = (y-z)(x-y)(x-z)$.

65. Complete the proof of Cramer's Rule for two equations containing two variables.

[**Hint:** In system (5), page 587, if $a = 0$, then $b \neq 0$ and $c \neq 0$, since $D = -bc \neq 0$. Now show that equation (6) provides a solution of the system when $a = 0$. Then three cases remain: $b = 0$, $c = 0$, and $d = 0$.]

66. Interchange columns 1 and 3 of a 3 by 3 determinant. Show that the value of the new determinant is -1 times the value of the original determinant.

67. Multiply each entry in row 2 of a 3 by 3 determinant by the number k, $k \neq 0$. Show that the value of the new determinant is k times the value of the original determinant.

68. Prove that a 3 by 3 determinant in which the entries in column 1 equal those in column 3 has the value 0.

69. Prove that if row 2 of a 3 by 3 determinant is multiplied by k, $k \neq 0$, and the result is added to the entries in row 1, there is no change in the value of the determinant.

Retain Your Knowledge

Problems 70–73 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

70. Find the real zeros of $f(x) = 3x^2 - 10x + 5$.

71. List the potential rational zeros of the polynomial function $P(x) = 2x^3 - 5x^2 + x - 10$.

72. Graph $f(x) = (x+1)^2 - 4$ using transformations (shifting, compressing, stretching, and/or reflecting).

73. Convert $5^x = y$ to an equivalent statement involving a logarithm.

8.4 Matrix Algebra

OBJECTIVES
1 Find the Sum and Difference of Two Matrices (p. 596)
2 Find Scalar Multiples of a Matrix (p. 598)
3 Find the Product of Two Matrices (p. 599)
4 Find the Inverse of a Matrix (p. 603)
5 Solve a System of Linear Equations Using an Inverse Matrix (p. 606)

Section 8.2 defined a matrix as a rectangular array of real numbers and used an augmented matrix to represent a system of linear equations. There is, however, a branch of mathematics, called **linear algebra**, that deals with matrices in such a way that an algebra of matrices is permitted. This section provides a survey of how this **matrix algebra** is developed.

Before getting started, recall the definition of a matrix.

DEFINITION

A **matrix** is a rectangular array of numbers:

$$\begin{array}{c c}
 & \begin{array}{cccccc} \text{Column 1} & \text{Column 2} & & \text{Column } j & & \text{Column } n \end{array} \\
\begin{array}{c} \text{Row 1} \\ \text{Row 2} \\ \vdots \\ \text{Row } i \\ \vdots \\ \text{Row } m \end{array} &
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}
\end{array}$$

Each number a_{ij} of the matrix has two indexes: the **row index** i and the **column index** j. The matrix shown here has m rows and n columns. The numbers a_{ij} are the **entries** of the matrix. For example, a_{23} refers to the entry in the second row, third column.

EXAMPLE 1 **Arranging Data in a Matrix**

In a survey of 900 people, the following information was obtained:

200 males	Thought federal defense spending was too high
150 males	Thought federal defense spending was too low
45 males	Had no opinion
315 females	Thought federal defense spending was too high
125 females	Thought federal defense spending was too low
65 females	Had no opinion

We can arrange these data in a rectangular array as follows:

	Too High	Too Low	No Opinion
Male	200	150	45
Female	315	125	65

or as the matrix

$$\begin{bmatrix} 200 & 150 & 45 \\ 315 & 125 & 65 \end{bmatrix}$$

This matrix has two rows (representing male and female) and three columns (representing "too high," "too low," and "no opinion"). ■

The matrix developed in Example 1 has 2 rows and 3 columns. In general, a matrix with m rows and n columns is called an $\boldsymbol{m}$ **by** $\boldsymbol{n}$ **matrix**. The matrix developed in Example 1 is a 2 by 3 matrix and contains $2 \cdot 3 = 6$ entries. An m by n matrix contains $m \cdot n$ entries.

If an m by n matrix has the same number of rows as columns, that is, if $m = n$, then the matrix is a **square matrix**.

EXAMPLE 2 Examples of Matrices

(a) $\begin{bmatrix} 5 & 0 \\ -6 & 1 \end{bmatrix}$ **A 2 by 2 square matrix** (b) $[1 \quad 0 \quad 3]$ **A 1 by 3 matrix**

(c) $\begin{bmatrix} 6 & -2 & 4 \\ 4 & 3 & 5 \\ 8 & 0 & 1 \end{bmatrix}$ **A 3 by 3 square matrix** ■

1 Find the Sum and Difference of Two Matrices

We begin our discussion of matrix algebra by first defining equivalent matrices and then defining the operations of addition and subtraction. It is important to note that these definitions require both matrices to have the same number of rows *and* the same number of columns as a condition for equality and for addition and subtraction.

Matrices usually are represented by capital letters, such as A, B, and C.

DEFINITION

Two matrices A and B are **equal**, written as

$$\boxed{A = B}$$

provided that A and B have the same number of rows and the same number of columns and each entry a_{ij} in A is equal to the corresponding entry b_{ij} in B. ■

For example,

$$\begin{bmatrix} 2 & 1 \\ 0.5 & -1 \end{bmatrix} = \begin{bmatrix} \sqrt{4} & 1 \\ \frac{1}{2} & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} \sqrt{9} & \sqrt{4} & 1 \\ 0 & 1 & \sqrt[3]{-8} \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 \\ 6 & 1 \end{bmatrix} \neq \begin{bmatrix} 4 & 0 \\ 6 & 1 \end{bmatrix}$$ **Because the entries in row 1, column 2 are not equal**

$$\begin{bmatrix} 4 & 1 & 2 \\ 6 & 1 & 2 \end{bmatrix} \neq \begin{bmatrix} 4 & 1 & 2 & 3 \\ 6 & 1 & 2 & 4 \end{bmatrix}$$ **Because the matrix on the left has 3 columns and the matrix on the right has 4 columns**

Suppose that A and B represent two m by n matrices. The **sum,** $\boldsymbol{A} + \boldsymbol{B}$, is defined as the m by n matrix formed by adding the corresponding entries a_{ij} of A and b_{ij} of B. The **difference,** $\boldsymbol{A} - \boldsymbol{B}$, is defined as the m by n matrix formed by subtracting the entries b_{ij} in B from the corresponding entries a_{ij} in A. Addition and subtraction of matrices are allowed only for matrices having the same number m of rows and the same number n of columns. For example, a 2 by 3 matrix and a 2 by 4 matrix cannot be added or subtracted.

EXAMPLE 3 Adding and Subtracting Matrices

Suppose that

$$A = \begin{bmatrix} 2 & 4 & 8 & -3 \\ 0 & 1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 4 & 0 & 1 \\ 6 & 8 & 2 & 0 \end{bmatrix}$$

Find: (a) $A + B$ (b) $A - B$

Algebraic Solution

(a) $$A + B = \begin{bmatrix} 2 & 4 & 8 & -3 \\ 0 & 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} -3 & 4 & 0 & 1 \\ 6 & 8 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2+(-3) & 4+4 & 8+0 & -3+1 \\ 0+6 & 1+8 & 2+2 & 3+0 \end{bmatrix}$$ **Add corresponding entries.**

$$= \begin{bmatrix} -1 & 8 & 8 & -2 \\ 6 & 9 & 4 & 3 \end{bmatrix}$$

(b) $$A - B = \begin{bmatrix} 2 & 4 & 8 & -3 \\ 0 & 1 & 2 & 3 \end{bmatrix} - \begin{bmatrix} -3 & 4 & 0 & 1 \\ 6 & 8 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2-(-3) & 4-4 & 8-0 & -3-1 \\ 0-6 & 1-8 & 2-2 & 3-0 \end{bmatrix}$$ **Subtract corresponding entries.**

$$= \begin{bmatrix} 5 & 0 & 8 & -4 \\ -6 & -7 & 0 & 3 \end{bmatrix}$$

Graphing Solution

Enter the matrices into a graphing utility. Name them $[A]$ and $[B]$. Figure 11 shows the results of adding and subtracting $[A]$ and $[B]$.

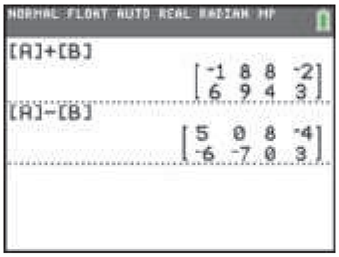


Figure 11 Matrix addition and subtraction

Now Work PROBLEM 9

Many of the algebraic properties of sums of real numbers are also true for sums of matrices. Suppose that A, B, and C are m by n matrices. Then matrix addition is **commutative**. That is,

Commutative Property of Matrix Addition

$$A + B = B + A$$

Matrix addition is also **associative**. That is,

Associative Property of Matrix Addition

$$(A + B) + C = A + (B + C)$$

Although we shall not prove these results, the proofs, as the following example illustrates, are based on the commutative and associative properties for real numbers.

EXAMPLE 4 Demonstrating the Commutative Property

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & 0 & 7 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 1 \\ 5 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 2+(-1) & 3+2 & -1+1 \\ 4+5 & 0+(-3) & 7+4 \end{bmatrix}$$

$$= \begin{bmatrix} -1+2 & 2+3 & 1+(-1) \\ 5+4 & -3+0 & 4+7 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 & 1 \\ 5 & -3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 & -1 \\ 4 & 0 & 7 \end{bmatrix}$$

A matrix whose entries are all equal to 0 is called a **zero matrix**. Each of the following matrices is a zero matrix.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \textbf{2 by 2 square zero matrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \textbf{2 by 3 zero matrix} \qquad \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \quad \textbf{1 by 3 zero matrix}$$

Zero matrices have properties similar to the real number 0. If A is an m by n matrix and 0 is the m by n zero matrix, then

$$\boxed{A + 0 = 0 + A = A}$$

In other words, a zero matrix is the additive identity in matrix algebra.

2 Find Scalar Multiples of a Matrix

We can also multiply a matrix by a real number. If k is a real number and A is an m by n matrix, the matrix kA is the m by n matrix formed by multiplying each entry a_{ij} in A by k. The number k is sometimes referred to as a **scalar**, and the matrix kA is called a **scalar multiple** of A.

EXAMPLE 5 Operations Using Matrices

Suppose that

$$A = \begin{bmatrix} 3 & 1 & 5 \\ -2 & 0 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 1 & 0 \\ 8 & 1 & -3 \end{bmatrix} \qquad C = \begin{bmatrix} 9 & 0 \\ -3 & 6 \end{bmatrix}$$

Find: (a) $4A$ (b) $\frac{1}{3}C$ (c) $3A - 2B$

Algebraic Solution

(a) $$4A = 4\begin{bmatrix} 3 & 1 & 5 \\ -2 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 4\cdot 3 & 4\cdot 1 & 4\cdot 5 \\ 4(-2) & 4\cdot 0 & 4\cdot 6 \end{bmatrix} = \begin{bmatrix} 12 & 4 & 20 \\ -8 & 0 & 24 \end{bmatrix}$$

(b) $$\frac{1}{3}C = \frac{1}{3}\begin{bmatrix} 9 & 0 \\ -3 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}\cdot 9 & \frac{1}{3}\cdot 0 \\ \frac{1}{3}(-3) & \frac{1}{3}\cdot 6 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$$

(c) $$3A - 2B = 3\begin{bmatrix} 3 & 1 & 5 \\ -2 & 0 & 6 \end{bmatrix} - 2\begin{bmatrix} 4 & 1 & 0 \\ 8 & 1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3\cdot 3 & 3\cdot 1 & 3\cdot 5 \\ 3(-2) & 3\cdot 0 & 3\cdot 6 \end{bmatrix} - \begin{bmatrix} 2\cdot 4 & 2\cdot 1 & 2\cdot 0 \\ 2\cdot 8 & 2\cdot 1 & 2(-3) \end{bmatrix}$$

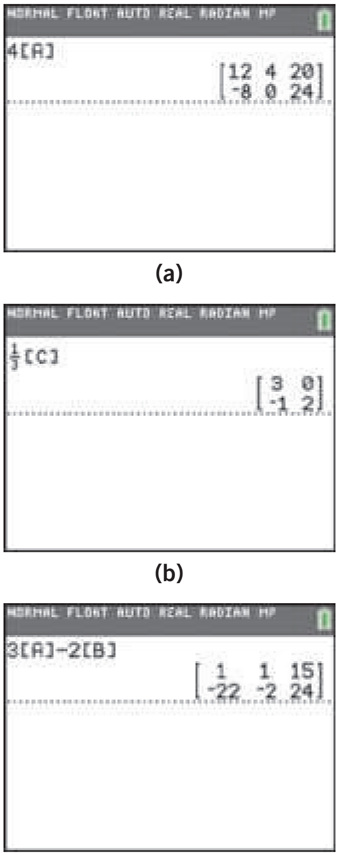
$$= \begin{bmatrix} 9 & 3 & 15 \\ -6 & 0 & 18 \end{bmatrix} - \begin{bmatrix} 8 & 2 & 0 \\ 16 & 2 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 9-8 & 3-2 & 15-0 \\ -6-16 & 0-2 & 18-(-6) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 15 \\ -22 & -2 & 24 \end{bmatrix}$$

Graphing Solution

Enter the matrices $[A]$, $[B]$, and $[C]$ into a graphing utility. Figure 12 shows the required computations.



(a) (b) (c)

Figure 12

Now Work PROBLEM 13

Some of the algebraic properties of scalar multiplication are listed next. Let h and k be real numbers, and let A and B be m by n matrices. Then

Properties of Scalar Multiplication

$$k(hA) = (kh)A$$
$$(k + h)A = kA + hA$$
$$k(A + B) = kA + kB$$

3 Find the Product of Two Matrices

Unlike the straightforward definition for adding two matrices, the definition for multiplying two matrices is not what might be expected. In preparation for this definition, we need the following definitions:

DEFINITION

A **row vector** $\boldsymbol{R}$ is a 1 by n matrix

$$R = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix}$$

A **column vector** $\boldsymbol{C}$ is an n by 1 matrix

$$C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

The **product** $\boldsymbol{RC}$ of R times C is defined as the number

$$RC = \begin{bmatrix} r_1 & r_2 \cdots r_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = r_1c_1 + r_2c_2 + \cdots + r_nc_n$$

Notice that a row vector and a column vector can be multiplied only if they contain the same number of entries.

EXAMPLE 6 The Product of a Row Vector and a Column Vector

If $R = \begin{bmatrix} 3 & -5 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix}$, then

$$RC = \begin{bmatrix} 3 & -5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix} = 3 \cdot 3 + (-5)4 + 2(-5) = 9 - 20 - 10 = -21$$

EXAMPLE 7 Using Matrices to Compute Revenue

A clothing store sells men's shirts for \$40, silk ties for \$20, and wool suits for \$400. Last month, the store had sales consisting of 100 shirts, 200 ties, and 50 suits. What was the total revenue due to these sales?

Solution Set up a row vector R to represent the prices of these three items and a column vector C to represent the corresponding number of items sold. Then

$$R = \begin{array}{c} \textbf{Prices} \\ \begin{array}{ccc} \textbf{Shirts} & \textbf{Ties} & \textbf{Suits} \end{array} \\ \begin{bmatrix} 40 & 20 & 400 \end{bmatrix} \end{array} \qquad C = \begin{array}{c} \textbf{Number sold} \\ \begin{bmatrix} 100 \\ 200 \\ 50 \end{bmatrix} \end{array} \begin{array}{l} \\ \textbf{Shirts} \\ \textbf{Ties} \\ \textbf{Suits} \end{array}$$

The total revenue obtained is the product RC. That is,

$$RC = \begin{bmatrix} 40 & 20 & 400 \end{bmatrix} \begin{bmatrix} 100 \\ 200 \\ 50 \end{bmatrix}$$

$$= \underbrace{40 \cdot 100}_{\textbf{Shirt revenue}} + \underbrace{20 \cdot 200}_{\textbf{Tie revenue}} + \underbrace{400 \cdot 50}_{\textbf{Suit revenue}} = \underbrace{\$28{,}000}_{\textbf{Total revenue}}$$

The definition for multiplying two matrices is based on the definition of a row vector times a column vector.

DEFINITION

Let A denote an m by r matrix and let B denote an r by n matrix. The **product** AB is defined as the m by n matrix whose entry in row i, column j is the product of the ith row of A and the jth column of B.

The definition of the product AB of two matrices A and B, in this order, requires that the number of columns of A equal the number of rows of B; otherwise, no product is defined.

In Words
To find the product AB, the number of columns in A must equal the number of rows in B.

$$\begin{array}{ccc} A & & B \\ m \text{ by } r & & r \text{ by } n \end{array}$$

Must be same for AB to be defined

AB is m by n.

An example will help to clarify the definition.

EXAMPLE 8 Multiplying Two Matrices

Find the product AB if

$$A = \begin{bmatrix} 2 & 4 & -1 \\ 5 & 8 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 5 & 1 & 4 \\ 4 & 8 & 0 & 6 \\ -3 & 1 & -2 & -1 \end{bmatrix}$$

Solution First, observe that A is 2 by 3 and B is 3 by 4. The number of columns in A equals the number of rows in B, so the product AB is defined and will be a 2 by 4 matrix.

Algebraic Solution

Suppose that we want the entry in row 2, column 3 of AB. To find it, find the product of the row vector from row 2 of A and the column vector from column 3 of B.

$$\begin{array}{cc} & \textbf{Column 3 of } B \\ \textbf{Row 2 of } A & \\ \begin{bmatrix} 5 & 8 & 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \end{array} = 5 \cdot 1 + 8 \cdot 0 + 0(-2) = 5$$

Graphing Solution

Enter the matrices A and B into a graphing utility. Figure 13 shows the product AB.

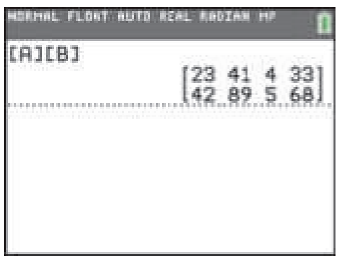


Figure 13 Matrix multiplication

So far, we have

$$AB = \begin{array}{c} \text{Column 3} \\ \downarrow \\ \begin{bmatrix} \_\_ & \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & 5 & \_\_ \end{bmatrix} \end{array} \leftarrow \text{Row 2}$$

Now, to find the entry in row 1, column 4 of AB, find the product of row 1 of A and column 4 of B.

$$\underset{\text{Row 1 of } A}{\begin{bmatrix} 2 & 4 & -1 \end{bmatrix}} \overset{\text{Column 4 of } B}{\begin{bmatrix} 4 \\ 6 \\ -1 \end{bmatrix}} = 2 \cdot 4 + 4 \cdot 6 + (-1)(-1) = 33$$

Continuing in this fashion, we find AB.

$$AB = \begin{bmatrix} 2 & 4 & -1 \\ 5 & 8 & 0 \end{bmatrix} \begin{bmatrix} 2 & 5 & 1 & 4 \\ 4 & 8 & 0 & 6 \\ -3 & 1 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \text{Row 1 of } A \text{ times column 1 of } B & \text{Row 1 of } A \text{ times column 2 of } B & \text{Row 1 of } A \text{ times column 3 of } B & \text{Row 1 of } A \text{ times column 4 of } B \\ \text{Row 2 of } A \text{ times column 1 of } B & \text{Row 2 of } A \text{ times column 2 of } B & \text{Row 2 of } A \text{ times column 3 of } B & \text{Row 2 of } A \text{ times column 4 of } B \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cdot 2 + 4 \cdot 4 + (-1)(-3) & 2 \cdot 5 + 4 \cdot 8 + (-1)1 & 2 \cdot 1 + 4 \cdot 0 + (-1)(-2) & 33 \text{ (from earlier)} \\ 5 \cdot 2 + 8 \cdot 4 + 0(-3) & 5 \cdot 5 + 8 \cdot 8 + 0 \cdot 1 & 5 \text{ (from earlier)} & 5 \cdot 4 + 8 \cdot 6 + 0(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 23 & 41 & 4 & 33 \\ 42 & 89 & 5 & 68 \end{bmatrix}$$

■

Now Work PROBLEM 27

Notice that for the matrices given in Example 8, the product BA is not defined because B is 3 by 4 and A is 2 by 3.

EXAMPLE 9 Multiplying Two Matrices

If

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$$

find: (a) AB (b) BA

Solution

(a) $$AB = \underset{2 \text{ by } 3}{\begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}} \underset{3 \text{ by } 2}{\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}} = \underset{2 \text{ by } 2}{\begin{bmatrix} 13 & 7 \\ -1 & -1 \end{bmatrix}}$$

(b) $$BA = \underset{3 \text{ by } 2}{\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}} \underset{2 \text{ by } 3}{\begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}} = \underset{3 \text{ by } 3}{\begin{bmatrix} 2 & 1 & 3 \\ 5 & 1 & 6 \\ 8 & 1 & 9 \end{bmatrix}}$$

■

Notice in Example 9 that AB is 2 by 2 and BA is 3 by 3. It is possible for both AB and BA to be defined and yet be unequal. In fact, even if A and B are both n by n matrices so that AB and BA are each defined and n by n, AB and BA will usually be unequal.

EXAMPLE 10 Multiplying Two Square Matrices

If

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix}$$

find: (a) AB (b) BA

Solution

(a) $$AB = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ 4 & 8 \end{bmatrix}$$

(b) $$BA = \begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -6 & 1 \\ 2 & 9 \end{bmatrix}$$

The preceding examples demonstrate that an important property of real numbers, the commutative property of multiplication, is not shared by matrices. In general:

THEOREM

Matrix multiplication is not commutative.

Now Work PROBLEMS 15 AND 19

Next, consider two of the properties of real numbers that *are* shared by matrices. Assuming that each product and sum is defined, the following is true:

Associative Property of Matrix Multiplication

$$\boxed{A(BC) = (AB)C}$$

Distributive Property

$$\boxed{A(B + C) = AB + AC}$$

For an n by n square matrix, the entries located in row i, column i, $1 \le i \le n$, are called the **diagonal entries** or **the main diagonal**. The n by n square matrix whose diagonal entries are 1's, and all other entries are 0's, is called the **identity matrix** $\boldsymbol{I_n}$. For example,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and so on.

EXAMPLE 11 Multiplication with an Identity Matrix

Let

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 \\ 4 & 6 \\ 5 & 2 \end{bmatrix}$$

Find: (a) AI_3 (b) I_2A (c) BI_2

Solution

(a) $$AI_3 = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} = A$$

(b) $$I_2A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} = A$$

(c) $$BI_2 = \begin{bmatrix} 3 & 2 \\ 4 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 6 \\ 5 & 2 \end{bmatrix} = B$$

Example 11 demonstrates the following property:

Identity Property

If A is an m by n matrix, then

$$I_mA = A \quad \text{and} \quad AI_n = A$$

If A is an n by n square matrix,

$$AI_n = I_nA = A$$

An identity matrix has properties similar to those of the real number 1. In other words, the identity matrix is a multiplicative identity in matrix algebra.

4 Find the Inverse of a Matrix

DEFINITION

Let A be a square n by n matrix. If there exists an n by n matrix A^{-1} (read as "A inverse") for which

$$AA^{-1} = A^{-1}A = I_n$$

then A^{-1} is called the **inverse** of the matrix A.

Note: If the determinant of A is zero, A is singular. (Refer to Section 8.3.)

Not every square matrix has an inverse. When a matrix A does have an inverse A^{-1}, then A is said to be **nonsingular**. If a matrix A has no inverse, it is called **singular**.

EXAMPLE 12 Multiplying a Matrix by Its Inverse

Show that the inverse of

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{is} \quad A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

Solution We need to show that $AA^{-1} = A^{-1}A = I_2$.

$$AA^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$A^{-1}A = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

The following discussion illustrates one way to find the inverse of

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

Suppose that A^{-1} is given by

$$A^{-1} = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \tag{1}$$

where x, y, z, and w are four variables. Based on the definition of an inverse, if A has an inverse, then

$$AA^{-1} = I_2$$
$$\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 3x + z & 3y + w \\ 2x + z & 2y + w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Because corresponding entries must be equal, it follows that this matrix equation is equivalent to two systems of linear equations.

$$\begin{cases} 3x + z = 1 \\ 2x + z = 0 \end{cases} \qquad \begin{cases} 3y + w = 0 \\ 2y + w = 1 \end{cases}$$

The augmented matrix of each system is

$$\left[\begin{array}{cc|c} 3 & 1 & 1 \\ 2 & 1 & 0 \end{array}\right] \qquad \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 2 & 1 & 1 \end{array}\right] \tag{2}$$

The usual procedure would be to transform each augmented matrix into reduced row echelon form. Notice, though, that the left sides of the augmented matrices are equal, so the same row operations (see Section 8.2) can be used to reduce each one. It is more efficient to combine the two augmented matrices (2) into a single matrix:

$$\left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right]$$

Next, use row operations to transform the matrix into reduced row echelon form.

$$\left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \underset{R_1 = -1r_2 + r_1}{\longrightarrow} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 2 & 1 & 0 & 1 \end{array}\right]$$

$$\underset{R_2 = -2r_1 + r_2}{\longrightarrow} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 3 \end{array}\right] \tag{3}$$

Matrix (3) is in reduced row echelon form.

Now reverse the earlier step of combining the two augmented matrices in (2), and write the single matrix (3) as two augmented matrices.

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \end{array}\right] \text{ and } \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 3 \end{array}\right]$$

The conclusion from these matrices is that $x = 1, z = -2$, and $y = -1, w = 3$. Substituting these values into matrix (1) results in

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

Notice in display (3) that the 2 by 2 matrix to the right of the vertical bar is, in fact, the inverse of A. Also notice that the identity matrix I_2 is the matrix that appears to the left of the vertical bar. These observations and the procedures used to get display (3) will work in general.

In Words

If A is nonsingular, begin with the matrix $[A|I_n]$, and after transforming it into reduced row echelon form, you end up with the matrix $[I_n|A^{-1}]$.

Procedure for Finding the Inverse of a Nonsingular Matrix*

To find the inverse of an n by n nonsingular matrix A, proceed as follows:

STEP 1: Form the matrix $[A|I_n]$.

STEP 2: Transform the matrix $[A|I_n]$ into reduced row echelon form.

STEP 3: The reduced row echelon form of $[A|I_n]$ contains the identity matrix I_n on the left of the vertical bar; the n by n matrix on the right of the vertical bar is the inverse of A.

EXAMPLE 13 Finding the Inverse of a Matrix

The matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{bmatrix}$$

is nonsingular. Find its inverse.

Algebraic Solution

First, form the matrix

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 3 & 4 & 0 & 1 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array}\right]$$

Next, use row operations to transform $[A|I_3]$ into reduced row echelon form.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 3 & 4 & 0 & 1 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array}\right] \underset{R_2 = r_1 + r_2}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 4 & 4 & 1 & 1 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array}\right] \underset{R_2 = \frac{1}{4}r_2}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{\substack{R_1 = -1r_2 + r_1 \\ R_3 = -4r_2 + r_3}}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right] \underset{R_3 = -1r_3}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

$$\underset{\substack{R_1 = r_3 + r_1 \\ R_2 = -1r_3 + r_2}}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{4} & \frac{3}{4} & -1 \\ 0 & 1 & 0 & -\frac{3}{4} & -\frac{3}{4} & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

Graphing Solution

Enter the matrix A into a graphing utility. Figure 14 shows A^{-1}.

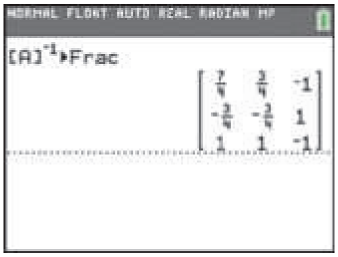


Figure 14 Inverse matrix

*For 2 by 2 matrices, there is a simple formula that can be used. See Problem 93.

The matrix $[A|I_3]$ is now in reduced row echelon form, and the identity matrix I_3 is on the left of the vertical bar. The inverse of A is

$$A^{-1} = \begin{bmatrix} \frac{7}{4} & \frac{3}{4} & -1 \\ -\frac{3}{4} & -\frac{3}{4} & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

You should verify that this is the correct inverse by showing that

$$AA^{-1} = A^{-1}A = I_3.$$

Now Work PROBLEM 37

If transforming the matrix $[A|I_n]$ into reduced row echelon form does not result in the identity matrix I_n to the left of the vertical bar, A is singular and has no inverse.

EXAMPLE 14 Showing That a Matrix Has No Inverse

Show that the matrix $A = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$ has no inverse.

Algebraic Solution

Begin by writing the matrix $[A|I_2]$.

$$[A|I_2] = \left[\begin{array}{cc|cc} 4 & 6 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right]$$

Then use row operations to transform $[A|I_2]$ into reduced row echelon form.

$$[A|I_2] = \left[\begin{array}{cc|cc} 4 & 6 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \xrightarrow[R_1 = \frac{1}{4}r_1]{} \left[\begin{array}{cc|cc} 1 & \frac{3}{2} & \frac{1}{4} & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \xrightarrow[R_2 = -2r_1 + r_2]{} \left[\begin{array}{cc|cc} 1 & \frac{3}{2} & \frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{2} & 1 \end{array}\right]$$

The matrix $[A|I_2]$ is sufficiently reduced for it to be clear that the identity matrix cannot appear to the left of the vertical bar, so A is singular and has no inverse.

Graphing Solution

Enter the matrix A. Figure 15 shows the result of trying to find its inverse. The ERROR comes about because A is singular.

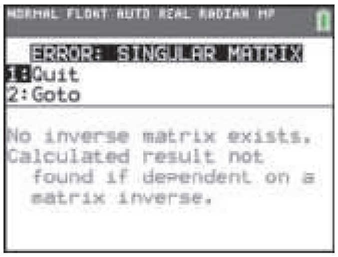


Figure 15

Seeing the Concept

Compute the determinant of A in Example 14 using a graphing utility. What is the result? Are you surprised?

Now Work PROBLEM 65

5 Solve a System of Linear Equations Using an Inverse Matrix

Inverse matrices can be used to solve systems of equations in which the number of equations is the same as the number of variables.

EXAMPLE 15 **Using the Inverse Matrix to Solve a System of Linear Equations**

Solve the system of equations: $\begin{cases} x + y = 3 \\ -x + 3y + 4z = -3 \\ 4y + 3z = 2 \end{cases}$

Solution Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix}$$

Then the original system of equations can be written compactly as the matrix equation

$$AX = B \tag{4}$$

From Example 13, the matrix A has the inverse A^{-1}. Multiply each side of equation (4) by A^{-1}.

$$\begin{aligned} AX &= B \\ A^{-1}(AX) &= A^{-1}B && \text{Multiply both sides by } A^{-1}. \\ (A^{-1}A)X &= A^{-1}B && \text{Associative Property of multiplication} \\ I_3X &= A^{-1}B && \text{Definition of an inverse matrix} \\ X &= A^{-1}B && \text{Property of the identity matrix} \end{aligned} \tag{5}$$

Now use (5) to find $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. This can be done either algebraically or with a graphing utility.

Algebraic Solution

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underset{\uparrow \atop \text{Example 13}}{A^{-1}B} = \begin{bmatrix} \frac{7}{4} & \frac{3}{4} & -1 \\ -\frac{3}{4} & -\frac{3}{4} & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

Graphing Solution

Enter the matrices A and B into a graphing utility. Figure 16 shows the solution to the system of equations.

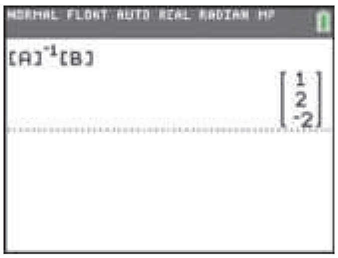


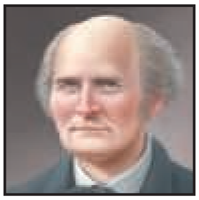
Figure 16

The solution is $x = 1$, $y = 2$, $z = -2$ or, using an ordered triplet, $(1, 2, -2)$. ■

The method used in Example 15 to solve a system of equations is particularly useful when it is necessary to solve several systems of equations in which the constants appearing to the right of the equal signs change, while the coefficients of the variables on the left side remain the same. See Problems 45–64 for some illustrations. Be careful; this method can be used only if the inverse exists. If it does not exist, row reduction must be used since the system is either inconsistent or dependent.

Now Work PROBLEM 49

Historical Feature



Arthur Cayley (1821–1895)

Matrices were invented in 1857 by Arthur Cayley (1821–1895) as a way of efficiently computing the result of substituting one linear system into another (see Historical Problem 3). The resulting system had incredible richness, in the sense that a wide variety of mathematical systems could be mimicked by the matrices. Cayley and his friend James J. Sylvester (1814–1897) spent much of the rest of their lives elaborating the theory. The torch was then passed to Georg Frobenius (1849–1917), whose deep investigations established a central place for matrices in modern mathematics. In 1924, rather to the surprise of physicists, it was found that matrices (with complex numbers in them) were exactly the right tool for describing the behavior of atomic systems. Today, matrices are used in a wide variety of applications.

Historical Problems

1. **Matrices and Complex Numbers** Frobenius emphasized in his research how matrices could be used to mimic other mathematical systems. Here, we mimic the behavior of complex numbers using matrices. Mathematicians call such a relationship an *isomorphism.*

Complex number $\longleftrightarrow$ Matrix

$$a + bi \longleftrightarrow \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

Note that the complex number can be read off the top line of the matrix. Then

$$2 + 3i \longleftrightarrow \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 & -2 \\ 2 & 4 \end{bmatrix} \longleftrightarrow 4 - 2i$$

(a) Find the matrices corresponding to $2 - 5i$ and $1 + 3i$.

(b) Multiply the two matrices.

(c) Find the corresponding complex number for the matrix found in part (b).

(d) Multiply $2 - 5i$ and $1 + 3i$. The result should be the same as that found in part (c).

The process also works for addition and subtraction. Try it for yourself.

2. Compute $(a + bi)(a - bi)$ using matrices. Interpret the result.

3. **Cayley's Definition of Matrix Multiplication** Cayley devised matrix multiplication to simplify the following problem:

$$\begin{cases} u = ar + bs \\ v = cr + ds \end{cases} \qquad \begin{cases} x = ku + lv \\ y = mu + nv \end{cases}$$

(a) Find x and y in terms of r and s by substituting u and v from the first system of equations into the second system of equations.

(b) Use the result of part (a) to find the 2 by 2 matrix A in

$$\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} r \\ s \end{bmatrix}$$

(c) Now look at the following way to do it. Write the equations in matrix form.

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k & l \\ m & n \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

so

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k & l \\ m & n \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}$$

Do you see how Cayley defined matrix multiplication?

8.4 Assess Your Understanding

Concepts and Vocabulary

1. A matrix that has the same number of rows as columns is called a(n) __________ matrix.
2. ***True or False*** Matrix addition is commutative.
3. ***True or False*** If A and B are square matrices, then $AB = BA$,
4. Suppose that A is a square n by n matrix that is nonsingular. The matrix B for which $AB = BA = I_n$ is called the __________ of the matrix A.
5. ***True or False*** The identity matrix has properties similar to those of the real number 1.
6. If $AX = B$ represents a matrix equation where A is a nonsingular matrix, then we can solve the equation using $X =$ ______.
7. To find the product AB of two matrices A and B, which of the following must be true?
   (a) The number of columns in A must equal the number of rows in B.
   (b) The number of rows in A must equal the number of columns in B.
   (c) A and B must have the same number of rows and the same number of columns.
   (d) A and B must both be square matrices.
8. A matrix that has no inverse is called which of the following?
   (a) zero matrix (b) nonsingular matrix
   (c) identity matrix (d) singular matrix

Skill Building

In Problems 9–26, use the following matrices. Determine whether the given expression is defined. If it is defined, express the result as a single matrix; if it is not, say "not defined."

$$A = \begin{bmatrix} 0 & 3 & -5 \\ 1 & 2 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 1 & 0 \\ -2 & 3 & -2 \end{bmatrix} \quad C = \begin{bmatrix} 4 & 1 \\ 6 & 2 \\ -2 & 3 \end{bmatrix}$$

9. $A + B$ **10.** $A - B$ **11.** $4A$ **12.** $-3B$

13. $3A - 2B$ **14.** $2A + 4B$ **15.** AC **16.** BC

17. AB **18.** BA **19.** CA **20.** CB

21. $C(A + B)$ **22.** $(A + B)C$ **23.** $AC - 3I_2$ **24.** $CA + 5I_3$

25. $CA - CB$ **26.** $AC + BC$

In Problems 27–34, determine whether the product is defined. If it is defined, find the product; if it is not, say "not defined."

27. $\begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 & 4 & 6 \\ 3 & -1 & 3 & 2 \end{bmatrix}$ **28.** $\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -6 & 6 & 1 & 0 \\ 2 & 5 & 4 & -1 \end{bmatrix}$ **29.** $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 4 \end{bmatrix}$ **30.** $\begin{bmatrix} 1 & -1 \\ -3 & 2 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 2 & 8 & -1 \\ 3 & 6 & 0 \end{bmatrix}$

31. $\begin{bmatrix} -4 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$ **32.** $\begin{bmatrix} 2 & -1 \\ 5 & 8 \\ -6 & 0 \end{bmatrix}\begin{bmatrix} 6 & 4 & 2 \\ -3 & 5 & -1 \\ 9 & 0 & 7 \end{bmatrix}$ **33.** $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 4 & 1 \\ 3 & 6 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 6 & 2 \\ 8 & -1 \end{bmatrix}$ **34.** $\begin{bmatrix} 4 & -2 & 3 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 6 \\ 1 & -1 \\ 0 & 2 \end{bmatrix}$

In Problems 35–44, each matrix is nonsingular. Find the inverse of each matrix.

35. $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ **36.** $\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$ **37.** $\begin{bmatrix} 6 & 5 \\ 2 & 2 \end{bmatrix}$ **38.** $\begin{bmatrix} -4 & 1 \\ 6 & -2 \end{bmatrix}$ **39.** $\begin{bmatrix} 2 & 1 \\ a & a \end{bmatrix}$ $a \neq 0$

40. $\begin{bmatrix} b & 3 \\ b & 2 \end{bmatrix}$ $b \neq 0$ **41.** $\begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{bmatrix}$ **42.** $\begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix}$ **43.** $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & -1 \\ 3 & 1 & 2 \end{bmatrix}$ **44.** $\begin{bmatrix} 3 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & -1 & 1 \end{bmatrix}$

In Problems 45–64, use the inverses found in Problems 35–44 to solve each system of equations.

45. $\begin{cases} 2x + y = 8 \\ x + y = 5 \end{cases}$ **46.** $\begin{cases} 3x - y = 8 \\ -2x + y = 4 \end{cases}$ **47.** $\begin{cases} 2x + y = 0 \\ x + y = 5 \end{cases}$ **48.** $\begin{cases} 3x - y = 4 \\ -2x + y = 5 \end{cases}$

49. $\begin{cases} 6x + 5y = 7 \\ 2x + 2y = 2 \end{cases}$ **50.** $\begin{cases} -4x + y = 0 \\ 6x - 2y = 14 \end{cases}$ **51.** $\begin{cases} 6x + 5y = 13 \\ 2x + 2y = 5 \end{cases}$ **52.** $\begin{cases} -4x + y = 5 \\ 6x - 2y = -9 \end{cases}$

53. $\begin{cases} 2x + y = -3 \\ ax + ay = -a \end{cases}$ $a \neq 0$ **54.** $\begin{cases} bx + 3y = 2b + 3 \\ bx + 2y = 2b + 2 \end{cases}$ $b \neq 0$ **55.** $\begin{cases} 2x + y = \dfrac{7}{a} \\ ax + ay = 5 \end{cases}$ $a \neq 0$ **56.** $\begin{cases} bx + 3y = 14 \\ bx + 2y = 10 \end{cases}$ $b \neq 0$

57. $\begin{cases} x - y + z = 0 \\ -2y + z = -1 \\ -2x - 3y = -5 \end{cases}$ **58.** $\begin{cases} x + 2z = 6 \\ -x + 2y + 3z = -5 \\ x - y = 6 \end{cases}$ **59.** $\begin{cases} x - y + z = 2 \\ -2y + z = 2 \\ -2x - 3y = \dfrac{1}{2} \end{cases}$ **60.** $\begin{cases} x + 2z = 2 \\ -x + 2y + 3z = -\dfrac{3}{2} \\ x - y = 2 \end{cases}$

61. $\begin{cases} x + y + z = 9 \\ 3x + 2y - z = 8 \\ 3x + y + 2z = 1 \end{cases}$ **62.** $\begin{cases} 3x + 3y + z = 8 \\ x + 2y + z = 5 \\ 2x - y + z = 4 \end{cases}$ **63.** $\begin{cases} x + y + z = 2 \\ 3x + 2y - z = \dfrac{7}{3} \\ 3x + y + 2z = \dfrac{10}{3} \end{cases}$ **64.** $\begin{cases} 3x + 3y + z = 1 \\ x + 2y + z = 0 \\ 2x - y + z = 4 \end{cases}$

In Problems 65–70, show that each matrix has no inverse.

65. $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$ **66.** $\begin{bmatrix} -3 & \frac{1}{2} \\ 6 & -1 \end{bmatrix}$ **67.** $\begin{bmatrix} 15 & 3 \\ 10 & 2 \end{bmatrix}$

68. $\begin{bmatrix} -3 & 0 \\ 4 & 0 \end{bmatrix}$ **69.** $\begin{bmatrix} -3 & 1 & -1 \\ 1 & -4 & -7 \\ 1 & 2 & 5 \end{bmatrix}$ **70.** $\begin{bmatrix} 1 & 1 & -3 \\ 2 & -4 & 1 \\ -5 & 7 & 1 \end{bmatrix}$

In Problems 71–74, use a graphing utility to find the inverse, if it exists, of each matrix. Round answers to two decimal places.

71. $\begin{bmatrix} 25 & 61 & -12 \\ 18 & -2 & 4 \\ 8 & 35 & 21 \end{bmatrix}$

72. $\begin{bmatrix} 18 & -3 & 4 \\ 6 & -20 & 14 \\ 10 & 25 & -15 \end{bmatrix}$

73. $\begin{bmatrix} 44 & 21 & 18 & 6 \\ -2 & 10 & 15 & 5 \\ 21 & 12 & -12 & 4 \\ -8 & -16 & 4 & 9 \end{bmatrix}$

74. $\begin{bmatrix} 16 & 22 & -3 & 5 \\ 21 & -17 & 4 & 8 \\ 2 & 8 & 27 & 20 \\ 5 & 15 & -3 & -10 \end{bmatrix}$

In Problems 75–78, use the idea behind Example 15 with a graphing utility to solve the following systems of equations. Round answers to two decimal places.

75. $\begin{cases} 25x + 61y - 12z = 10 \\ 18x - 12y + 7y = -9 \\ 3x + 4y - z = 12 \end{cases}$

76. $\begin{cases} 25x + 61y - 12z = 15 \\ 18x - 12y + 7z = -3 \\ 3x + 4y - z = 12 \end{cases}$

77. $\begin{cases} 25x + 61y - 12z = 21 \\ 18x - 12y + 7z = 7 \\ 3x + 4y - z = -2 \end{cases}$

78. $\begin{cases} 25x + 61y - 12z = 25 \\ 18x - 12y + 7z = 10 \\ 3x + 4y - z = -4 \end{cases}$

Mixed Practice

In Problems 79–86, algebraically solve each system of equations using any method you wish.

79. $\begin{cases} 2x + 3y = 11 \\ 5x + 7y = 24 \end{cases}$

80. $\begin{cases} 2x + 8y = -8 \\ x + 7y = -13 \end{cases}$

81. $\begin{cases} x - 2y + 4z = 2 \\ -3x + 5y - 2z = 17 \\ 4x - 3y = -22 \end{cases}$

82. $\begin{cases} 2x + 3y - z = -2 \\ 4x + 3z = 6 \\ 6y - 2z = 2 \end{cases}$

83. $\begin{cases} 5x - y + 4z = 2 \\ -x + 5y - 4z = 3 \\ 7x + 13y - 4z = 17 \end{cases}$

84. $\begin{cases} 3x + 2y - z = 2 \\ 2x + y + 6z = -7 \\ 2x + 2y - 14z = 17 \end{cases}$

85. $\begin{cases} 2x - 3y + z = 4 \\ -3x + 2y - z = -3 \\ -5y + z = 6 \end{cases}$

86. $\begin{cases} -4x + 3y + 2z = 6 \\ 3x + y - z = -2 \\ x + 9y + z = 6 \end{cases}$

Applications and Extensions

87. College Tuition Nikki and Joe take classes at a community college, LCCC, and a local university, SIUE. The number of credit hours taken and the cost per credit hour (2015–2016 academic year, tuition and approximate fees) are as follows:

	LCCC	SIUE
Nikki	6	9
Joe	3	12

	Cost per Credit Hour
LCCC	\$128.00
SIUE	\$341.60

(a) Write a matrix A for the credit hours taken by each student and a matrix B for the cost per credit hour.
(b) Compute AB and interpret the results.

Sources: *lc.edu, siue.edu*

88. School Loan Interest Jamal and Stephanie both have school loans issued from the same two banks. The amounts borrowed and the monthly interest rates are given next (interest is compounded monthly).

	Lender 1	Lender 2
Jamal	\$4000	\$3000
Stephanie	\$2500	\$3800

	Monthly Interest Rate
Lender 1	0.011 (1.1%)
Lender 2	0.006 (0.6%)

(a) Write a matrix A for the amounts borrowed by each student and a matrix B for the monthly interest rates.
(b) Compute AB and interpret the result.
(c) Let $C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Compute $A(C + B)$ and interpret the result.

89. Computing the Cost of Production The Acme Steel Company is a producer of stainless steel and aluminum containers. On a certain day, the following stainless steel containers were manufactured: 500 with 10-gallon (gal) capacity, 350 with 5-gal capacity, and 400 with 1-gal capacity. On the same day, the following aluminum containers were manufactured: 700 with 10-gal capacity, 500 with 5-gal capacity, and 850 with 1-gal capacity.
(a) Find a 2 by 3 matrix representing these data. Find a 3 by 2 matrix to represent the same data.
(b) If the amount of material used in the 10-gal containers is 15 pounds (lb), the amount used in the 5-gal containers is 8 lb, and the amount used in the 1-gal containers is 3 lb, find a 3 by 1 matrix representing the amount of material used.
(c) Multiply the 2 by 3 matrix found in part (a) and the 3 by 1 matrix found in part (b) to get a 2 by 1 matrix showing the day's usage of material.
(d) If stainless steel costs Acme \$0.10 per pound and aluminum costs \$0.05 per pound, find a 1 by 2 matrix representing cost.
(e) Multiply the matrices found in parts (c) and (d) to determine the total cost of the day's production.

90. Computing Profit Rizza Ford has two locations, one in the city and the other in the suburbs. In January, the city location sold 400 subcompacts, 250 intermediate-size cars, and 50 SUVs; in February, it sold 350 subcompacts, 100 intermediates, and 30 SUVs. At the suburban location in January, 450 subcompacts, 200 intermediates, and 140 SUVs were sold. In February, the suburban location sold 350 subcompacts, 300 intermediates, and 100 SUVs.
(a) Find 2 by 3 matrices that summarize the sales data for each location for January and February (one matrix for each month).

(b) Use matrix addition to obtain total sales for the 2-month period.

(c) The profit on each kind of car is \$100 per subcompact, \$150 per intermediate, and \$200 per SUV. Find a 3 by 1 matrix representing this profit.

(d) Multiply the matrices found in parts (b) and (c) to get a 2 by 1 matrix showing the profit at each location.

91. Cryptography One method of encryption is to use a matrix to encrypt the message and then use the corresponding inverse matrix to decode the message. The encrypted matrix, E, is obtained by multiplying the message matrix, M, by a key matrix, K. The original message can be retrieved by multiplying the encrypted matrix by the inverse of the key matrix. That is, $E = M \cdot K$ and $M = E \cdot K^{-1}$.

(a) Given the key matrix $K = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, find its inverse, K^{-1}. [**Note:** This key matrix is known as the Q_2^3 Fibonacci encryption matrix.]

(b) Use your result from part (a) to decode the encrypted matrix $E = \begin{bmatrix} 47 & 34 & 33 \\ 44 & 36 & 27 \\ 47 & 41 & 20 \end{bmatrix}$.

(c) Each entry in your result for part (b) represents the position of a letter in the English alphabet $(A = 1, B = 2, C = 3$, and so on $)$. What is the original message?

Source: *goldenmuseum.com*

92. Economic Mobility The relative income of a child (low, medium, or high) generally depends on the relative income of the child's parents. The matrix P, given by

$$\begin{array}{c} \text{Parent's Income} \\ \begin{array}{ccc} \text{L} & \text{M} & \text{H} \end{array} \end{array}$$

$$P = \begin{bmatrix} 0.4 & 0.2 & 0.1 \\ 0.5 & 0.6 & 0.5 \\ 0.1 & 0.2 & 0.4 \end{bmatrix} \begin{array}{l} \text{L} \\ \text{M} \\ \text{H} \end{array} \quad \text{Child's income}$$

is called a *left stochastic transition matrix*. For example, the entry $P_{21} = 0.5$ means that 50% of the children of low-relative-income parents will transition to the medium level of income. The diagonal entry P_{ii} represents the percent of children who remain in the same income level as their parents. Assuming that the transition matrix is valid from one generation to the next, compute and interpret P^2.

Source: *Understanding Mobility in America, April 2006*

93. Consider the 2 by 2 square matrix

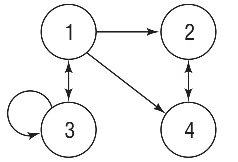
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If $D = ad - bc \neq 0$, show that A is nonsingular and that

$$A^{-1} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Use the following discussion for Problems 94 and 95.

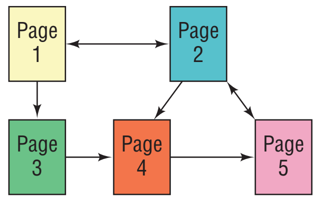
In graph theory, an ***adjacency matrix****, A, is a way of representing which nodes (or vertices) are connected. For a simple directed graph, each entry, a_{ij}, is either 1 (if a direct path exists from node i to node j) or 0 (if no direct path exists from node i to node j). For example, consider the following graph and corresponding adjacency matrix.*



$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The entry a_{14} is 1 because a direct path exists from node 1 to node 4. However, the entry a_{41} is 0 because no path exists from node 4 to node 1. The entry a_{33} is 1 because a direct path exists from node 3 to itself. The matrix $B_k = A + A^2 + \cdots + A^k$ indicates the number of ways to get from node i to node j within k moves (steps).

94. Website Map A content map can be used to show how different pages on a website are connected. For example, the following content map shows the relationship among the five pages of a certain website with links between pages represented by arrows.



The content map can be represented by a 5 by 5 adjacency matrix where each entry, a_{ij}, is either 1 (if a link exists from page i to page j) or 0 (if no link exists from page i to page j).

(a) Write the 5 by 5 adjacency matrix that represents the given content map.

(b) Explain the significance of the entries on the main diagonal in your result from part (a).

(c) Find and interpret A^2.

95. Three-Click Rule An unofficial, and often contested, guideline for website design is to make all website content available to a user within three clicks. The webpage adjacency matrix for a certain website is given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find B_3. Does this website adhere to the Three-Click Rule?

(b) Which page can be reached the greatest number of ways from page 1 within three clicks?

96. Computer Graphics: Translating An important aspect of computer graphics is the ability to transform the coordinates of points within a graphic. For transformation purposes, a point (x, y) is represented as the column matrix $X = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$.

To translate a point (x, y) horizontally h units and vertically k units, we use the translation matrix $S = \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$ and compute the matrix product SX. The translation is to the right for $h > 0$ and to the left for $h < 0$. Likewise, the translation is up for $k > 0$ and down for $k < 0$. The transformed coordinates are the first two entries in the resulting column matrix.

(a) Write the translation matrix needed to translate a point 3 units to the left and 5 units up.

(b) Find and interpret S^{-1}.

97. Computer Graphics: Rotating Besides translating a point, it is also important in computer graphics to be able to rotate a point. This is achieved by multiplying a point's column matrix (see Problem 96) by an appropriate rotation matrix R to form the matrix product RX. For example, to rotate a point 60°, the rotation matrix is $R = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) Write the coordinates of the point $(6, 4)$ after it has been rotated 60°.

(b) Find and interpret R^{-1}.

Explaining Concepts: Discussion and Writing

98. Create a situation different from any found in the text that can be represented by a matrix.

99. Explain why the number of columns in matrix A must equal the number of rows in matrix B when finding the product AB.

100. If $a, b,$ and $c \neq 0$ are real numbers with $ac = bc$, then $a = b$. Does this same property hold for matrices? In other words, if A, B, and C are matrices and $AC = BC$, must $A = B$?

101. What is the solution of the system of equations $AX = 0$ if A^{-1} exists? Discuss the solution of $AX = 0$ if A^{-1} does not exist.

Retain Your Knowledge

Problems 102–105 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

102. Write a polynomial with minimum degree and leading coefficient 1 that has zeros $x = 3$ (multiplicity 2), $x = 0$ (multiplicity 3), and $x = -2$ (multiplicity 1).

103. A function $f(x)$ has as an average rate of change of $\frac{3}{8}$ over the interval [0, 12]. If $f(0) = \frac{1}{2}$, find $f(12)$.

104. Solve: $\dfrac{5x}{x+2} = \dfrac{x}{x-2}$

105. The demand function for a certain product is $D(x) = 3500 - x^2, 0 \le x \le 59.16$, where x is the price per unit and D is in thousands of units sold. Determine the number of units sold if the price per unit is \$32.

8.5 Partial Fraction Decomposition

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Identity (Section 1.2, p. 99)
- Proper and Improper Rational Functions (Section 5.4, p. 377)
- Factoring Polynomials (Chapter R, Review, Section R.5, pp. 51–56)
- Complex Zeros; Fundamental Theorem of Algebra (Section 5.3, pp. 366–370)

Now Work the **'Are You Prepared?'** problems on page 618.

OBJECTIVES
1. Decompose $\frac{P}{Q}$ Where Q Has Only Nonrepeated Linear Factors (p. 613)
2. Decompose $\frac{P}{Q}$ Where Q Has Repeated Linear Factors (p. 615)
3. Decompose $\frac{P}{Q}$ Where Q Has a Nonrepeated Irreducible Quadratic Factor (p. 617)
4. Decompose $\frac{P}{Q}$ Where Q Has a Repeated Irreducible Quadratic Factor (p. 618)

Consider the problem of adding two rational expressions:

$$\frac{3}{x+4} \quad \text{and} \quad \frac{2}{x-3}$$

The result is

$$\frac{3}{x+4} + \frac{2}{x-3} = \frac{3(x-3) + 2(x+4)}{(x+4)(x-3)} = \frac{5x-1}{x^2+x-12}$$

The reverse procedure, starting with the rational expression $\frac{5x-1}{x^2+x-12}$ and writing it as the sum (or difference) of the two simpler fractions $\frac{3}{x+4}$ and $\frac{2}{x-3}$, is referred to as **partial fraction decomposition**, and the two simpler fractions are called **partial fractions**. Decomposing a rational expression into a sum of partial fractions is important in solving certain types of calculus problems. This section presents a systematic way to decompose rational expressions.

Recall that a rational expression is the ratio of two polynomials, say P and $Q \neq 0$. Assume that P and Q have no common factors. Recall also that a rational expression $\frac{P}{Q}$ is called **proper** if the degree of the polynomial in the numerator is less than the degree of the polynomial in the denominator. Otherwise, the rational expression is called **improper**.

Because any improper rational expression can be reduced by long division to a mixed form consisting of the sum of a polynomial and a proper rational expression, we shall restrict the discussion that follows to proper rational expressions.

The partial fraction decomposition of the rational expression $\frac{P}{Q}$, in lowest terms, depends on the factors of the denominator Q. Recall from Section 5.3 that any polynomial whose coefficients are real numbers can be factored (over the real numbers) into products of linear and/or irreducible quadratic factors. This means that the denominator Q of the rational expression $\frac{P}{Q}$ will contain only factors of one or both of the following types:

- *Linear factors* of the form $x - a$, where a is a real number.
- *Irreducible quadratic factors* of the form $ax^2 + bx + c$, where $a, b,$ and c are real numbers, $a \neq 0$, and $b^2 - 4ac < 0$ (which guarantees that $ax^2 + bx + c$ cannot be written as the product of two linear factors with real coefficients).

As it turns out, there are four cases to be examined. We begin with the case for which Q has only nonrepeated linear factors. Throughout we assume the rational expression $\frac{P}{Q}$ is in lowest terms.

1 Decompose $\frac{P}{Q}$ Where Q Has Only Nonrepeated Linear Factors

Case 1: *Q* has only nonrepeated linear factors.

Under the assumption that Q has only nonrepeated linear factors, the polynomial Q has the form

$$\boxed{Q(x) = (x - a_1)(x - a_2) \cdot \cdots \cdot (x - a_n)}$$

where no two of the numbers $a_1, a_2, \ldots, a_n$ are equal. In this case, the partial fraction decomposition of $\frac{P}{Q}$ is of the form

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_n}{x-a_n} \tag{1}$$

where the numbers $A_1, A_2, \ldots, A_n$ are to be determined.

The example shows how to find these numbers.

EXAMPLE 1 Nonrepeated Linear Factors

Find the partial fraction decomposition of $\dfrac{x}{x^2 - 5x + 6}$.

Solution First, factor the denominator,

$$x^2 - 5x + 6 = (x - 2)(x - 3)$$

and notice that the denominator contains only nonrepeated linear factors. Then decompose the rational expression according to equation (1):

$$\frac{x}{x^2 - 5x + 6} = \frac{A}{x - 2} + \frac{B}{x - 3} \qquad \textbf{(2)}$$

where A and B are to be determined. To find A and B, clear the fractions by multiplying each side by $(x - 2)(x - 3) = x^2 - 5x + 6$. The result is

$$x = A(x - 3) + B(x - 2) \qquad \textbf{(3)}$$

or

$$x = (A + B)x + (-3A - 2B)$$

This equation is an identity in x. Equate the coefficients of like powers of x to get

$$\begin{cases} 1 = A + B & \textbf{Equate the coefficients of } x\textbf{: } 1x = (A + B)x. \\ 0 = -3A - 2B & \textbf{Equate the constants: } 0 = -3A - 2B. \end{cases}$$

This system of two equations containing two variables, A and B, can be solved using whatever method you wish. Solving it yields

$$A = -2 \qquad B = 3$$

From equation (2), the partial fraction decomposition is

$$\frac{x}{x^2 - 5x + 6} = \frac{-2}{x - 2} + \frac{3}{x - 3}$$

Check: The decomposition can be checked by adding the rational expressions.

$$\frac{-2}{x - 2} + \frac{3}{x - 3} = \frac{-2(x - 3) + 3(x - 2)}{(x - 2)(x - 3)} = \frac{x}{(x - 2)(x - 3)}$$
$$= \frac{x}{x^2 - 5x + 6}$$

■

The numbers to be found in the partial fraction decomposition can sometimes be found more readily by using suitable choices for x (which may include complex numbers) in the identity obtained after fractions have been cleared. In Example 1, the identity after clearing fractions is equation (3):

$$x = A(x - 3) + B(x - 2)$$

Let $x = 2$ in this expression, and the term containing B drops out, leaving $2 = A(-1)$, or $A = -2$. Similarly, let $x = 3$, and the term containing A drops out, leaving $3 = B$. As before, $A = -2$ and $B = 3$.

Now Work PROBLEM 13

2 Decompose $\frac{P}{Q}$ Where Q Has Repeated Linear Factors

Case 2: *Q* has repeated linear factors.

If the polynomial Q has a repeated linear factor, say $(x-a)^n$, $n \geq 2$ an integer, then, in the partial fraction decomposition of $\frac{P}{Q}$, allow for the terms

$$\boxed{\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}}$$

where the numbers $A_1, A_2, \ldots, A_n$ are to be determined.

EXAMPLE 2 **Repeated Linear Factors**

Find the partial fraction decomposition of $\dfrac{x+2}{x^3 - 2x^2 + x}$.

Solution First, factor the denominator,

$$x^3 - 2x^2 + x = x(x^2 - 2x + 1) = x(x-1)^2$$

and notice that the denominator has the nonrepeated linear factor x and the repeated linear factor $(x-1)^2$. By Case 1, the term $\frac{A}{x}$ must be in the decomposition; and by Case 2, the terms $\frac{B}{x-1} + \frac{C}{(x-1)^2}$ must be in the decomposition.

Now write

$$\frac{x+2}{x^3 - 2x^2 + x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \tag{4}$$

Again, clear fractions by multiplying each side by $x^3 - 2x^2 + x = x(x-1)^2$. The result is the identity

$$x + 2 = A(x-1)^2 + Bx(x-1) + Cx \tag{5}$$

Let $x = 0$ in this expression and the terms containing B and C drop out, leaving $2 = A(-1)^2$, or $A = 2$. Similarly, let $x = 1$, and the terms containing A and B drop out, leaving $3 = C$. Then equation (5) becomes

$$x + 2 = 2(x-1)^2 + Bx(x-1) + 3x$$

Let $x = 2$ (any choice other than 0 or 1 will work as well). The result is

$$\begin{aligned} 4 &= 2(1)^2 + B(2)(1) + 3(2) \\ 4 &= 2 + 2B + 6 \\ 2B &= -4 \\ B &= -2 \end{aligned}$$

Therefore, $A = 2$, $B = -2$, and $C = 3$.

From equation (4), the partial fraction decomposition is

$$\frac{x+2}{x^3 - 2x^2 + x} = \frac{2}{x} + \frac{-2}{x-1} + \frac{3}{(x-1)^2}$$

EXAMPLE 3 **Repeated Linear Factors**

Find the partial fraction decomposition of $\dfrac{x^3 - 8}{x^2(x-1)^3}$.

Solution The denominator contains the repeated linear factors x^2 and $(x-1)^3$. The partial fraction decomposition takes the form

$$\frac{x^3 - 8}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3} \tag{6}$$

As before, clear fractions and obtain the identity

$$x^3 - 8 = Ax(x-1)^3 + B(x-1)^3 + Cx^2(x-1)^2 + Dx^2(x-1) + Ex^2 \tag{7}$$

Let $x = 0$. (Do you see why this choice was made?) Then

$$\begin{aligned} -8 &= B(-1) \\ B &= 8 \end{aligned}$$

Let $x = 1$ in equation (7). Then

$$-7 = E$$

Use $B = 8$ and $E = -7$ in equation (7), and collect like terms.

$$\begin{aligned} x^3 - 8 &= Ax(x-1)^3 + 8(x-1)^3 \\ &\quad + Cx^2(x-1)^2 + Dx^2(x-1) - 7x^2 \\ x^3 - 8 - 8(x^3 - 3x^2 + 3x - 1) + 7x^2 &= Ax(x-1)^3 \\ &\quad + Cx^2(x-1)^2 + Dx^2(x-1) \\ -7x^3 + 31x^2 - 24x &= x(x-1)[A(x-1)^2 + Cx(x-1) + Dx] \\ x(x-1)(-7x+24) &= x(x-1)[A(x-1)^2 + Cx(x-1) + Dx] \\ -7x + 24 &= A(x-1)^2 + Cx(x-1) + Dx \end{aligned} \tag{8}$$

Now work with equation (8). Let $x = 0$. Then

$$24 = A$$

Let $x = 1$ in equation (8). Then

$$17 = D$$

Use $A = 24$ and $D = 17$ in equation (8).

$$-7x + 24 = 24(x-1)^2 + Cx(x-1) + 17x$$

Let $x = 2$ and simplify.

$$\begin{aligned} -14 + 24 &= 24 + C(2) + 34 \\ -48 &= 2C \\ -24 &= C \end{aligned}$$

The numbers A, B, C, D, and E are all now known. So, from equation (6),

$$\frac{x^3 - 8}{x^2(x-1)^3} = \frac{24}{x} + \frac{8}{x^2} + \frac{-24}{x-1} + \frac{17}{(x-1)^2} + \frac{-7}{(x-1)^3}$$

Now Work Example 3 by solving the system of five equations containing five variables that the expansion of equation (7) leads to.

Now Work PROBLEM 19

The final two cases involve irreducible quadratic factors. A quadratic factor is irreducible if it cannot be factored into linear factors with real coefficients.

A quadratic expression $ax^2 + bx + c$ is irreducible whenever $b^2 - 4ac < 0$. For example, $x^2 + x + 1$ and $x^2 + 4$ are irreducible.

3 Decompose $\dfrac{P}{Q}$ Where Q Has a Nonrepeated Irreducible Quadratic Factor

Case 3: Q contains a nonrepeated irreducible quadratic factor.

If Q contains a nonrepeated irreducible quadratic factor of the form $ax^2 + bx + c$, then, in the partial fraction decomposition of $\dfrac{P}{Q}$, allow for the term

$$\boxed{\frac{Ax + B}{ax^2 + bx + c}}$$

where the numbers A and B are to be determined.

EXAMPLE 4 **Nonrepeated Irreducible Quadratic Factor**

Find the partial fraction decomposition of $\dfrac{3x - 5}{x^3 - 1}$.

Solution Factor the denominator,

$$x^3 - 1 = (x - 1)(x^2 + x + 1)$$

Notice that it has a nonrepeated linear factor $x - 1$ and a nonrepeated irreducible quadratic factor $x^2 + x + 1$. Allow for the term $\dfrac{A}{x - 1}$ by Case 1, and allow for the term $\dfrac{Bx + C}{x^2 + x + 1}$ by Case 3. Then

$$\frac{3x - 5}{x^3 - 1} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1} \tag{9}$$

Multiply each side of equation (9) by $x^3 - 1 = (x - 1)(x^2 + x + 1)$ to obtain the identity

$$3x - 5 = A(x^2 + x + 1) + (Bx + C)(x - 1) \tag{10}$$

Expand the identity in (10) to obtain

$$3x - 5 = (A + B)x^2 + (A - B + C)x + (A - C)$$

This identity leads to the system of equations

$$\begin{cases} A + B \qquad\;\; = 0 & (1) \\ A - B + C = 3 & (2) \\ A \qquad - C = -5 & (3) \end{cases}$$

The solution of this system is $A = -\dfrac{2}{3}$, $B = \dfrac{2}{3}$, $C = \dfrac{13}{3}$. Then, from equation (9),

$$\frac{3x - 5}{x^3 - 1} = \frac{-\frac{2}{3}}{x - 1} + \frac{\frac{2}{3}x + \frac{13}{3}}{x^2 + x + 1}$$

■

Now Work Example 4 using equation (10) and assigning values to x.

Now Work PROBLEM 21

4 Decompose $\frac{P}{Q}$ Where Q Has a Repeated Irreducible Quadratic Factor

Case 4: Q contains a repeated irreducible quadratic factor.

If the polynomial Q contains a repeated irreducible quadratic factor $(ax^2 + bx + c)^n$, $n \geq 2$, n an integer, then, in the partial fraction decomposition of $\frac{P}{Q}$, allow for the terms

$$\boxed{\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_nx + B_n}{(ax^2 + bx + c)^n}}$$

where the numbers $A_1, B_1, A_2, B_2, \ldots, A_n, B_n$ are to be determined.

EXAMPLE 5 **Repeated Irreducible Quadratic Factor**

Find the partial fraction decomposition of $\frac{x^3 + x^2}{(x^2 + 4)^2}$.

Solution The denominator contains the repeated irreducible quadratic factor $(x^2 + 4)^2$, so write

$$\frac{x^3 + x^2}{(x^2 + 4)^2} = \frac{Ax + B}{x^2 + 4} + \frac{Cx + D}{(x^2 + 4)^2} \tag{11}$$

Clear fractions to obtain

$$x^3 + x^2 = (Ax + B)(x^2 + 4) + Cx + D$$

Collecting like terms yields the identity

$$x^3 + x^2 = Ax^3 + Bx^2 + (4A + C)x + 4B + D$$

Equating coefficients results in the system

$$\begin{cases} A = 1 \\ B = 1 \\ 4A + C = 0 \\ 4B + D = 0 \end{cases}$$

The solution is $A = 1, B = 1, C = -4, D = -4$. From equation (11),

$$\frac{x^3 + x^2}{(x^2 + 4)^2} = \frac{x + 1}{x^2 + 4} + \frac{-4x - 4}{(x^2 + 4)^2}$$

Now Work PROBLEM 35

8.5 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. ***True or False*** The equation $(x - 1)^2 - 1 = x(x - 2)$ is an example of an identity. (p. 99)
2. ***True or False*** The rational expression $\frac{5x^2 - 1}{x^3 + 1}$ is proper. (p. 377)
3. Factor completely: $3x^4 + 6x^3 + 3x^2$ (pp. 51–56)
4. ***True or False*** Every polynomial with real numbers as coefficients can be factored into products of linear and/or irreducible quadratic factors. (p. 369)

Skill Building

In Problems 5–12, tell whether the given rational expression is proper or improper. If improper, rewrite it as the sum of a polynomial and a proper rational expression.

5. $\dfrac{x}{x^2-1}$ **6.** $\dfrac{5x+2}{x^3-1}$ **7.** $\dfrac{x^2+5}{x^2-4}$ **8.** $\dfrac{3x^2-2}{x^2-1}$

9. $\dfrac{5x^3+2x-1}{x^2-4}$ **10.** $\dfrac{3x^4+x^2-2}{x^3+8}$ **11.** $\dfrac{x(x-1)}{(x+4)(x-3)}$ **12.** $\dfrac{2x(x^2+4)}{x^2+1}$

In Problems 13–46, find the partial fraction decomposition of each rational expression.

13. $\dfrac{4}{x(x-1)}$ **14.** $\dfrac{3x}{(x+2)(x-1)}$ **15.** $\dfrac{1}{x(x^2+1)}$ **16.** $\dfrac{1}{(x+1)(x^2+4)}$

17. $\dfrac{x}{(x-1)(x-2)}$ **18.** $\dfrac{3x}{(x+2)(x-4)}$ **19.** $\dfrac{x^2}{(x-1)^2(x+1)}$ **20.** $\dfrac{x+1}{x^2(x-2)}$

21. $\dfrac{1}{x^3-8}$ **22.** $\dfrac{2x+4}{x^3-1}$ **23.** $\dfrac{x^2}{(x-1)^2(x+1)^2}$ **24.** $\dfrac{x+1}{x^2(x-2)^2}$

25. $\dfrac{x-3}{(x+2)(x+1)^2}$ **26.** $\dfrac{x^2+x}{(x+2)(x-1)^2}$ **27.** $\dfrac{x+4}{x^2(x^2+4)}$ **28.** $\dfrac{10x^2+2x}{(x-1)^2(x^2+2)}$

29. $\dfrac{x^2+2x+3}{(x+1)(x^2+2x+4)}$ **30.** $\dfrac{x^2-11x-18}{x(x^2+3x+3)}$ **31.** $\dfrac{x}{(3x-2)(2x+1)}$ **32.** $\dfrac{1}{(2x+3)(4x-1)}$

33. $\dfrac{x}{x^2+2x-3}$ **34.** $\dfrac{x^2-x-8}{(x+1)(x^2+5x+6)}$ **35.** $\dfrac{x^2+2x+3}{(x^2+4)^2}$ **36.** $\dfrac{x^3+1}{(x^2+16)^2}$

37. $\dfrac{7x+3}{x^3-2x^2-3x}$ **38.** $\dfrac{x^3+1}{x^5-x^4}$ **39.** $\dfrac{x^2}{x^3-4x^2+5x-2}$ **40.** $\dfrac{x^2+1}{x^3+x^2-5x+3}$

41. $\dfrac{x^3}{(x^2+16)^3}$ **42.** $\dfrac{x^2}{(x^2+4)^3}$ **43.** $\dfrac{4}{2x^2-5x-3}$ **44.** $\dfrac{4x}{2x^2+3x-2}$

45. $\dfrac{2x+3}{x^4-9x^2}$ **46.** $\dfrac{x^2+9}{x^4-2x^2-8}$

Mixed Practice

In Problems 47–54, use the division algorithm to rewrite each improper rational expression as the sum of a polynomial and a proper rational expression. Find the partial fraction decomposition of the proper rational expression. Finally, express the improper rational expression as the sum of a polynomial and the partial fraction decomposition.

47. $\dfrac{x^3+x^2-3}{x^2+3x-4}$ **48.** $\dfrac{x^3-3x^2+1}{x^2+5x+6}$ **49.** $\dfrac{x^3}{x^2+1}$ **50.** $\dfrac{x^3+x}{x^2+4}$

51. $\dfrac{x^4-5x^2+x-4}{x^2+4x+4}$ **52.** $\dfrac{x^4+x^3-x+2}{x^2-2x+1}$ **53.** $\dfrac{x^5+x^4-x^2+2}{x^4-2x^2+1}$ **54.** $\dfrac{x^5-x^3+x^2+1}{x^4+6x^2+9}$

Retain Your Knowledge

Problems 55–58 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

55. Credit Card Balance Nick has a credit card balance of \$4200. If the credit card company charges 18% interest compounded daily, and Nick does not make any payments on the account, how long will it take for his balance to double? Round to two decimal places.

56. Given $f(x) = x + 4$ and $g(x) = x^2 - 3x$, find $(g \circ f)(-3)$.

57. Determine whether $f(x) = -3x^2 + 120x + 50$ has a maximum or a minimum value, and then find the value.

58. Given $f(x) = \dfrac{x+1}{x-2}$ and $g(x) = 3x - 4$, find $f \circ g$.

'Are You Prepared?' Answers

1. True **2.** True **3.** $3x^2(x+1)^2$ **4.** True

8.6 Systems of Nonlinear Equations

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Lines (Section 2.2, pp. 173–184)
- Circles (Section 2.3, pp. 189–193)
- Parabolas (Section 7.2, pp. 515–521)
- Ellipses (Section 7.3, pp. 525–533)
- Hyperbolas (Section 7.4, pp. 536–546)

Now Work the **'Are You Prepared?'** problems on page 626.

OBJECTIVES 1 Solve a System of Nonlinear Equations Using Substitution (p. 620)
2 Solve a System of Nonlinear Equations Using Elimination (p. 621)

1 Solve a System of Nonlinear Equations Using Substitution

In Section 8.1 we observed that the solution to a system of linear equations could be found geometrically by determining the point(s) of intersection (if any) of the equations in the system. Similarly, in solving systems of nonlinear equations, the solution(s) also represent(s) the point(s) of intersection (if any) of the graphs of the equations.

There is no general methodology for solving a system of nonlinear equations. Sometimes substitution is best; at other times elimination is best; and sometimes neither of these methods works. Experience and a certain degree of imagination are your allies here.

Before we begin, two comments are in order.

- If the system contains two variables and if the equations in the system are easy to graph, then graph them. By graphing each equation in the system, you can get an idea of how many solutions a system has and approximately where they are located.
- Extraneous solutions can creep in when solving nonlinear systems, so it is imperative to check all apparent solutions.

EXAMPLE 1 **Solving a System of Nonlinear Equations**

Solve the following system of equations:

$$\begin{cases} 3x - y = -2 & \textbf{(1)} \quad \textbf{A line} \\ 2x^2 - y = 0 & \textbf{(2)} \quad \textbf{A parabola} \end{cases}$$

Algebraic Solution Using Substitution

First, notice that the system contains two variables and that we know how to graph each equation by hand. See Figure 17. The system apparently has two solutions.

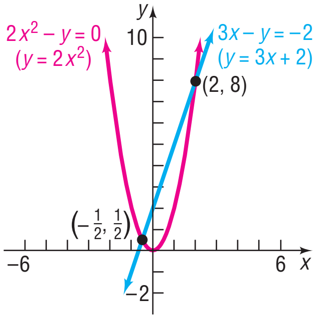


Figure 17

To use substitution to solve the system, we choose to solve equation (1) for y.

$$3x - y = -2 \qquad \textbf{Equation (1)}$$
$$y = 3x + 2$$

Graphing Solution

Use a graphing utility to graph $Y_1 = 3x + 2$ and $Y_2 = 2x^2$. From Figure 18 observe that the system apparently has two solutions. Use INTERSECT to find that the solutions to the system of equations are $(-0.5, 0.5)$ and $(2, 8)$.

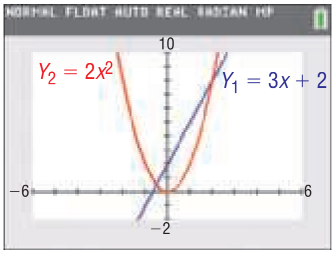


Figure 18

Substitute this expression for y in equation (2). The result is an equation containing just the variable x, which can then be solved.

$$\begin{aligned} 2x^2 - y &= 0 && \textbf{Equation (2)} \\ 2x^2 - (3x + 2) &= 0 && \textbf{Substitute } 3x + 2 \textbf{ for } y. \\ 2x^2 - 3x - 2 &= 0 && \textbf{Remove parentheses.} \\ (2x + 1)(x - 2) &= 0 && \textbf{Factor.} \\ 2x + 1 = 0 \quad \text{or} \quad x - 2 &= 0 && \textbf{Apply the Zero-Product Property.} \\ x = -\frac{1}{2} \quad \text{or } x &= 2 \end{aligned}$$

Use these values for x in $y = 3x + 2$ to find

$$y = 3\left(-\frac{1}{2}\right) + 2 = \frac{1}{2} \quad \text{or} \quad y = 3(2) + 2 = 8$$

The apparent solutions are $x = -\frac{1}{2}, y = \frac{1}{2}$ and $x = 2, y = 8$.

Check: For $x = -\frac{1}{2}, y = \frac{1}{2}$:

$$\begin{cases} 3\left(-\frac{1}{2}\right) - \frac{1}{2} = -\frac{3}{2} - \frac{1}{2} = -2 & \textbf{(1)} \\ 2\left(-\frac{1}{2}\right)^2 - \frac{1}{2} = 2\left(\frac{1}{4}\right) - \frac{1}{2} = 0 & \textbf{(2)} \end{cases}$$

For $x = 2, y = 8$:

$$\begin{cases} 3(2) - 8 = 6 - 8 = -2 & \textbf{(1)} \\ 2(2)^2 - 8 = 2(4) - 8 = 0 & \textbf{(2)} \end{cases}$$

Each solution checks. The graphs of the two equations intersect at the points $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $(2, 8)$, as shown in Figure 17. ■

Now Work PROBLEM 15 USING SUBSTITUTION

2 Solve a System of Nonlinear Equations Using Elimination

EXAMPLE 2 **Solving a System of Nonlinear Equations**

Solve: $\begin{cases} x^2 + y^2 = 13 & \textbf{(1)} \quad \textbf{A circle} \\ x^2 - y = 7 & \textbf{(2)} \quad \textbf{A parabola} \end{cases}$

Algebraic Solution Using Elimination

First graph each equation, as shown in Figure 19 on the next page. Based on the graph, four solutions are expected. Notice that subtracting equation (2) from equation (1) eliminates the variable x.

$$\begin{aligned} &\begin{cases} x^2 + y^2 = 13 \\ x^2 - y = 7 \end{cases} \\ &\overline{\quad y^2 + y = 6 \quad} && \textbf{Subtract.} \end{aligned}$$

Graphing Solution

Use a graphing utility to graph $x^2 + y^2 = 13$ and $x^2 - y = 7$. (Remember that to graph $x^2 + y^2 = 13$ requires two functions, $Y_1 = \sqrt{13 - x^2}$ and $Y_2 = -\sqrt{13 - x^2}$, and a square screen.) From Figure 20 on the next page, observe that the system apparently has four solutions.

This quadratic equation in y can be solved by factoring.

$$y^2 + y - 6 = 0$$
$$(y + 3)(y - 2) = 0$$
$$y = -3 \quad \text{or} \quad y = 2$$

Use these values for y in equation (2) to find x.

If $y = 2$, then $x^2 = y + 7 = 9$, so $x = 3$ or -3.

If $y = -3$, then $x^2 = y + 7 = 4$, so $x = 2$ or -2.

There are four solutions: $x = 3, y = 2$; $x = -3, y = 2$; $x = 2, y = -3$; and $x = -2, y = -3$.

You should verify that, in fact, these four solutions also satisfy equation (1), so all four are solutions of the system. The four points, $(3, 2)$, $(-3, 2)$, $(2, -3)$, and $(-2, -3)$, are the points of intersection of the graphs. Look again at Figure 19.

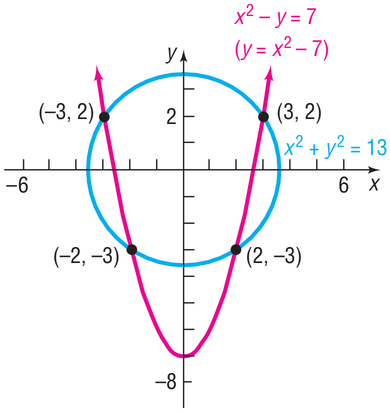


Figure 19

Use INTERSECT to find that the solutions to the system of equations are $(-3, 2)$, $(3, 2)$, $(-2, -3)$, and $(2, -3)$.

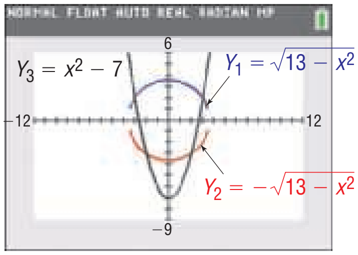


Figure 20

Now Work PROBLEM 13 USING ELIMINATION

EXAMPLE 3 Solving a System of Nonlinear Equations

Solve: $\begin{cases} x^2 - y^2 = 4 & \textbf{(1)} \quad \textbf{A hyperbola} \\ y = x^2 & \textbf{(2)} \quad \textbf{A parabola} \end{cases}$

Algebraic Solution Using Substitution

Either substitution or elimination can be used here. To use substitution, replace x^2 by y in equation (1).

$$x^2 - y^2 = 4 \quad \textbf{Equation (1)}$$
$$y - y^2 = 4 \quad y = x^2$$
$$y^2 - y + 4 = 0 \quad \textbf{Place in standard form.}$$

This is a quadratic equation whose discriminant is $(-1)^2 - 4 \cdot 1 \cdot 4 = 1 - 4 \cdot 4 = -15 < 0$. The equation has no real solutions, so the system is inconsistent. The graphs of these two equations do not intersect. See Figure 21.

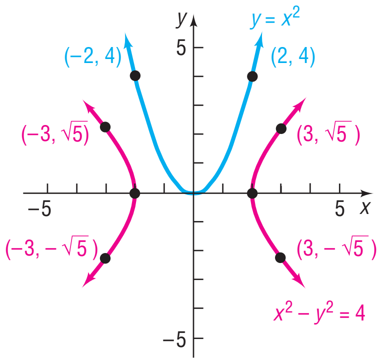


Figure 21

Graphing Solution

Graph $Y_1 = x^2$ and $x^2 - y^2 = 4$, as shown in Figure 22. To graph $x^2 - y^2 = 4$, use two functions:

$$Y_2 = \sqrt{x^2 - 4} \text{ and } Y_3 = -\sqrt{x^2 - 4}$$

From Figure 22, observe that the graphs of these two equations do not intersect. The system is inconsistent.

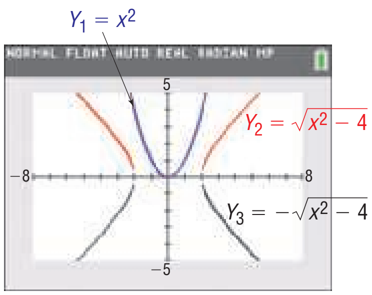


Figure 22

EXAMPLE 4 Solving a System of Nonlinear Equations

$$\text{Solve: } \begin{cases} x^2 + x + y^2 - 3y + 2 = 0 & \textbf{(1)} \\ x + 1 + \dfrac{y^2 - y}{x} = 0 & \textbf{(2)} \end{cases}$$

Algebraic Solution Using Elimination

Because it is not straightforward how to graph the equations in the system, we proceed directly to use the method of elimination.

First, multiply equation (2) by x to eliminate the fraction. The result is an equivalent system because x cannot be 0. [Look at the original equation (2) to see why.]

$$\begin{cases} x^2 + x + y^2 - 3y + 2 = 0 & \textbf{(1)} \\ x^2 + x + y^2 - y = 0 & \textbf{(2)} \quad x \neq 0 \end{cases}$$

Now subtract equation (2) from equation (1) to eliminate x. The result is

$$-2y + 2 = 0$$

$$y = 1 \quad \textbf{Solve for } y.$$

To find x, back-substitute $y = 1$ in equation (1).

$$x^2 + x + y^2 - 3y + 2 = 0 \quad \textbf{Equation (1)}$$

$$x^2 + x + (1)^2 - 3(1) + 2 = 0 \quad \textbf{Substitute 1 for } y \textbf{ in (1).}$$

$$x^2 + x = 0 \quad \textbf{Simplify.}$$

$$x(x+1) = 0 \quad \textbf{Factor.}$$

$$x = 0 \text{ or } x = -1 \quad \textbf{Apply the Zero-Product Property.}$$

Because x cannot be 0, the value $x = 0$ is extraneous, so discard it.

Check: Check $x = -1, y = 1$:

$$\begin{cases} (-1)^2 + (-1) + 1^2 - 3(1) + 2 = 1 - 1 + 1 - 3 + 2 = 0 & \textbf{(1)} \\ -1 + 1 + \dfrac{1^2 - 1}{-1} = 0 + \dfrac{0}{-1} = 0 & \textbf{(2)} \end{cases}$$

The solution is $x = -1, y = 1$. The point of intersection of the graphs of the equations is $(-1, 1)$. ■

Graphing Solution

First, multiply equation (2) by x to eliminate the fraction. The result is an equivalent system because x cannot be 0 [look at the original equation (2) to see why]:

$$\begin{cases} x^2 + x + y^2 - 3y + 2 = 0 & \textbf{(1)} \\ x^2 + x + y^2 - y = 0 & \textbf{(2)} \quad x \neq 0 \end{cases}$$

Solve each equation for y. First, solve equation (1) for y:

$$x^2 + x + y^2 - 3y + 2 = 0 \quad \textbf{Equation (1)}$$

$$y^2 - 3y = -x^2 - x - 2 \quad \textbf{Rearrange so that terms involving } y \textbf{ are on left side.}$$

$$y^2 - 3y + \frac{9}{4} = -x^2 - x - 2 + \frac{9}{4} \quad \textbf{Complete the square involving } y.$$

$$\left(y - \frac{3}{2}\right)^2 = -x^2 - x + \frac{1}{4} \quad \textbf{Factor and simplify.}$$

$$y - \frac{3}{2} = \pm\sqrt{-x^2 - x + \frac{1}{4}} \quad \textbf{Square Root Method}$$

$$y = \frac{3}{2} \pm \sqrt{-x^2 - x + \frac{1}{4}} \quad \textbf{Solve for } y.$$

Now solve equation (2) for y:

$$x^2 + x + y^2 - y = 0 \quad \textbf{Equation (2)}$$

$$y^2 - y = -x^2 - x \quad \textbf{Rearrange so that terms involving } y \textbf{ are on left side.}$$

$$y^2 - y + \frac{1}{4} = -x^2 - x + \frac{1}{4} \quad \textbf{Complete the square involving } y.$$

$$\left(y - \frac{1}{2}\right)^2 = -x^2 - x + \frac{1}{4} \quad \textbf{Factor.}$$

$$y - \frac{1}{2} = \pm\sqrt{-x^2 - x + \frac{1}{4}} \quad \textbf{Square Root Method}$$

$$y = \frac{1}{2} \pm \sqrt{-x^2 - x + \frac{1}{4}} \quad \textbf{Solve for } y.$$

Now graph each equation using a graphing utility. See Figure 23 on the next page.

Use INTERSECT to find that the points of intersection are $(-1, 1)$ and $(0, 1)$. Since $x \neq 0$ [look back at the original equation (2)], the graph of Y_3 has a hole at the point $(0, 1)$ and Y_4 has a hole at $(0, 0)$. The value $x = 0$ is extraneous, so discard it. The only solution is $x = -1, y = 1$.

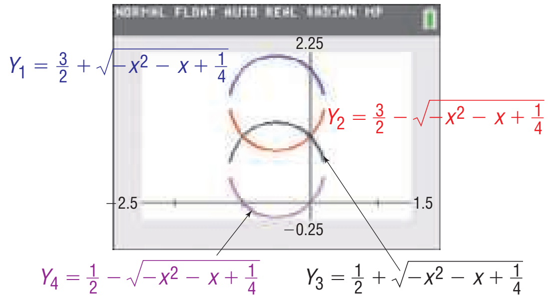


Figure 23

In Problem 55 you are asked to graph the equations given in Example 4 by hand. Be sure to show holes in the graph of equation (2) at $x = 0$.

Now Work PROBLEMS 29 AND 49

EXAMPLE 5 Solving a System of Nonlinear Equations

Solve: $\begin{cases} 3xy - 2y^2 = -2 & \textbf{(1)} \\ 9x^2 + 4y^2 = 10 & \textbf{(2)} \end{cases}$

Algebraic Solution

Multiply equation (1) by 2, and add the result to equation (2) to eliminate the y^2 terms.

$$\begin{cases} 6xy - 4y^2 = -4 & \textbf{(1)} \\ 9x^2 + 4y^2 = 10 & \textbf{(2)} \end{cases}$$

$$9x^2 + 6xy = 6 \quad \textbf{Add.}$$

$$3x^2 + 2xy = 2 \quad \textbf{Divide each side by 3.}$$

Since $x \neq 0$ (do you see why?), solve for y in this equation to get

$$y = \frac{2 - 3x^2}{2x} \quad x \neq 0 \quad \textbf{(3)}$$

Now substitute for y in equation (2) of the system.

$$9x^2 + 4y^2 = 10 \quad \textbf{Equation (2)}$$

$$9x^2 + 4\left(\frac{2 - 3x^2}{2x}\right)^2 = 10 \quad \textbf{Substitute } y = \frac{2 - 3x^2}{2x} \textbf{ in (2).}$$

$$9x^2 + \frac{4 - 12x^2 + 9x^4}{x^2} = 10$$

$$9x^4 + 4 - 12x^2 + 9x^4 = 10x^2 \quad \textbf{Multiply both sides by } x^2.$$

$$18x^4 - 22x^2 + 4 = 0 \quad \textbf{Subtract } 10x^2 \textbf{ from both sides.}$$

$$9x^4 - 11x^2 + 2 = 0 \quad \textbf{Divide both sides by 2.}$$

Graphing Solution

To graph $3xy - 2y^2 = -2$, solve for y. In this instance, it is easier to view the equation as a quadratic equation in the variable y.

$$3xy - 2y^2 = -2$$

$$2y^2 - 3xy - 2 = 0 \quad \textbf{Place in standard form.}$$

$$y = \frac{-(-3x) \pm \sqrt{(-3x)^2 - 4(2)(-2)}}{2(2)}$$

Use the quadratic formula with $a = 2$, $b = -3x$, $c = -2$.

$$y = \frac{3x \pm \sqrt{9x^2 + 16}}{4} \quad \textbf{Simplify.}$$

Using a graphing utility, graph

$Y_1 = \frac{3x + \sqrt{9x^2 + 16}}{4}$ and $Y_2 = \frac{3x - \sqrt{9x^2 + 16}}{4}$.

From equation (2), graph $Y_3 = \frac{\sqrt{10 - 9x^2}}{2}$ and $Y_4 = \frac{-\sqrt{10 - 9x^2}}{2}$. See Figure 24.

This quadratic equation (in x^2) can be factored:

$$(9x^2 - 2)(x^2 - 1) = 0$$

$$9x^2 - 2 = 0 \quad \text{or} \quad x^2 - 1 = 0$$

$$x^2 = \frac{2}{9} \qquad\qquad x^2 = 1$$

$$x = \pm\sqrt{\frac{2}{9}} = \pm\frac{\sqrt{2}}{3} \qquad\qquad x = \pm 1$$

To find y, use equation (3).

$$\text{If } x = \frac{\sqrt{2}}{3}: \quad y = \frac{2 - 3x^2}{2x} = \frac{2 - \frac{2}{3}}{2\left(\frac{\sqrt{2}}{3}\right)} = \frac{4}{2\sqrt{2}} = \sqrt{2}$$

$$\text{If } x = -\frac{\sqrt{2}}{3}: \quad y = \frac{2 - 3x^2}{2x} = \frac{2 - \frac{2}{3}}{2\left(-\frac{\sqrt{2}}{3}\right)} = \frac{4}{-2\sqrt{2}} = -\sqrt{2}$$

$$\text{If } x = 1: \quad y = \frac{2 - 3x^2}{2x} = \frac{2 - 3(1)^2}{2} = -\frac{1}{2}$$

$$\text{If } x = -1: \quad y = \frac{2 - 3x^2}{2x} = \frac{2 - 3(-1)^2}{-2} = \frac{1}{2}$$

The system has four solutions: $\left(\frac{\sqrt{2}}{3}, \sqrt{2}\right)$, $\left(-\frac{\sqrt{2}}{3}, -\sqrt{2}\right)$, $\left(1, -\frac{1}{2}\right)$, and $\left(-1, \frac{1}{2}\right)$. Check them for yourself. ■

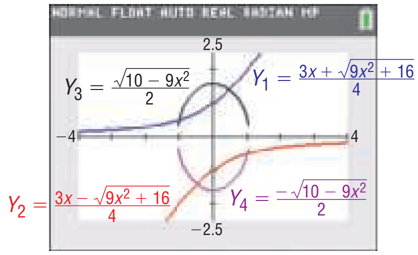


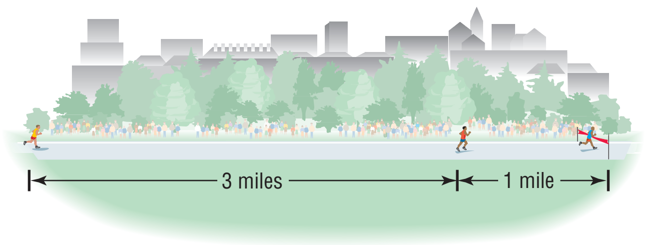
Figure 24

Use INTERSECT to find that the solutions to the system of equations are $(-1, 0.5)$, $(0.47, 1.41)$, $(1, -0.5)$, and $(-0.47, -1.41)$, each rounded to two decimal places. ■

Now Work PROBLEM 47

The next example illustrates an imaginative solution to a system of nonlinear equations.

EXAMPLE 6 Running a Long-Distance Race



In a 50-mile race, the winner crosses the finish line 1 mile ahead of the second-place runner and 4 miles ahead of the third-place runner. Assuming that each runner maintains a constant speed throughout the race, by how many miles does the second-place runner beat the third-place runner?

Solution Let v_1, v_2, and v_3 denote the speeds of the first-, second-, and third-place runners, respectively. Let t_1 and t_2 denote the times (in hours) required for the first-place runner and the second-place runner to finish the race. Then the following system of equations results:

$$\begin{cases} 50 = v_1 t_1 & \text{(1) First-place runner goes 50 miles in } t_1 \text{ hours.} \\ 49 = v_2 t_1 & \text{(2) Second-place runner goes 49 miles in } t_1 \text{ hours.} \\ 46 = v_3 t_1 & \text{(3) Third-place runner goes 46 miles in } t_1 \text{ hours.} \\ 50 = v_2 t_2 & \text{(4) Second-place runner goes 50 miles in } t_2 \text{ hours.} \end{cases}$$

We seek the distance d of the third-place runner from the finish at time t_2. At time t_2, the third-place runner has gone a distance of v_3t_2 miles, so the distance d remaining is $50 - v_3t_2$. Now

$$\begin{aligned}
d &= 50 - v_3t_2 \\
&= 50 - v_3\left(t_1 \cdot \frac{t_2}{t_1}\right) \\
&= 50 - (v_3t_1) \cdot \frac{t_2}{t_1} \\
&= 50 - 46 \cdot \frac{\frac{50}{v_2}}{\frac{50}{v_1}} && \left\{\begin{array}{l}\text{From (3), } v_3t_1 = 46 \\ \text{From (4), } t_2 = \dfrac{50}{v_2} \\ \text{From (1), } t_1 = \dfrac{50}{v_1}\end{array}\right. \\
&= 50 - 46 \cdot \frac{v_1}{v_2} \\
&= 50 - 46 \cdot \frac{50}{49} && \text{From the quotient of (1) and (2).} \\
&\approx 3.06 \text{ miles}
\end{aligned}$$

Historical Feature

In the beginning of this section, it was stated that imagination and experience are important in solving systems of nonlinear equations. Indeed, these kinds of problems lead into some of the deepest and most difficult parts of modern mathematics. Look again at the graphs in Examples 1 and 2 of this section (Figures 17 and 19). Example 1 has two solutions, and Example 2 has four solutions. We might conjecture that the number of solutions is equal to the product of the degrees of the equations involved. This conjecture was indeed made by Étienne Bézout (1730–1783), but working out the details took about 150 years. It turns out that arriving at the correct number of intersections requires counting not only the complex number intersections, but also those intersections that, in a certain sense, lie at infinity. For example, a parabola and a line lying on the axis of the parabola intersect at the vertex and at infinity. This topic is part of the study of algebraic geometry.

Historical Problem

A papyrus dating back to 1950 BC contains the following problem: "A given surface area of 100 units of area shall be represented as the sum of two squares whose sides are to each other as 1 is to $\frac{3}{4}$."

Solve for the sides by solving the system of equations

$$\begin{cases} x^2 + y^2 = 100 \\ x = \frac{3}{4}y \end{cases} \qquad x = 6 \text{ units, } y = 8 \text{ units}$$

8.6 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Graph the equation: $y = 3x + 2$ (pp. 173–184)
2. Graph the equation: $y + 4 = x^2$ (pp. 515–519)
3. Graph the equation: $y^2 = x^2 - 1$ (pp. 536–543)
4. Graph the equation: $x^2 + 4y^2 = 4$ (pp. 525–530)

Skill Building

In Problems 5–24, graph each equation of the system. Then solve the system to find the points of intersection.

5. $\begin{cases} y = x^2 + 1 \\ y = x + 1 \end{cases}$

6. $\begin{cases} y = x^2 + 1 \\ y = 4x + 1 \end{cases}$

7. $\begin{cases} y = \sqrt{36 - x^2} \\ y = 8 - x \end{cases}$

8. $\begin{cases} y = \sqrt{4 - x^2} \\ y = 2x + 4 \end{cases}$

9. $\begin{cases} y = \sqrt{x} \\ y = 2 - x \end{cases}$

10. $\begin{cases} y = \sqrt{x} \\ y = 6 - x \end{cases}$

11. $\begin{cases} x = 2y \\ x = y^2 - 2y \end{cases}$

12. $\begin{cases} y = x - 1 \\ y = x^2 - 6x + 9 \end{cases}$

13. $\begin{cases} x^2 + y^2 = 4 \\ x^2 + 2x + y^2 = 0 \end{cases}$

14. $\begin{cases} x^2 + y^2 = 8 \\ x^2 + y^2 + 4y = 0 \end{cases}$

15. $\begin{cases} y = 3x - 5 \\ x^2 + y^2 = 5 \end{cases}$

16. $\begin{cases} x^2 + y^2 = 10 \\ y = x + 2 \end{cases}$

17. $\begin{cases} x^2 + y^2 = 4 \\ y^2 - x = 4 \end{cases}$ **18.** $\begin{cases} x^2 + y^2 = 16 \\ x^2 - 2y = 8 \end{cases}$ **19.** $\begin{cases} xy = 4 \\ x^2 + y^2 = 8 \end{cases}$ **20.** $\begin{cases} x^2 = y \\ xy = 1 \end{cases}$

21. $\begin{cases} x^2 + y^2 = 4 \\ y = x^2 - 9 \end{cases}$ **22.** $\begin{cases} xy = 1 \\ y = 2x + 1 \end{cases}$ **23.** $\begin{cases} y = x^2 - 4 \\ y = 6x - 13 \end{cases}$ **24.** $\begin{cases} x^2 + y^2 = 10 \\ xy = 3 \end{cases}$

In Problems 25–54, solve each system. Use any method you wish.

25. $\begin{cases} 2x^2 + y^2 = 18 \\ xy = 4 \end{cases}$ **26.** $\begin{cases} x^2 - y^2 = 21 \\ x + y = 7 \end{cases}$ **27.** $\begin{cases} y = 2x + 1 \\ 2x^2 + y^2 = 1 \end{cases}$

28. $\begin{cases} x^2 - 4y^2 = 16 \\ 2y - x = 2 \end{cases}$ **29.** $\begin{cases} x + y + 1 = 0 \\ x^2 + y^2 + 6y - x = -5 \end{cases}$ **30.** $\begin{cases} 2x^2 - xy + y^2 = 8 \\ xy = 4 \end{cases}$

31. $\begin{cases} 4x^2 - 3xy + 9y^2 = 15 \\ 2x + 3y = 5 \end{cases}$ **32.** $\begin{cases} 2y^2 - 3xy + 6y + 2x + 4 = 0 \\ 2x - 3y + 4 = 0 \end{cases}$ **33.** $\begin{cases} x^2 - 4y^2 + 7 = 0 \\ 3x^2 + y^2 = 31 \end{cases}$

34. $\begin{cases} 3x^2 - 2y^2 + 5 = 0 \\ 2x^2 - y^2 + 2 = 0 \end{cases}$ **35.** $\begin{cases} 7x^2 - 3y^2 + 5 = 0 \\ 3x^2 + 5y^2 = 12 \end{cases}$ **36.** $\begin{cases} x^2 - 3y^2 + 1 = 0 \\ 2x^2 - 7y^2 + 5 = 0 \end{cases}$

37. $\begin{cases} x^2 + 2xy = 10 \\ 3x^2 - xy = 2 \end{cases}$ **38.** $\begin{cases} 5xy + 13y^2 + 36 = 0 \\ xy + 7y^2 = 6 \end{cases}$ **39.** $\begin{cases} 2x^2 + y^2 = 2 \\ x^2 - 2y^2 + 8 = 0 \end{cases}$

40. $\begin{cases} y^2 - x^2 + 4 = 0 \\ 2x^2 + 3y^2 = 6 \end{cases}$ **41.** $\begin{cases} x^2 + 2y^2 = 16 \\ 4x^2 - y^2 = 24 \end{cases}$ **42.** $\begin{cases} 4x^2 + 3y^2 = 4 \\ 2x^2 - 6y^2 = -3 \end{cases}$

43. $\begin{cases} \dfrac{5}{x^2} - \dfrac{2}{y^2} + 3 = 0 \\ \dfrac{3}{x^2} + \dfrac{1}{y^2} = 7 \end{cases}$ **44.** $\begin{cases} \dfrac{2}{x^2} - \dfrac{3}{y^2} + 1 = 0 \\ \dfrac{6}{x^2} - \dfrac{7}{y^2} + 2 = 0 \end{cases}$ **45.** $\begin{cases} \dfrac{1}{x^4} + \dfrac{6}{y^4} = 6 \\ \dfrac{2}{x^4} - \dfrac{2}{y^4} = 19 \end{cases}$

46. $\begin{cases} \dfrac{1}{x^4} - \dfrac{1}{y^4} = 1 \\ \dfrac{1}{x^4} + \dfrac{1}{y^4} = 4 \end{cases}$ **47.** $\begin{cases} x^2 - 3xy + 2y^2 = 0 \\ x^2 + xy = 6 \end{cases}$ **48.** $\begin{cases} x^2 - xy - 2y^2 = 0 \\ xy + x + 6 = 0 \end{cases}$

49. $\begin{cases} y^2 + y + x^2 - x - 2 = 0 \\ y + 1 + \dfrac{x - 2}{y} = 0 \end{cases}$ **50.** $\begin{cases} x^3 - 2x^2 + y^2 + 3y - 4 = 0 \\ x - 2 + \dfrac{y^2 - y}{x^2} = 0 \end{cases}$ **51.** $\begin{cases} \log_x y = 3 \\ \log_x (4y) = 5 \end{cases}$

52. $\begin{cases} \log_x (2y) = 3 \\ \log_x (4y) = 2 \end{cases}$ **53.** $\begin{cases} \ln x = 4 \ln y \\ \log_3 x = 2 + 2 \log_3 y \end{cases}$ **54.** $\begin{cases} \ln x = 5 \ln y \\ \log_2 x = 3 + 2 \log_2 y \end{cases}$

55. Graph the equations given in Example 4.

56. Graph the equations given in Problem 49.

In Problems 57–64, use a graphing utility to solve each system of equations. Express the solution(s) rounded to two decimal places.

57. $\begin{cases} y = x^{2/3} \\ y = e^{-x} \end{cases}$ **58.** $\begin{cases} y = x^{3/2} \\ y = e^{-x} \end{cases}$ **59.** $\begin{cases} x^2 + y^3 = 2 \\ x^3 y = 4 \end{cases}$ **60.** $\begin{cases} x^3 + y^2 = 2 \\ x^2 y = 4 \end{cases}$

61. $\begin{cases} x^4 + y^4 = 12 \\ xy^2 = 2 \end{cases}$ **62.** $\begin{cases} x^4 + y^4 = 6 \\ xy = 1 \end{cases}$ **63.** $\begin{cases} xy = 2 \\ y = \ln x \end{cases}$ **64.** $\begin{cases} x^2 + y^2 = 4 \\ y = \ln x \end{cases}$

Mixed Practice

In Problems 65–70, graph each equation and find the point(s) of intersection, if any.

65. The line $x + 2y = 0$ and the circle $(x - 1)^2 + (y - 1)^2 = 5$

66. The line $x + 2y + 6 = 0$ and the circle $(x + 1)^2 + (y + 1)^2 = 5$

67. The circle $(x - 1)^2 + (y + 2)^2 = 4$ and the parabola $y^2 + 4y - x + 1 = 0$

68. The circle $(x + 2)^2 + (y - 1)^2 = 4$ and the parabola $y^2 - 2y - x - 5 = 0$

69. $y = \dfrac{4}{x - 3}$ and the circle $x^2 - 6x + y^2 + 1 = 0$

70. $y = \dfrac{4}{x + 2}$ and the circle $x^2 + 4x + y^2 - 4 = 0$

Applications and Extensions

71. The difference of two numbers is 2 and the sum of their squares is 10. Find the numbers.

72. The sum of two numbers is 7 and the difference of their squares is 21. Find the numbers.

73. The product of two numbers is 4 and the sum of their squares is 8. Find the numbers.

74. The product of two numbers is 10 and the difference of their squares is 21. Find the numbers.

75. The difference of two numbers is the same as their product, and the sum of their reciprocals is 5. Find the numbers.

76. The sum of two numbers is the same as their product, and the difference of their reciprocals is 3. Find the numbers.

77. The ratio of a to b is $\frac{2}{3}$. The sum of a and b is 10. What is the ratio of $a + b$ to $b - a$?

78. The ratio of a to b is 4:3. The sum of a and b is 14. What is the ratio of $a - b$ to $a + b$?

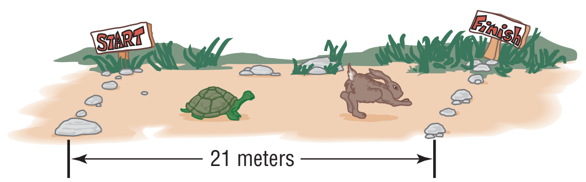
79. Geometry The perimeter of a rectangle is 16 inches and its area is 15 square inches. What are its dimensions?

80. Geometry An area of 52 square feet is to be enclosed by two squares whose sides are in the ratio of 2:3. Find the sides of the squares.

81. Geometry Two circles have circumferences that add up to 12π centimeters and areas that add up to 20π square centimeters. Find the radius of each circle.

82. Geometry The altitude of an isosceles triangle drawn to its base is 3 centimeters, and its perimeter is 18 centimeters. Find the length of its base.

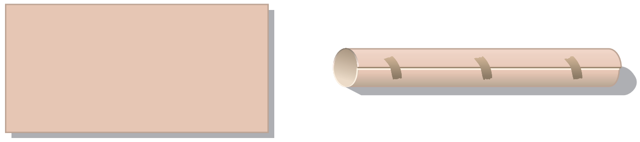
83. The Tortoise and the Hare In a 21-meter race between a tortoise and a hare, the tortoise leaves 9 minutes before the hare. The hare, by running at an average speed of 0.5 meter per hour faster than the tortoise, crosses the finish line 3 minutes before the tortoise. What are the average speeds of the tortoise and the hare?



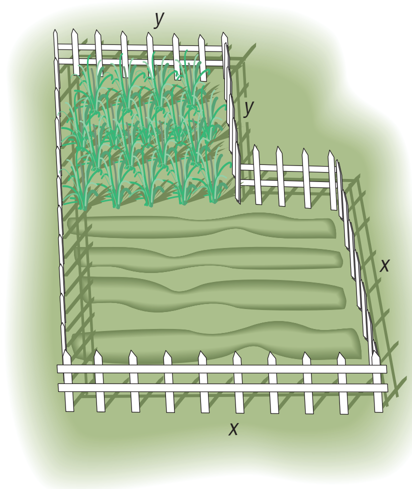
84. Running a Race In a 1-mile race, the winner crosses the finish line 10 feet ahead of the second-place runner and 20 feet ahead of the third-place runner. Assuming that each runner maintains a constant speed throughout the race, by how many feet does the second-place runner beat the third-place runner?

85. Constructing a Box A rectangular piece of cardboard, whose area is 216 square centimeters, is made into an open box by cutting a 2-centimeter square from each corner and turning up the sides. See the figure. If the box is to have a volume of 224 cubic centimeters, what size cardboard should you start with?

86. Constructing a Cylindrical Tube A rectangular piece of cardboard, whose area is 216 square centimeters, is made into a cylindrical tube by joining together two sides of the rectangle. See the figure. If the tube is to have a volume of 224 cubic centimeters, what size cardboard should you start with?



87. Fencing A farmer has 300 feet of fence available to enclose a 4500-square-foot region in the shape of adjoining squares, with sides of length x and y. See the figure. Find x and y.

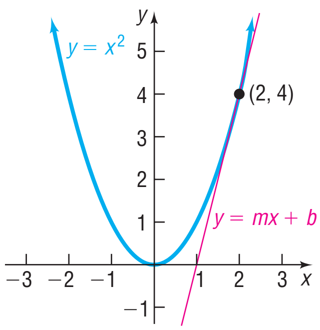


88. Bending Wire A wire 60 feet long is cut into two pieces. Is it possible to bend one piece into the shape of a square and the other into the shape of a circle so that the total area enclosed by the two pieces is 100 square feet? If this is possible, find the length of the side of the square and the radius of the circle.

89. Geometry Find formulas for the length l and width w of a rectangle in terms of its area A and perimeter P.

90. Geometry Find formulas for the base b and one of the equal sides l of an isosceles triangle in terms of its altitude h and perimeter P.

91. Descartes' Method of Equal Roots Descartes' method for finding tangents depends on the idea that, for many graphs, the tangent line at a given point is the *unique* line that intersects the graph at that point only. We apply his method to find an equation of the tangent line to the parabola $y = x^2$ at the point $(2, 4)$. See the figure.



First, we know that the equation of the tangent line must be in the form $y = mx + b$. Using the fact that the point $(2, 4)$ is on the line, we can solve for b in terms of m and get the equation $y = mx + (4 - 2m)$. Now we want $(2, 4)$ to be the *unique* solution to the system

$$\begin{cases} y = x^2 \\ y = mx + 4 - 2m \end{cases}$$

From this system, we get $x^2 - mx + (2m - 4) = 0$. By using the quadratic formula, we get

$$x = \frac{m \pm \sqrt{m^2 - 4(2m - 4)}}{2}$$

To obtain a unique solution for x, the two roots must be equal; in other words, the discriminant $m^2 - 4(2m - 4)$ must be 0. Complete the work to get m, and write an equation of the tangent line.

In Problems 92–98, use Descartes' method from Problem 91 to find an equation of the line tangent to each graph at the given point.

92. $x^2 + y^2 = 10$; at $(1, 3)$

93. $y = x^2 + 2$; at $(1, 3)$

94. $x^2 + y = 5$; at $(-2, 1)$

95. $2x^2 + 3y^2 = 14$; at $(1, 2)$

96. $3x^2 + y^2 = 7$; at $(-1, 2)$

97. $x^2 - y^2 = 3$; at $(2, 1)$

98. $2y^2 - x^2 = 14$; at $(2, 3)$

99. If r_1 and r_2 are two solutions of a quadratic equation $ax^2 + bx + c = 0$, it can be shown that

$$r_1 + r_2 = -\frac{b}{a} \quad \text{and} \quad r_1 r_2 = \frac{c}{a}$$

Solve this system of equations for r_1 and r_2.

Explaining Concepts: Discussion and Writing

100. A circle and a line intersect at most twice. A circle and a parabola intersect at most four times. Deduce that a circle and the graph of a polynomial of degree 3 intersect at most six times. What do you conjecture about a polynomial of degree 4? What about a polynomial of degree n? Can you explain your conclusions using an algebraic argument?

101. Suppose that you are the manager of a sheet metal shop. A customer asks you to manufacture 10,000 boxes, each box being open on top. The boxes are required to have a square base and a 9-cubic-foot capacity. You construct the boxes by cutting out a square from each corner of a square piece of sheet metal and folding along the edges.

(a) What are the dimensions of the square to be cut if the area of the square piece of sheet metal is 100 square feet?

(b) Could you make the box using a smaller piece of sheet metal? Make a list of the dimensions of the box for various pieces of sheet metal.

Retain Your Knowledge

Problems 102–105 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

102. Solve using the quadratic formula: $7x^2 = 8 - 6x$

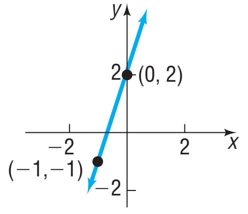
103. Find an equation of the line with slope $-\frac{2}{5}$ that passes through the point $(10, -7)$.

104. Find an equation of the line that contains the point $(-3, 7)$ perpendicular to the line $y = -4x - 5$.

105. Determine the interest rate required for an investment of \$1500 to be worth \$1800 after 3 years if interest is compounded quarterly. Round your answer to two decimal places.

'Are You Prepared?' Answers

1.



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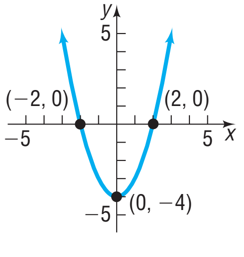

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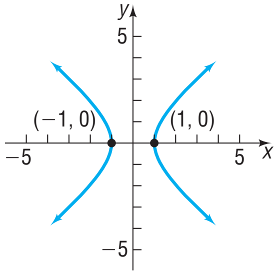

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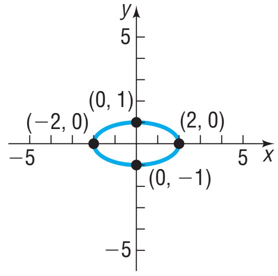


PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Solving Linear Inequalities (Section 1.7, pp. 150–151)
- Lines (Section 2.2, pp. 173–184)
- Circles (Section 2.3, pp. 189–193)
- Graphing Techniques: Transformations (Section 3.5, pp. 256–264)

Now Work the **'Are You Prepared?'** problems on page 636.

OBJECTIVES
1 Graph an Inequality by Hand (p. 630)
2 Graph an Inequality Using a Graphing Utility (p. 632)
3 Graph a System of Inequalities (p. 633)

Section 1.7 discussed inequalities in one variable. This section discusses inequalities in two variables.

EXAMPLE 1 **Examples of Inequalities in Two Variables**

(a) $3x + y \le 6$ (b) $x^2 + y^2 < 4$ (c) $y^2 > x$

1 Graph an Inequality by Hand

An inequality in two variables x and y is **satisfied** by an ordered pair (a, b) if, when x is replaced by a and y by b, a true statement results. The **graph of an inequality in two variables** x and y consists of all points (x, y) whose coordinates satisfy the inequality.

EXAMPLE 2 **Graphing an Inequality by Hand**

Graph the linear inequality: $3x + y \le 6$

Solution Begin by graphing the equation

$$3x + y = 6$$

formed by replacing (for now) the $\le$ symbol with an $=$ sign. The graph of the equation is a line. See Figure 25(a). This line is part of the graph of the inequality because the inequality is nonstrict, so the line is drawn as a solid line. (Do you see why? We are seeking points for which $3x + y$ is less than *or equal to* 6.)

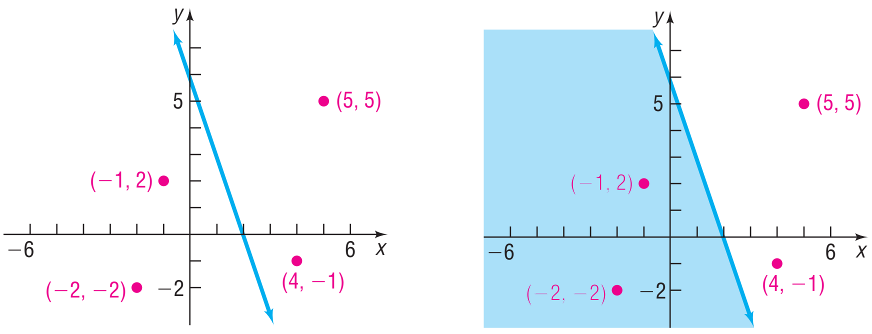


Figure 25 (a) $3x + y = 6$ (b) Graph of $3x + y \le 6$

Now test a few randomly selected points to see whether they belong to the graph of the inequality.

	$3x + y \le 6$	Conclusion
$(4, -1)$	$3(4) + (-1) = 11 > 6$	Does not belong to the graph
$(5, 5)$	$3(5) + 5 = 20 > 6$	Does not belong to the graph
$(-1, 2)$	$3(-1) + 2 = -1 \le 6$	Belongs to the graph
$(-2, -2)$	$3(-2) + (-2) = -8 \le 6$	Belongs to the graph

Look again at Figure 25(a). Notice that the two points that belong to the graph both lie on the same side of the line, and the two points that do not belong to the graph lie on the opposite side. As it turns out, all the points that satisfy the inequality will lie on one side of the line or on the line itself. All the points that do not satisfy the inequality will lie on the other side. The graph we seek consists of all points that lie on the line or on the same side of the line as $(-1, 2)$ and $(-2, -2)$. This graph is shown as the shaded region in Figure 25(b). ■

Now Work PROBLEM 15

The graph of any inequality in two variables may be obtained in a like way. The steps to follow are given next.

Note: The strict inequalities are $<$ and $>$. The nonstrict inequalities are $\le$ and $\ge$. ■

Steps for Graphing an Inequality by Hand

STEP 1: Replace the inequality symbol by an equal sign, and graph the resulting equation. If the inequality is strict, use dashes; if it is nonstrict, use a solid mark. This graph separates the xy-plane into two or more regions.

STEP 2: In each region, select a test point P.

(a) If the coordinates of P satisfy the inequality, so do all the points in that region. Indicate this by shading the region.

(b) If the coordinates of P do not satisfy the inequality, none of the points in that region does.

EXAMPLE 3 **Graphing an Inequality by Hand**

Graph: $x^2 + y^2 \le 4$

Solution

STEP 1: Graph the equation $x^2 + y^2 = 4$, a circle of radius 2, center at the origin. A solid circle is used because the inequality is not strict.

STEP 2: Use two test points, one inside the circle, the other outside.

Inside	$(0, 0)$:	$x^2 + y^2 = 0^2 + 0^2 = 0 \le 4$	**Belongs to the graph**
Outside	$(4, 0)$:	$x^2 + y^2 = 4^2 + 0^2 = 16 > 4$	**Does not belong to the graph**

All the points inside and on the circle satisfy the inequality. See Figure 26. ■

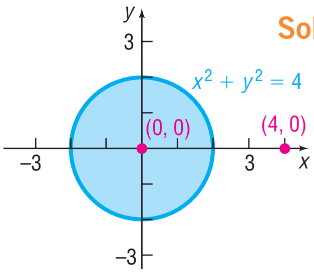


Figure 26 $x^2 + y^2 \le 4$

Now Work PROBLEM 17

Linear Inequalities

Linear inequalities are inequalities equivalent to one of the forms

$$Ax + By < C \qquad Ax + By > C \qquad Ax + By \le C \qquad Ax + By \ge C$$

where A and B are not both zero.

The graph of the corresponding equation of a linear inequality is a line that separates the xy-plane into two regions called **half-planes**. See Figure 27.

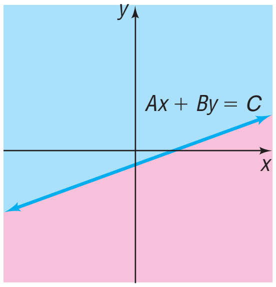


Figure 27

As shown, $Ax + By = C$ is the equation of the boundary line, and it divides the plane into two half-planes: one for which $Ax + By < C$ and the other for which $Ax + By > C$. Because of this, for linear inequalities, only one test point is required.

EXAMPLE 4 Graphing Linear Inequalities by Hand

Graph: (a) $y < 2$ (b) $y \geq 2x$

Solution (a) Points on the horizontal line $y = 2$ are not part of the graph of the inequality, so the graph is shown as a dashed line. Since $(0, 0)$ satisfies the inequality, the graph consists of the half-plane below the line $y = 2$. See Figure 28.

(b) Points on the line $y = 2x$ are part of the graph of the inequality, so the graph is shown as a solid line. Use $(3, 0)$ as a test point. It does not satisfy the inequality $[0 < 2 \cdot 3]$. Points in the half-plane on the opposite side of $(3, 0)$ satisfy the inequality. See Figure 29.

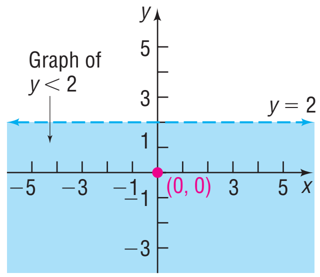


Figure 28 $y < 2$

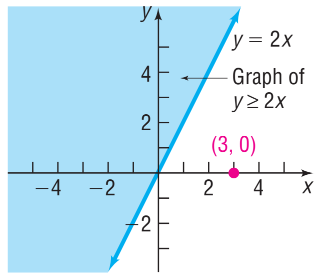


Figure 29 $y \geq 2x$

Now Work PROBLEM 13

2 Graph an Inequality Using a Graphing Utility

Graphing utilities can also be used to graph inequalities. The steps to follow are given next.

Steps for Graphing an Inequality Using a Graphing Utility

STEP 1: Replace the inequality symbol by an equal sign and graph the resulting equation. This graph separates the *xy*-plane into two or more regions.

STEP 2: Select a test point P in each region.

(a) Use a graphing utility to determine if the test point P satisfies the inequality. If the test point satisfies the inequality, then so do all the points in this region. Indicate this by using the graphing utility to shade the region.

(b) If the coordinates of P do not satisfy the inequality, then none of the points in that region does.

EXAMPLE 5 Graphing an Inequality Using a Graphing Utility

Use a graphing utility to graph $3x + y \leq 6$.

Solution

STEP 1: Begin by graphing the equation $3x + y = 6$ $(Y_1 = -3x + 6)$. See Figure 30.

STEP 2: Select a test point in one of the regions and determine whether it satisfies the inequality. To test the point $(-1, 2)$, for example, enter $3(-1) + 2 \leq 6$. See Figure 31(a). The 1 that appears indicates that the statement entered (the inequality) is true. When the point $(5, 5)$ is tested, a 0 appears, indicating that the statement entered is false. So $(-1, 2)$ is part of the graph of the inequality and

$(5, 5)$ is not. Shade the region containing the point $(-1, 2)$ that is below Y_1. Figure 31(b) shows the graph of the inequality on a TI-84 Plus C.

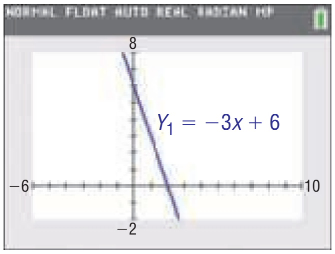
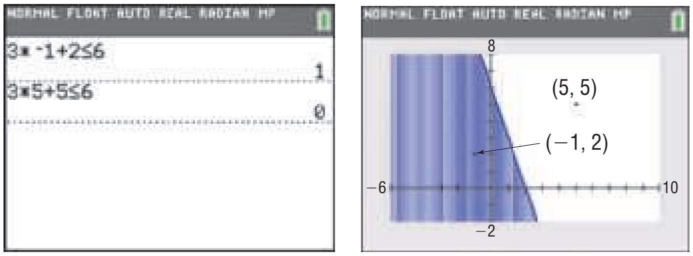


Figure 30 $3x + y = 6$



(a) (b)

Figure 31 $3x + y \le 6$

Note: A second approach that could be used in Example 5 is to solve the inequality $3x + y \le 6$ for y, $y \le -3x + 6$, graph the corresponding line $y = -3x + 6$, and then shade below since the inequality is of the form $y <$ or $y \le$ (shade above if of the form $y >$ or $y \ge$).

Now Work PROBLEM 15 USING A GRAPHING UTILITY

3 Graph a System of Inequalities

The **graph of a system of inequalities** in two variables x and y is the set of all points (x, y) that simultaneously satisfy *each* inequality in the system. The graph of a system of inequalities can be obtained by graphing each inequality individually and then determining where, if at all, they intersect.

EXAMPLE 6 Graphing a System of Linear Inequalities by Hand

Graph the system: $\begin{cases} x + y \ge 2 \\ 2x - y \le 4 \end{cases}$

Solution Begin by graphing the lines $x + y = 2$ and $2x - y = 4$ using solid lines since the inequalities are nonstrict. Use the test point $(0, 0)$ on each inequality. For example, $(0, 0)$ does not satisfy $x + y \ge 2$, so shade above the line $x + y = 2$. See Figure 32(a). Also, $(0, 0)$ does satisfy $2x - y \le 4$, so shade above the line $2x - y = 4$. See Figure 32(b). The intersection of the shaded regions (in purple) gives the result presented in Figure 32(c).

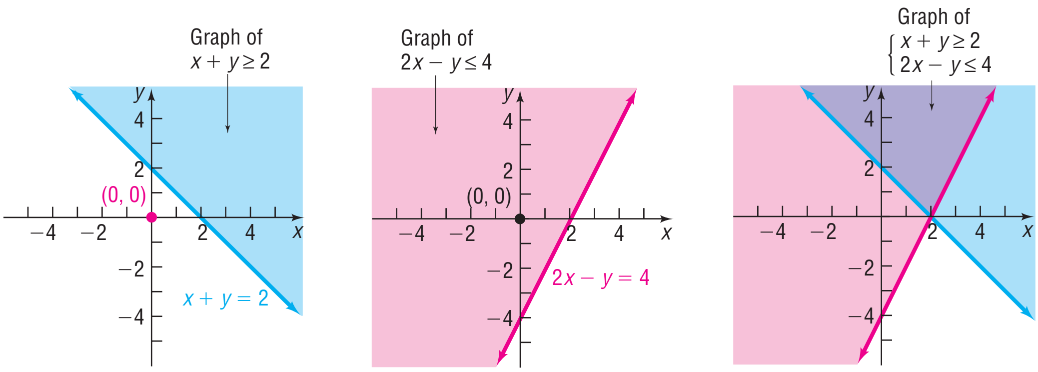


Figure 32 (a) (b) (c)

EXAMPLE 7 Graphing a System of Linear Inequalities Using a Graphing Utility

Graph the system: $\begin{cases} x + y \ge 2 \\ 2x - y \le 4 \end{cases}$

Solution First, graph the lines $x + y = 2$ $(Y_1 = -x + 2)$ and $2x - y = 4$ $(Y_2 = 2x - 4)$. See Figure 33 on the next page.

Notice that the graphs divide the viewing window into four regions. Select a test point for each region, and determine whether the point makes *both*

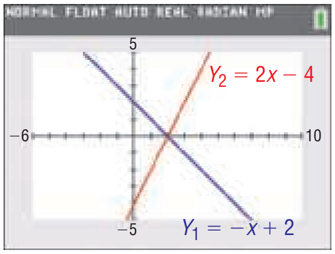


Figure 33

inequalities true. We choose to test $(0, 0)$, $(2, 3)$, $(4, 0)$, and $(2, -2)$. Figure 34(a) shows that $(2, 3)$ makes both inequalities true. The graph is shown in Figure 34(b).

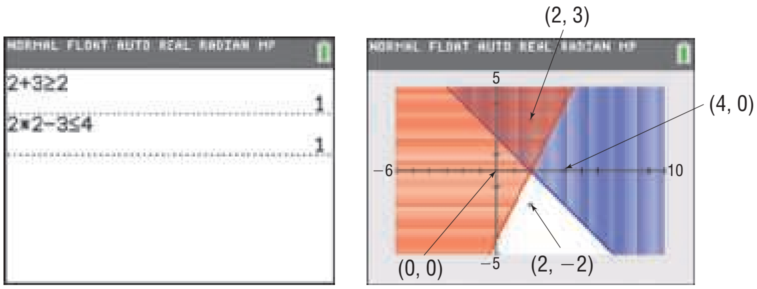


Figure 34 **(a)** **(b)**

Rather than testing four points, we could test just point $(0, 0)$ on each inequality. For example, $(0, 0)$ does not satisfy $x + y \geq 2$, so shade above the line $x + y = 2$. In addition, $(0, 0)$ does satisfy $2x - y \leq 4$, so shade above the line $2x - y = 4$. The intersection of the shaded regions gives the result presented in Figure 34(b).

Now Work PROBLEM 23

EXAMPLE 8 Graphing a System of Linear Inequalities by Hand

Graph the system: $\begin{cases} x + y \leq 2 \\ x + y \geq 0 \end{cases}$

Solution See Figure 35. The overlapping purple-shaded region between the two boundary lines is the graph of the system.

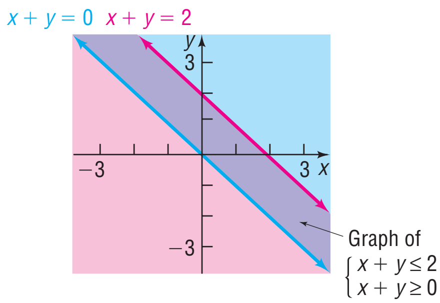


Figure 35

Now Work PROBLEM 29

EXAMPLE 9 Graphing a System of Linear Inequalities by Hand

Graph the systems:

(a) $\begin{cases} 2x - y \geq 0 \\ 2x - y \geq 2 \end{cases}$ (b) $\begin{cases} x + 2y \leq 2 \\ x + 2y \geq 6 \end{cases}$

Solution (a) See Figure 36. The overlapping purple-shaded region is the graph of the system. Note that the graph of the system is identical to the graph of the single inequality $2x - y \geq 2$.

(b) See Figure 37. Here, because no overlapping region results, there are no points in the xy-plane that simultaneously satisfy each inequality. The system has no solution.

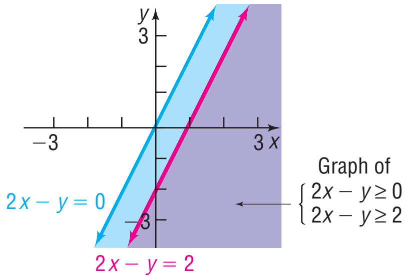


Figure 36

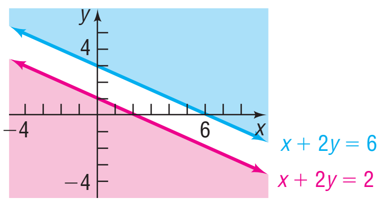


Figure 37

EXAMPLE 10 Graphing a System of Nonlinear Inequalities by Hand

Graph the region below the graph of $x + y = 2$ and above the graph of $y = x^2 - 4$ by graphing the system:

$$\begin{cases} y \geq x^2 - 4 \\ x + y \leq 2 \end{cases}$$

Label all points of intersection.

Solution Figure 38 shows the graph of the region above the graph of the parabola $y = x^2 - 4$ and below the graph of the line $x + y = 2$. The points of intersection are found by solving the system of equations

$$\begin{cases} y = x^2 - 4 \\ x + y = 2 \end{cases}$$

Use substitution to find

$$\begin{aligned} x + (x^2 - 4) &= 2 \\ x^2 + x - 6 &= 0 \\ (x + 3)(x - 2) &= 0 \\ x = -3 \quad \text{or} \quad x &= 2 \end{aligned}$$

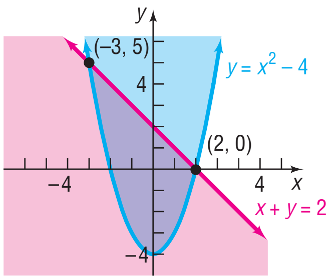


Figure 38

The two points of intersection are $(-3, 5)$ and $(2, 0)$.

Now Work PROBLEM 37

EXAMPLE 11 Graphing a System of Four Linear Inequalities by Hand

Graph the system:
$$\begin{cases} x + y \geq 3 \\ 2x + y \geq 4 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

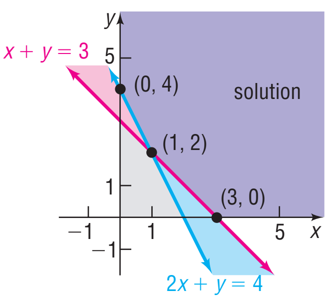


Figure 39

Solution See Figure 39. The two inequalities $x \geq 0$ and $y \geq 0$ require the graph of the system to be in quadrant I, which is shaded light gray. Concentrate on the remaining two inequalities. The intersection of the graphs of these two inequalities and quadrant I is shown in dark purple.

EXAMPLE 12 Financial Planning

A retired couple can invest up to \$25,000. As their financial adviser, you recommend that they place at least \$15,000 in Treasury bills yielding 2% and at most \$5000 in corporate bonds yielding 3%.

(a) Using x to denote the amount of money invested in Treasury bills and y the amount invested in corporate bonds, write a system of linear inequalities that describes the possible amounts of each investment. Assume that x and y are in thousands of dollars.

(b) Graph the system.

Solution (a) The system of linear inequalities is

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ x + y \leq 25 \\ x \geq 15 \\ y \leq 5 \end{cases}$$

x and y are nonnegative variables since they represent money invested, in thousands of dollars.
The total of the two investments, $x + y$, cannot exceed \$25,000.
At least \$15,000 in Treasury bills
At most \$5000 in corporate bonds

(b) See the shaded region in Figure 40. Note that the inequalities $x \geq 0$ and $y \geq 0$ require that the graph of the system be in quadrant I.

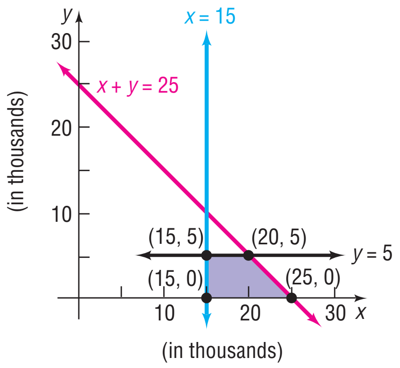


Figure 40

The graph of the system of linear inequalities in Figure 40 is **bounded**, because it can be contained within some circle of sufficiently large radius. A graph that cannot be contained in any circle is **unbounded**. For example, the graph of the system of linear inequalities in Figure 39 is unbounded, since it extends indefinitely in the positive x and positive y directions.

Notice in Figures 39 and 40 that those points that belong to the graph and are also points of intersection of boundary lines have been plotted. Such points are referred to as **vertices** or **corner points** of the graph. The system graphed in Figure 39 has three corner points: $(0, 4)$, $(1, 2)$, and $(3, 0)$. The system graphed in Figure 40 has four corner points: $(15, 0)$, $(25, 0)$, $(20, 5)$, and $(15, 5)$.

These ideas will be used in the next section in developing a method for solving linear programming problems, an important application of linear inequalities.

Now Work PROBLEM 45

8.7 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Solve the inequality: $3x + 4 < 8 - x$ (pp. 150–151)
2. Graph the equation: $3x - 2y = 6$ (pp. 173–184)
3. Graph the equation: $x^2 + y^2 = 9$ (pp. 189–193)
4. Graph the equation: $y = x^2 + 4$ (pp. 256–264)
5. ***True or False*** The lines $2x + y = 4$ and $4x + 2y = 0$ are parallel. (pp. 181–182)
6. The graph of $y = (x - 2)^2$ may be obtained by shifting the graph of ________ to the (left/right) a distance of ________ units. (pp. 256–264)

Concepts and Vocabulary

7. When graphing an inequality in two variables, use __________ if the inequality is strict; if the inequality is nonstrict, use a __________ mark.

8. The graph of the corresponding equation of a linear inequality is a line that separates the xy-plane into two regions. The two regions are called ______________.

9. ***True or False*** The graph of a system of inequalities must have an overlapping region.

10. If a graph of a system of linear inequalities cannot be contained in any circle, then it is ______________.

Skill Building

In Problems 11–22, graph each inequality.

11. $x \geq 0$
12. $y \geq 0$
13. $x \geq 4$
14. $y \leq 2$
15. $2x + y \geq 6$
16. $3x + 2y \leq 6$
17. $x^2 + y^2 > 1$
18. $x^2 + y^2 \leq 9$
19. $y \leq x^2 - 1$
20. $y > x^2 + 2$
21. $xy \geq 4$
22. $xy \leq 1$

In Problems 23–34, graph each system of linear inequalities.

23. $\begin{cases} x + y \leq 2 \\ 2x + y \geq 4 \end{cases}$
24. $\begin{cases} 3x - y \geq 6 \\ x + 2y \leq 2 \end{cases}$
25. $\begin{cases} 2x - y \leq 4 \\ 3x + 2y \geq -6 \end{cases}$
26. $\begin{cases} 4x - 5y \leq 0 \\ 2x - y \geq 2 \end{cases}$
27. $\begin{cases} 2x - 3y \leq 0 \\ 3x + 2y \leq 6 \end{cases}$
28. $\begin{cases} 4x - y \geq 2 \\ x + 2y \geq 2 \end{cases}$
29. $\begin{cases} x - 2y \leq 6 \\ 2x - 4y \geq 0 \end{cases}$
30. $\begin{cases} x + 4y \leq 8 \\ x + 4y \geq 4 \end{cases}$
31. $\begin{cases} 2x + y \geq -2 \\ 2x + y \geq 2 \end{cases}$
32. $\begin{cases} x - 4y \leq 4 \\ x - 4y \geq 0 \end{cases}$
33. $\begin{cases} 2x + 3y \geq 6 \\ 2x + 3y \leq 0 \end{cases}$
34. $\begin{cases} 2x + y \geq 0 \\ 2x + y \geq 2 \end{cases}$

In Problems 35–42, graph each system of inequalities.

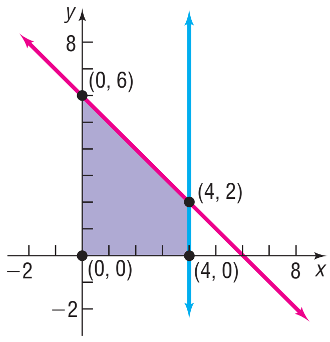
35. $\begin{cases} x^2 + y^2 \leq 9 \\ x + y \geq 3 \end{cases}$
36. $\begin{cases} x^2 + y^2 \geq 9 \\ x + y \leq 3 \end{cases}$
37. $\begin{cases} y \geq x^2 - 4 \\ y \leq x - 2 \end{cases}$
38. $\begin{cases} y^2 \leq x \\ y \geq x \end{cases}$
39. $\begin{cases} x^2 + y^2 \leq 16 \\ y \geq x^2 - 4 \end{cases}$
40. $\begin{cases} x^2 + y^2 \leq 25 \\ y \leq x^2 - 5 \end{cases}$
41. $\begin{cases} xy \geq 4 \\ y \geq x^2 + 1 \end{cases}$
42. $\begin{cases} y + x^2 \leq 1 \\ y \geq x^2 - 1 \end{cases}$

In Problems 43–52, graph each system of linear inequalities. Tell whether the graph is bounded or unbounded, and label the corner points.

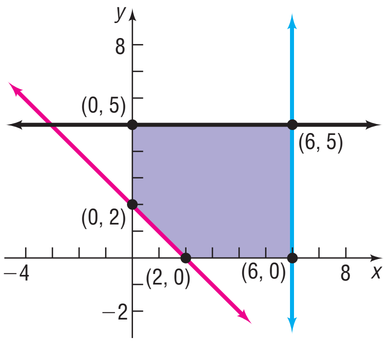
43. $\begin{cases} x \geq 0 \\ y \geq 0 \\ 2x + y \leq 6 \\ x + 2y \leq 6 \end{cases}$
44. $\begin{cases} x \geq 0 \\ y \geq 0 \\ x + y \geq 4 \\ 2x + 3y \geq 6 \end{cases}$
45. $\begin{cases} x \geq 0 \\ y \geq 0 \\ x + y \geq 2 \\ 2x + y \geq 4 \end{cases}$
46. $\begin{cases} x \geq 0 \\ y \geq 0 \\ 3x + y \leq 6 \\ 2x + y \leq 2 \end{cases}$
47. $\begin{cases} x \geq 0 \\ y \geq 0 \\ x + y \geq 2 \\ 2x + 3y \leq 12 \\ 3x + y \leq 12 \end{cases}$
48. $\begin{cases} x \geq 0 \\ y \geq 0 \\ x + y \geq 1 \\ x + y \leq 7 \\ 2x + y \leq 10 \end{cases}$
49. $\begin{cases} x \geq 0 \\ y \geq 0 \\ x + y \geq 2 \\ x + y \leq 8 \\ 2x + y \leq 10 \end{cases}$
50. $\begin{cases} x \geq 0 \\ y \geq 0 \\ x + y \geq 2 \\ x + y \leq 8 \\ x + 2y \geq 1 \end{cases}$
51. $\begin{cases} x \geq 0 \\ y \geq 0 \\ x + 2y \geq 1 \\ x + 2y \leq 10 \end{cases}$
52. $\begin{cases} x \geq 0 \\ y \geq 0 \\ x + 2y \geq 1 \\ x + 2y \leq 10 \\ x + y \geq 2 \\ x + y \leq 8 \end{cases}$

In Problems 53–56, write a system of linear inequalities for the given graph.

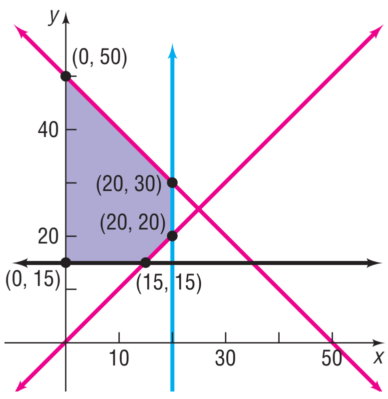
53.



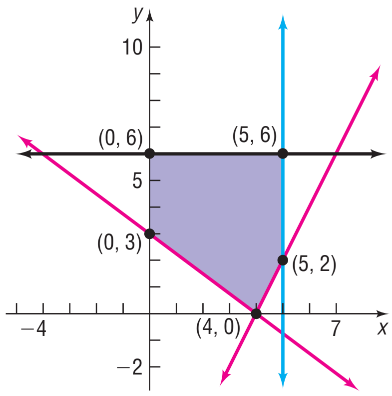
54.



55.



56.



Applications and Extensions

57. Financial Planning A retired couple has up to \$50,000 to invest. As their financial adviser, you recommend that they place at least \$35,000 in Treasury bills yielding 1% and at most \$10,000 in corporate bonds yielding 3%.

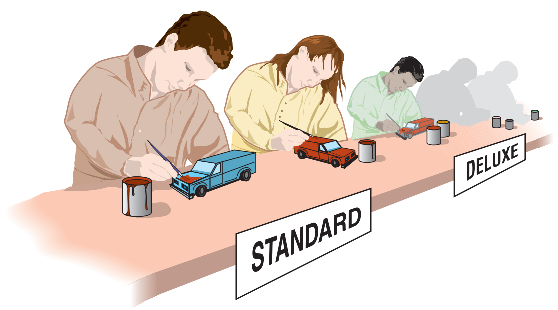
(a) Using x to denote the amount of money invested in Treasury bills and y to denote the amount invested in corporate bonds, write a system of linear inequalities that describes the possible amounts of each investment.

(b) Graph the system and label the corner points.

58. Manufacturing Trucks Mike's Toy Truck Company manufactures two models of toy trucks, a standard model and a deluxe model. Each standard model requires 2 hours (h) for painting and 3 h for detail work; each deluxe model requires 3 h for painting and 4 h for detail work. Two painters and three detail workers are employed by the company, and each works 40 h per week.

(a) Using x to denote the number of standard-model trucks and y to denote the number of deluxe-model trucks, write a system of linear inequalities that describes the possible numbers of each model of truck that can be manufactured in a week.

(b) Graph the system and label the corner points.



59. Blending Coffee Bill's Coffee House, a store that specializes in coffee, has available 75 pounds (lb) of *A* grade coffee and 120 lb of *B* grade coffee. These will be blended into 1-lb packages as follows: an economy blend that contains 4 ounces (oz) of *A* grade coffee and 12 oz of *B* grade coffee, and a superior blend that contains 8 oz of *A* grade coffee and 8 oz of *B* grade coffee.

(a) Using x to denote the number of packages of the economy blend and y to denote the number of packages of the superior blend, write a system of linear inequalities that describes the possible numbers of packages of each kind of blend.

(b) Graph the system and label the corner points.

60. Mixed Nuts Nola's Nuts, a store that specializes in selling nuts, has available 90 pounds (lb) of cashews and 120 lb of peanuts. These are to be mixed in 12-ounce (oz) packages as follows: a lower-priced package containing 8 oz of peanuts and 4 oz of cashews, and a quality package containing 6 oz of peanuts and 6 oz of cashews.

(a) Use x to denote the number of lower-priced packages, and use y to denote the number of quality packages. Write a system of linear inequalities that describes the possible numbers of each kind of package.

(b) Graph the system and label the corner points.

61. Transporting Goods A small truck can carry no more than 1600 pounds (lb) of cargo and no more than 150 cubic feet (ft^3) of cargo. A printer weighs 20 lb and occupies 3 ft^3 of space. A microwave oven weighs 30 lb and occupies 2 ft^3 of space.

(a) Using x to represent the number of microwave ovens and y to represent the number of printers, write a system of linear inequalities that describes the number of ovens and printers that can be hauled by the truck.

(b) Graph the system and label the corner points.

Retain Your Knowledge

Problems 62–65 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

62. Solve $2(x+1)^2 + 8 = 0$ in the complex number system.

63. Let $A = (7, -8)$ and $B = (0, -3)$ be points in the xy-plane. Find the distance d between the points, and find the midpoint of the line segment connecting the points.

64. Use the Intermediate Value Theorem to show that $f(x) = 6x^2 + 5x - 6$ has a real zero on the interval $[-1, 2]$.

65. Find any vertical or horizontal asymptotes for the graph of $f(x) = \dfrac{5x-2}{x+3}$.

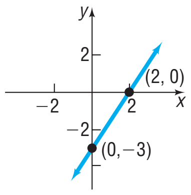
'Are You Prepared?' Answers

1. $\{x \mid x < 1\}$ or $(-\infty, 1)$

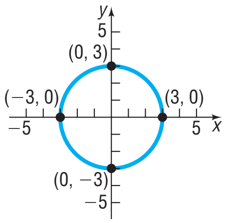
5. True

6. $y = x^2$; right; 2

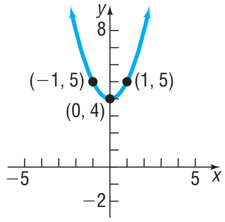
2.



3.



4.



8.8 Linear Programming

OBJECTIVES 1 Set Up a Linear Programming Problem (p. 639)
2 Solve a Linear Programming Problem (p. 640)

Historically, linear programming evolved as a technique for solving problems involving resource allocation of goods and materials for the U.S. Air Force during World War II. Today, linear programming techniques are used to solve a wide variety of problems, such as optimizing airline scheduling and establishing telephone lines. Although most practical linear programming problems involve systems of several hundred linear inequalities containing several hundred variables, we limit our discussion to problems containing only two variables, because we can solve such problems using graphing techniques.*

1 Set Up a Linear Programming Problem

Let's begin by returning to Example 12 of the previous section.

EXAMPLE 1 **Financial Planning**

A retired couple has up to \$25,000 to invest. As their financial adviser, you recommend that they place at least \$15,000 in Treasury bills yielding 2% and at most \$5000 in corporate bonds yielding 3%. Develop a model that can be used to determine how much money they should place in each investment so that income is maximized.

*The **simplex method** is a way to solve linear programming problems involving many inequalities and variables. This method was developed by George Dantzig in 1946 and is particularly well suited for computerization. In 1984, Narendra Karmarkar of Bell Laboratories discovered a way of solving large linear programming problems that improves on the simplex method.

Solution The problem is typical of a *linear programming problem*. The problem requires that a certain linear expression, the income, be maximized. If I represents income, x the amount invested in Treasury bills at 2%, and y the amount invested in corporate bonds at 3%, then

$$I = 0.02x + 0.03y$$

Assume, as before, that I, x, and y are in thousands of dollars.

The linear expression $I = 0.02x + 0.03y$ is called the **objective function**. Further, the problem requires that the maximum income be achieved under certain conditions, or **constraints**, each of which is a linear inequality involving the variables. (See Example 12 in Section 8.7.) The linear programming problem may be modeled as

$$\text{Maximize} \qquad I = 0.02x + 0.03y$$

subject to the conditions that

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ x + y \leq 25 \\ x \geq 15 \\ y \leq 5 \end{cases}$$

In general, every linear programming problem has two components:

1. A linear objective function that is to be maximized or minimized
2. A collection of linear inequalities that must be satisfied simultaneously

DEFINITION

A **linear programming problem** in two variables x and y consists of maximizing (or minimizing) a linear objective function

$$\boxed{z = Ax + By \qquad A \text{ and } B \text{ are real numbers, not both } 0}$$

subject to certain conditions, or constraints, expressible as linear inequalities in x and y.

2 Solve a Linear Programming Problem

To maximize (or minimize) the quantity $z = Ax + By$, we need to identify points (x, y) that make the expression for z the largest (or smallest) possible. But not all points (x, y) are eligible; only those that also satisfy each linear inequality (constraint) can be used. Each point (x, y) that satisfies the system of linear inequalities (the constraints) is a **feasible point**. Linear programming problems seek the feasible point(s) that maximizes (or minimizes) the objective function.

Look again at the linear programming problem in Example 1.

EXAMPLE 2 Analyzing a Linear Programming Problem

Consider the linear programming problem

$$\text{Maximize} \qquad I = 0.02x + 0.03y$$

subject to the conditions that

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ x + y \leq 25 \\ x \geq 15 \\ y \leq 5 \end{cases}$$

Graph the constraints. Then graph the objective function for $I = 0, 0.3, 0.45, 0.55$, and 0.6.

Solution Figure 41 shows the graph of the constraints. We superimpose on this graph the graph of the objective function for the given values of I.

For $I = 0$, the objective function is the line $0 = 0.02x + 0.03y$.

For $I = 0.3$, the objective function is the line $0.3 = 0.02x + 0.03y$.

For $I = 0.45$, the objective function is the line $0.45 = 0.02x + 0.03y$.

For $I = 0.55$, the objective function is the line $0.55 = 0.02x + 0.03y$.

For $I = 0.6$, the objective function is the line $0.6 = 0.02x + 0.03y$.

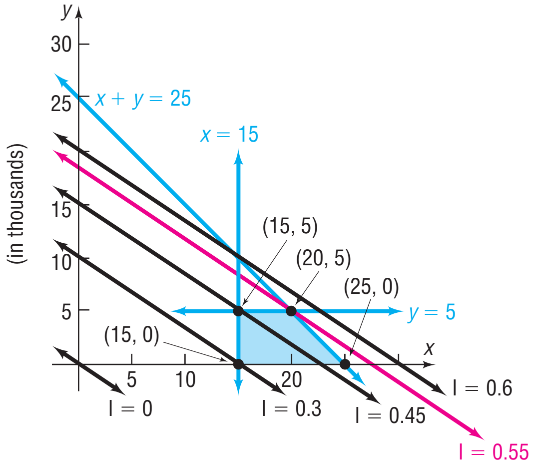


Figure 41

DEFINITION

A **solution** to a linear programming problem consists of a feasible point that maximizes (or minimizes) the objective function, together with the corresponding value of the objective function.

One condition for a linear programming problem in two variables to have a solution is that the graph of the feasible points be bounded. (Refer to page 636.)

If none of the feasible points maximizes (or minimizes) the objective function or if there are no feasible points, the linear programming problem has no solution.

Consider the linear programming problem posed in Example 2, and look again at Figure 41. The feasible points are the points that lie in the shaded region. For example, $(20, 3)$ is a feasible point, as are $(15, 5)$, $(20, 5)$, $(18, 4)$, and so on. To find the solution of the problem requires finding a feasible point (x, y) that makes $I = 0.02x + 0.03y$ as large as possible. Notice that as I increases in value from $I = 0$ to $I = 0.3$ to $I = 0.45$ to $I = 0.55$ to $I = 0.6$, the result is a collection of parallel lines. Further, notice that the largest value of I that can be obtained using feasible points is $I = 0.55$, which corresponds to the line $0.55 = 0.02x + 0.03y$. Any larger value of I results in a line that does not pass through any feasible points. Finally, notice that the feasible point that yields $I = 0.55$ is the point $(20, 5)$, a corner point. These observations form the basis of the following results, which are stated without proof.

THEOREM

Location of the Solution of a Linear Programming Problem

If a linear programming problem has a solution, it is located at a corner point of the graph of the feasible points.

If a linear programming problem has multiple solutions, at least one of them is located at a corner point of the graph of the feasible points.

In either case, the corresponding value of the objective function is unique. ■

We shall not consider linear programming problems that have no solution. As a result, we can outline the procedure for solving a linear programming problem as follows:

Procedure for Solving a Linear Programming Problem

STEP 1: Write an expression for the quantity to be maximized (or minimized). This expression is the objective function.

STEP 2: Write all the constraints as a system of linear inequalities, and graph the system.

STEP 3: List the corner points of the graph of the feasible points.

STEP 4: List the corresponding values of the objective function at each corner point. The largest (or smallest) of these is the solution.

EXAMPLE 3 Solving a Minimum Linear Programming Problem

Minimize the expression

$$z = 2x + 3y$$

subject to the constraints

$$y \le 5 \qquad x \le 6 \qquad x + y \ge 2 \qquad x \ge 0 \qquad y \ge 0$$

Solution

STEP 1: The objective function is $z = 2x + 3y$.

STEP 2: We seek the smallest value of z that can occur if x and y are solutions of the system of linear inequalities

$$\begin{cases} y \le 5 \\ x \le 6 \\ x + y \ge 2 \\ x \ge 0 \\ y \ge 0 \end{cases}$$

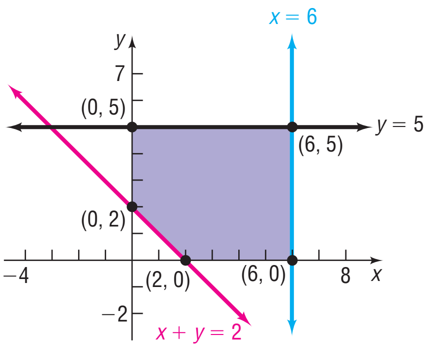


Figure 42

STEP 3: The graph of this system (the set of feasible points) is shown as the shaded region in Figure 42. The corner points have also been plotted.

STEP 4: Table 1 lists the corner points and the corresponding values of the objective function. From the table, the minimum value of z is 4, and it occurs at the point $(2, 0)$.

Table 1

Corner Point (x, y)	Value of the Objective Function $z = 2x + 3y$
$(0, 2)$	$z = 2(0) + 3(2) = 6$
$(0, 5)$	$z = 2(0) + 3(5) = 15$
$(6, 5)$	$z = 2(6) + 3(5) = 27$
$(6, 0)$	$z = 2(6) + 3(0) = 12$
$(2, 0)$	$z = 2(2) + 3(0) = 4$

Now Work PROBLEMS 5 AND 11

EXAMPLE 4 Maximizing Profit

At the end of every month, after filling orders for its regular customers, a coffee company has some pure Colombian coffee and some special-blend coffee remaining. The practice of the company has been to package a mixture of the two coffees into 1-pound (lb) packages as follows: a low-grade mixture containing 4 ounces (oz) of Colombian coffee and 12 oz of special-blend coffee, and a high-grade mixture containing 8 oz of Colombian and 8 oz of special-blend coffee. A profit of \$0.30 per package is made on the low-grade mixture, whereas a profit of \$0.40 per package is made on the high-grade mixture. This month, 120 lb of special-blend coffee and 100 lb of pure Colombian coffee remain. How many packages of each mixture should be prepared to achieve a maximum profit? Assume that all packages prepared can be sold.

Solution

Step 1: Begin by assigning symbols for the two variables.

$$x = \text{Number of packages of the low-grade mixture}$$
$$y = \text{Number of packages of the high-grade mixture}$$

If P denotes the profit, then

$$P = \$0.30x + \$0.40y \qquad \textbf{Objective function}$$

Step 2: The goal is to maximize P subject to certain constraints on x and y. Because x and y represent numbers of packages, the only meaningful values for x and y are nonnegative integers. This yields the two constraints

$$x \geq 0 \qquad y \geq 0 \qquad \textbf{Nonnegative constraints}$$

There is only so much of each type of coffee available. For example, the total amount of Colombian coffee used in the two mixtures cannot exceed 100 lb, or 1600 oz. Because 4 oz are used in each low-grade package and 8 oz are used in each high-grade package, this leads to the constraint

$$4x + 8y \leq 1600 \qquad \textbf{Colombian coffee constraint}$$

Similarly, the supply of 120 lb, or 1920 oz, of special-blend coffee leads to the constraint

$$12x + 8y \leq 1920 \qquad \textbf{Special-blend coffee constraint}$$

The linear programming problem may be stated as

$$\text{Maximize} \qquad P = 0.3x + 0.4y$$

subject to the constraints

$$x \geq 0 \qquad y \geq 0 \qquad 4x + 8y \leq 1600 \qquad 12x + 8y \leq 1920$$

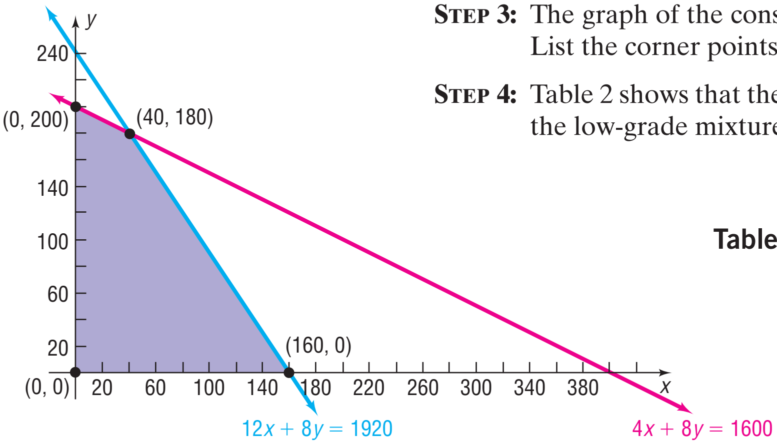


Figure 43

Step 3: The graph of the constraints (the feasible points) is illustrated in Figure 43. List the corner points, and evaluate the objective function at each point.

Step 4: Table 2 shows that the maximum profit, \$84, is achieved with 40 packages of the low-grade mixture and 180 packages of the high-grade mixture.

Table 2

Corner Point (x, y)	Value of Profit $P = 0.3x + 0.4y$
$(0, 0)$	$P = 0$
$(0, 200)$	$P = 0.3(0) + 0.4(200) = \$80$
$(40, 180)$	$P = 0.3(40) + 0.4(180) = \$84$
$(160, 0)$	$P = 0.3(160) + 0.4(0) = \$48$

Now Work PROBLEM 19

8.8 Assess Your Understanding

Concepts and Vocabulary

1. A linear programming problem requires that a linear expression, called the ____________ ____________, be maximized or minimized.

2. ***True or False*** If a linear programming problem has a solution, it is located at a corner point of the graph of the feasible points.

Skill Building

In Problems 3–8, find the maximum and minimum value of the given objective function of a linear programming problem. The figure illustrates the graph of the feasible points.

3. $z = x + y$

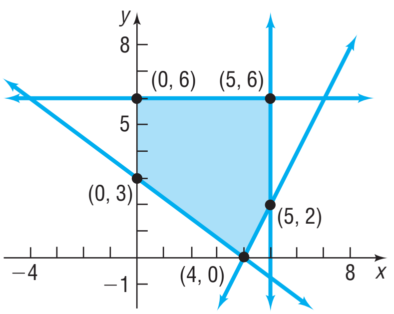
4. $z = 2x + 3y$

5. $z = x + 10y$

6. $z = 10x + y$

7. $z = 5x + 7y$

8. $z = 7x + 5y$



In Problems 9–18, solve each linear programming problem.

9. Maximize $z = 2x + y$ subject to $x \ge 0,\ y \ge 0,\ x + y \le 6,\ x + y \ge 1$

10. Maximize $z = x + 3y$ subject to $x \ge 0,\ y \ge 0,\ x + y \ge 3,\ x \le 5,\ y \le 7$

11. Minimize $z = 2x + 5y$ subject to $x \ge 0,\ y \ge 0,\ x + y \ge 2,\ x \le 5,\ y \le 3$

12. Minimize $z = 3x + 4y$ subject to $x \ge 0,\ y \ge 0,\ 2x + 3y \ge 6,\ x + y \le 8$

13. Maximize $z = 3x + 5y$ subject to $x \ge 0,\ y \ge 0,\ x + y \ge 2,\ 2x + 3y \le 12,\ 3x + 2y \le 12$

14. Maximize $z = 5x + 3y$ subject to $x \ge 0,\ y \ge 0,\ x + y \ge 2,\ x + y \le 8,\ 2x + y \le 10$

15. Minimize $z = 5x + 4y$ subject to $x \ge 0,\ y \ge 0,\ x + y \ge 2,\ 2x + 3y \le 12,\ 3x + y \le 12$

16. Minimize $z = 2x + 3y$ subject to $x \ge 0,\ y \ge 0,\ x + y \ge 3,\ x + y \le 9,\ x + 3y \ge 6$

17. Maximize $z = 5x + 2y$ subject to $x \ge 0,\ y \ge 0,\ x + y \le 10,\ 2x + y \ge 10,\ x + 2y \ge 10$

18. Maximize $z = 2x + 4y$ subject to $x \ge 0,\ y \ge 0,\ 2x + y \ge 4,\ x + y \le 9$

Applications and Extensions

19. **Maximizing Profit** A manufacturer of skis produces two types: downhill and cross-country. Use the following table to determine how many of each kind of ski should be produced to achieve a maximum profit. What is the maximum profit? What would the maximum profit be if the time available for manufacturing were increased to 48 hours?

	Downhill	Cross-country	Time Available
Manufacturing time per ski	2 hours	1 hour	40 hours
Finishing time per ski	1 hour	1 hour	32 hours
Profit per ski	\$70	\$50	

20. **Farm Management** A farmer has 70 acres of land available for planting either soybeans or wheat. The cost of preparing the soil, the workdays required, and the expected profit per acre planted for each type of crop are given in the following table.

	Soybeans	Wheat
Preparation cost per acre	\$60	\$30
Workdays required per acre	3	4
Profit per acre	\$180	\$100

The farmer cannot spend more than \$1800 in preparation costs and cannot use a total of more than 120 workdays. How many acres of each crop should be planted to maximize the profit? What is the maximum profit? What is the maximum profit if the farmer is willing to spend no more than \$2400 on preparation?

21. **Banquet Seating** A banquet hall offers two types of tables for rent: 6-person rectangular tables at a cost of \$28 each and 10-person round tables at a cost of \$52 each. Kathleen would like to rent the hall for a wedding banquet and needs tables for 250 people. The hall can have a maximum of 35 tables, and the hall has only 15 rectangular tables available. How many of each type of table should be rented to minimize cost and what is the minimum cost?

Source: *facilities.princeton.edu*

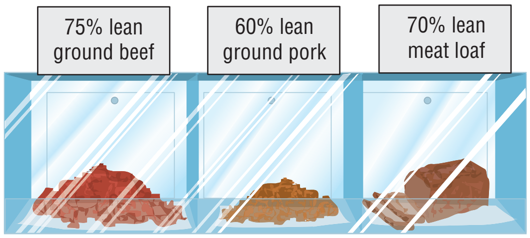
22. **Spring Break** The student activities department of a community college plans to rent buses and vans for a spring-break trip. Each bus has 40 regular seats and 1 special seat designed to accommodate travelers with disabilities. Each van has 8 regular seats and 3 special seats. The rental cost is \$350 for each van and \$975 for each bus. If 320 regular and 36 special seats are required for the trip, how many vehicles of each type should be rented to minimize cost?

Source: *www.busrates.com*

23. **Return on Investment** An investment broker is instructed by her client to invest up to \$20,000, some in a junk bond yielding 9% per annum and some in Treasury bills yielding 7% per annum. The client wants to invest at least \$8000 in T-bills and no more than \$12,000 in the junk bond.
    (a) How much should the broker recommend that the client place in each investment to maximize income if the client insists that the amount invested in T-bills must equal or exceed the amount placed in the junk bond?
    (b) How much should the broker recommend that the client place in each investment to maximize income if the client insists that the amount invested in T-bills must not exceed the amount placed in the junk bond?

24. **Production Scheduling** In a factory, machine 1 produces 8-inch (in.) pliers at the rate of 60 units per hour (h) and 6-in. pliers at the rate of 70 units/h. Machine 2 produces 8-in. pliers at the rate of 40 units/h and 6-in. pliers at the rate of 20 units/h. It costs \$50/h to operate machine 1, and machine 2 costs \$30/h to operate. The production schedule requires that at least 240 units of 8-in. pliers and at least 140 units of 6-in. pliers be produced during each 10-h day. Which combination of machines will cost the least money to operate?

25. **Managing a Meat Market** A meat market combines ground beef and ground pork in a single package for meat loaf. The ground beef is 75% lean (75% beef, 25% fat) and costs the market \$0.75 per pound (lb). The ground pork is 60% lean and costs the market \$0.45/lb. The meat loaf must be at least 70% lean. If the market wants to use at least 50 lb of its available pork, but no more than 200 lb of its available ground beef, how much ground beef should be mixed with ground pork so that the cost is minimized?



26. **Ice Cream** The Mom and Pop Ice Cream Company makes two kinds of chocolate ice cream: regular and premium. The properties of 1 gallon (gal) of each type are shown in the table:

	Regular	Premium
Flavoring	24 oz	20 oz
Milk-fat products	12 oz	20 oz
Shipping weight	5 lbs	6 lbs
Profit	\$0.75	\$0.90

In addition, current commitments require the company to make at least 1 gal of premium for every 4 gal of regular. Each day, the company has available 725 pounds (lb) of flavoring and 425 lb of milk-fat products. If the company can ship no more than 3000 lb of product per day, how many gallons of each type should be produced daily to maximize profit?

Source: *www.scitoys.com/ingredients/ice_cream.html*

27. **Maximizing Profit on Ice Skates** A factory manufactures two kinds of ice skates: racing skates and figure skates. The racing skates require 6 work-hours in the fabrication department, whereas the figure skates require 4 work-hours there. The racing skates require 1 work-hour in the finishing

department, whereas the figure skates require 2 work-hours there. The fabricating department has available at most 120 work-hours per day, and the finishing department has no more than 40 work-hours per day available. If the profit on each racing skate is \$10 and the profit on each figure skate is \$12, how many of each should be manufactured each day to maximize profit? (Assume that all skates made are sold.)

28. **Financial Planning** A retired couple have up to \$50,000 to place in fixed-income securities. Their financial adviser suggests two securities to them: one is an AAA bond that yields 8% per annum; the other is a certificate of deposit (CD) that yields 4%. After careful consideration of the alternatives, the couple decide to place at most \$20,000 in the AAA bond and at least \$15,000 in the CD. They also instruct the financial adviser to place at least as much in the CD as in the AAA bond. How should the financial adviser proceed to maximize the return on their investment?

29. **Product Design** An entrepreneur is having a design group produce at least six samples of a new kind of fastener that he wants to market. It costs \$9.00 to produce each metal fastener and \$4.00 to produce each plastic fastener. He wants to have at least two of each version of the fastener and needs to have all the samples 24 hours (h) from now. It takes 4 h to produce each metal sample and 2 h to produce each plastic sample. To minimize the cost of the samples, how many of each kind should the entrepreneur order? What will be the cost of the samples?

30. **Animal Nutrition** Kevin's dog Amadeus likes two kinds of canned dog food. Gourmet Dog costs 40 cents a can and has 20 units of a vitamin complex; the calorie content is 75 calories. Chow Hound costs 32 cents a can and has 35 units of vitamins and 50 calories. Kevin likes Amadeus to have at least 1175 units of vitamins a month and at least 2375 calories during the same time period. Kevin has space to store only 60 cans of dog food at a time. How much of each kind of dog food should Kevin buy each month to minimize his cost?

31. **Airline Revenue** An airline has two classes of service: first class and coach. Management's experience has been that each aircraft should have at least 8 but no more than 16 first-class seats and at least 80 but no more than 120 coach seats.
   (a) If management decides that the ratio of first-class seats to coach seats should never exceed 1:12, with how many of each type of seat should an aircraft be configured to maximize revenue?
   (b) If management decides that the ratio of first-class seats to coach seats should never exceed 1:8, with how many of each type of seat should an aircraft be configured to maximize revenue?
   (c) If you were management, what would you do?
   [**Hint:** Assume that the airline charges \$$C$ for a coach seat and \$$F$ for a first-class seat; $C > 0$, $F > C$.]

Explaining Concepts: Discussion and Writing

32. Explain in your own words what a linear programming problem is and how it can be solved.

Retain Your Knowledge

Problems 33–36 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

33. Solve: $2m^{2/5} - m^{1/5} = 1$

34. Solve: $25^{x+3} = 5^{x-4}$

35. **Radioactive Decay** The half-life of titanium-44 is 63 years. How long will it take 200 grams to decay to 75 grams? Round to one decimal place.

36. Find the equation of the line that is parallel to $y = 3x + 11$ and passes through the point $(-2, 1)$.

Chapter Review

Things to Know

Systems of equations (pp. 554–556)

Systems with no solutions are inconsistent. Systems with a solution are consistent.

Consistent systems of linear equations have either a unique solution (independent) or an infinite number of solutions (dependent).

Matrix (p. 569) Rectangular array of numbers, called entries

Augmented matrix (p. 569)

Row operations (p. 571)

Row echelon form (p. 573)

Determinants and Cramer's Rule (pp. 585, 587, 588–589, and 590)

Matrix (p. 595)

m by n matrix (p. 596)	Matrix with m rows and n columns
Identity matrix I_n (p. 602)	An n by n square matrix whose diagonal entries are 1's, while all other entries are 0's
Inverse of a matrix (p. 603)	A^{-1} is the inverse of A if $AA^{-1} = A^{-1}A = I_n$.
Nonsingular matrix (p. 603)	A square matrix that has an inverse

Linear programming problem (p. 640)

Maximize (or minimize) a linear objective function, $z = Ax + By$, subject to certain conditions, or constraints, expressible as linear inequalities in x and y. A feasible point (x, y) is a point that satisfies the constraints (linear inequalities) of a linear programming problem.

Location of solution (p. 642)

If a linear programming problem has a solution, it is located at a corner point of the graph of the feasible points. If a linear programming problem has multiple solutions, at least one of them is located at a corner point of the graph of the feasible points. In either case, the corresponding value of the objective function is unique.

Objectives

Section		You should be able to ...	Example(s)	Review Exercises
8.1	1	Solve systems of equations by substitution (p. 556)	4	1–7, 56, 59, 60
	2	Solve systems of equations by elimination (p. 557)	5, 6	1–7, 56, 59, 60
	3	Identify inconsistent systems of equations containing two variables (p. 559)	7	5, 54
	4	Express the solution of a system of dependent equations containing two variables (p. 559)	8	7, 53
	5	Solve systems of three equations containing three variables (p. 560)	9	8–10, 55, 57
	6	Identify inconsistent systems of equations containing three variables (p. 562)	10	10
	7	Express the solution of a system of dependent equations containing three variables (p. 563)	11	9
8.2	1	Write the augmented matrix of a system of linear equations (p. 569)	1	20–25
	2	Write the system of equations from the augmented matrix (p. 571)	2	11, 12
	3	Perform row operations on a matrix (p. 571)	3, 4	20–25
	4	Solve a system of linear equations using matrices (p. 572)	5–10	20–25
8.3	1	Evaluate 2 by 2 determinants (p. 585)	1	26
	2	Use Cramer's Rule to solve a system of two equations containing two variables (p. 586)	2	29, 30
	3	Evaluate 3 by 3 determinants (p. 588)	4	27, 28
	4	Use Cramer's Rule to solve a system of three equations containing three variables (p. 590)	5	31
	5	Know properties of determinants (p. 591)	6–9	32, 33
8.4	1	Find the sum and difference of two matrices (p. 596)	3, 4	13
	2	Find scalar multiples of a matrix (p. 598)	5	14
	3	Find the product of two matrices (p. 599)	6–11	15, 16
	4	Find the inverse of a matrix (p. 603)	12–14	17–19
	5	Solve a system of linear equations using an inverse matrix (p. 606)	15	20–25
8.5	1	Decompose $\frac{P}{Q}$, where Q has only nonrepeated linear factors (p. 613)	1	34
	2	Decompose $\frac{P}{Q}$, where Q has repeated linear factors (p. 615)	2, 3	35
	3	Decompose $\frac{P}{Q}$, where Q has a nonrepeated irreducible quadratic factor (p. 617)	4	36, 38
	4	Decompose $\frac{P}{Q}$, where Q has a repeated irreducible quadratic factor (p. 618)	5	37

Section	You should be able to ...	Example(s)	Review Exercises
8.6	1 Solve a system of nonlinear equations using substitution (p. 620)	1, 3	39–43
	2 Solve a system of nonlinear equations using elimination (p. 621)	2, 4, 5	39–43
8.7	1 Graph an inequality by hand (p. 630)	2–4	44, 45
	2 Graph an inequality using a graphing utility (p. 632)	5, 7	44, 45
	3 Graph a system of inequalities (p. 633)	6–12	46–50, 58
8.8	1 Set up a linear programming problem (p. 639)	1	61
	2 Solve a linear programming problem (p. 640)	2–4	51, 52, 61

Review Exercises

In Problems 1–10, solve each system of equations using the method of substitution or the method of elimination. If the system has no solution, say that it is inconsistent.

1. $\begin{cases} 2x - y = 5 \\ 5x + 2y = 8 \end{cases}$

2. $\begin{cases} 3x - 4y = 4 \\ x - 3y = \dfrac{1}{2} \end{cases}$

3. $\begin{cases} x - 2y - 4 = 0 \\ 3x + 2y - 4 = 0 \end{cases}$

4. $\begin{cases} y = 2x - 5 \\ x = 3y + 4 \end{cases}$

5. $\begin{cases} x - 3y + 4 = 0 \\ \dfrac{1}{2}x - \dfrac{3}{2}y + \dfrac{4}{3} = 0 \end{cases}$

6. $\begin{cases} 2x + 3y - 13 = 0 \\ 3x - 2y = 0 \end{cases}$

7. $\begin{cases} 2x + 5y = 10 \\ 4x + 10y = 20 \end{cases}$

8. $\begin{cases} x + 2y - z = 6 \\ 2x - y + 3z = -13 \\ 3x - 2y + 3z = -16 \end{cases}$

9. $\begin{cases} 2x - 4y + z = -15 \\ x + 2y - 4z = 27 \\ 5x - 6y - 2z = -3 \end{cases}$

10. $\begin{cases} x - 4y + 3z = 15 \\ -3x + y - 5z = -5 \\ -7x - 5y - 9z = 10 \end{cases}$

In Problems 11 and 12, write the system of equations that corresponds to the given augmented matrix.

11. $\left[\begin{array}{cc|c} 3 & 2 & 8 \\ 1 & 4 & -1 \end{array}\right]$

12. $\left[\begin{array}{ccc|c} 1 & 2 & 5 & -2 \\ 5 & 0 & -3 & 8 \\ 2 & -1 & 0 & 0 \end{array}\right]$

In Problems 13–16, use the following matrices to compute each expression.

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ -1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 4 & -3 & 0 \\ 1 & 1 & -2 \end{bmatrix} \quad C = \begin{bmatrix} 3 & -4 \\ 1 & 5 \\ 5 & 2 \end{bmatrix}$$

13. $A + C$

14. $6A$

15. AB

16. BC

In Problems 17–19, find the inverse, if there is one, of each matrix. If there is not an inverse, say that the matrix is singular.

17. $\begin{bmatrix} 4 & 6 \\ 1 & 3 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$

19. $\begin{bmatrix} 4 & -8 \\ -1 & 2 \end{bmatrix}$

In Problems 20–25, solve each system of equations using matrices. If the system has no solution, say that it is inconsistent.

20. $\begin{cases} 3x - 2y = 1 \\ 10x + 10y = 5 \end{cases}$

21. $\begin{cases} 5x - 6y - 3z = 6 \\ 4x - 7y - 2z = -3 \\ 3x + y - 7z = 1 \end{cases}$

22. $\begin{cases} 2x + y + z = 5 \\ 4x - y - 3z = 1 \\ 8x + y - z = 5 \end{cases}$

23. $\begin{cases} x - 2z = 1 \\ 2x + 3y = -3 \\ 4x - 3y - 4z = 3 \end{cases}$

24. $\begin{cases} x - y + z = 0 \\ x - y - 5z - 6 = 0 \\ 2x - 2y + z - 1 = 0 \end{cases}$

25. $\begin{cases} x - y - z - t = 1 \\ 2x + y - z + 2t = 3 \\ x - 2y - 2z - 3t = 0 \\ 3x - 4y + z + 5t = -3 \end{cases}$

In Problems 26–28, find the value of each determinant.

26. $\begin{vmatrix} 3 & 4 \\ 1 & 3 \end{vmatrix}$

27. $\begin{vmatrix} 1 & 4 & 0 \\ -1 & 2 & 6 \\ 4 & 1 & 3 \end{vmatrix}$

28. $\begin{vmatrix} 2 & 1 & -3 \\ 5 & 0 & 1 \\ 2 & 6 & 0 \end{vmatrix}$

In Problems 29–31, use Cramer's Rule, if applicable, to solve each system.

29. $\begin{cases} x - 2y = 4 \\ 3x + 2y = 4 \end{cases}$

30. $\begin{cases} 2x + 3y - 13 = 0 \\ 3x - 2y = 0 \end{cases}$

31. $\begin{cases} x + 2y - z = 6 \\ 2x - y + 3z = -13 \\ 3x - 2y + 3z = -16 \end{cases}$

In Problems 32 and 33, use properties of determinants to find the value of each determinant if it is known that $\begin{vmatrix} x & y \\ a & b \end{vmatrix} = 8$.

32. $\begin{vmatrix} 2x & y \\ 2a & b \end{vmatrix}$

33. $\begin{vmatrix} y & x \\ b & a \end{vmatrix}$

In Problems 34–38, write the partial fraction decomposition of each rational expression.

34. $\dfrac{6}{x(x-4)}$

35. $\dfrac{x-4}{x^2(x-1)}$

36. $\dfrac{x}{(x^2+9)(x+1)}$

37. $\dfrac{x^3}{(x^2+4)^2}$

38. $\dfrac{x^2}{(x^2+1)(x^2-1)}$

In Problems 39–43, solve each system of equations.

39. $\begin{cases} 2x + y + 3 = 0 \\ x^2 + y^2 = 5 \end{cases}$

40. $\begin{cases} 2xy + y^2 = 10 \\ 3y^2 - xy = 2 \end{cases}$

41. $\begin{cases} x^2 + y^2 = 6y \\ x^2 = 3y \end{cases}$

42. $\begin{cases} 3x^2 + 4xy + 5y^2 = 8 \\ x^2 + 3xy + 2y^2 = 0 \end{cases}$

43. $\begin{cases} x^2 - 3x + y^2 + y = -2 \\ \dfrac{x^2 - x}{y} + y + 1 = 0 \end{cases}$

In Problems 44 and 45, graph each inequality by hand. Verify your results using a graphing utility.

44. $3x + 4y \le 12$

45. $y \le x^2$

In Problems 46–48, graph each system of inequalities by hand. Tell whether the graph is bounded or unbounded, and label the corner points.

46. $\begin{cases} -2x + y \le 2 \\ x + y \ge 2 \end{cases}$

47. $\begin{cases} x \ge 0 \\ y \ge 0 \\ x + y \le 4 \\ 2x + 3y \le 6 \end{cases}$

48. $\begin{cases} x \ge 0 \\ y \ge 0 \\ 2x + y \le 8 \\ x + 2y \ge 2 \end{cases}$

In Problems 49 and 50, graph each system of inequalities by hand.

49. $\begin{cases} x^2 + y^2 \le 16 \\ x + y \ge 2 \end{cases}$

50. $\begin{cases} x^2 + y^2 \le 25 \\ xy \le 4 \end{cases}$

In Problems 51 and 52, solve each linear programming problem.

51. Maximize $z = 3x + 4y$ subject to $x \ge 0, y \ge 0, 3x + 2y \ge 6, x + y \le 8$

52. Minimize $z = 3x + 5y$ subject to $x \ge 0, y \ge 0, x + y \ge 1, 3x + 2y \le 12, x + 3y \le 12$

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53. Find A so that the system of equations has infinitely many solutions.

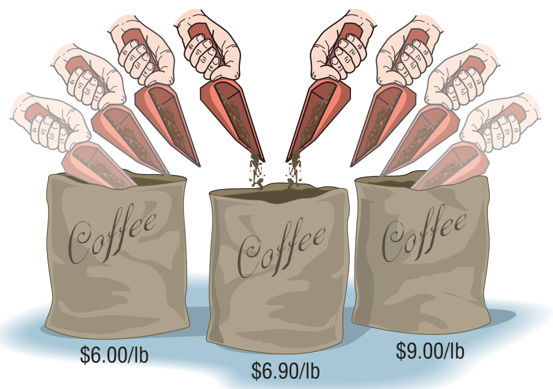
$$\begin{cases} 2x + 5y = 5 \\ 4x + 10y = A \end{cases}$$

54. Find A so that the system in Problem 53 is inconsistent.

55. Curve Fitting Find the quadratic function $y = ax^2 + bx + c$ that passes through the three points $(0, 1)$, $(1, 0)$, and $(-2, 1)$.

56. Blending Coffee A coffee distributor is blending a new coffee that will cost \$6.90 per pound. It will consist of a blend of \$6.00-per-pound coffee and \$9.00-per-pound coffee. What amounts of each type of coffee should be mixed to achieve the desired blend?

[**Hint:** Assume that the weight of the blended coffee is 100 pounds.]



57. Cookie Orders A cookie company makes three kinds of cookies (oatmeal raisin, chocolate chip, and shortbread) packaged in small, medium, and large boxes. The small box contains 1 dozen oatmeal raisin and 1 dozen chocolate chip; the medium box has 2 dozen oatmeal raisin, 1 dozen chocolate chip, and 1 dozen shortbread; the large box contains 2 dozen oatmeal raisin, 2 dozen chocolate chip, and 3 dozen shortbread. If you require exactly 15 dozen oatmeal raisin, 10 dozen chocolate chip, and 11 dozen shortbread, how many of each size box should you buy?

58. Mixed Nuts A store that specializes in selling nuts has available 72 pounds (lb) of cashews and 120 lb of peanuts. These are to be mixed in 12-ounce (oz) packages as follows: a lower-priced package containing 8 oz of peanuts and 4 oz of cashews, and a quality package containing 6 oz of peanuts and 6 oz of cashews.

(a) Use x to denote the number of lower-priced packages, and use y to denote the number of quality packages. Write a system of linear inequalities that describes the possible numbers of each kind of package.

(b) Graph the system and label the corner points.

59. Determining the Speed of the Current of the Aguarico River On a recent trip to the Cuyabeno Wildlife Reserve in the Amazon region of Ecuador, Mike took a 100-kilometer trip by speedboat down the Aguarico River from Chiritza to the Flotel Orellana. As Mike watched the Amazon unfold, he wondered how fast the speedboat was going and how fast the current of the white-water Aguarico River was. Mike timed the trip downstream at 2.5 hours and the return trip at 3 hours. What were the two speeds?

60. Constant Rate Jobs If Bruce and Bryce work together for 1 hour and 20 minutes, they will finish a certain job. If Bryce and Marty work together for 1 hour and 36 minutes, the same job can be finished. If Marty and Bruce work together, they can complete this job in 2 hours and 40 minutes. How long would it take each of them, working alone, to finish the job?

61. Minimizing Production Cost A factory produces gasoline engines and diesel engines. Each week the factory is obligated to deliver at least 20 gasoline engines and at least 15 diesel engines. Due to physical limitations, however, the factory cannot make more than 60 gasoline engines or more than 40 diesel engines in any given week. Finally, to prevent layoffs, a total of at least 50 engines must be produced. If gasoline engines cost \$450 each to produce and diesel engines cost \$550 each to produce, how many of each should be produced per week to minimize the cost? What is the excess capacity of the factory? That is, how many of each kind of engine are being produced in excess of the number that the factory is obligated to deliver?

62. Describe four ways of solving a system of three linear equations containing three variables. Which method do you prefer? Why?

Chapter Test

CHAPTER **Test Prep** VIDEOS

The Chapter Test Prep Videos are step-by-step solutions available in MyMathLab®, or on this text's YouTube™ Channel. Flip back to the Resources for Success page for a link to this text's YouTube channel.

In Problems 1–4, solve each system of equations using the method of substitution or the method of elimination. If the system has no solution, say that it is inconsistent.

1. $\begin{cases} -2x + y = -7 \\ 4x + 3y = 9 \end{cases}$

2. $\begin{cases} \frac{1}{3}x - 2y = 1 \\ 5x - 30y = 18 \end{cases}$

3. $\begin{cases} x - y + 2z = 5 \\ 3x + 4y - z = -2 \\ 5x + 2y + 3z = 8 \end{cases}$

4. $\begin{cases} 3x + 2y - 8z = -3 \\ -x - \frac{2}{3}y + z = 1 \\ 6x - 3y + 15z = 8 \end{cases}$

5. Write the augmented matrix corresponding to the system of equations: $\begin{cases} 4x - 5y + z = 0 \\ -2x - y + 6 = -19 \\ x + 5y - 5z = 10 \end{cases}$

6. Write the system of equations corresponding to the augmented matrix: $\left[\begin{array}{ccc|c} 3 & 2 & 4 & -6 \\ 1 & 0 & 8 & 2 \\ -2 & 1 & 3 & -11 \end{array}\right]$

In Problems 7–10, use the given matrices to compute each expression.

$$A = \begin{bmatrix} 1 & -1 \\ 0 & -4 \\ 3 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -2 & 5 \\ 0 & 3 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 4 & 6 \\ 1 & -3 \\ -1 & 8 \end{bmatrix}$$

7. $2A + C$

8. $A - 3C$

9. CB

10. BA

In Problems 11 and 12, find the inverse of each nonsingular matrix.

11. $A = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$

12. $B = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 5 & -1 \\ 2 & 3 & 0 \end{bmatrix}$

In Problems 13–16, solve each system of equations using matrices. If the system has no solution, say that it is inconsistent.

13. $\begin{cases} 6x + 3y = 12 \\ 2x - y = -2 \end{cases}$

14. $\begin{cases} x + \frac{1}{4}y = 7 \\ 8x + 2y = 56 \end{cases}$

15. $\begin{cases} x + 2y + 4z = -3 \\ 2x + 7y + 15z = -12 \\ 4x + 7y + 13z = -10 \end{cases}$

16. $\begin{cases} 2x + 2y - 3z = 5 \\ x - y + 2z = 8 \\ 3x + 5y - 8z = -2 \end{cases}$

In Problems 17 and 18, find the value of each determinant.

17. $\begin{vmatrix} -2 & 5 \\ 3 & 7 \end{vmatrix}$

18. $\begin{vmatrix} 2 & -4 & 6 \\ 1 & 4 & 0 \\ -1 & 2 & -4 \end{vmatrix}$

In Problems 19 and 20, use Cramer's Rule, if possible, to solve each system.

19. $\begin{cases} 4x + 3y = -23 \\ 3x - 5y = 19 \end{cases}$

20. $\begin{cases} 4x - 3y + 2z = 15 \\ -2x + y - 3z = -15 \\ 5x - 5y + 2z = 18 \end{cases}$

In Problems 21 and 22, solve each system of equations.

21. $\begin{cases} 3x^2 + y^2 = 12 \\ y^2 = 9x \end{cases}$

22. $\begin{cases} 2y^2 - 3x^2 = 5 \\ y - x = 1 \end{cases}$

23. Graph the system of inequalities: $\begin{cases} x^2 + y^2 \le 100 \\ 4x - 3y \ge 0 \end{cases}$

In Problems 24 and 25, write the partial fraction decomposition of each rational expression.

24. $\dfrac{3x + 7}{(x + 3)^2}$

25. $\dfrac{4x^2 - 3}{x(x^2 + 3)^2}$

26. Graph the system of inequalities. Tell whether the graph is bounded or unbounded, and label all corner points.

$$\begin{cases} x \ge 0 \\ y \ge 0 \\ x + 2y \ge 8 \\ 2x - 3y \ge 2 \end{cases}$$

27. Maximize $z = 5x + 8y$ subject to $x \ge 0, 2x + y \le 8$, and $x - 3y \le -3$.

28. Megan went clothes shopping and bought 2 pairs of flare jeans, 2 camisoles, and 4 T-shirts for \$90.00. At the same store, Paige bought one pair of flare jeans and 3 T-shirts for \$42.50, while Kara bought 1 pair of flare jeans, 3 camisoles, and 2 T-shirts for \$62.00. Determine the price of each clothing item.

Cumulative Review

In Problems 1–6, solve each equation.

1. $2x^2 - x = 0$

2. $\sqrt{3x + 1} = 4$

3. $2x^3 - 3x^2 - 8x - 3 = 0$

4. $3^x = 9^{x+1}$

5. $\log_3(x - 1) + \log_3(2x + 1) = 2$

6. $3^x = e$

7. Determine whether the function $g(x) = \dfrac{2x^3}{x^4 + 1}$ is even, odd, or neither. Is the graph of g symmetric with respect to the x-axis, y-axis, or origin?

8. Find the center and radius of the circle $x^2 + y^2 - 2x + 4y - 11 = 0$. Graph the circle.

9. Graph $f(x) = 3^{x-2} + 1$ using transformations. What is the domain, range, and horizontal asymptote of f?

10. The function $f(x) = \dfrac{5}{x + 2}$ is one-to-one. Find f^{-1}. Find the domain and the range of f and the domain and the range of f^{-1}.

11. Graph each equation.
(a) $y = 3x + 6$ (b) $x^2 + y^2 = 4$
(c) $y = x^3$ (d) $y = \dfrac{1}{x}$
(e) $y = \sqrt{x}$ (f) $y = e^x$
(g) $y = \ln x$ (h) $2x^2 + 5y^2 = 1$
(i) $x^2 - 3y^2 = 1$ (j) $x^2 - 2x - 4y + 1 = 0$

12. $f(x) = x^3 - 3x + 5$
(a) Using a graphing utility, graph f and approximate the zero(s) of f.
(b) Using a graphing utility, approximate the local maxima and local minima.
(c) Determine the intervals on which f is increasing.

Chapter Projects

I. Markov Chains A **Markov chain** (or process) is one in which future outcomes are determined by a current state. Future outcomes are based on probabilities. The probability of moving to a certain state depends only on the state previously occupied and does not vary with time. An example of a Markov chain is the maximum education achieved by children based on the highest educational level attained by their parents, where the states are (1) earned college degree, (2) high school diploma only, (3) elementary school only. If p_{ij} is the probability of moving from state i to state j, the **transition matrix** is the $m \times m$ matrix

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ \vdots & \vdots & & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}$$

The table represents the probabilities for the highest educational level of children based on the highest educational level of their parents. For example, the table shows that the probability p_{21} is 40% that parents with a high-school education (row 2) will have children with a college education (column 1).

Highest Educational Level of Parents	Maximum Education That Children Achieve		
	College	High School	Elementary
College	80%	18%	2%
High school	40%	50%	10%
Elementary	20%	60%	20%

1. Convert the percentages to decimals.
2. What is the transition matrix?
3. Sum across the rows. What do you notice? Why do you think that you obtained this result?
4. If P is the transition matrix of a Markov chain, the (i, j)th entry of P^n (nth power of P) gives the probability of passing from state i to state j in n stages. What is the probability that the grandchild of a college graduate is a college graduate?
5. What is the probability that the grandchild of a high school graduate finishes college?
6. The row vector $v^{(0)} = [0.319 \quad 0.564 \quad 0.117]$ represents the proportion of the U.S. population 25 years or older that has college, high school, and elementary school, respectively, as the highest educational level in 2014.* In a Markov chain the probability distribution $v^{(k)}$ after k stages is $v^{(k)} = v^{(0)}P^k$, where P^k is the kth power of the transition matrix. What will be the distribution of highest educational attainment of the grandchildren of the current population?
7. Calculate $P^3, P^4, P^5, \ldots$. Continue until the matrix does not change. This is called the long-run or steady-state distribution. What is the long-run distribution of highest educational attainment of the population?

*****Source:*** *U.S. Census Bureau.*

The following projects are available at the Instructor's Resource Center (IRC).

II. Project at Motorola: ***Error Control Coding*** The high-powered engineering needed to ensure that wireless communications are transmitted correctly is analyzed using matrices to control coding errors.

III. Using Matrices to Find the Line of Best Fit Have you wondered how our calculators get a line of best fit? See how to find the line by solving a matrix equation.

IV. CBL Experiment Simulate two people walking toward each other at a constant rate. Then solve the resulting system of equations to determine when and where they will meet.

9 Sequences; Induction; the Binomial Theorem

UN Projects World Population Will Hit 9.6 Billion by 2050

The population of the planet is expected to reach 9.6 billion by 2050, according to a new UN report—a slightly larger number than anticipated, because fertility projections have increased in nations where women have the most children.

More than half of this projected demographic growth will be in Africa, which continues to add people even as population growth in the world at large slows down.

"Although population growth has slowed for the world as a whole, this report reminds us that some developing countries, especially in Africa, are still growing rapidly," said Under-Secretary-General for Economic and Social Affairs Wu Hongbo, in the UN report.

World Population Prospects: The 2012 Revision notes that the population in the developed regions of the world should remain stable at around 1.3 billion until 2050 thanks to trends of low fertility, while the world's 49 least-developed countries are projected to double in population size.

The new figures do *not* mean that world population growth has started to speed up again. As countries industrialize, they tend to undergo "demographic transition," wherein high death and birth rates are slowly replaced with low birth and death rates. It's a demographic shift often helped along by an increase in the granting of rights for women.

Some experts suspect that the world population will plateau around 2060, and there's a possibility that—after a transitional period of a higher death rate than birth rate—world birth and death rates could actually even out, keeping the human population stable.

Source: Faine Greenwood, June 14, 2013, 11:36. GlobalPost®(www.globalpost.com/dispatch/news/science/130614/un-projects-world-population-will-hit-96-billion-2050)

—See the Internet-based Chapter Project I—

••• A Look Back, A Look Ahead •••

This chapter may be divided into three independent parts: Sections 9.1–9.3, Section 9.4, and Section 9.5.

In Chapter 3, we defined a function and its domain, which was usually some set of real numbers. In Sections 9.1–9.3, we discuss a sequence, which is a function whose domain is the set of positive integers.

Throughout this text, where it seemed appropriate, we have given proofs of many of the results. In Section 9.4, a technique for proving theorems involving natural numbers is discussed.

In Chapter R, Review, Section R.4, there are formulas for expanding $(x+a)^2$ and $(x+a)^3$. In Section 9.5, we discuss the Binomial Theorem, a formula for the expansion of $(x+a)^n$, where n is any positive integer.

The topics introduced in this chapter are covered in more detail in courses titled *Discrete Mathematics*. Applications of these topics can be found in the fields of computer science, engineering, business and economics, the social sciences, and the physical and biological sciences.

Outline

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Functions (Section 3.1, pp. 207–213)
- Compound Interest (Section 6.7, pp. 475–480)

Now Work the 'Are You Prepared?' problems on page 663.

OBJECTIVES
1. Write the First Several Terms of a Sequence (p. 654)
2. Write the Terms of a Sequence Defined by a Recursive Formula (p. 657)
3. Use Summation Notation (p. 658)
4. Find the Sum of a Sequence Algebraically and Using a Graphing Utility (p. 659)
5. Solve Annuity and Amortization Problems (p. 661)

When you hear the word *sequence* as it is used in the phrase "a sequence of events," you probably think of a collection of events, one of which happens first, another second, and so on. In mathematics, the word *sequence* also refers to outcomes that are first, second, and so on.

DEFINITION

A **sequence** is a function whose domain is the set of positive integers.

In a sequence, then, the inputs are 1, 2, 3, ... Because a sequence is a function, it will have a graph. Figure 1(a) shows the graph of the function $f(x) = \frac{1}{x}, x > 0$. If all the points on this graph were removed except those whose x-coordinates are positive integers—that is, if all points were removed except $(1, 1)$, $\left(2, \frac{1}{2}\right)$, $\left(3, \frac{1}{3}\right)$, and so on—the remaining points would be the graph of the sequence $f(n) = \frac{1}{n}$, as shown in Figure 1(b). Note that n is used to represent the independent variable in a sequence. This serves to remind us that n is a positive integer.

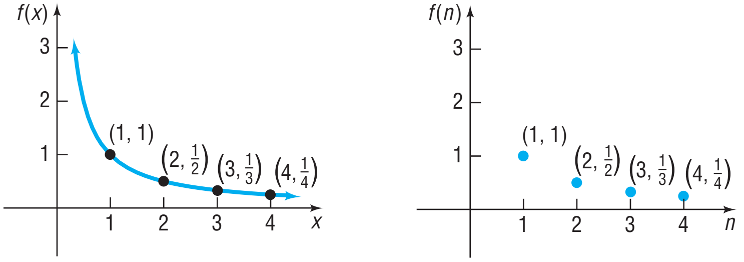


Figure 1 (a) $f(x) = \frac{1}{x}, x > 0$ (b) $f(n) = \frac{1}{n}$, n a positive integer

1 Write the First Several Terms of a Sequence

A sequence is usually represented by listing its values in order. For example, the sequence whose graph is given in Figure 1(b) might be represented as

$$f(1), f(2), f(3), f(4), \ldots \quad \text{or} \quad 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

The list never ends, as the ellipsis indicates. The numbers in this ordered list are called the **terms** of the sequence.

In dealing with sequences, subscripted letters are used such as a_1 to represent the first term, a_2 for the second term, a_3 for the third term, and so on.

For the sequence $f(n) = \frac{1}{n}$, this means

$$\underbrace{a_1 = f(1) = 1}_{\text{first term}} \quad \underbrace{a_2 = f(2) = \frac{1}{2}}_{\text{second term}} \quad \underbrace{a_3 = f(3) = \frac{1}{3}}_{\text{third term}} \quad \underbrace{a_4 = f(4) = \frac{1}{4}}_{\text{fourth term}} \ldots \underbrace{a_n = f(n) = \frac{1}{n}}_{n\text{th term}} \ldots$$

In other words, the traditional function notation $f(n)$ is not used for sequences. For this particular sequence, we have a rule for the nth term, which is $a_n = \frac{1}{n}$, so it is easy to find any term of the sequence.

When a formula for the nth term (sometimes called the **general term**) of a sequence is known, the entire sequence can be represented by placing braces around the formula for the nth term. For example, the sequence whose nth term is $b_n = \left(\frac{1}{2}\right)^n$ may be represented as

$$\{b_n\} = \left\{\left(\frac{1}{2}\right)^n\right\}$$

or by

$$b_1 = \frac{1}{2}, \quad b_2 = \frac{1}{4}, \quad b_3 = \frac{1}{8}, \ldots, \quad b_n = \left(\frac{1}{2}\right)^n, \ldots$$

EXAMPLE 1 Writing the First Several Terms of a Sequence

Write down the first six terms of the following sequence and graph it.

$$\{a_n\} = \left\{\frac{n-1}{n}\right\}$$

Algebraic Solution

The first six terms of the sequence are

$$a_1 = \frac{1-1}{1} = 0, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{2}{3},$$

$$a_4 = \frac{3}{4}, \quad a_5 = \frac{4}{5}, \quad a_6 = \frac{5}{6}$$

See Figure 2 for the graph.

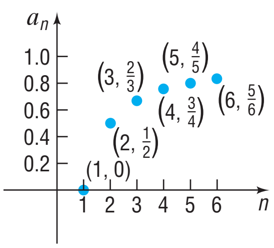


Figure 2 $\{a_n\} = \left\{\dfrac{n-1}{n}\right\}$

Graphing Solution

Figure 3 shows the sequence generated on a TI-84 Plus C graphing calculator. We can see the first few terms of the sequence on the screen.

We could also obtain the terms of the sequence using the TABLE feature. First, put the graphing utility in SEQuence mode. Press $Y=$ and enter the formula for the sequence into the graphing utility. See Figure 4. Set up the table with TblStart = 1 and ΔTbl = 1. See Table 1. Finally, we can graph the sequence. See Figure 5. Notice that the first term of the sequence is hard to see since it lies on the x-axis. TRACEing the graph allows you to determine the terms of the sequence.

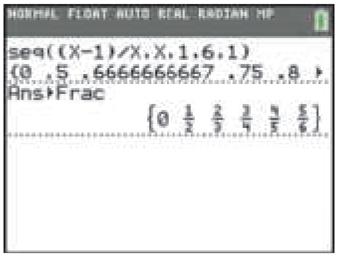


Figure 3

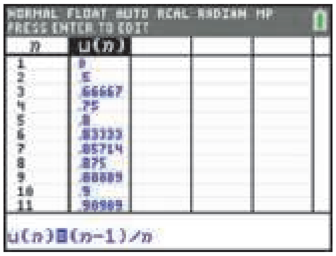


Table 1

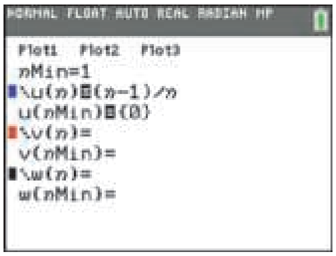


Figure 4

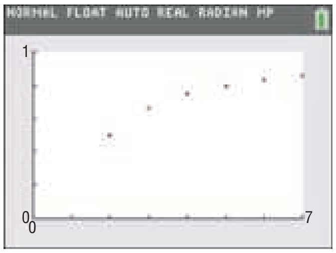


Figure 5

Now Work PROBLEM 19

We will usually provide solutions done by hand. The reader is encouraged to verify solutions using a graphing utility.

EXAMPLE 2 Writing the First Several Terms of a Sequence

Write down the first six terms of the following sequence and graph it.

$$\{b_n\} = \left\{(-1)^{n+1}\left(\frac{2}{n}\right)\right\}$$

Solution The first six terms of the sequence are

$$b_1 = (-1)^{1+1}\left(\frac{2}{1}\right) = 2 \quad b_2 = (-1)^{2+1}\left(\frac{2}{2}\right) = -1 \quad b_3 = (-1)^{3+1}\left(\frac{2}{3}\right) = \frac{2}{3}$$

$$b_4 = -\frac{1}{2} \qquad b_5 = \frac{2}{5} \qquad b_6 = -\frac{1}{3}$$

See Figure 6 for the graph. ■

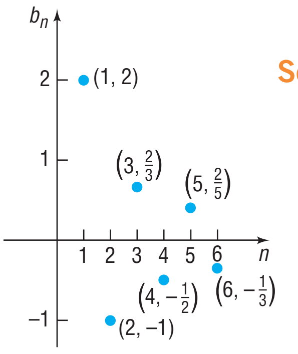


Figure 6 $\{b_n\} = \left\{(-1)^{n+1}\left(\frac{2}{n}\right)\right\}$

Notice in the sequence $\{b_n\}$ in Example 2 that the signs of the terms **alternate**. This occurs when we use factors such as $(-1)^{n+1}$, which equals 1 if n is odd and -1 if n is even, or $(-1)^n$, which equals -1 if n is odd and 1 if n is even.

EXAMPLE 3 Writing the First Several Terms of a Sequence

Write down the first six terms of the following sequence and graph it.

$$\{c_n\} = \left\{\begin{matrix} n & \text{if } n \text{ is even} \\ \frac{1}{n} & \text{if } n \text{ is odd} \end{matrix}\right\}$$

Solution The first six terms of the sequence are

$$c_1 = 1 \quad c_2 = 2 \quad c_3 = \frac{1}{3} \quad c_4 = 4 \quad c_5 = \frac{1}{5} \quad c_6 = 6$$

See Figure 7 for the graph. ■

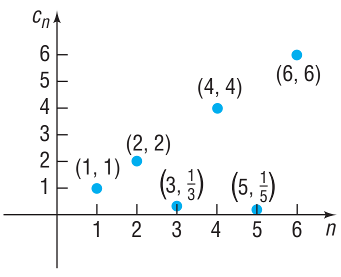


Figure 7 $\{c_n\} = \left\{\begin{matrix} n & \text{if } n \text{ is even} \\ \frac{1}{n} & \text{if } n \text{ is odd} \end{matrix}\right\}$

Now Work PROBLEM 21

Note that the formulas that generate the terms of a sequence are not unique. For example, the terms of the sequence in Example 3 could also be found using

$$\{d_n\} = \{n^{(-1)^n}\}$$

Sometimes a sequence is indicated by an observed pattern in the first few terms that makes it possible to infer the makeup of the nth term. In the example that follows, a sufficient number of terms of the sequence is given so that a natural choice for the nth term is suggested.

EXAMPLE 4 Determining a Sequence from a Pattern

(a) $e, \frac{e^2}{2}, \frac{e^3}{3}, \frac{e^4}{4}, \ldots \qquad a_n = \frac{e^n}{n}$

(b) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots \qquad b_n = \frac{1}{3^{n-1}}$

(c) $1, 3, 5, 7, \ldots \qquad c_n = 2n - 1$

(d) $1, 4, 9, 16, 25, \ldots \qquad d_n = n^2$

(e) $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots \qquad e_n = (-1)^{n+1}\left(\frac{1}{n}\right)$ ■

Now Work PROBLEM 29

The Factorial Symbol

Some sequences in mathematics involve a special product called a *factorial*.

DEFINITION

If $n \geq 0$ is an integer, the **factorial symbol $n!$** is defined as follows:

$$0! = 1 \qquad 1! = 1$$
$$n! = n(n-1)\cdot\cdots\cdot 3\cdot 2\cdot 1 \qquad \text{if } n \geq 2$$

Table 2

n	$n!$
0	1
1	1
2	2
3	6
4	24
5	120
6	720

For example, $2! = 2\cdot 1 = 2$, $3! = 3\cdot 2\cdot 1 = 6$, $4! = 4\cdot 3\cdot 2\cdot 1 = 24$, and so on. Table 2 lists the values of $n!$ for $0 \leq n \leq 6$.

Because

$$n! = \underbrace{n(n-1)(n-2)\cdot\ldots\cdot 3\cdot 2\cdot 1}_{(n-1)!}$$

the formula

$$\boxed{n! = n\cdot(n-1)!}$$

can be used to find successive factorials. For example, because $6! = 720$,

$$7! = 7\cdot 6! = 7(720) = 5040$$

and

$$8! = 8\cdot 7! = 8(5040) = 40{,}320$$

Exploration

Use your calculator's factorial key to see how fast factorials increase in value. Find the value of 69!. What happens when you try to find 70!? In fact, 70! is larger than 10^{100} (a googol), the largest number most calculators can display.

Now Work PROBLEM 13

2 Write the Terms of a Sequence Defined by a Recursive Formula

A second way of defining a sequence is to assign a value to the first (or the first few) term(s) and specify the nth term by a formula or equation that involves one or more of the terms preceding it. Such sequences are said to be defined **recursively**, and the rule or formula is called a **recursive formula**.

EXAMPLE 5 Writing the Terms of a Recursively Defined Sequence

Write down the first five terms of the following recursively defined sequence.

$$s_1 = 1, \qquad s_n = ns_{n-1}$$

Algebraic Solution

The first term is given as $s_1 = 1$. To get the second term, use $n = 2$ in the formula $s_n = ns_{n-1}$ to get $s_2 = 2s_1 = 2\cdot 1 = 2$. To get the third term, use $n = 3$ in the formula to get $s_3 = 3s_2 = 3\cdot 2 = 6$. To get a new term requires knowing the value of the preceding term. The first five terms are

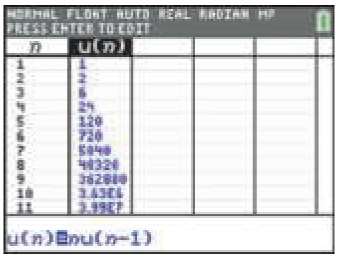
$$s_1 = 1$$
$$s_2 = 2\cdot 1 = 2$$
$$s_3 = 3\cdot 2 = 6$$
$$s_4 = 4\cdot 6 = 24$$
$$s_5 = 5\cdot 24 = 120$$

Do you recognize this sequence? $s_n = n!$.

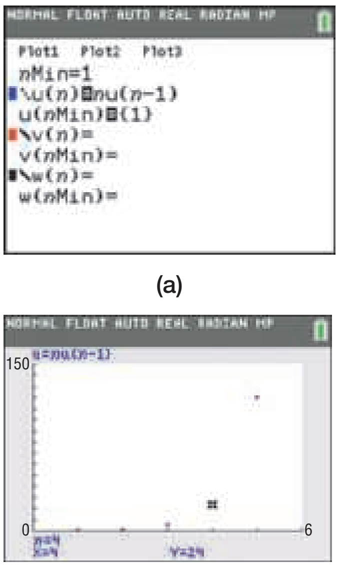
Graphing Solution

First, put the graphing utility into SEQuence mode. Press $Y=$ and enter the recursive formula into the graphing utility to generate the desired sequence. See Figure 8(a). Next, set up the viewing window. Finally, graph the recursive relation and use TRACE to determine the terms in the sequence. See Figure 8(b). For example, the fourth term of the sequence is 24. Table 3 also shows the terms of the sequence.

Table 3



n	u(n)
1	1
2	2
3	6
4	24
5	120
6	720
7	5040
8	40320
9	362880
10	3.63E6
11	3.99E7



(a)

(b)

Figure 8

EXAMPLE 6 **Writing the Terms of a Recursively Defined Sequence**

Write down the first five terms of the following recursively defined sequence.

$$u_1 = 1 \qquad u_2 = 1 \qquad u_n = u_{n-2} + u_{n-1}$$

Solution The first two terms are given. Finding each successive term requires knowing the previous two terms. That is,

$$\begin{aligned} u_1 &= 1 \\ u_2 &= 1 \\ u_3 &= u_1 + u_2 = 1 + 1 = 2 \\ u_4 &= u_2 + u_3 = 1 + 2 = 3 \\ u_5 &= u_3 + u_4 = 2 + 3 = 5 \end{aligned}$$

Now Work PROBLEMS 37 AND 45

The sequence given in Example 6 is called the **Fibonacci sequence**, and the terms of this sequence are called **Fibonacci numbers**. These numbers appear in a wide variety of applications (see Problems 91–94).

3 Use Summation Notation

It is often important to find the sum of the first n terms of a sequence $\{a_n\}$—that is,

$$a_1 + a_2 + a_3 + \cdots + a_n$$

Rather than writing down all these terms, we can use **summation notation** to express the sum more concisely:

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

The symbol Σ (the Greek letter sigma, which is an S in our alphabet) is simply an instruction to sum, or add up, the terms. The integer k is called the **index** of the sum; it tells where to start the sum and where to end it. The expression

$$\sum_{k=1}^{n} a_k$$

is an instruction to add the terms a_k of the sequence $\{a_n\}$ starting with $k = 1$ and ending with $k = n$. The expression is read as "the sum of a_k from $k = 1$ to $k = n$."

EXAMPLE 7 **Expanding Summation Notation**

Write out each sum.

(a) $\displaystyle\sum_{k=1}^{n} \frac{1}{k}$ (b) $\displaystyle\sum_{k=1}^{n} k!$

Solution (a) $\displaystyle\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ (b) $\displaystyle\sum_{k=1}^{n} k! = 1! + 2! + \cdots + n!$

Now Work PROBLEM 53

EXAMPLE 8 **Writing a Sum in Summation Notation**

Express each sum using summation notation.

(a) $1^2 + 2^2 + 3^2 + \cdots + 9^2$ (b) $1 + \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \cdots + \dfrac{1}{2^{n-1}}$

Solution (a) The sum $1^2 + 2^2 + 3^2 + \cdots + 9^2$ has 9 terms, each of the form k^2, and starts at $k = 1$ and ends at $k = 9$:

$$1^2 + 2^2 + 3^2 + \cdots + 9^2 = \sum_{k=1}^{9} k^2$$

(b) The sum

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}}$$

has n terms, each of the form $\frac{1}{2^{k-1}}$, and starts at $k = 1$ and ends at $k = n$:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} = \sum_{k=1}^{n} \frac{1}{2^{k-1}}$$

Now Work PROBLEM 63

The index of summation need not always begin at 1 or end at n; for example, the sum in Example 8(b) could also be expressed as

$$\sum_{k=0}^{n-1} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$$

Letters other than k may be used as the index. For example,

$$\sum_{j=1}^{n} j! \quad \text{and} \quad \sum_{i=1}^{n} i!$$

both represent the same sum given in Example 7(b).

4 Find the Sum of a Sequence Algebraically and Using a Graphing Utility

The following theorem lists some properties of sequences using summation notation. These properties are useful for adding the terms of a sequence algebraically.

THEOREM

Properties of Sequences

If $\{a_n\}$ and $\{b_n\}$ are two sequences and c is a real number, then:

$$\sum_{k=1}^{n} (ca_k) = ca_1 + ca_2 + \cdots + ca_n = c(a_1 + a_2 + \cdots + a_n) = c\sum_{k=1}^{n} a_k \quad \textbf{(1)}$$

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \quad \textbf{(2)}$$

$$\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k \quad \textbf{(3)}$$

$$\sum_{k=j+1}^{n} a_k = \sum_{k=1}^{n} a_k - \sum_{k=1}^{j} a_k \quad \text{where } 0 < j < n \quad \textbf{(4)}$$

The proof of property (1) follows from the distributive property of real numbers. The proofs of properties 2 and 3 are based on the commutative and associative properties of real numbers. Property (4) states that the sum from $j + 1$ to n equals the sum from 1 to n minus the sum from 1 to j. It can be helpful to employ this property when the index of summation begins at a number larger than 1.

THEOREM

Formulas for Sums of Sequences

$$\sum_{k=1}^{n} c = \underbrace{c + c + \cdots + c}_{n \text{ terms}} = cn \qquad c \text{ is a real number} \tag{5}$$

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \tag{6}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{7}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 \tag{8}$$

The proof of formula (5) follows from the definition of summation notation. You are asked to prove formula (6) in Problem 98. The proofs of formulas (7) and (8) require mathematical induction, which is discussed in Section 9.4.

Notice the difference between formulas (5) and (6). In (5), the constant c is being summed from 1 to n, while in (6) the index of summation k is being summed from 1 to n.

EXAMPLE 9 Finding the Sum of a Sequence

Find the sum of each sequence.

(a) $\sum_{k=1}^{5} (3k)$ (b) $\sum_{k=1}^{10} (k^3 + 1)$

(c) $\sum_{k=1}^{24} (k^2 - 7k + 2)$ (d) $\sum_{k=6}^{20} (4k^2)$

Algebraic Solution

(a)
$$\begin{aligned} \sum_{k=1}^{5} (3k) &= 3\sum_{k=1}^{5} k && \textbf{Property (1)} \\ &= 3\left(\frac{5(5+1)}{2}\right) && \textbf{Formula (6)} \\ &= 3(15) \\ &= 45 \end{aligned}$$

(b)
$$\begin{aligned} \sum_{k=1}^{10} (k^3 + 1) &= \sum_{k=1}^{10} k^3 + \sum_{k=1}^{10} 1 && \textbf{Property (2)} \\ &= \left(\frac{10(10+1)}{2}\right)^2 + 1(10) && \textbf{Formulas (8) and (5)} \\ &= 3025 + 10 \\ &= 3035 \end{aligned}$$

Graphing Solution

(a) Figure 9 shows the solution using a TI-84 Plus C graphing calculator. So $\sum_{k=1}^{5} (3k) = 45$.

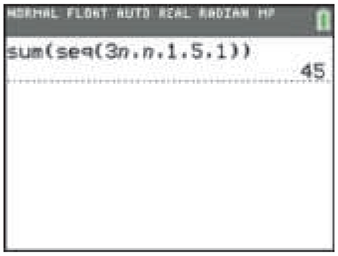


Figure 9

(b) Figure 10 shows the solution using a TI-84 Plus C graphing calculator. So $\sum_{k=1}^{10} (k^3 + 1) = 3035$.

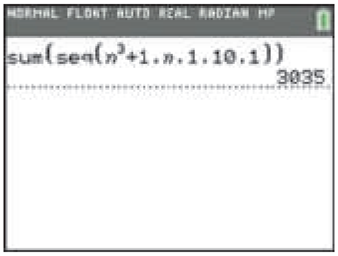


Figure 10

(c)
$$\sum_{k=1}^{24}(k^2 - 7k + 2) = \sum_{k=1}^{24} k^2 - \sum_{k=1}^{24}(7k) + \sum_{k=1}^{24} 2 \qquad \text{Properties (2) and (3)}$$
$$= \sum_{k=1}^{24} k^2 - 7\sum_{k=1}^{24} k + \sum_{k=1}^{24} 2 \qquad \text{Property (1)}$$
$$= \frac{24(24+1)(2\cdot 24+1)}{6} - 7\left(\frac{24(24+1)}{2}\right) + 2(24) \qquad \text{Formulas (7), (6), (5)}$$
$$= 4900 - 2100 + 48$$
$$= 2848$$

(d) Notice that the index of summation starts at 6. Use property (4) as follows:

$$\sum_{k=6}^{20}(4k^2) \underset{\text{Property (1)}}{=} 4\sum_{k=6}^{20} k^2 \underset{\text{Property (4)}}{=} 4\left[\sum_{k=1}^{20} k^2 - \sum_{k=1}^{5} k^2\right] \underset{\text{Formula (7)}}{=} 4\left[\frac{20(21)(41)}{6} - \frac{5(6)(11)}{6}\right]$$
$$= 4[2870 - 55] = 11{,}260$$

(c) Figure 11 shows the solution using a TI-84 Plus C graphing calculator.

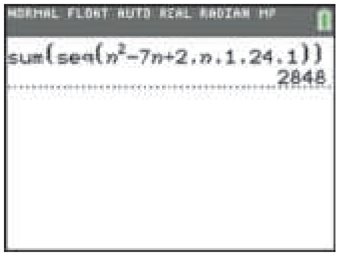


Figure 11

(d) Figure 12 shows the solution using a TI-84 Plus C graphing calculator.

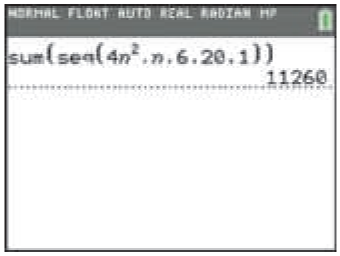


Figure 12

Now Work PROBLEM 75

5 Solve Annuity and Amortization Problems

In Section 6.7 we developed the compound interest formula, which gives the future value when a fixed amount of money is deposited in an account that pays interest compounded periodically. Often, though, money is invested in small amounts at periodic intervals. An **annuity** is a sequence of equal periodic deposits. The periodic deposits may be made annually, quarterly, monthly, or daily.

When deposits are made at the same time that the interest is credited, the annuity is called **ordinary**. We will only deal with ordinary annuities here. The **amount of an annuity** is the sum of all deposits made plus all interest paid.

Suppose that the initial amount deposited in an annuity is $\$M$, the periodic deposit is $\$P$, and the per annum rate of interest is $r\%$ (expressed as a decimal) compounded N times per year. The periodic deposit is made at the same time that the interest is credited, so N deposits are made per year. The amount A_n of the annuity after n deposits will equal A_{n-1}, the amount of the annuity after $n-1$ deposits, plus the interest earned on this amount, plus P, the periodic deposit. That is,

$$\underset{\substack{\text{Amount}\\ \text{after}\\ n\text{ deposits}}}{A_n} = \underset{\substack{\text{Amount}\\ \text{in previous}\\ \text{period}}}{A_{n-1}} + \underset{\substack{\text{Interest}\\ \text{earned}}}{\frac{r}{N}A_{n-1}} + \underset{\substack{\text{Periodic}\\ \text{deposit}}}{P} = \left(1 + \frac{r}{N}\right)A_{n-1} + P$$

We have established the following result:

THEOREM

Annuity Formula

If $A_0 = M$ represents the initial amount deposited in an annuity that earns $r\%$ per annum compounded N times per year, and if P is the periodic deposit made at each payment period, then the amount A_n of the annuity after n deposits is given by the recursive sequence

$$\boxed{A_0 = M \qquad A_n = \left(1 + \frac{r}{N}\right)A_{n-1} + P \qquad n \geq 1 \qquad \textbf{(9)}}$$

Formula (9) may be explained as follows: the money in the account initially, A_0, is $\$M$; the money in the account after $n - 1$ payments, A_{n-1}, earns interest $\frac{r}{N}A_{n-1}$ during the nth period; so when the periodic payment of P dollars is added, the amount after n payments, A_n, is obtained.

EXAMPLE 10 Saving for Spring Break

A trip to Cancun during spring break will cost \$450 and full payment is due March 2. To have the money, a student, on September 1, deposits \$100 in a savings account that pays 4% per annum compounded monthly. On the first of each month, the student deposits \$50 in this account.

(a) Find a recursive sequence that explains how much is in the account after n months.

(b) Use the TABLE feature to list the amounts of the annuity for the first 6 months.

(c) After the deposit on March 1 is made, is there enough in the account to pay for the Cancun trip?

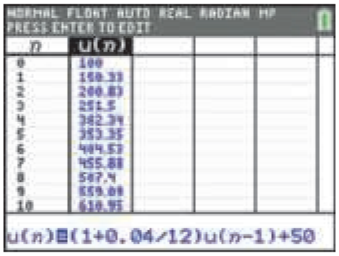
(d) If the student deposits \$60 each month, will there be enough for the trip after the March 1 deposit?

Solution

(a) The initial amount deposited in the account is $A_0 = \$100$. The monthly deposit is $P = \$50$, and the per annum rate of interest is $r = 0.04$ compounded $N = 12$ times per year. The amount A_n in the account after n monthly deposits is given by the recursive sequence

$$A_0 = 100 \qquad A_n = \left(1 + \frac{r}{N}\right)A_{n-1} + P = \left(1 + \frac{0.04}{12}\right)A_{n-1} + 50$$

Table 4

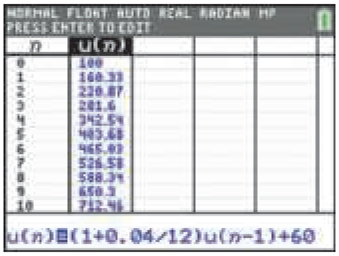


(b) In SEQuence mode on a TI-84 Plus C, enter the sequence $\{A_n\}$ and create Table 4. On September 1 $(n = 0)$, there is \$100 in the account. After the first payment on October 1, the value of the account is \$150.33. After the second payment on November 1, the value of the account is \$200.83. After the third payment on December 1, the value of the account is \$251.50, and so on.

(c) On March 1 $(n = 6)$, there is only \$404.53, not enough to pay for the trip to Cancun.

(d) If the periodic deposit, P, is \$60, then on March 1, there is \$465.03 in the account, enough for the trip. See Table 5. ■

Table 5



Now Work PROBLEM 85

Recursive sequences can also be used to compute information about loans. When equal periodic payments are made to pay off a loan, the loan is said to be **amortized**.

THEOREM

Amortization Formula

If $\$B$ is borrowed at an interest rate of $r\%$ (expressed as a decimal) per annum compounded monthly, the balance A_n due after n monthly payments of $\$P$ is given by the recursive sequence

$$A_0 = B \qquad A_n = \left(1 + \frac{r}{12}\right)A_{n-1} - P \qquad n \geq 1 \qquad \textbf{(10)}$$

Formula (10) may be explained as follows: The initial loan balance is $\$B$. The balance due after n payments, A_n, will equal the balance due previously, A_{n-1}, plus the interest charged on that amount, reduced by the periodic payment P.

EXAMPLE 11 **Mortgage Payments**

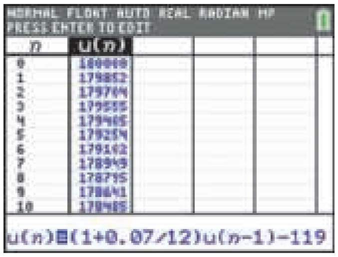
John and Wanda borrowed \$180,000 at 7% per annum compounded monthly for 30 years to purchase a home. Their monthly payment is determined to be \$1197.54.

(a) Find a recursive formula that represents their balance after each payment of \$1197.54 has been made.

(b) Determine their balance after the first payment is made.

(c) When will their balance be below \$170,000?

Table 6



n	u(n)
0	180000
1	179852
2	179704
3	179555
4	179405
5	179254
6	179102
7	178949
8	178795
9	178641
10	178485

Solution

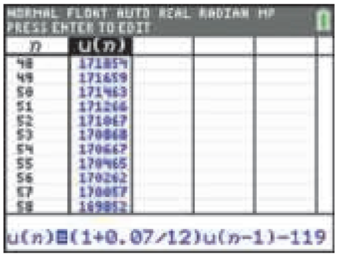
(a) Use formula (10) with $A_0 = 180{,}000$, $r = 0.07$, and $P = \$1197.54$. Then

$$A_0 = 180{,}000 \qquad A_n = \left(1 + \frac{0.07}{12}\right)A_{n-1} - 1197.54$$

(b) In SEQuence mode on a TI-84 Plus C, enter the sequence $\{A_n\}$ and create Table 6. After the first payment is made, the balance is $A_1 = \$179{,}852$.

(c) Scroll down until the balance is below \$170,000. See Table 7. After the fifty-eighth payment is made $(n = 58)$, or 4 years, 10 months, the balance is below \$170,000. ■

Table 7



n	u(n)
48	171854
49	171659
50	171463
51	171266
52	171067
53	170868
54	170667
55	170465
56	170262
57	170057
58	169852

Now Work PROBLEM 87

9.1 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. For the function $f(x) = \dfrac{x-1}{x}$, find $f(2)$ and $f(3)$. (pp. 210–213)

2. ***True or False*** A function is a relation between two sets D and R so that each element x in the first set D is related to exactly one element y in the second set R. (pp. 207–210)

3. If \$1000 is invested at 4% per annum compounded semiannually, how much is in the account after 2 years? (pp. 475–477)

4. How much do you need to invest now at 5% per annum compounded monthly so that in 1 year you will have \$10,000? (p. 479)

Concepts and Vocabulary

5. A(n) ________ is a function whose domain is the set of positive integers.

6. ***True or False*** The notation a_5 represents the fifth term of a sequence.

7. If $n \geq 0$ is an integer, then $n! =$ ______________ when $n \geq 2$.

8. The sequence $a_1 = 5$, $a_n = 3a_{n-1}$ is an example of a(n) ________ sequence.

(a) alternating (b) recursive
(c) Fibonacci (d) summation

9. The notation $a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$ is an example of ________ notation.

10. $\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n =$ ________.

(a) $n!$ (b) $\dfrac{n(n+1)}{2}$

(c) nk (d) $\dfrac{n(n+1)(2n+1)}{6}$

Skill Building

In Problems 11–16, evaluate each factorial expression.

11. $10!$ **12.** $9!$ **13.** $\dfrac{9!}{6!}$ **14.** $\dfrac{12!}{10!}$ **15.** $\dfrac{3!\,7!}{4!}$ **16.** $\dfrac{5!\,8!}{3!}$

In Problems 17–28, write down the first five terms of each sequence.

17. $\{s_n\} = \{n\}$ **18.** $\{s_n\} = \{n^2 + 1\}$ **19.** $\{a_n\} = \left\{\dfrac{n}{n+2}\right\}$ **20.** $\{b_n\} = \left\{\dfrac{2n+1}{2n}\right\}$

21. $\{c_n\} = \{(-1)^{n+1}n^2\}$ **22.** $\{d_n\} = \left\{(-1)^{n-1}\left(\frac{n}{2n-1}\right)\right\}$ **23.** $\{s_n\} = \left\{\frac{2^n}{3^n+1}\right\}$ **24.** $\{s_n\} = \left\{\left(\frac{4}{3}\right)^n\right\}$

25. $\{t_n\} = \left\{\frac{(-1)^n}{(n+1)(n+2)}\right\}$ **26.** $\{a_n\} = \left\{\frac{3^n}{n}\right\}$ **27.** $\{b_n\} = \left\{\frac{n}{e^n}\right\}$ **28.** $\{c_n\} = \left\{\frac{n^2}{2^n}\right\}$

In Problems 29–36, the given pattern continues. Write down the nth term of a sequence $\{a_n\}$ suggested by the pattern.

29. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ **30.** $\frac{1}{1\cdot 2}, \frac{1}{2\cdot 3}, \frac{1}{3\cdot 4}, \frac{1}{4\cdot 5}, \ldots$ **31.** $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ **32.** $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \ldots$

33. $1, -1, 1, -1, 1, -1, \ldots$ **34.** $1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, 7, \frac{1}{8}, \ldots$ **35.** $1, -2, 3, -4, 5, -6, \ldots$ **36.** $2, -4, 6, -8, 10, \ldots$

In Problems 37–50, a sequence is defined recursively. Write down the first five terms.

37. $a_1 = 2; \quad a_n = 3 + a_{n-1}$ **38.** $a_1 = 3; \quad a_n = 4 - a_{n-1}$ **39.** $a_1 = -2; \quad a_n = n + a_{n-1}$

40. $a_1 = 1; \quad a_n = n - a_{n-1}$ **41.** $a_1 = 5; \quad a_n = 2a_{n-1}$ **42.** $a_1 = 2; \quad a_n = -a_{n-1}$

43. $a_1 = 3; \quad a_n = \frac{a_{n-1}}{n}$ **44.** $a_1 = -2; \quad a_n = n + 3a_{n-1}$ **45.** $a_1 = 1; \quad a_2 = 2; \quad a_n = a_{n-1}\cdot a_{n-2}$

46. $a_1 = -1; \quad a_2 = 1; \quad a_n = a_{n-2} + na_{n-1}$ **47.** $a_1 = A; \quad a_n = a_{n-1} + d$ **48.** $a_1 = A; \quad a_n = ra_{n-1}, \quad r \neq 0$

49. $a_1 = \sqrt{2}; \quad a_n = \sqrt{2 + a_{n-1}}$ **50.** $a_1 = \sqrt{2}; \quad a_n = \sqrt{\frac{a_{n-1}}{2}}$

In Problems 51–60, write out each sum.

51. $\sum_{k=1}^{n} (k+2)$ **52.** $\sum_{k=1}^{n} (2k+1)$ **53.** $\sum_{k=1}^{n} \frac{k^2}{2}$ **54.** $\sum_{k=1}^{n} (k+1)^2$ **55.** $\sum_{k=0}^{n} \frac{1}{3^k}$

56. $\sum_{k=0}^{n} \left(\frac{3}{2}\right)^k$ **57.** $\sum_{k=0}^{n-1} \frac{1}{3^{k+1}}$ **58.** $\sum_{k=0}^{n-1} (2k+1)$ **59.** $\sum_{k=2}^{n} (-1)^k \ln k$ **60.** $\sum_{k=3}^{n} (-1)^{k+1}2^k$

In Problems 61–70, express each sum using summation notation.

61. $1 + 2 + 3 + \cdots + 20$ **62.** $1^3 + 2^3 + 3^3 + \cdots + 8^3$

63. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{13}{13+1}$ **64.** $1 + 3 + 5 + 7 + \cdots + [2(12) - 1]$

65. $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots + (-1)^6\left(\frac{1}{3^6}\right)$ **66.** $\frac{2}{3} - \frac{4}{9} + \frac{8}{27} - \cdots + (-1)^{12}\left(\frac{2}{3}\right)^{11}$

67. $3 + \frac{3^2}{2} + \frac{3^3}{3} + \cdots + \frac{3^n}{n}$ **68.** $\frac{1}{e} + \frac{2}{e^2} + \frac{3}{e^3} + \cdots + \frac{n}{e^n}$

69. $a + (a+d) + (a+2d) + \cdots + (a+nd)$ **70.** $a + ar + ar^2 + \cdots + ar^{n-1}$

In Problems 71–82, find the sum of each sequence.

71. $\sum_{k=1}^{40} 5$ **72.** $\sum_{k=1}^{50} 8$ **73.** $\sum_{k=1}^{40} k$ **74.** $\sum_{k=1}^{24} (-k)$

75. $\sum_{k=1}^{20} (5k+3)$ **76.** $\sum_{k=1}^{26} (3k-7)$ **77.** $\sum_{k=1}^{16} (k^2+4)$ **78.** $\sum_{k=0}^{14} (k^2-4)$

79. $\sum_{k=10}^{60} (2k)$ **80.** $\sum_{k=8}^{40} (-3k)$ **81.** $\sum_{k=5}^{20} k^3$ **82.** $\sum_{k=4}^{24} k^3$

Applications and Extensions

83. Trout Population A pond currently has 2000 trout in it. A fish hatchery decides to add an additional 20 trout each month. In addition, it is known that the trout population is growing 3% per month. The size of the population after n months is given by the recursively defined sequence

$$p_0 = 2000, \qquad p_n = 1.03p_{n-1} + 20$$

(a) How many trout are in the pond at the end of the second month? That is, what is p_2?
(b) Using a graphing utility, determine how long it will be before the trout population reaches 5000.

84. Environmental Control The Environmental Protection Agency (EPA) determines that Maple Lake has 250 tons of pollutants as a result of industrial waste and that 10% of the pollutant present is neutralized by solar oxidation every year. The EPA imposes new pollution control laws that result in 15 tons of new pollutant entering the lake each year. The amount of pollutant in the lake at the end of each year is given by the recursively defined sequence

$$p_0 = 250, \qquad p_n = 0.9p_{n-1} + 15$$

(a) Determine the amount of pollutant in the lake at the end of the second year. That is, determine p_2.
(b) Using a graphing utility, provide pollutant amounts for the next 20 years.
(c) What is the equilibrium level of pollution in Maple Lake? That is, what is $\lim_{n\to\infty} p_n$?

85. Roth IRA On January 1, Liam deposits $1500 into a Roth individual retirement account (IRA) and decides to deposit an additional $750 at the end of each quarter into the account.

(a) Find a recursive formula that represents Liam's balance at the end of each quarter if the rate of return is assumed to be 5% per annum compounded quarterly.
(b) Use a graphing utility to determine how long it will be before the value of the account exceeds $150,000.
(c) What will be the value of the account in 30 years, when Liam retires?

86. Education Savings Account On January 1, Aubrey's parents deposit $4000 in an education savings account and decide to place an additional $75 into the account at the end of each month.

(a) Find a recursive formula that represents the balance at the end of each month if the rate of return is assumed to be 1.5% per annum compounded monthly.
(b) Use a graphing utility to determine how long it will be before the value of the account exceeds $10,000.
(c) What will be the value of the account in 16 years when Aubrey goes to college?

87. Credit Card Debt John has a balance of $3000 on his Discover card that charges 1% interest per month on any unpaid balance from the previous month. John pays $100 toward the balance each month. The payment is made at the time the interest is charged. His balance each month after making a $100 payment is given by the recursively defined sequence

$$B_0 = \$3000, \qquad B_n = 1.01B_{n-1} - 100$$

(a) Determine John's balance after making the first payment. That is, determine B_1.
(b) Using a graphing utility, determine when John's balance will be below $2000. How many payments of $100 have been made?
(c) Using a graphing utility, determine when John will pay off the balance. What is the total of all the payments?
(d) What was John's interest expense?

88. Car Loans Phil bought a car by taking out a loan for $18,500 at 0.5% interest per month. Phil's normal monthly payment is $434.47 per month, but he decides that he can afford to pay $100 extra toward the balance each month. His balance each month is given by the recursively defined sequence

$$B_0 = \$18{,}500, \qquad B_n = 1.005B_{n-1} - 534.47$$

(a) Determine Phil's balance after making the first payment. That is, determine B_1.
(b) Using a graphing utility, determine when Phil's balance will be below $10,000. How many payments of $534.47 have been made?
(c) Using a graphing utility, determine when Phil will pay off the balance. What is the total of all the payments?
(d) What was Phil's interest expense?

89. Home Loan Bill and Laura borrowed $150,000 at 6% per annum compounded monthly for 30 years to purchase a home. Their monthly payment is determined to be $899.33.

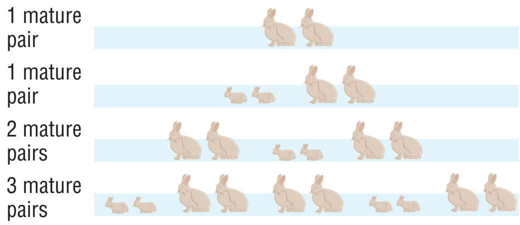
(a) Find a recursive formula for their balance after each monthly payment has been made.
(b) Determine Bill and Laura's balance after the first payment.
(c) Using a graphing utility, create a table showing Bill and Laura's balance after each monthly payment.
(d) Using a graphing utility, determine when Bill and Laura's balance will be below $140,000.
(e) Using a graphing utility, determine when Bill and Laura will pay off the balance.
(f) Determine Bill and Laura's interest expense when the loan is paid.
(g) Suppose that Bill and Laura decide to pay an additional $100 each month on their loan. Answer parts (a) to (f) under this scenario.
(h) Is it worthwhile for Bill and Laura to pay the additional $100? Explain.

90. Home Loan Jodi and Jeff borrowed $120,000 at 6.5% per annum compounded monthly for 30 years to purchase a home. Their monthly payment is determined to be $758.48.

(a) Find a recursive formula for their balance after each monthly payment has been made.
(b) Determine Jodi and Jeff's balance after the first payment.
(c) Using a graphing utility, create a table showing Jodi and Jeff's balance after each monthly payment.
(d) Using a graphing utility, determine when Jodi and Jeff's balance will be below $100,000.
(e) Using a graphing utility, determine when Jodi and Jeff will pay off the balance.
(f) Determine Jodi and Jeff's interest expense when the loan is paid.
(g) Suppose that Jodi and Jeff decide to pay an additional $100 each month on their loan. Answer parts (a) to (f) under this scenario.
(h) Is it worthwhile for Jodi and Jeff to pay the additional $100? Explain.

91. Growth of a Rabbit Colony A colony of rabbits begins with one pair of mature rabbits, which will produce a pair of offspring (one male, one female) each month. Assume that all rabbits mature in 1 month and produce a pair of offspring (one male, one female) after 2 months. If no rabbits ever die, how many pairs of mature rabbits are there after 7 months?

[**Hint:** The Fibonacci sequence models this colony. Do you see why?]



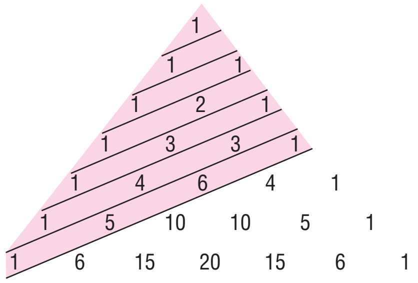
92. Fibonacci Sequence Let

$$u_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$$

define the nth term of a sequence.

(a) Show that $u_1 = 1$ and $u_2 = 1$.
(b) Show that $u_{n+2} = u_{n+1} + u_n$.
(c) Draw the conclusion that $\{u_n\}$ is the Fibonacci sequence.

93. The Pascal Triangle Divide the triangular array shown (called the Pascal triangle) using diagonal lines as indicated. Find the sum of the numbers in each of these diagonal rows. Do you recognize this sequence?



94. Fibonacci Sequence Use the result of Problem 92 to do the following problems:

(a) Write the first 10 terms of the Fibonacci sequence.
(b) Compute the ratio $\dfrac{u_{n+1}}{u_n}$ for the first 10 terms.
(c) As n gets large, what number does the ratio approach? This number is referred to as the **golden ratio**. Rectangles whose sides are in this ratio were considered pleasing to the eye by the Greeks. For example, the facade of the Parthenon was constructed using the golden ratio.
(d) Compute the ratio $\dfrac{u_n}{u_{n+1}}$ for the first 10 terms.
(e) As n gets large, what number does the ratio approach? This number is also referred to as the **conjugate golden ratio**. This ratio is believed to have been used in the construction of the Great Pyramid in Egypt. The ratio equals the sum of the areas of the four face triangles divided by the total surface area of the Great Pyramid.

95. Approximating $f(x) = e^x$ In calculus, it can be shown that

$$f(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

We can approximate the value of $f(x) = e^x$ for any x using the following sum:

$$f(x) = e^x \approx \sum_{k=0}^{n} \frac{x^k}{k!}$$

for some n.

(a) Approximate $f(1.3)$ with $n = 4$
(b) Approximate $f(1.3)$ with $n = 7$.
(c) Use a calculator to approximate $f(1.3)$.
(d) Using trial and error along with a graphing utility's SEQuence mode, determine the value of n required to approximate $f(1.3)$ correct to eight decimal places.

96. Approximating $f(x) = e^x$ Refer to Problem 95.

(a) Approximate $f(-2.4)$ with $n = 3$.
(b) Approximate $f(-2.4)$ with $n = 6$.
(c) Use a calculator to approximate $f(-2.4)$.
(d) Using trial and error along with a graphing utility's SEQuence mode, determine the value of n required to approximate $f(-2.4)$ correct to eight decimal places.

97. Bode's Law In 1772, Johann Bode published the following formula for predicting the mean distances, in astronomical units (AU), of the planets from the sun:

$$a_1 = 0.4 \qquad \{a_n\} = \{0.4 + 0.3 \cdot 2^{n-2}\}, n \geq 2$$

where n is the number of the planet from the sun.

(a) Determine the first eight terms of this sequence.
(b) At the time of Bode's publication, the known planets were Mercury (0.39 AU), Venus (0.72 AU), Earth (1 AU), Mars (1.52 AU), Jupiter (5.20 AU), and Saturn (9.54 AU). How do the actual distances compare to the terms of the sequence?
(c) The planet Uranus was discovered in 1781, and the asteroid Ceres was discovered in 1801. The mean orbital distances from the sun to Uranus and Ceres* are 19.2 AU and 2.77 AU, respectively. How well do these values fit within the sequence?
(d) Determine the ninth and tenth terms of Bode's sequence.
(e) The planets Neptune and Pluto* were discovered in 1846 and 1930, respectively. Their mean orbital distances from the sun are 30.07 AU and 39.44 AU, respectively. How do these actual distances compare to the terms of the sequence?
(f) On July 29, 2005, NASA announced the discovery of a dwarf planet $(n = 11)$, which has been named Eris.* Use Bode's Law to predict the mean orbital distance of Eris from the sun. Its actual mean distance is not yet known, but Eris is currently about 97 astronomical units from the sun.

Source: *NASA*

98. Show that

$$1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$$

(continued)

*Ceres, Haumea, Makemake, Pluto, and Eris are referred to as dwarf planets.

[**Hint:** Let

$$S = 1 + 2 + \cdots + (n-1) + n$$
$$S = n + (n-1) + (n-2) + \cdots + 1$$

Add these equations. Then

$$2S = \underbrace{[1+n] + [2+(n-1)] + \cdots + [n+1]}_{n \text{ terms in brackets}}$$

Now complete the derivation.]

Computing Square Roots *A method for approximating $\sqrt{p}$ can be traced back to the Babylonians. The formula is given by the recursively defined sequence*

$$a_0 = k \qquad a_n = \frac{1}{2}\left(a_{n-1} + \frac{p}{a_{n-1}}\right)$$

where k is an initial guess as to the value of the square root. Use this recursive formula to approximate the following square roots by finding a_5. Compare this result to the value provided by your calculator.

99. $\sqrt{5}$

100. $\sqrt{8}$

101. $\sqrt{21}$

102. $\sqrt{89}$

103. Triangular Numbers A **triangular number** is a term of the sequence

$$u_1 = 1 \qquad u_{n+1} = u_n + (n+1)$$

Write down the first seven triangular numbers.

104. For the sequence given in Problem 103, show that

$$u_{n+1} = \frac{(n+1)(n+2)}{2}.$$

105. For the sequence given in Problem 103, show that

$$u_{n+1} + u_n = (n+1)^2.$$

Explaining Concepts: Discussion and Writing

106. Investigate various applications that lead to a Fibonacci sequence, such as art, architecture, or financial markets. Write an essay on these applications.

107. Write a paragraph that explains why the numbers found in Problem 103 are called triangular.

Retain Your Knowledge

Problems 108–111 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

108. If \$2500 is invested at 3% compounded monthly, find the amount that results after a period of 2 years.

109. Solve the inequality: $x^3 + x^2 - 16x - 16 \geq 0$

110. Find the horizontal and vertical asymptotes of $R(x) = \dfrac{2x^2 - 50}{x^2 - 3x - 10}$.

111. Find an equation of the parabola with vertex $(-3, 4)$ and focus $(1, 4)$.

'Are You Prepared?' Answers

1. $f(2) = \frac{1}{2}; f(3) = \frac{2}{3}$ **2.** True **3.** \$1082.43 **4.** \$9513.28

9.2 Arithmetic Sequences

OBJECTIVES
1 Determine Whether a Sequence Is Arithmetic (p. 667)
2 Find a Formula for an Arithmetic Sequence (p. 668)
3 Find the Sum of an Arithmetic Sequence (p. 669)

1 Determine Whether a Sequence Is Arithmetic

When the difference between successive terms of a sequence is always the same number, the sequence is called **arithmetic**.

DEFINITION

An **arithmetic sequence*** may be defined recursively as $a_1 = a$, $a_n - a_{n-1} = d$, or as

$$\boxed{a_1 = a \qquad a_n = a_{n-1} + d} \qquad \textbf{(1)}$$

where $a_1 = a$ and d are real numbers. The number a is the first term, and the number d is called the **common difference**.

*Sometimes called an **arithmetic progression**.

The terms of an arithmetic sequence with first term a_1 and common difference d follow the pattern

$$a_1 \quad a_1 + d \quad a_1 + 2d \quad a_1 + 3d \ldots$$

EXAMPLE 1 **Determining Whether a Sequence Is Arithmetic**

The sequence

$$4, \quad 6, \quad 8, \quad 10, \ldots$$

is arithmetic since the difference of successive terms is 2. The first term is $a_1 = 4$, and the common difference is $d = 2$. ■

EXAMPLE 2 **Determining Whether a Sequence Is Arithmetic**

Show that the following sequence is arithmetic. Find the first term and the common difference.

$$\{s_n\} = \{3n + 5\}$$

Solution The first term is $s_1 = 3 \cdot 1 + 5 = 8$. The nth term and the $(n-1)$st term of the sequence $\{s_n\}$ are

$$s_n = 3n + 5 \quad \text{and} \quad s_{n-1} = 3(n-1) + 5 = 3n + 2$$

Their difference d is

$$d = s_n - s_{n-1} = (3n + 5) - (3n + 2) = 5 - 2 = 3$$

Since the difference of any two successive terms is the constant 3, the sequence $\{s_n\}$ is arithmetic, and the common difference is 3. ■

EXAMPLE 3 **Determining Whether a Sequence Is Arithmetic**

Show that the sequence $\{t_n\} = \{4 - n\}$ is arithmetic. Find the first term and the common difference.

Solution The first term is $t_1 = 4 - 1 = 3$. The nth term and the $(n-1)$st term are

$$t_n = 4 - n \quad \text{and} \quad t_{n-1} = 4 - (n-1) = 5 - n$$

Their difference d is

$$d = t_n - t_{n-1} = (4 - n) - (5 - n) = 4 - 5 = -1$$

Since the difference of any two successive terms is the constant -1, $\{t_n\}$ is an arithmetic sequence whose common difference is -1. ■

Now Work PROBLEM 9

2 Find a Formula for an Arithmetic Sequence

Suppose that a is the first term of an arithmetic sequence whose common difference is d. We seek a formula for the nth term, a_n. To see the pattern, consider the first few terms.

$$\begin{aligned}
a_1 &= a \\
a_2 &= a_1 + d = a_1 + 1 \cdot d \\
a_3 &= a_2 + d = (a_1 + d) + d = a_1 + 2 \cdot d \\
a_4 &= a_3 + d = (a_1 + 2 \cdot d) + d = a_1 + 3 \cdot d \\
a_5 &= a_4 + d = (a_1 + 3 \cdot d) + d = a_1 + 4 \cdot d \\
&\vdots \\
a_n &= a_{n-1} + d = [a_1 + (n-2)d] + d = a_1 + (n-1)d
\end{aligned}$$

This leads to the following result:

THEOREM

nth Term of an Arithmetic Sequence

For an arithmetic sequence $\{a_n\}$ whose first term is a_1 and whose common difference is d, the nth term is determined by the formula

$$a_n = a_1 + (n - 1)d \tag{2}$$

EXAMPLE 4 **Finding a Particular Term of an Arithmetic Sequence**

Find the 41st term of the arithmetic sequence: 2, 6, 10, 14, 18, . . .

Solution The first term of this arithmetic sequence is $a_1 = 2$, and the common difference is $d = 4$. By formula (2), the nth term is

$$a_n = 2 + (n - 1)4 \quad a_n = a_1 + (n-1)d;\ a_1 = 2,\ d = 4$$

The 41st term is

$$a_{41} = 2 + (41 - 1) \cdot 4 = 162$$

Now Work PROBLEM 25

EXAMPLE 5 **Finding a Recursive Formula for an Arithmetic Sequence**

The 8th term of an arithmetic sequence is 75, and the 20th term is 39.

(a) Find the first term and the common difference.
(b) Give a recursive formula for the sequence.
(c) What is the nth term of the sequence?

Solution (a) Formula (2) states that $a_n = a_1 + (n - 1)d$. As a result,

$$\begin{cases} a_8 = a_1 + 7d = 75 \\ a_{20} = a_1 + 19d = 39 \end{cases}$$

This is a system of two linear equations containing two variables, a_1 and d, which can be solved by elimination. Subtracting the second equation from the first gives

$$-12d = 36$$
$$d = -3$$

With $d = -3$, use $a_1 + 7d = 75$ to find that $a_1 = 75 - 7d = 75 - 7(-3) = 96$. The first term is $a_1 = 96$, and the common difference is $d = -3$.

(b) Using formula (1), a recursive formula for this sequence is

$$a_1 = 96 \qquad a_n = a_{n-1} - 3$$

(c) Using formula (2), a formula for the nth term of the sequence $\{a_n\}$ is

$$a_n = a_1 + (n - 1)d = 96 + (n - 1)(-3) = 99 - 3n$$

Exploration

Graph the recursive formula from Example 5, $a_1 = 96$, $a_n = a_{n-1} - 3$, using a graphing utility. Conclude that the graph of the recursive formula behaves like the graph of a linear function. How is d, the common difference, related to m, the slope of a line?

Now Work PROBLEMS 17 AND 31

3 Find the Sum of an Arithmetic Sequence

The next result gives two formulas for finding the sum of the first n terms of an arithmetic sequence.

THEOREM

Sum of the First n Terms of an Arithmetic Sequence

Let $\{a_n\}$ be an arithmetic sequence with first term a_1 and common difference d. The sum S_n of the first n terms of $\{a_n\}$ may be found in two ways:

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} [a_1 + (k-1)d]$$

$$= \frac{n}{2}[2a_1 + (n-1)d] \qquad \textbf{(3)}$$

$$= \frac{n}{2}(a_1 + a_n) \qquad \textbf{(4)}$$

Proof

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n \qquad \text{Sum of first } n \text{ terms}$$

$$= a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + [a_1 + (n-1)d] \qquad \text{Formula (2)}$$

$$= \underbrace{(a_1 + a_1 + \cdots + a_1)}_{n \text{ terms}} + [d + 2d + \cdots + (n-1)d] \qquad \text{Rearrange terms.}$$

$$= na_1 + d[1 + 2 + \cdots + (n-1)]$$

$$= na_1 + d\left[\frac{(n-1)n}{2}\right] \qquad \text{Formula 6, Section 9.1}$$

$$= na_1 + \frac{n}{2}(n-1)d$$

$$= \frac{n}{2}[2a_1 + (n-1)d] \qquad \text{Factor out } \frac{n}{2}\text{; this is formula (3).}$$

$$= \frac{n}{2}[a_1 + a_1 + (n-1)d]$$

$$= \frac{n}{2}(a_1 + a_n) \qquad \text{Use formula (2); this is formula (4).}$$

■

Notice that formula (3) involves the first term and common difference, whereas formula (4) involves the first term and the nth term. Use whichever form is easier.

EXAMPLE 6 **Finding the Sum of an Arithmetic Sequence**

Find the sum S_n of the first n terms of the sequence $\{a_n\} = \{3n + 5\}$; that is, find

$$8 + 11 + 14 + \cdots + (3n + 5) = \sum_{k=1}^{n} (3k + 5)$$

Solution The sequence $\{a_n\} = \{3n + 5\}$ is an arithmetic sequence with first term $a_1 = 8$ and nth term $a_n = 3n + 5$. To find the sum S_n, use formula (4).

$$S_n = \sum_{k=1}^{n} (3k + 5) \underset{\uparrow}{=} \frac{n}{2}[8 + (3n + 5)] = \frac{n}{2}(3n + 13)$$

$$S_n = \frac{n}{2}(a_1 + a_n)$$

■

Now Work PROBLEM 39

EXAMPLE 7 Finding the Sum of an Arithmetic Sequence

Find the sum: $60 + 64 + 68 + 72 + \cdots + 120$

Solution This is the sum S_n of an arithmetic sequence $\{a_n\}$ whose first term is $a_1 = 60$ and whose common difference is $d = 4$. The nth term is $a_n = 120$. Use formula (2) to find n.

$$\begin{aligned} a_n &= a_1 + (n-1)d && \textbf{Formula (2)} \\ 120 &= 60 + (n-1)\cdot 4 && a_n = 120, a_1 = 60, d = 4 \\ 60 &= 4(n-1) && \textbf{Simplify.} \\ 15 &= n - 1 && \textbf{Simplify.} \\ n &= 16 && \textbf{Solve for } n. \end{aligned}$$

Now use formula (4) to find the sum S_{16}.

$$60 + 64 + 68 + \cdots + 120 = \underset{\uparrow \atop S_n = \frac{n}{2}(a_1 + a_n)}{S_{16}} = \frac{16}{2}(60 + 120) = 1440$$

Now Work PROBLEM 43

EXAMPLE 8 Creating a Floor Design

A ceramic tile floor is designed in the shape of a trapezoid 20 feet wide at the base and 10 feet wide at the top. See Figure 13. The tiles, which measure 12 inches by 12 inches, are to be placed so that each successive row contains one fewer tile than the preceding row. How many tiles will be required?

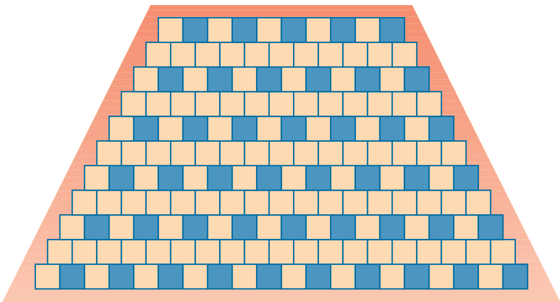


Figure 13

Solution The bottom row requires 20 tiles and the top row, 10 tiles. Since each successive row requires one fewer tile, the total number of tiles required is

$$S = 20 + 19 + 18 + \cdots + 11 + 10$$

This is the sum of an arithmetic sequence; the common difference is -1. The number of terms to be added is $n = 11$, with the first term $a_1 = 20$ and the last term $a_{11} = 10$. The sum S is

$$S = \frac{n}{2}(a_1 + a_{11}) = \frac{11}{2}(20 + 10) = 165$$

In all, 165 tiles will be required.

9.2 Assess Your Understanding

Concepts and Vocabulary

1. In a(n) __________ sequence, the difference between successive terms is a constant.
2. ***True or False*** For an arithmetic sequence $\{a_n\}$ whose first term is a_1 and whose common difference is d, the nth term is determined by the formula $a_n = a_1 + nd$.
3. If the 5th term of an arithmetic sequence is 12 and the common difference is 5, then the 6th term of the sequence is ___.
4. ***True or False*** The sum S_n of the first n terms of an arithmetic sequence $\{a_n\}$ whose first term is a_1 can be found using the formula $S_n = \frac{n}{2}(a_1 + a_n)$.
5. An arithmetic sequence can always be expressed as a(n) ________ sequence.
   (a) Fibonacci (b) alternating
   (c) geometric (d) recursive
6. If $a_n = -2n + 7$ is the nth term of an arithmetic sequence, the first term is ________.
   (a) -2 (b) 0
   (c) 5 (d) 7

Skill Building

In Problems 7–16, show that each sequence is arithmetic. Find the common difference and write out the first four terms.

7. $\{s_n\} = \{n + 4\}$
8. $\{s_n\} = \{n - 5\}$
9. $\{a_n\} = \{2n - 5\}$
10. $\{b_n\} = \{3n + 1\}$
11. $\{c_n\} = \{6 - 2n\}$
12. $\{a_n\} = \{4 - 2n\}$
13. $\{t_n\} = \left\{\frac{1}{2} - \frac{1}{3}n\right\}$
14. $\{t_n\} = \left\{\frac{2}{3} + \frac{n}{4}\right\}$
15. $\{s_n\} = \{\ln 3^n\}$
16. $\{s_n\} = \{e^{\ln n}\}$

In Problems 17–24, find the nth term of the arithmetic sequence $\{a_n\}$ whose initial term a_1 and common difference d are given. What is the 51st term?

17. $a_1 = 2; \quad d = 3$
18. $a_1 = -2; \quad d = 4$
19. $a_1 = 5; \quad d = -3$
20. $a_1 = 6; \quad d = -2$
21. $a_1 = 0; \quad d = \frac{1}{2}$
22. $a_1 = 1; \quad d = -\frac{1}{3}$
23. $a_1 = \sqrt{2}; \quad d = \sqrt{2}$
24. $a_1 = 0; \quad d = \pi$

In Problems 25–30, find the indicated term in each arithmetic sequence.

25. 100th term of 2, 4, 6, ...
26. 80th term of $-1, 1, 3, \ldots$
27. 90th term of $1, -2, -5, \ldots$
28. 80th term of $5, 0, -5, \ldots$
29. 80th term of $2, \frac{5}{2}, 3, \frac{7}{2}, \ldots$
30. 70th term of $2\sqrt{5}, 4\sqrt{5}, 6\sqrt{5}, \ldots$

In Problems 31–38, find the first term and the common difference of the arithmetic sequence described. Give a recursive formula for the sequence. Find a formula for the nth term.

31. 8th term is 8; 20th term is 44
32. 4th term is 3; 20th term is 35
33. 9th term is -5; 15th term is 31
34. 8th term is 4; 18th term is -96
35. 15th term is 0; 40th term is -50
36. 5th term is -2; 13th term is 30
37. 14th term is -1; 18th term is -9
38. 12th term is 4; 18th term is 28

In Problems 39–56, find each sum.

39. $1 + 3 + 5 + \cdots + (2n - 1)$
40. $2 + 4 + 6 + \cdots + 2n$
41. $7 + 12 + 17 + \cdots + (2 + 5n)$
42. $-1 + 3 + 7 + \cdots + (4n - 5)$
43. $2 + 4 + 6 + \cdots + 70$
44. $1 + 3 + 5 + \cdots + 59$
45. $5 + 9 + 13 + \cdots + 49$
46. $2 + 5 + 8 + \cdots + 41$
47. $73 + 78 + 83 + 88 + \cdots + 558$
48. $7 + 1 - 5 - 11 - \cdots - 299$
49. $4 + 4.5 + 5 + 5.5 + \cdots + 100$
50. $8 + 8\frac{1}{4} + 8\frac{1}{2} + 8\frac{3}{4} + 9 + \cdots + 50$
51. $\sum_{n=1}^{80}(2n - 5)$
52. $\sum_{n=1}^{90}(3 - 2n)$
53. $\sum_{n=1}^{100}\left(6 - \frac{1}{2}n\right)$
54. $\sum_{n=1}^{80}\left(\frac{1}{3}n + \frac{1}{2}\right)$
55. The sum of the first 120 terms of the sequence

    14, 16, 18, 20, ...

56. The sum of the first 46 terms of the sequence

    $2, -1, -4, -7, \ldots$

Applications and Extensions

57. Find x so that $x + 3, 2x + 1$, and $5x + 2$ are consecutive terms of an arithmetic sequence.

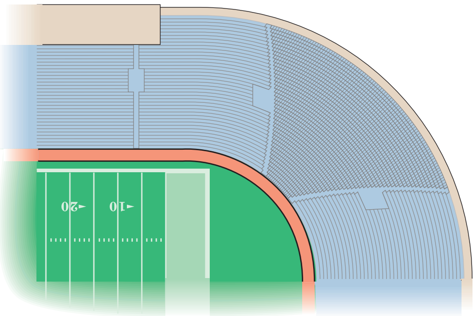
58. Find x so that $2x, 3x + 2$, and $5x + 3$ are consecutive terms of an arithmetic sequence.

59. How many terms must be added in an arithmetic sequence whose first term is 11 and whose common difference is 3 to obtain a sum of 1092?

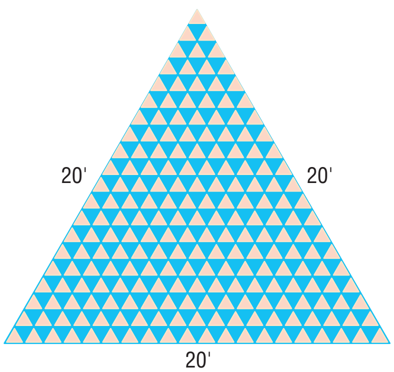
60. How many terms must be added in an arithmetic sequence whose first term is 78 and whose common difference is -4 to obtain a sum of 702?

61. Drury Lane Theater The Drury Lane Theater has 25 seats in the first row and 30 rows in all. Each successive row contains one additional seat. How many seats are in the theater?

62. Football Stadium The corner section of a football stadium has 15 seats in the first row and 40 rows in all. Each successive row contains two additional seats. How many seats are in this section?



63. Creating a Mosaic A mosaic is designed in the shape of an equilateral triangle, 20 feet on each side. Each tile in the mosaic is in the shape of an equilateral triangle, 12 inches to a side. The tiles are to alternate in color as shown in the illustration. How many tiles of each color will be required?



64. Constructing a Brick Staircase A brick staircase has a total of 30 steps. The bottom step requires 100 bricks. Each successive step requires two fewer bricks than the prior step.

(a) How many bricks are required for the top step?

(b) How many bricks are required to build the staircase?

65. Cooling Air As a parcel of air rises (for example, as it is pushed over a mountain), it cools at the *dry adiabatic lapse rate* of 5.5°F per 1000 feet until it reaches its dew point. If the ground temperature is 67°F, write a formula for the sequence of temperatures, $\{T_n\}$, of a parcel of air that has risen n thousand feet. What is the temperature of a parcel of air if it has risen 5000 feet?

Source: *National Aeronautics and Space Administration*

66. Citrus Ladders Ladders used by fruit pickers are typically tapered with a wide bottom for stability and a narrow top for ease of picking. If the bottom rung of such a ladder is 49 inches wide and the top rung is 24 inches wide, how many rungs does the ladder have if each rung is 2.5 inches shorter than the one below it? How much material would be needed to make the rungs for the ladder described?

Source: *www.stokesladders.com*

67. Seats in an Amphitheater An outdoor amphitheater has 35 seats in the first row, 37 in the second row, 39 in the third row, and so on. There are 27 rows altogether. How many can the amphitheater seat?

68. Stadium Construction How many rows are in the corner section of a stadium containing 2040 seats if the first row has 10 seats and each successive row has 4 additional seats?

69. Salary If you take a job with a starting salary of \$35,000 per year and a guaranteed raise of \$1400 per year, how many years will it be before your aggregate salary is \$280,000?

[**Hint:** Remember that your aggregate salary after 2 years is \$35,000 + (\$35,000 + \$1400).]

Explaining Concepts: Discussion and Writing

70. Make up an arithmetic sequence. Give it to a friend and ask for its 20th term.

71. Describe the similarities and differences between arithmetic sequences and linear functions.

Retain Your Knowledge

Problems 72–75 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

72. If a credit card charges 15.3% interest compounded monthly, find the effective rate of interest.

73. Determine whether $x + 7$ is a factor $x^4 + 5x^3 - 19x^2 - 29x + 42$.

74. Analyze and graph the equation: $25x^2 + 4y^2 = 100$

75. Find the inverse of the matrix $\begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}$, if there is one; otherwise, state that the matrix is singular.

9.3 Geometric Sequences; Geometric Series

OBJECTIVES

1 Determine Whether a Sequence Is Geometric

When the ratio of successive terms of a sequence is always the same nonzero number, the sequence is called **geometric**.

DEFINITION

A **geometric sequence*** may be defined recursively as $a_1 = a$, $\dfrac{a_n}{a_{n-1}} = r$, or as

$$a_1 = a \qquad a_n = ra_{n-1} \qquad \textbf{(1)}$$

where $a_1 = a$ and $r \neq 0$ are real numbers. The number a_1 is the first term, and the nonzero number r is called the **common ratio**.

The terms of a geometric sequence with first term a_1 and common ratio r follow the pattern

$$a_1 \quad a_1 r \quad a_1 r^2 \quad a_1 r^3 \ldots$$

EXAMPLE 1 Determining Whether a Sequence Is Geometric

The sequence

$$2,\ 6,\ 18,\ 54,\ 162, \ldots$$

is geometric because the ratio of successive terms is 3; $\left(\dfrac{6}{2} = \dfrac{18}{6} = \dfrac{54}{18} = \cdots = 3\right)$. The first term is $a_1 = 2$, and the common ratio is 3.

EXAMPLE 2 Determining Whether a Sequence Is Geometric

Show that the following sequence is geometric.

$$\{s_n\} = \{2^{-n}\}$$

Find the first term and the common ratio.

*Sometimes called a **geometric progression**.

Solution The first term is $s_1 = 2^{-1} = \frac{1}{2}$. The nth term and the $(n-1)$st term of the sequence $\{s_n\}$ are

$$s_n = 2^{-n} \quad \text{and} \quad s_{n-1} = 2^{-(n-1)}$$

Their ratio is

$$\frac{s_n}{s_{n-1}} = \frac{2^{-n}}{2^{-(n-1)}} = 2^{-n+(n-1)} = 2^{-1} = \frac{1}{2}$$

Because the ratio of successive terms is the nonzero constant $\frac{1}{2}$, the sequence $\{s_n\}$ is geometric with common ratio $\frac{1}{2}$.

EXAMPLE 3 **Determining Whether a Sequence Is Geometric**

Show that the following sequence is geometric.

$$\{t_n\} = \{3 \cdot 4^n\}$$

Find the first term and the common ratio.

Solution The first term is $t_1 = 3 \cdot 4^1 = 12$. The nth term and the $(n-1)$st term are

$$t_n = 3 \cdot 4^n \quad \text{and} \quad t_{n-1} = 3 \cdot 4^{n-1}$$

Their ratio is

$$\frac{t_n}{t_{n-1}} = \frac{3 \cdot 4^n}{3 \cdot 4^{n-1}} = 4^{n-(n-1)} = 4$$

The sequence, $\{t_n\}$, is a geometric sequence with common ratio 4.

Now Work PROBLEM 11

2 Find a Formula for a Geometric Sequence

Suppose that a_1 is the first term of a geometric sequence with common ratio $r \neq 0$. We seek a formula for the nth term, a_n. To see the pattern, consider the first few terms:

$$\begin{aligned}
a_1 &= a_1 \cdot 1 = a_1 r^0 \\
a_2 &= r a_1 = a_1 r^1 \\
a_3 &= r a_2 = r(a_1 r) = a_1 r^2 \\
a_4 &= r a_3 = r(a_1 r^2) = a_1 r^3 \\
a_5 &= r a_4 = r(a_1 r^3) = a_1 r^4 \\
&\vdots \\
a_n &= r a_{n-1} = r(a_1 r^{n-2}) = a_1 r^{n-1}
\end{aligned}$$

This leads to the following result:

THEOREM ***n*th Term of a Geometric Sequence**

For a geometric sequence $\{a_n\}$ whose first term is a_1 and whose common ratio is r, the nth term is determined by the formula

$$a_n = a_1 r^{n-1} \qquad r \neq 0 \qquad \textbf{(2)}$$

EXAMPLE 4 **Finding a Particular Term of a Geometric Sequence**

(a) Find the nth term of the geometric sequence: $10, 9, \frac{81}{10}, \frac{729}{100}, \ldots$

(b) Find the 9th term of this sequence.

(c) Find a recursive formula for this sequence.

Solution (a) The first term of this geometric sequence is $a_1 = 10$, and the common ratio is $\frac{9}{10}$. (Use $\frac{9}{10}$ or $\frac{\frac{81}{10}}{9} = \frac{9}{10}$ or any two successive terms.) Then, by formula (2), the nth term is

$$a_n = 10\left(\frac{9}{10}\right)^{n-1} \qquad a_n = a_1 r^{n-1};\ a_1 = 10,\ r = \frac{9}{10}$$

(b) The 9th term is

$$a_9 = 10\left(\frac{9}{10}\right)^{9-1} = 10\left(\frac{9}{10}\right)^{8} = 4.3046721$$

(c) The first term in the sequence is 10, and the common ratio is $r = \frac{9}{10}$. Using formula (1), the recursive formula is $a_1 = 10$, $a_n = \frac{9}{10}a_{n-1}$. ■

Exploration

Use a graphing utility to find the ninth term of the sequence given in Example 4. Use it to find the 20th and 50th terms. Now use a graphing utility to graph the recursive formula found in Example 4(c). Conclude that the graph of the recursive formula behaves like the graph of an exponential function. How is r, the common ratio, related to a, the base of the exponential function $y = a^x$? ■

Now Work PROBLEMS 19, 27, AND 35

3 Find the Sum of a Geometric Sequence

THEOREM

Sum of the First n Terms of a Geometric Sequence

Let $\{a_n\}$ be a geometric sequence with first term a_1 and common ratio r, where $r \neq 0$, $r \neq 1$. The sum S_n of the first n terms of $\{a_n\}$ is

$$\boxed{\begin{aligned} S_n &= a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1} = \sum_{k=1}^{n} a_1 r^{k-1} \\ &= a_1 \cdot \frac{1 - r^n}{1 - r} \qquad r \neq 0, 1 \qquad (3) \end{aligned}}$$

■

Proof The sum S_n of the first n terms of $\{a_n\} = \{a_1 r^{n-1}\}$ is

$$S_n = a_1 + a_1 r + \cdots + a_1 r^{n-1} \qquad (4)$$

Multiply each side by r to obtain

$$rS_n = a_1 r + a_1 r^2 + \cdots + a_1 r^n \qquad (5)$$

Now, subtract (5) from (4). The result is

$$S_n - rS_n = a_1 - a_1 r^n$$
$$(1 - r)S_n = a_1(1 - r^n)$$

Since $r \neq 1$, solve for S_n.

$$S_n = a_1 \cdot \frac{1 - r^n}{1 - r}$$

■

EXAMPLE 5 **Finding the Sum of the First n Terms of a Geometric Sequence**

Find the sum S_n of the first n terms of the sequence $\left\{\left(\frac{1}{2}\right)^n\right\}$; that is, find

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \left(\frac{1}{2}\right)^n = \sum_{k=1}^{n} \frac{1}{2}\left(\frac{1}{2}\right)^{k-1}$$

Solution The sequence $\left\{\left(\frac{1}{2}\right)^n\right\}$ is a geometric sequence with $a_1 = \frac{1}{2}$ and $r = \frac{1}{2}$. Use formula (3) to get

$$S_n = \sum_{k=1}^{n} \frac{1}{2}\left(\frac{1}{2}\right)^{k-1} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \left(\frac{1}{2}\right)^n$$

$$= \frac{1}{2}\left[\frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}}\right] \qquad \text{Formula (3); } a_1 = \frac{1}{2},\ r = \frac{1}{2}$$

$$= \frac{1}{2}\left[\frac{1 - \left(\frac{1}{2}\right)^n}{\frac{1}{2}}\right]$$

$$= 1 - \left(\frac{1}{2}\right)^n$$

Now Work PROBLEM 41

EXAMPLE 6 Using a Graphing Utility to Find the Sum of a Geometric Sequence

Use a graphing utility to find the sum of the first 15 terms of the sequence $\left\{\left(\frac{1}{3}\right)^n\right\}$; that is, find

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \left(\frac{1}{3}\right)^{15} = \sum_{k=1}^{15} \frac{1}{3}\left(\frac{1}{3}\right)^{k-1}$$

Solution Figure 14 shows the result using a TI-84 Plus C graphing calculator. The sum of the first 15 terms of the sequence $\left\{\left(\frac{1}{3}\right)^n\right\}$ is approximately 0.4999999652.

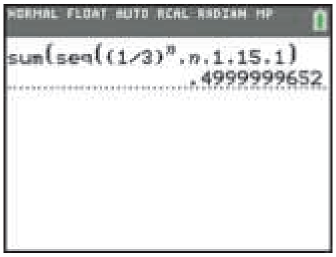


Figure 14

Now Work PROBLEM 47

4 Determine Whether a Geometric Series Converges or Diverges

DEFINITION

An infinite sum of the form

$$\boxed{a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1} + \cdots}$$

with first term a_1 and common ratio r, is called an **infinite geometric series** and is denoted by

$$\boxed{\sum_{k=1}^{\infty} a_1 r^{k-1}}$$

Based on formula (3), the sum S_n of the first n terms of a geometric series is

$$S_n = a_1 \cdot \frac{1 - r^n}{1 - r} = \frac{a_1}{1 - r} - \frac{a_1 r^n}{1 - r} \qquad \textbf{(6)}$$

Note: In calculus, limit notation is used, and the sum is written

$$L = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_1 r^{k-1} = \sum_{k=1}^{\infty} a_1 r^{k-1}$$

If this finite sum S_n approaches a number L as $n \to \infty$, then the infinite geometric series $\sum_{k=1}^{\infty} a_1 r^{k-1}$ **converges** to L and L is called the **sum of the infinite geometric series**. The sum is written as

$$L = \sum_{k=1}^{\infty} a_1 r^{k-1}$$

A series that does not converge is called a **divergent series**.

THEOREM

Convergence of an Infinite Geometric Series

If $|r| < 1$, the infinite geometric series $\sum_{k=1}^{\infty} a_1 r^{k-1}$ converges. Its sum is

$$\boxed{\sum_{k=1}^{\infty} a_1 r^{k-1} = \frac{a_1}{1-r} \qquad \textbf{(7)}}$$

Intuitive Proof Since $|r| < 1$, it follows that $|r^n|$ approaches 0 as $n \to \infty$. Then, based on formula (6), the term $\dfrac{a_1 r^n}{1-r}$ approaches 0, so the sum S_n approaches $\dfrac{a_1}{1-r}$ as $n \to \infty$. ■

EXAMPLE 7 Determining Whether a Geometric Series Converges or Diverges

Determine whether the geometric series

$$\sum_{k=1}^{\infty} 2\left(\frac{2}{3}\right)^{k-1} = 2 + \frac{4}{3} + \frac{8}{9} + \cdots$$

converges or diverges. If it converges, find its sum.

Solution Comparing $\sum_{k=1}^{\infty} 2\left(\frac{2}{3}\right)^{k-1}$ to $\sum_{k=1}^{\infty} a_1 r^{k-1}$, the first term is $a_1 = 2$ and the common ratio is $r = \frac{2}{3}$. Since $|r| < 1$, the series converges. Use formula (7) to find its sum:

$$\sum_{k=1}^{\infty} 2\left(\frac{2}{3}\right)^{k-1} = 2 + \frac{4}{3} + \frac{8}{9} + \cdots = \frac{2}{1-\frac{2}{3}} = 6$$

Now Work PROBLEM 53

EXAMPLE 8 Repeating Decimals

Show that the repeating decimal 0.999 . . . equals 1.

Solution The decimal $0.999\ldots = 0.9 + 0.09 + 0.009 + \cdots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$ is an infinite geometric series. Write it in the form $\sum_{k=1}^{\infty} a_1 r^{k-1}$ and use formula (7).

$$0.999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots = \sum_{k=1}^{\infty} \frac{9}{10^k} = \sum_{k=1}^{\infty} \frac{9}{10 \cdot 10^{k-1}} = \sum_{k=1}^{\infty} \frac{9}{10}\left(\frac{1}{10}\right)^{k-1}$$

Compare this series to $\sum_{k=1}^{\infty} a_1 r^{k-1}$ and note that $a_1 = \frac{9}{10}$ and $r = \frac{1}{10}$. Since $|r| < 1$, the series converges and its sum is

$$0.999\ldots = \frac{\frac{9}{10}}{1 - \frac{1}{10}} = \frac{\frac{9}{10}}{\frac{9}{10}} = 1$$

The repeating decimal 0.999. . . equals 1. ■

EXAMPLE 9 **Pendulum Swings**

Initially, a pendulum swings through an arc of 18 inches. See Figure 15. On each successive swing, the length of the arc is 0.98 of the previous length.

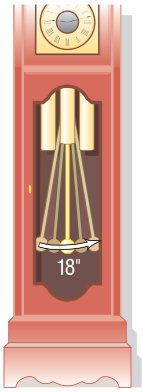


Figure 15

(a) What is the length of the arc of the 10th swing?
(b) On which swing is the length of the arc first less than 12 inches?
(c) After 15 swings, what total distance will the pendulum have swung?
(d) When it stops, what total distance will the pendulum have swung?

Solution

(a) The length of the first swing is 18 inches.
The length of the second swing is $0.98(18)$ inches.
The length of the third swing is $0.98(0.98)(18) = 0.98^2(18)$ inches.
The length of the arc of the 10th swing is

$$(0.98)^9(18) \approx 15.007 \text{ inches}$$

(b) The length of the arc of the nth swing is $(0.98)^{n-1}(18)$. For this to be exactly 12 inches requires that

$$(0.98)^{n-1}(18) = 12$$

$$(0.98)^{n-1} = \frac{12}{18} = \frac{2}{3} \qquad \text{Divide both sides by 18.}$$

$$n - 1 = \log_{0.98}\left(\frac{2}{3}\right) \qquad \text{Express as a logarithm.}$$

$$n = 1 + \frac{\ln\left(\frac{2}{3}\right)}{\ln 0.98} \approx 1 + 20.07 = 21.07 \qquad \text{Solve for } n\text{; use the Change of Base Formula.}$$

The length of the arc of the pendulum exceeds 12 inches on the 21st swing and is first less than 12 inches on the 22nd swing.

(c) After 15 swings, the pendulum will have swung the following total distance L:

$$L = \underset{\text{1st}}{18} + \underset{\text{2nd}}{0.98(18)} + \underset{\text{3rd}}{(0.98)^2(18)} + \underset{\text{4th}}{(0.98)^3(18)} + \cdots + \underset{\text{15th}}{(0.98)^{14}(18)}$$

This is the sum of a geometric sequence. The common ratio is 0.98; the first term is 18. The sum has 15 terms, so

$$L = 18 \cdot \frac{1 - 0.98^{15}}{1 - 0.98} \approx 18(13.07) \approx 235.3 \text{ inches}$$

The pendulum will have swung through approximately 235.3 inches after 15 swings.

(d) When the pendulum stops, it will have swung the following total distance T:

$$T = 18 + 0.98(18) + (0.98)^2(18) + (0.98)^3(18) + \cdots$$

This is the sum of an infinite geometric series. The common ratio is $r = 0.98$; the first term is $a_1 = 18$. Since $|r| < 1$, the series converges. Its sum is

$$T = \frac{a_1}{1 - r} = \frac{18}{1 - 0.98} = 900$$

The pendulum will have swung a total of 900 inches when it finally stops. ■

Now Work PROBLEM 87

Historical Feature

Fibonacci

Sequences are among the oldest objects of mathematical investigation, having been studied for over 3500 years. After the initial steps, however, little progress was made until about 1600.

Arithmetic and geometric sequences appear in the Rhind papyrus, a mathematical text containing 85 problems copied around 1650 BC by the Egyptian scribe Ahmes from an earlier work (see Historical Problem 1). Fibonacci (AD 1220) wrote about problems similar to those found in the Rhind papyrus, leading one to suspect that Fibonacci may have had material available that is now lost. This material would have been in the non-Euclidean Greek tradition of Heron (about AD 75) and Diophantus (about AD 250). One problem, again modified slightly, is still with us in the familiar puzzle rhyme "As I was going to St. Ives . . . " (see Historical Problem 2).

The Rhind papyrus indicates that the Egyptians knew how to add up the terms of an arithmetic or geometric sequence, as did the Babylonians. The rule for summing up a geometric sequence is found in Euclid's *Elements* (Book IX, 35, 36), where, like all Euclid's algebra, it is presented in a geometric form.

Investigations of other kinds of sequences began in the 1500s, when algebra became sufficiently developed to handle the more complicated problems. The development of calculus in the 1600s added a powerful new tool, especially for finding the sum of an infinite series, and the subject continues to flourish today.

Historical Problems

1. *Arithmetic sequence problem from the Rhind papyrus (statement modified slightly for clarity)* One hundred loaves of bread are to be divided among five people so that the amounts that they receive form an arithmetic sequence. The first two together receive one-seventh of what the last three receive. How many loaves does each receive?

   [*Partial answer:* First person receives $1\frac{2}{3}$ loaves.]

2. The following old English children's rhyme resembles one of the Rhind papyrus problems.

   As I was going to St. Ives
   I met a man with seven wives
   Each wife had seven sacks
   Each sack had seven cats
   Each cat had seven kits [kittens]
   Kits, cats, sacks, wives
   How many were going to St. Ives?

   (a) Assuming that the speaker and the cat fanciers met by traveling in opposite directions, what is the answer?

   (b) How many kittens are being transported?

   (c) Kits, cats, sacks, wives; how many?

9.3 Assess Your Understanding

Concepts and Vocabulary

1. The formula for the nth term of a geometric sequence is ________.

2. If the eighth term of a geometric sequence is $-\frac{8}{9}$ and the common ratio is $\frac{3}{4}$, then the ninth term of the sequence is _____.

3. In a geometric sequence, the ________ of successive terms is a constant.
   (a) difference (b) product (c) ratio (d) sum

4. If $|r| < 1$, the sum of the geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ is ________.

5. If a series does not converge, it is called ________.
   (a) arithmetic (b) divergent (c) geometric (d) insurgent

6. ***True or False*** A geometric sequence may be defined recursively.

7. ***True or False*** In a geometric sequence, the common ratio is always a positive number.

8. ***True or False*** For a geometric sequence with first term a_1 and common ratio r, where $r \neq 0$, $r \neq 1$, the sum of the first n terms is $S_n = a_1 \cdot \frac{1 - r^n}{1 - r}$.

Skill Building

In Problems 9–18, show that each sequence is geometric. Then find the common ratio and write out the first four terms.

9. $\{s_n\} = \{3^n\}$ **10.** $\{s_n\} = \{(-5)^n\}$ **11.** $\{a_n\} = \left\{-3\left(\frac{1}{2}\right)^n\right\}$ **12.** $\{b_n\} = \left\{\left(\frac{5}{2}\right)^n\right\}$ **13.** $\{c_n\} = \left\{\frac{2^{n-1}}{4}\right\}$

14. $\{d_n\} = \left\{\frac{3^n}{9}\right\}$ **15.** $\{e_n\} = \{2^{n/3}\}$ **16.** $\{f_n\} = \{3^{2n}\}$ **17.** $\{t_n\} = \left\{\frac{3^{n-1}}{2^n}\right\}$ **18.** $\{u_n\} = \left\{\frac{2^n}{3^{n-1}}\right\}$

In Problems 19–26, find the fifth term and the nth term of the geometric sequence whose initial term a_1 and common ratio r are given.

19. $a_1 = 2;\quad r = 3$

20. $a_1 = -2;\quad r = 4$

21. $a_1 = 5;\quad r = -1$

22. $a_1 = 6;\quad r = -2$

23. $a_1 = 0;\quad r = \frac{1}{2}$

24. $a_1 = 1;\quad r = -\frac{1}{3}$

25. $a_1 = \sqrt{2};\quad r = \sqrt{2}$

26. $a_1 = 0;\quad r = \frac{1}{\pi}$

In Problems 27–32, find the indicated term of each geometric sequence.

27. 7th term of $1, \frac{1}{2}, \frac{1}{4}, \ldots$

28. 8th term of $1, 3, 9, \ldots$

29. 9th term of $1, -1, 1, \ldots$

30. 10th term of $-1, 2, -4, \ldots$

31. 8th term of $0.4, 0.04, 0.004, \ldots$

32. 7th term of $0.1, 1.0, 10.0, \ldots$

In Problems 33–40, find the nth term a_n of each geometric sequence. When given, r is the common ratio.

33. $7, 14, 28, 56, \ldots$

34. $5, 10, 20, 40, \ldots$

35. $-3, 1, -\frac{1}{3}, \frac{1}{9}, \ldots$

36. $4, 1, \frac{1}{4}, \frac{1}{16}, \ldots$

37. $a_6 = 243;\quad r = -3$

38. $a_2 = 7;\quad r = \frac{1}{3}$

39. $a_2 = 7;\quad a_4 = 1575$

40. $a_3 = \frac{1}{3};\quad a_6 = \frac{1}{81}$

In Problems 41–46, find each sum.

41. $\frac{1}{4} + \frac{2}{4} + \frac{2^2}{4} + \frac{2^3}{4} + \cdots + \frac{2^{n-1}}{4}$

42. $\frac{3}{9} + \frac{3^2}{9} + \frac{3^3}{9} + \cdots + \frac{3^n}{9}$

43. $\sum_{k=1}^{n} \left(\frac{2}{3}\right)^k$

44. $\sum_{k=1}^{n} 4 \cdot 3^{k-1}$

45. $-1 - 2 - 4 - 8 - \cdots - (2^{n-1})$

46. $2 + \frac{6}{5} + \frac{18}{25} + \cdots + 2\left(\frac{3}{5}\right)^{n-1}$

For Problems 47–52, use a graphing utility to find the sum of each geometric sequence.

47. $\frac{1}{4} + \frac{2}{4} + \frac{2^2}{4} + \frac{2^3}{4} + \cdots + \frac{2^{14}}{4}$

48. $\frac{3}{9} + \frac{3^2}{9} + \frac{3^3}{9} + \cdots + \frac{3^{15}}{9}$

49. $\sum_{n=1}^{15} \left(\frac{2}{3}\right)^n$

50. $\sum_{n=1}^{15} 4 \cdot 3^{n-1}$

51. $-1 - 2 - 4 - 8 - \cdots - 2^{14}$

52. $2 + \frac{6}{5} + \frac{18}{25} + \cdots + 2\left(\frac{3}{5}\right)^{15}$

In Problems 53–68, determine whether each infinite geometric series converges or diverges. If it converges, find its sum.

53. $1 + \frac{1}{3} + \frac{1}{9} + \cdots$

54. $2 + \frac{4}{3} + \frac{8}{9} + \cdots$

55. $8 + 4 + 2 + \cdots$

56. $6 + 2 + \frac{2}{3} + \cdots$

57. $2 - \frac{1}{2} + \frac{1}{8} - \frac{1}{32} + \cdots$

58. $1 - \frac{3}{4} + \frac{9}{16} - \frac{27}{64} + \cdots$

59. $8 + 12 + 18 + 27 + \cdots$

60. $9 + 12 + 16 + \frac{64}{3} + \cdots$

61. $\sum_{k=1}^{\infty} 5\left(\frac{1}{4}\right)^{k-1}$

62. $\sum_{k=1}^{\infty} 8\left(\frac{1}{3}\right)^{k-1}$

63. $\sum_{k=1}^{\infty} \frac{1}{2} \cdot 3^{k-1}$

64. $\sum_{k=1}^{\infty} 3\left(\frac{3}{2}\right)^{k-1}$

65. $\sum_{k=1}^{\infty} 6\left(-\frac{2}{3}\right)^{k-1}$

66. $\sum_{k=1}^{\infty} 4\left(-\frac{1}{2}\right)^{k-1}$

67. $\sum_{k=1}^{\infty} 3\left(\frac{2}{3}\right)^k$

68. $\sum_{k=1}^{\infty} 2\left(\frac{3}{4}\right)^k$

Mixed Practice

In Problems 69–82, determine whether the given sequence is arithmetic, geometric, or neither. If the sequence is arithmetic, find the common difference; if it is geometric, find the common ratio. If the sequence is arithmetic or geometric, find the sum of the first 50 terms.

69. $\{n + 2\}$

70. $\{2n - 5\}$

71. $\{4n^2\}$

72. $\{5n^2 + 1\}$

73. $\left\{3 - \frac{2}{3}n\right\}$

74. $\left\{8 - \frac{3}{4}n\right\}$

75. $1, 3, 6, 10, \ldots$

76. $2, 4, 6, 8, \ldots$

77. $\left\{\left(\frac{2}{3}\right)^n\right\}$

78. $\left\{\left(\frac{5}{4}\right)^n\right\}$

79. $-1, 2, -4, 8, \ldots$

80. $1, 1, 2, 3, 5, 8, \ldots$

81. $\{3^{n/2}\}$

82. $\{(-1)^n\}$

Applications and Extensions

83. Find x so that $x, x + 2$, and $x + 3$ are consecutive terms of a geometric sequence.

84. Find x so that $x - 1, x$, and $x + 2$ are consecutive terms of a geometric sequence.

85. Salary Increases If you have been hired at an annual salary of \$42,000 and expect to receive annual increases of 3%, what will your salary be when you begin your fifth year?

86. Equipment Depreciation A new piece of equipment cost a company \$15,000. Each year, for tax purposes, the company depreciates the value by 15%. What value should the company give the equipment after 5 years?

87. Pendulum Swings Initially, a pendulum swings through an arc of 2 feet. On each successive swing, the length of the arc is 0.9 of the previous length.

(a) What is the length of the arc of the 10th swing?

(b) On which swing is the length of the arc first less than 1 foot?

(c) After 15 swings, what total length will the pendulum have swung?

(d) When it stops, what total length will the pendulum have swung?

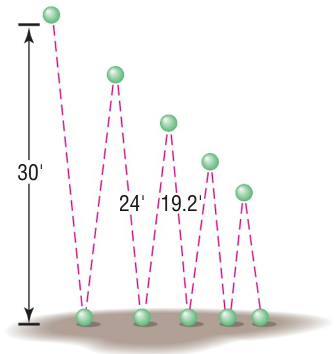
88. Bouncing Balls A ball is dropped from a height of 30 feet. Each time it strikes the ground, it bounces up to 0.8 of the previous height.

(a) What height will the ball bounce up to after it strikes the ground for the third time?

(b) How high will it bounce after it strikes the ground for the nth time?

(c) How many times does the ball need to strike the ground before its bounce is less than 6 inches?

(d) What total vertical distance does the ball travel before it stops bouncing?



Amount of an Annuity *Suppose that P is the deposit in dollars made at the end of each payment period for an annuity (see Section 9.1) paying $i = \frac{r}{N}$ percent interest per payment period. The amount A of the annuity after n deposits is given by*

$$A = P\frac{(1 + i)^n - 1}{i}$$

Use this result to answer Problems 89–94.

89. Retirement Christine contributes \$100 each month to her 401(k). What will be the value of Christine's 401(k) after the 360th deposit (30 years) if the per annum rate of return is assumed to be 12% compounded monthly?

90. Saving for a Home Jolene wants to purchase a new home. Suppose that she invests \$400 per month into a mutual fund. If the per annum rate of return of the mutual fund is assumed to be 10% compounded monthly, how much will Jolene have for a down payment after the 36th deposit (3 years)?

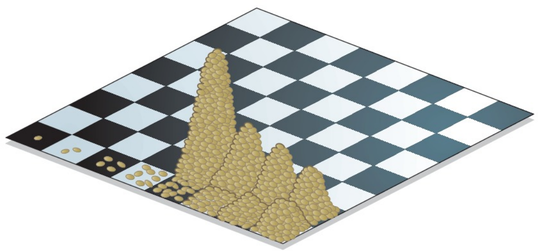
91. Tax-Sheltered Annuity Don contributes \$500 at the end of each quarter to a tax-sheltered annuity (TSA). What will the value of the TSA be after the 80th deposit (20 years) if the per annum rate of return is assumed to be 8% compounded quarterly?

92. Retirement Ray contributes \$1000 to an individual retirement account (IRA) semiannually. What will the value of the IRA be when Ray makes his 30th deposit (after 15 years) if the per annum rate of return is assumed to be 10% compounded semiannually?

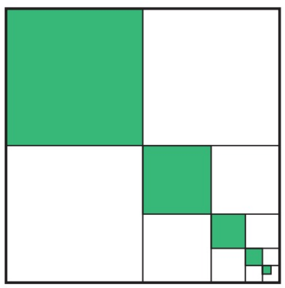
93. Sinking Fund Scott and Alice want to purchase a vacation home in 10 years and need \$50,000 for a down payment. How much should they place in a savings account each month if the per annum rate of return is assumed to be 6% compounded monthly?

94. Sinking Fund For a child born in 2015, the cost of a 4-year college education at a public university is projected to be \$200,000. Assuming an 8% per annum rate of return compounded monthly, how much must be contributed to a college fund every month to have \$200,000 in 18 years when the child begins college?

95. Grains of Wheat on a Chess Board In an old fable, a commoner who had saved the king's life was told he could ask the king for any just reward. Being a shrewd man, the commoner said, "A simple wish, sire. Place one grain of wheat on the first square of a chessboard, two grains on the second square, four grains on the third square, continuing until you have filled the board. This is all I seek." Compute the total number of grains needed to do this to see why the request, seemingly simple, could not be granted. (A chessboard consists of $8 \times 8 = 64$ squares.)



96. Look at the figure. What fraction of the square is eventually shaded if the indicated shading process continues indefinitely?



97. Multiplier Suppose that, throughout the U.S. economy, individuals spend 90% of every additional dollar that they earn. Economists would say that an individual's **marginal propensity to consume** is 0.90. For example, if Jane earns an additional dollar, she will spend $0.9(1) = \$0.90$ of it. The individual who earns \$0.90 (from Jane) will spend 90% of it, or \$0.81. This process of spending continues and results in an infinite geometric series as follows:

$$1, 0.90, 0.90^2, 0.90^3, 0.90^4, \ldots$$

The sum of this infinite geometric series is called the **multiplier**. What is the multiplier if individuals spend 90% of every additional dollar that they earn?

98. Multiplier Refer to Problem 97. Suppose that the marginal propensity to consume throughout the U.S. economy is 0.95. What is the multiplier for the U.S. economy?

99. Stock Price One method of pricing a stock is to discount the stream of future dividends of the stock. Suppose that a stock pays $\$P$ per year in dividends, and historically, the dividend has been increased $i\%$ per year. If you desire an annual rate of return of $r\%$, this method of pricing a stock states that the price that you should pay is the present value of an infinite stream of payments:

$$\text{Price} = P + P \cdot \frac{1+i}{1+r} + P \cdot \left(\frac{1+i}{1+r}\right)^2 + P \cdot \left(\frac{1+i}{1+r}\right)^3 + \cdots$$

The price of the stock is the sum of an infinite geometric series. Suppose that a stock pays an annual dividend of \$4.00, and historically, the dividend has been increased 3% per year. You desire an annual rate of return of 9%. What is the most you should pay for the stock?

100. Stock Price Refer to Problem 99. Suppose that a stock pays an annual dividend of \$2.50, and historically, the dividend has increased 4% per year. You desire an annual rate of return of 11%. What is the most that you should pay for the stock?

101. A Rich Man's Promise A rich man promises to give you \$1000 on September 1, 2015. Each day thereafter he will give you $\frac{9}{10}$ of what he gave you the previous day. What is the first date on which the amount you receive is less than 1¢? How much have you received when this happens?

102. Show that the "Amount of an Annuity" formula that you used in Problems 89–94 results from summing a geometric sequence.

103. Seating Revenue A special section in the end zone of a football stadium has 2 seats in the first row and 14 rows total. Each successive row has 2 seats more than the row before. In this particular section, the first seat is sold for 1 cent, and each following seat sells for 5% more than the previous seat. Find the total revenue generated if every seat in the section is sold. Round only the final answer, and state the final answer in dollars rounded to two decimal places. (JJC)†

†Courtesy of the Joliet Junior College Mathematics Department.

Explaining Concepts: Discussion and Writing

104. Critical Thinking You are interviewing for a job and receive two offers for a five-year contract:

A: \$40,000 to start, with guaranteed annual increases of 6% for the first 5 years

B: \$44,000 to start, with guaranteed annual increases of 3% for the first 5 years

Which offer is better if your goal is to be making as much as possible after 5 years? Which is better if your goal is to make as much money as possible over the contract (5 years)?

105. Critical Thinking Which of the following choices, *A* or *B*, results in more money?

A: To receive \$1000 on day 1, \$999 on day 2, \$998 on day 3, with the process to end after 1000 days

B: To receive \$1 on day 1, \$2 on day 2, \$4 on day 3, for 19 days

106. Critical Thinking You have just signed a 7-year professional football league contract with a beginning salary of \$2,000,000 per year. Management gives you the following options with regard to your salary over the 7 years.

1. A bonus of \$100,000 each year
2. An annual increase of 4.5% per year beginning after 1 year
3. An annual increase of \$95,000 per year beginning after 1 year

Which option provides the most money over the 7-year period? Which the least? Which would you choose? Why?

107. Critical Thinking Suppose you were offered a job in which you would work 8 hours per day for 5 workdays per week for 1 month at hard manual labor. Your pay the first day would be 1 penny. On the second day your pay would be two pennies; the third day 4 pennies. Your pay would double on each successive workday. There are 22 workdays in the month. There will be no sick days. If you miss a day of work, there is no pay or pay increase. How much do you get paid if you work all 22 days? How much do you get paid for the 22nd workday? What risks do you run if you take this job offer? Would you take the job?

108. Can a sequence be both arithmetic and geometric? Give reasons for your answer.

109. Make up a geometric sequence. Give it to a friend and ask for its 20th term.

110. Make up two infinite geometric series, one that has a sum and one that does not. Give them to a friend and ask for the sum of each series.

111. Describe the similarities and differences between geometric sequences and exponential functions.

Retain Your Knowledge

Problems 112–115 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

112. Use the Change-of-Base Formula and a calculator to evaluate $\log_7 62$. Round the answer to three decimal places.

113. Find the remainder when $P(x) = 8x^4 - 2x^3 + x - 8$ is divided by $x + 2$.

114. Find the equation of the hyperbola with vertices at $(-2, 0)$ and $(2, 0)$ and a focus at $(4, 0)$.

115. Find the value of the determinant: $\begin{vmatrix} 3 & 1 & 0 \\ 0 & -2 & 6 \\ 4 & -1 & -2 \end{vmatrix}$.

9.4 Mathematical Induction

OBJECTIVE 1 Prove Statements Using Mathematical Induction (p. 684)

1 Prove Statements Using Mathematical Induction

Mathematical induction is a method for proving that statements involving natural numbers are true for all natural numbers.*

For example, the statement "$2n$ is always an even integer" can be proved for all natural numbers n by using mathematical induction. Also, the statement "the sum of the first n positive odd integers equals n^2," that is,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2 \tag{1}$$

can be proved for all natural numbers n by using mathematical induction.

Before stating the method of mathematical induction, let's try to gain a sense of the power of the method. We shall use the statement in equation (1) for this purpose by restating it for various values of $n = 1, 2, 3, \ldots$.

$n = 1$ The sum of the first positive odd integer is 1^2; $1 = 1^2$.

$n = 2$ The sum of the first 2 positive odd integers is 2^2; $1 + 3 = 4 = 2^2$.

$n = 3$ The sum of the first 3 positive odd integers is 3^2; $1 + 3 + 5 = 9 = 3^2$.

$n = 4$ The sum of the first 4 positive odd integers is 4^2; $1 + 3 + 5 + 7 = 16 = 4^2$.

Although from this pattern we might conjecture that statement (1) is true for any choice of n, can we really be sure that it does not fail for some choice of n? The method of proof by mathematical induction will, in fact, prove that the statement is true for all n.

THEOREM

The Principle of Mathematical Induction

Suppose that the following two conditions are satisfied with regard to a statement about natural numbers:

CONDITION I: The statement is true for the natural number 1.

CONDITION II: If the statement is true for some natural number k, it is also true for the next natural number $k + 1$.

Then the statement is true for all natural numbers. ■

*Recall that the natural numbers are the numbers $1, 2, 3, 4, \ldots$. In other words, the terms *natural numbers* and *positive integers* are synonymous.

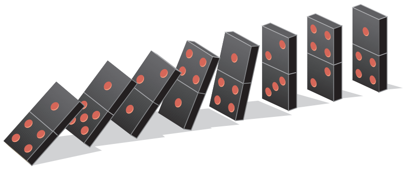


Figure 16

We shall not prove this principle. However, the following physical interpretation illustrates why the principle works. Think of a collection of natural numbers obeying a statement as a collection of infinitely many dominoes. See Figure 16.

Now, suppose that two facts are given:

1. The first domino is pushed over.
2. If one domino falls over, say the kth domino, so will the next one, the $(k+1)$st domino.

Is it safe to conclude that *all* the dominoes fall over? The answer is yes, because if the first one falls (Condition I), the second one does also (by Condition II); and if the second one falls, so does the third (by Condition II); and so on.

EXAMPLE 1 Using Mathematical Induction

Show that the following statement is true for all natural numbers n.

$$1+3+5+\cdots+(2n-1)=n^2 \tag{2}$$

Solution First show that statement (2) holds for $n=1$. Because $1=1^2$, statement (2) is true for $n=1$. Condition I holds.

Next, show that Condition II holds. From statement (2), assume that

$$1+3+5+\cdots+(2k-1)=k^2 \tag{3}$$

is true for some natural number k.

Now show that, based on equation (3), statement (2) holds for $k+1$. Look at the sum of the first $k+1$ positive odd integers to determine whether this sum equals $(k+1)^2$.

$$\begin{aligned}1+3+5+\cdots+[2(k+1)-1] &= \underbrace{[1+3+5+\cdots+(2k-1)]}_{=\,k^2 \text{ by equation (3)}}+(2k+1)\\ &= k^2+(2k+1)\\ &= k^2+2k+1=(k+1)^2\end{aligned}$$

Conditions I and II are satisfied; by the Principle of Mathematical Induction, statement (2) is true for all natural numbers n. ■

EXAMPLE 2 Using Mathematical Induction

Show that the following statement is true for all natural numbers n.

$$2^n > n$$

Solution First, show that the statement $2^n > n$ holds when $n=1$. Because $2^1 = 2 > 1$, the inequality is true for $n=1$. Condition I holds.

Next, assume, for some natural number k, that $2^k > k$. Now show that the formula holds for $k+1$; that is, show that $2^{k+1} > k+1$.

$$2^{k+1} = 2\cdot 2^k \underset{\substack{\uparrow\\ \text{We know that}\\ 2^k>k.}}{>} 2\cdot k = k+k \underset{\substack{\uparrow\\ k\geq 1}}{\geq} k+1$$

If $2^k > k$, then $2^{k+1} > k+1$, so Condition II of the Principle of Mathematical Induction is satisfied. The statement $2^n > n$ is true for all natural numbers n. ■

EXAMPLE 3 **Using Mathematical Induction**

Show that the following formula is true for all natural numbers n.

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \qquad \textbf{(4)}$$

Solution First, show that formula (4) is true when $n = 1$. Because

$$\frac{1(1+1)}{2} = \frac{1(2)}{2} = 1$$

Condition I of the Principle of Mathematical Induction holds.

Next, assume that formula (4) holds for some k, and determine whether the formula then holds for $k + 1$. Assume that

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} \quad \text{for some } k \qquad \textbf{(5)}$$

Now show that

$$1 + 2 + 3 + \cdots + (k+1) = \frac{(k+1)[(k+1)+1]}{2} = \frac{(k+1)(k+2)}{2}$$

as follows:

$$\begin{aligned} 1 + 2 + 3 + \cdots + (k+1) &= \underbrace{[1 + 2 + 3 + \cdots + k]}_{= \frac{k(k+1)}{2} \text{ by equation (5)}} + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k^2 + k + 2k + 2}{2} \\ &= \frac{k^2 + 3k + 2}{2} = \frac{(k+1)(k+2)}{2} \end{aligned}$$

Condition II also holds. As a result, formula (4) is true for all natural numbers n. ■

Now Work PROBLEM 1

EXAMPLE 4 **Using Mathematical Induction**

Show that $3^n - 1$ is divisible by 2 for all natural numbers n.

Solution First, show that the statement is true when $n = 1$. Because $3^1 - 1 = 3 - 1 = 2$ is divisible by 2, the statement is true when $n = 1$. Condition I is satisfied.

Next, assume that the statement holds for some k, and determine whether the statement holds for $k + 1$. Assume that $3^k - 1$ is divisible by 2 for some k. Now show that $3^{k+1} - 1$ is divisible by 2.

$$\begin{aligned} 3^{k+1} - 1 &= 3^{k+1} - 3^k + 3^k - 1 && \text{Subtract and add } 3^k. \\ &= 3^k(3-1) + (3^k - 1) = 3^k \cdot 2 + (3^k - 1) \end{aligned}$$

Because $3^k \cdot 2$ is divisible by 2 and $3^k - 1$ is divisible by 2, it follows that $3^k \cdot 2 + (3^k - 1) = 3^{k+1} - 1$ is divisible by 2. Condition II is also satisfied. As a result, the statement "$3^n - 1$ is divisible by 2" is true for all natural numbers n. ■

Now Work PROBLEM 19

WARNING The conclusion that a statement involving natural numbers is true for all natural numbers is made only after *both* Conditions I and II of the Principle of Mathematical Induction have been satisfied. Problem 28 demonstrates a statement for which only Condition I holds, and the statement is *not* true for all natural numbers. Problem 29 demonstrates a statement for which only Condition II holds, and the statement is *not* true for any natural number. ■

9.4 Assess Your Understanding

Skill Building

In Problems 1–22, use the Principle of Mathematical Induction to show that the given statement is true for all natural numbers n.

1. $2 + 4 + 6 + \cdots + 2n = n(n+1)$

2. $1 + 5 + 9 + \cdots + (4n-3) = n(2n-1)$

3. $3 + 4 + 5 + \cdots + (n+2) = \frac{1}{2}n(n+5)$

4. $3 + 5 + 7 + \cdots + (2n+1) = n(n+2)$

5. $2 + 5 + 8 + \cdots + (3n-1) = \frac{1}{2}n(3n+1)$

6. $1 + 4 + 7 + \cdots + (3n-2) = \frac{1}{2}n(3n-1)$

7. $1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$

8. $1 + 3 + 3^2 + \cdots + 3^{n-1} = \frac{1}{2}(3^n - 1)$

9. $1 + 4 + 4^2 + \cdots + 4^{n-1} = \frac{1}{3}(4^n - 1)$

10. $1 + 5 + 5^2 + \cdots + 5^{n-1} = \frac{1}{4}(5^n - 1)$

11. $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

12. $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

13. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$

14. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$

15. $4 + 3 + 2 + \cdots + (5-n) = \frac{1}{2}n(9-n)$

16. $-2 - 3 - 4 - \cdots - (n+1) = -\frac{1}{2}n(n+3)$

17. $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$

18. $1\cdot 2 + 3\cdot 4 + 5\cdot 6 + \cdots + (2n-1)(2n) = \frac{1}{3}n(n+1)(4n-1)$

19. $n^2 + n$ is divisible by 2.

20. $n^3 + 2n$ is divisible by 3.

21. $n^2 - n + 2$ is divisible by 2.

22. $n(n+1)(n+2)$ is divisible by 6.

Applications and Extensions

In Problems 23–27, prove each statement.

23. If $x > 1$, then $x^n > 1$.

24. If $0 < x < 1$, then $0 < x^n < 1$.

25. $a - b$ is a factor of $a^n - b^n$.
[**Hint:** $a^{k+1} - b^{k+1} = a(a^k - b^k) + b^k(a - b)$]

26. $a + b$ is a factor of $a^{2n+1} + b^{2n+1}$.

27. $(1 + a)^n \ge 1 + na$, for $a > 0$

28. Show that the statement "$n^2 - n + 41$ is a prime number" is true for $n = 1$ but is not true for $n = 41$.

29. Show that the formula

$$2 + 4 + 6 + \cdots + 2n = n^2 + n + 2$$

obeys Condition II of the Principle of Mathematical Induction. That is, show that if the formula is true for some k, it is also true for $k + 1$. Then show that the formula is false for $n = 1$ (or for any other choice of n).

30. Use mathematical induction to prove that if $r \ne 1$, then

$$a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{1 - r^n}{1 - r}$$

31. Use mathematical induction to prove that

$$a + (a + d) + (a + 2d) + \cdots + [a + (n-1)d] = na + d\frac{n(n-1)}{2}$$

32. Extended Principle of Mathematical Induction The Extended Principle of Mathematical Induction states that if Conditions I and II hold, that is,

(I) A statement is true for a natural number j.

(II) If the statement is true for some natural number $k \ge j$, then it is also true for the next natural number $k + 1$.

then the statement is true for all natural numbers $\ge j$. Use the Extended Principle of Mathematical Induction to show that the number of diagonals in a convex polygon of n sides is $\frac{1}{2}n(n-3)$.

[**Hint:** Begin by showing that the result is true when $n = 4$ (Condition I).]

33. Geometry Use the Extended Principle of Mathematical Induction to show that the sum of the interior angles of a convex polygon of n sides equals $(n - 2)\cdot 180°$.

Explaining Concepts: Discussion and Writing

34. How would you explain the Principle of Mathematical Induction to a friend?

Retain Your Knowledge

Problems 35–38 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

35. Solve: $\log_2 \sqrt{x+5} = 4$

36. If $A = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & -5 \\ 7 & -3 & 1 \end{bmatrix}$, find $-2A + B$.

37. Solve the system: $\begin{cases} 4x + 3y = -7 \\ 2x - 5y = 16 \end{cases}$

38. For $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ 1 & 0 \\ -2 & 2 \end{bmatrix}$, find $A \cdot B$.

9.5 The Binomial Theorem

OBJECTIVES **1** Evaluate $\binom{n}{j}$ (p. 688)

2 Use the Binomial Theorem (p. 690)

Formulas have been given for expanding $(x+a)^n$ for $n = 2$ and $n = 3$. The *Binomial Theorem** is a formula for the expansion of $(x+a)^n$ for any positive integer n. If $n = 1, 2, 3$, and 4, the expansion of $(x+a)^n$ is straightforward.

$(x+a)^1 = x + a$ — **Two terms, beginning with x^1 and ending with a^1**

$(x+a)^2 = x^2 + 2ax + a^2$ — **Three terms, beginning with x^2 and ending with a^2**

$(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$ — **Four terms, beginning with x^3 and ending with a^3**

$(x+a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4$ — **Five terms, begining with x^4 and ending with a^4**

Notice that each expansion of $(x+a)^n$ begins with x^n and ends with a^n. From left to right, the powers of x are decreasing by 1, while the powers of a are increasing by 1. Also, the number of terms equals $n + 1$. Notice, too, that the degree of each monomial in the expansion equals n. For example, in the expansion of $(x+a)^3$, each monomial $(x^3, 3ax^2, 3a^2x, a^3)$ is of degree 3. As a result, it is reasonable to conjecture that the expansion of $(x+a)^n$ would look like this:

$$(x+a)^n = x^n + \underline{\quad\quad} ax^{n-1} + \underline{\quad\quad} a^2x^{n-2} + \cdots + \underline{\quad\quad} a^{n-1}x + a^n$$

where the blanks are numbers to be found. This is in fact the case, as will be seen shortly.

Before we can fill in the blanks, we need to introduce the symbol $\binom{n}{j}$.

1 Evaluate $\binom{n}{j}$

COMMENT On a graphing calculator, the symbol $\binom{n}{j}$ may be denoted by the key $\boxed{nCr}$. ■

The symbol $\binom{n}{j}$, read " n taken j at a time," is defined next.

*The name *binomial* is derived from the fact that $x + a$ is a binomial; that is, it contains two terms.

DEFINITION

If j and n are integers with $0 \le j \le n$, the symbol $\binom{n}{j}$ is defined as

$$\binom{n}{j} = \frac{n!}{j!\,(n-j)!} \tag{1}$$

EXAMPLE 1 Evaluating $\binom{n}{j}$

Find:

(a) $\binom{3}{1}$ (b) $\binom{4}{2}$ (c) $\binom{8}{7}$ (d) $\binom{65}{15}$

Solution

(a) $\binom{3}{1} = \dfrac{3!}{1!\,(3-1)!} = \dfrac{3!}{1!\,2!} = \dfrac{3\cdot 2\cdot 1}{1\,(2\cdot 1)} = \dfrac{6}{2} = 3$

(b) $\binom{4}{2} = \dfrac{4!}{2!\,(4-2)!} = \dfrac{4!}{2!\,2!} = \dfrac{4\cdot 3\cdot 2\cdot 1}{(2\cdot 1)\,(2\cdot 1)} = \dfrac{24}{4} = 6$

(c) $\binom{8}{7} = \dfrac{8!}{7!\,(8-7)!} = \underset{\uparrow\; 8! = 8\cdot 7!}{\dfrac{8!}{7!\,1!}} = \dfrac{8\cdot \cancel{7!}}{\cancel{7!}\cdot 1!} = \dfrac{8}{1} = 8$

(d) Figure 17 shows the solution using a TI-84 Plus C graphing calculator. So

$$\binom{65}{15} \approx 2.073746998 \times 10^{14}$$

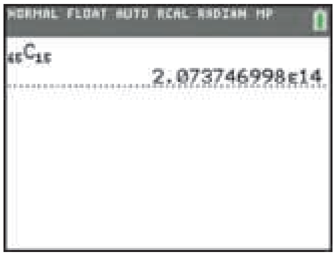


Figure 17

Now Work PROBLEM 5

Four useful formulas involving the symbol $\binom{n}{j}$ are

$$\binom{n}{0} = 1 \qquad \binom{n}{1} = n \qquad \binom{n}{n-1} = n \qquad \binom{n}{n} = 1$$

Proof $\binom{n}{0} = \dfrac{n!}{0!\,(n-0)!} = \dfrac{\cancel{n!}}{0!\,\cancel{n!}} = \dfrac{1}{1} = 1$

$$\binom{n}{1} = \frac{n!}{1!\,(n-1)!} = \frac{n!}{(n-1)!} = \frac{n\cancel{(n-1)!}}{\cancel{(n-1)!}} = n$$

You are asked to prove the remaining two formulas in Problem 45.

Suppose that the values of the symbol $\binom{n}{j}$ are arranged in a triangular display, as shown next and in Figure 18.

$$\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}$$

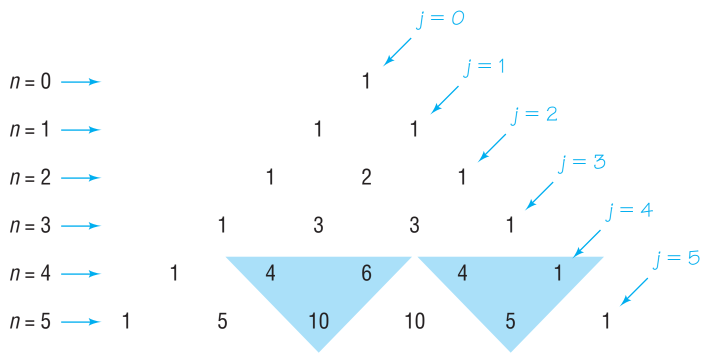


Figure 18 The Pascal Triangle

This display is called the **Pascal triangle**, named after Blaise Pascal (1623–1662), a French mathematician.

The Pascal triangle has 1's down the sides. To get any other entry, add the two nearest entries in the row above it. The shaded triangles in Figure 18 illustrate this feature of the Pascal triangle. Based on this feature, the row corresponding to $n = 6$ is found as follows:

$$\begin{array}{l} \boldsymbol{n = 5 \rightarrow} \quad 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\ \boldsymbol{n = 6 \rightarrow} \quad 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \end{array}$$

This addition always works (see the theorem on page 692).

Although the Pascal triangle provides an interesting and organized display of the symbol $\binom{n}{j}$, in practice it is not all that helpful. For example, if you wanted to know the value of $\binom{12}{5}$, you would need to produce 13 rows of the triangle before seeing the answer. It is much faster to use definition (1).

2 Use the Binomial Theorem

THEOREM

Binomial Theorem

Let x and a be real numbers. For any positive integer n, we have

$$\begin{aligned} (x+a)^n &= \binom{n}{0}x^n + \binom{n}{1}ax^{n-1} + \cdots + \binom{n}{j}a^j x^{n-j} + \cdots + \binom{n}{n}a^n \\ &= \sum_{j=0}^{n} \binom{n}{j} x^{n-j} a^j \end{aligned} \qquad \textbf{(2)}$$

Now you know why it was necessary to introduce the symbol $\binom{n}{j}$; these symbols are the numerical coefficients that appear in the expansion of $(x + a)^n$. Because of this, the symbol $\binom{n}{j}$ is called a **binomial coefficient**.

EXAMPLE 2 **Expanding a Binomial**

Use the Binomial Theorem to expand $(x + 2)^5$.

Solution In the Binomial Theorem, let $a = 2$ and $n = 5$. Then

$$(x+2)^5 = \binom{5}{0}x^5 + \binom{5}{1}2x^4 + \binom{5}{2}2^2x^3 + \binom{5}{3}2^3x^2 + \binom{5}{4}2^4x + \binom{5}{5}2^5$$

↑ **Use equation (2).**

$$= 1 \cdot x^5 + 5 \cdot 2x^4 + 10 \cdot 4x^3 + 10 \cdot 8x^2 + 5 \cdot 16x + 1 \cdot 32$$

↑ **Use row $n = 5$ of the Pascal triangle or definition (1) for $\binom{n}{j}$.**

$$= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$$

EXAMPLE 3 **Expanding a Binomial**

Expand $(2y - 3)^4$ using the Binomial Theorem.

Solution First, rewrite the expression $(2y - 3)^4$ as $[2y + (-3)]^4$. Now use the Binomial Theorem with $n = 4$, $x = 2y$, and $a = -3$.

$$\begin{aligned}[2y + (-3)]^4 &= \binom{4}{0}(2y)^4 + \binom{4}{1}(-3)(2y)^3 + \binom{4}{2}(-3)^2(2y)^2 \\ &\quad + \binom{4}{3}(-3)^3(2y) + \binom{4}{4}(-3)^4 \\ &= 1 \cdot 16y^4 + 4(-3)8y^3 + 6 \cdot 9 \cdot 4y^2 + 4(-27)2y + 1 \cdot 81 \\ &\uparrow \\ &\text{Use row } n = 4 \text{ of the Pascal triangle or definition (1) for } \binom{n}{j}. \\ &= 16y^4 - 96y^3 + 216y^2 - 216y + 81\end{aligned}$$

In this expansion, note that the signs alternate because $a = -3 < 0$. ■

Now Work PROBLEM 21

EXAMPLE 4 **Finding a Particular Coefficient in a Binomial Expansion**

Find the coefficient of y^8 in the expansion of $(2y + 3)^{10}$.

Solution Write out the expansion using the Binomial Theorem.

$$\begin{aligned}(2y + 3)^{10} &= \binom{10}{0}(2y)^{10} + \binom{10}{1}(2y)^9(3)^1 + \binom{10}{2}(2y)^8(3)^2 + \binom{10}{3}(2y)^7(3)^3 \\ &\quad + \binom{10}{4}(2y)^6(3)^4 + \cdots + \binom{10}{9}(2y)(3)^9 + \binom{10}{10}(3)^{10}\end{aligned}$$

From the third term in the expansion, the coefficient of y^8 is

$$\binom{10}{2}(2)^8(3)^2 = \frac{10!}{2!\,8!} \cdot 2^8 \cdot 9 = \frac{10 \cdot 9 \cdot \cancel{8!}}{2 \cdot \cancel{8!}} \cdot 2^8 \cdot 9 = 103{,}680$$ ■

As this solution demonstrates, the Binomial Theorem can be used to find a particular term in an expansion without writing the entire expansion.

Based on the expansion of $(x + a)^n$, the term containing x^j is

$$\binom{n}{n-j} a^{n-j} x^j \qquad \textbf{(3)}$$

Example 4 can be solved by using formula (3) with $n = 10$, $a = 3$, $x = 2y$, and $j = 8$. Then the term containing y^8 is

$$\begin{aligned}\binom{10}{10-8} 3^{10-8}(2y)^8 &= \binom{10}{2} \cdot 3^2 \cdot 2^8 \cdot y^8 = \frac{10!}{2!\,8!} \cdot 9 \cdot 2^8 y^8 \\ &= \frac{10 \cdot 9 \cdot \cancel{8!}}{2 \cdot \cancel{8!}} \cdot 9 \cdot 2^8\, y^8 = 103{,}680y^8\end{aligned}$$

EXAMPLE 5 **Finding a Particular Term in a Binomial Expansion**

Find the 6th term in the expansion of $(x+2)^9$.

Solution A Expand using the Binomial Theorem until the 6th term is reached.

$$(x+2)^9 = \binom{9}{0}x^9 + \binom{9}{1}x^8 \cdot 2 + \binom{9}{2}x^7 \cdot 2^2 + \binom{9}{3}x^6 \cdot 2^3 + \binom{9}{4}x^5 \cdot 2^4 + \binom{9}{5}x^4 \cdot 2^5 + \cdots$$

The 6th term is

$$\binom{9}{5}x^4 \cdot 2^5 = \frac{9!}{5!\,4!} \cdot x^4 \cdot 32 = 4032x^4$$

Solution B The 6th term in the expansion of $(x+2)^9$, which has 10 terms total, contains x^4. (Do you see why?) By formula (3), the 6th term is

$$\binom{9}{9-4}2^{9-4}\,x^4 = \binom{9}{5}2^5\,x^4 = \frac{9!}{5!\,4!} \cdot 32x^4 = 4032x^4$$

Now Work PROBLEMS 29 AND 35

The following theorem shows that the *triangular addition* feature of the Pascal triangle illustrated in Figure 18 always works.

THEOREM

If n and j are integers with $1 \le j \le n$, then

$$\binom{n}{j-1} + \binom{n}{j} = \binom{n+1}{j} \qquad \textbf{(4)}$$

Proof

$$\binom{n}{j-1} + \binom{n}{j} = \frac{n!}{(j-1)!\,[n-(j-1)]!} + \frac{n!}{j!\,(n-j)!}$$

$$= \frac{n!}{(j-1)!\,(n-j+1)!} + \frac{n!}{j!\,(n-j)!}$$

$$= \frac{jn!}{j(j-1)!\,(n-j+1)!} + \frac{(n-j+1)n!}{j!\,(n-j+1)\,(n-j)!}$$

Multiply the first term by $\frac{j}{j}$ and the second term by $\frac{n-j+1}{n-j+1}$ to make the denominators equal.

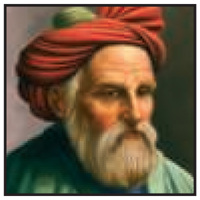
$$= \frac{jn!}{j!\,(n-j+1)!} + \frac{(n-j+1)n!}{j!\,(n-j+1)!}$$

$$= \frac{jn! + (n-j+1)n!}{j!\,(n-j+1)!}$$

$$= \frac{n!\,(j+n-j+1)}{j!\,(n-j+1)!}$$

$$= \frac{n!\,(n+1)}{j!\,(n-j+1)!} = \frac{(n+1)!}{j!\,[(n+1)-j]!} = \binom{n+1}{j}$$

Historical Feature



Omar Khayyám (1048–1131)

The case $n = 2$ of the Binomial Theorem, $(a + b)^2$, was known to Euclid in 300 BC, but the general law seems to have been discovered by the Persian mathematician and astronomer Omar Khayyám (1048–1131), who is also well known as the author of the *Rubáiyát*, a collection of four-line poems making observations on the human condition. Omar Khayyám did not state the Binomial Theorem explicitly, but he claimed to have a method for extracting third, fourth, and fifth roots, and so on. A little study shows that one must know the Binomial Theorem to create such a method.

The heart of the Binomial Theorem is the formula for the numerical coefficients, and, as we saw, they can be written in a symmetric triangular form. The Pascal triangle appears first in the books of Yang Hui (about 1270) and Chu Shih-chieh (1303). Pascal's name is attached to the triangle because of the many applications he made of it, especially to counting and probability. In establishing these results, he was one of the earliest users of mathematical induction.

Many people worked on the proof of the Binomial Theorem, which was finally completed for all n (including complex numbers) by Niels Abel (1802–1829).

9.5 Assess Your Understanding

Concepts and Vocabulary

1. The ________ ________ is a triangular display of the binomial coefficients.

2. $\binom{n}{0} =$ __ and $\binom{n}{1} =$ __ .

3. ***True or False*** $\binom{n}{j} = \dfrac{j!}{(n-j)!\,n!}$

4. The ________ ________ can be used to expand expressions like $(2x + 3)^6$.

Skill Building

In Problems 5–16, evaluate each expression.

5. $\binom{5}{3}$ **6.** $\binom{7}{3}$ **7.** $\binom{7}{5}$ **8.** $\binom{9}{7}$

9. $\binom{50}{49}$ **10.** $\binom{100}{98}$ **11.** $\binom{1000}{1000}$ **12.** $\binom{1000}{0}$

13. $\binom{55}{23}$ **14.** $\binom{60}{20}$ **15.** $\binom{47}{25}$ **16.** $\binom{37}{19}$

In Problems 17–28, expand each expression using the Binomial Theorem.

17. $(x + 1)^5$ **18.** $(x - 1)^5$ **19.** $(x - 2)^6$ **20.** $(x + 3)^5$

21. $(3x + 1)^4$ **22.** $(2x + 3)^5$ **23.** $(x^2 + y^2)^5$ **24.** $(x^2 - y^2)^6$

25. $(\sqrt{x} + \sqrt{2})^6$ **26.** $(\sqrt{x} - \sqrt{3})^4$ **27.** $(ax + by)^5$ **28.** $(ax - by)^4$

In Problems 29–42, use the Binomial Theorem to find the indicated coefficient or term.

29. The coefficient of x^6 in the expansion of $(x + 3)^{10}$

30. The coefficient of x^3 in the expansion of $(x - 3)^{10}$

31. The coefficient of x^7 in the expansion of $(2x - 1)^{12}$

32. The coefficient of x^3 in the expansion of $(2x + 1)^{12}$

33. The coefficient of x^7 in the expansion of $(2x + 3)^9$

34. The coefficient of x^2 in the expansion of $(2x - 3)^9$

35. The 5th term in the expansion of $(x + 3)^7$

36. The 3rd term in the expansion of $(x - 3)^7$

37. The 3rd term in the expansion of $(3x - 2)^9$

38. The 6th term in the expansion of $(3x + 2)^8$

39. The coefficient of x^0 in the expansion of $\left(x^2 + \dfrac{1}{x}\right)^{12}$

40. The coefficient of x^0 in the expansion of $\left(x - \dfrac{1}{x^2}\right)^9$

41. The coefficient of x^4 in the expansion of $\left(x - \dfrac{2}{\sqrt{x}}\right)^{10}$

42. The coefficient of x^2 in the expansion of $\left(\sqrt{x} + \dfrac{3}{\sqrt{x}}\right)^8$

Applications and Extensions

43. Use the Binomial Theorem to find the numerical value of $(1.001)^5$ correct to five decimal places.
[**Hint:** $(1.001)^5 = (1 + 10^{-3})^5$]

44. Use the Binomial Theorem to find the numerical value of $(0.998)^6$ correct to five decimal places.

45. Show that $\binom{n}{n-1} = n$ and $\binom{n}{n} = 1$.

46. Show that if n and j are integers with $0 \le j \le n$, then,

$$\binom{n}{j} = \binom{n}{n-j}$$

Conclude that the Pascal triangle is symmetric with respect to a vertical line drawn from the topmost entry.

47. If n is a positive integer, show that

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

[**Hint:** $2^n = (1 + 1)^n$; now use the Binomial Theorem.]

48. If n is a positive integer, show that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n} = 0$$

49. $\binom{5}{0}\left(\frac{1}{4}\right)^5 + \binom{5}{1}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right) + \binom{5}{2}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2$
$+ \binom{5}{3}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^3 + \binom{5}{4}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^4 + \binom{5}{5}\left(\frac{3}{4}\right)^5 = ?$

50. Stirling's Formula An approximation for $n!$, when n is large, is given by

$$n! \approx \sqrt{2n\pi}\left(\frac{n}{e}\right)^n\left(1 + \frac{1}{12n - 1}\right)$$

Calculate $12!$, $20!$, and $25!$ on your calculator. Then use Stirling's formula to approximate $12!$, $20!$, and $25!$.

Retain Your Knowledge

Problems 51–54 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

51. Solve $6^x = 5^{x+1}$. Express the answer both in exact form and as a decimal rounded to three decimal places.

52. Given that $(f \circ g)(x) = x^2 - 8x + 19$ and $f(x) = x^2 + 3$, find $g(x)$.

53. Solve the system of equations:

$$\begin{cases} x - y - z = 0 \\ 2x + y + 3z = -1 \\ 4x + 2y - z = 12 \end{cases}$$

54. Graph the system of inequalities. Tell whether the graph is bounded or unbounded, and label the corner points.

$$\begin{cases} x \ge 0 \\ y \ge 0 \\ x + y \le 6 \\ 2x + y \le 10 \end{cases}$$

Chapter Review

Things to Know

Sequence (p. 654)	A function whose domain is the set of positive integers
Factorials (p. 657)	$0! = 1, 1! = 1, n! = n(n-1)\cdot\cdots\cdot 3\cdot 2\cdot 1$ if $n \ge 2$ is an integer
Arithmetic sequence (pp. 667 and 669)	$a_1 = a,\ a_n = a_{n-1} + d$, where $a_1 = a$ = first term, d = common difference $a_n = a_1 + (n-1)d$
Sum of the first n terms of an arithmetic sequence (p. 670)	$S_n = \frac{n}{2}[2a_1 + (n-1)d] = \frac{n}{2}(a_1 + a_n)$
Geometric sequence (pp. 674 and 675)	$a_1 = a,\ a_n = ra_{n-1}$, where $a_1 = a$ = first term, r = common ratio $a_n = a_1 r^{n-1} \quad r \ne 0$
Sum of the first n terms of a geometric sequence (p. 676)	$S_n = a_1 \frac{1 - r^n}{1 - r} \quad r \ne 0, 1$
Infinite geometric series (p. 677)	$a_1 + a_1 r + \cdots + a_1 r^{n-1} + \cdots = \sum_{k=1}^{\infty} a_1 r^{k-1}$
Sum of a convergent infinite geometric series (p. 678)	If $\lvert r\rvert < 1$, $\sum_{k=1}^{\infty} a_1 r^{k-1} = \frac{a_1}{1 - r}$

Principle of Mathematical Induction (p. 684)	If the following two conditions are satisfied, Condition I: The statement is true for the natural number 1. Condition II: If the statement is true for some natural number k, it is also true for $k+1$. then the statement is true for all natural numbers.
Binomial coefficient (p. 689)	$\binom{n}{j} = \frac{n!}{j!\,(n-j)!}$
The Pascal triangle (p. 690)	See Figure 18.
Binomial Theorem (p. 690)	$(x+a)^n = \binom{n}{0}x^n + \binom{n}{1}ax^{n-1} + \cdots + \binom{n}{j}a^j x^{n-j} + \cdots + \binom{n}{n}a^n = \sum_{j=0}^{n}\binom{n}{j}x^{n-j}a^j$

Objectives

Section		You should be able to . . .	Examples	Review Exercises
9.1	1	Write the first several terms of a sequence (p. 654)	1–4	1, 2
	2	Write the terms of a sequence defined by a recursive formula (p. 657)	5, 6	3, 4
	3	Use summation notation (p. 658)	7, 8	5, 6
	4	Find the sum of a sequence algebraically and using a graphing utility (p. 659)	9	13, 14
	5	Solve annuity and amortization problems (p. 661)	10, 11	37
9.2	1	Determine whether a sequence is arithmetic (p. 667)	1–3	7–12
	2	Find a formula for an arithmetic sequence (p. 668)	4, 5	17, 19–21, 34(a)
	3	Find the sum of an arithmetic sequence (p. 669)	6–8	7, 10, 14, 34(b), 35
9.3	1	Determine whether a sequence is geometric (p. 674)	1–3	7–12
	2	Find a formula for a geometric sequence (p. 675)	4	11, 18, 36(a)–(c), 38
	3	Find the sum of a geometric sequence (p. 676)	5, 6	9, 11, 15, 16
	4	Determine whether a geometric series converges or diverges (p. 677)	7–9	22–25, 36(d)
9.4	1	Prove statements using mathematical induction (p. 684)	1–4	26–28
9.5	1	Evaluate $\binom{n}{j}$ (p. 688)	1	29
	2	Use the Binomial Theorem (p. 690)	2–5	30–33

Review Exercises

In Problems 1–4, write down the first five terms of each sequence.

1. $\{a_n\} = \left\{(-1)^n\left(\frac{n+3}{n+2}\right)\right\}$ **2.** $\{c_n\} = \left\{\frac{2^n}{n^2}\right\}$ **3.** $a_1 = 3;\quad a_n = \frac{2}{3}a_{n-1}$ **4.** $a_1 = 2;\quad a_n = 2 - a_{n-1}$

5. Write out $\sum_{k=1}^{4}(4k+2)$.

6. Express $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{13}$ using summation notation.

In Problems 7–12, determine whether the given sequence is arithmetic, geometric, or neither. If the sequence is arithmetic, find the common difference and the sum of the first n terms. If the sequence is geometric, find the common ratio and the sum of the first n terms.

7. $\{a_n\} = \{n+5\}$ **8.** $\{c_n\} = \{2n^3\}$ **9.** $\{s_n\} = \{2^{3n}\}$

10. $0, 4, 8, 12, \ldots$ **11.** $3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \ldots$ **12.** $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$

In Problems 13–16, find each sum.

13. $\sum_{k=1}^{30}(k^2+2)$ **14.** $\sum_{k=1}^{40}(-2k+8)$ **15.** $\sum_{k=1}^{7}\left(\frac{1}{3}\right)^k$ **16.** $\sum_{k=1}^{10}(-2)^k$

In Problems 17–19, find the indicated term in each sequence. [**Hint:** *Find the general term first.*]

17. 9th term of 3, 7, 11, 15, . . . **18.** 11th term of $1, \frac{1}{10}, \frac{1}{100}, \ldots$ **19.** 9th term of $\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, \ldots$

In Problems 20 and 21, find a general formula for each arithmetic sequence.

20. 7th term is 31; 20th term is 96

21. 10th term is 0; 18th term is 8

In Problems 22–25, determine whether each infinite geometric series converges or diverges. If it converges, find its sum.

22. $3+1+\frac{1}{3}+\frac{1}{9}+\cdots$ **23.** $2-1+\frac{1}{2}-\frac{1}{4}+\cdots$

24. $\frac{1}{2}+\frac{3}{4}+\frac{9}{8}+\cdots$ **25.** $\sum_{k=1}^{\infty}4\left(\frac{1}{2}\right)^{k-1}$

In Problems 26–28, use the Principle of Mathematical Induction to show that the given statement is true for all natural numbers.

26. $3+6+9+\cdots+3n=\frac{3n}{2}(n+1)$ **27.** $2+6+18+\cdots+2\cdot 3^{n-1}=3^n-1$

28. $1^2+4^2+7^2+\cdots+(3n-2)^2=\frac{1}{2}n(6n^2-3n-1)$

29. Evaluate: $\binom{5}{2}$

In Problems 30 and 31, expand each expression using the Binomial Theorem.

30. $(x+2)^5$ **31.** $(3x-4)^4$

---

32. Find the coefficient of x^7 in the expansion of $(x+2)^9$.

33. Find the coefficient of x^2 in the expansion of $(2x+1)^7$.

34. Constructing a Brick Staircase A brick staircase has a total of 25 steps. The bottom step requires 80 bricks. Each step thereafter requires three fewer bricks than the prior step.

(a) How many bricks are required for the top step?

(b) How many bricks are required to build the staircase?

35. Creating a Floor Design A mosaic tile floor is designed in the shape of a trapezoid 30 feet wide at the base and 15 feet wide at the top. The tiles, 12 inches by 12 inches, are to be placed so that each successive row contains one fewer tile than the row below. How many tiles will be required?

36. Bouncing Balls A ball is dropped from a height of 20 feet. Each time it strikes the ground, it bounces up to three-quarters of the height of the previous bounce.

(a) What height will the ball bounce up to after it strikes the ground for the 3rd time?

(b) How high will it bounce after it strikes the ground for the nth time?

(c) How many times does the ball need to strike the ground before its bounce is less than 6 inches?

(d) What total distance does the ball travel before it stops bouncing?

37. Retirement Planning Chris gets paid once a month and contributes \$200 each pay period into his 401(k). If Chris plans on retiring in 20 years, what will be the value of his 401(k) if the per annum rate of return of the 401(k) is 10% compounded monthly?

38. Salary Increases Your friend has just been hired at an annual salary of \$50,000. If she expects to receive annual increases of 4%, what will be her salary as she begins her 5th year?

Chapter Test

CHAPTER Test Prep VIDEOS

The Chapter Test Prep Videos are step-by-step solutions available in MyMathLab®, or on this text's YouTube Channel. Flip back to the Resources for Success page for a link to this text's YouTube channel.

In Problems 1 and 2, write down the first five terms of each sequence.

1. $\{s_n\} = \left\{\dfrac{n^2 - 1}{n + 8}\right\}$

2. $a_1 = 4,\ a_n = 3a_{n-1} + 2$

In Problems 3 and 4, write out each sum. Evaluate each sum.

3. $\displaystyle\sum_{k=1}^{3} (-1)^{k+1}\left(\frac{k+1}{k^2}\right)$

4. $\displaystyle\sum_{k=1}^{4}\left[\left(\frac{2}{3}\right)^k - k\right]$

5. Write the following sum using summation notation.

$$-\frac{2}{5} + \frac{3}{6} - \frac{4}{7} + \cdots + \frac{11}{14}$$

In Problems 6–11, determine whether the given sequence is arithmetic, geometric, or neither. If the sequence is arithmetic, find the common difference and the sum of the first n terms. If the sequence is geometric, find the common ratio and the sum of the first n terms.

6. $6, 12, 36, 144, \ldots$

7. $\left\{-\dfrac{1}{2}\cdot 4^n\right\}$

8. $-2, -10, -18, -26, \ldots$

9. $\left\{-\dfrac{n}{2} + 7\right\}$

10. $25, 10, 4, \dfrac{8}{5}, \ldots$

11. $\left\{\dfrac{2n - 3}{2n + 1}\right\}$

---

12. Determine whether the infinite geometric series

$$256 - 64 + 16 - 4 + \cdots$$

converges or diverges. If it converges, find its sum.

13. Expand $(3m + 2)^5$ using the Binomial Theorem.

14. Use the Principle of Mathematical Induction to show that the given statement is true for all natural numbers.

$$\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\cdots\left(1 + \frac{1}{n}\right) = n + 1$$

15. A new car sold for \$31,000. If the vehicle loses 15% of its value each year, how much will it be worth after 10 years?

16. A weightlifter begins his routine by benching 100 pounds and increases the weight by 30 pounds for each set. If he does 10 repetitions in each set, what is the total weight lifted after 5 sets?

Cumulative Review

1. Find all the solutions, real and complex, of the equation

$$|x^2| = 9$$

2. (a) Graph the circle $x^2 + y^2 = 100$ and the parabola $y = 3x^2$.

(b) Solve the system of equations: $\begin{cases} x^2 + y^2 = 100 \\ y = 3x^2 \end{cases}$

(c) Where do the circle and the parabola intersect?

3. Solve the equation: $2e^x = 5$

4. Find an equation of the line with slope 5 and x-intercept 2.

5. Find the standard equation of the circle whose center is the point $(-1, 2)$ if $(3, 5)$ is a point on the circle.

6. $f(x) = \dfrac{3x}{x - 2}$ and $g(x) = 2x + 1$

Find:

(a) $(f \circ g)(2)$

(b) $(g \circ f)(4)$

(c) $(f \circ g)(x)$

(d) The domain of $(f \circ g)(x)$

(e) $(g \circ f)(x)$

(f) The domain of $(g \circ f)(x)$

(g) The function g^{-1} and its domain

(h) The function f^{-1} and its domain

7. Find the equation of an ellipse with center at the origin, a focus at $(0, 3)$, and a vertex at $(0, 4)$.

8. Find the equation of a parabola with vertex at $(-1, 2)$ and focus at $(-1, 3)$.

Chapter Projects

I. Population Growth The size of the population of the United States essentially depends on its current population, the birth and death rates of the population, and immigration. Let b represent the birth rate of the U.S. population, and let d represent its death rate. Then $r = b - d$ represents the growth rate of the population, where r varies from year to year. The U.S. population after n years can be modeled using the recursive function

$$p_n = (1 + r)p_{n-1} + I$$

where I represents net immigration into the United States.

1. Using data from the CIA World Factbook at *https://www.cia.gov/library/publications/the-world-factbook/* determine the birth and death rates in the United States for the most recent year that data are available. Birth rates and death rates are given as the number of live births per 1000 population. Each must be computed as the number of births (deaths) per individual. For example, in 2014, the birth rate was 13.42 per 1000 and the death rate was 8.15 per 1000, so

$$b = \frac{13.42}{1000} = 0.01342, \quad \text{and} \quad d = \frac{8.15}{1000} = 0.00815.$$

   Next, using data from the Immigration and Naturalization Service at *www.fedstats.gov*, determine the net immigration into the United States for the same year used to obtain b and d.

2. Determine the value of r, the growth rate of the population.
3. Find a recursive formula for the population of the United States.
4. Use the recursive formula to predict the population of the United States in the following year. In other words, if data are available for the year 2015, predict the U.S. population in 2016.
5. Does your prediction seem reasonable? Explain.
6. Repeat Problems 1–5 for Uganda using the CIA World Factbook (in 2014, the birth rate was 47.17 per 1000 and the death rate was 10.97 per 1000).
7. Do your results for the United States (a developed country) and Uganda (a developing country) seem in line with the article in the chapter opener? Explain.
8. Do you think the recursive formula found in Problem 3 will be useful in predicting future populations? Why or why not?

The following projects are available at the Instructor's Resource Center (IRC):

II. Project at Motorola ***Digital Wireless Communication*** Cell phones take speech and change it into digital code using only zeros and ones. See how the code length can be modeled using a mathematical sequence.

III. Economics Economists use the current price of a good and a recursive model to predict future consumer demand and to determine future production.

IV. Standardized Tests Many tests of intelligence, aptitude, and achievement contain questions asking for the terms of a mathematical sequence.

10 Counting and Probability

Purchasing a Lottery Ticket

In recent years, the jackpot prizes for the nation's two major multistate lotteries, Mega Millions and Powerball, have climbed to all-time highs. This has happened since Mega Millions (in October 2013) and Powerball (in October 2015) made it more difficult to win their top prizes. The probability of winning the Mega Millions jackpot is now about 1 in 259 million, and the probability for Powerball is about 1 in 292 million.

With such improbable chances of winning the jackpots, one might wonder if there *ever* comes a point when purchasing a lottery ticket is worthwhile. One important consideration in making this determination is the **expected profit**. For a game of chance, the expected profit is a measure of how much a player will profit (or lose) if she or he plays the game a large number of times.

The project at the end of this chapter explores the expected profits from playing Mega Millions and Powerball and examines how the expected profit is related to the jackpot amounts.

—See Chapter Project I—

••• A Look Back

We introduced sets in Chapter R, Review, and have been using them to represent solutions of equations and inequalities and to represent the domain and range of functions.

A Look Ahead •••

Here we discuss methods for counting the number of elements in a set and consider the role of sets in probability.

Outline

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Sets (Chapter R, Review, Section R.1, pp. 2–3)

Now Work the 'Are You Prepared?' problems on page 704.

OBJECTIVES
1 Find All the Subsets of a Set (p. 700)
2 Count the Number of Elements in a Set (p. 700)
3 Solve Counting Problems Using the Multiplication Principle (p. 702)

Counting plays a major role in many diverse areas, such as probability, statistics, and computer science; counting techniques are a part of a branch of mathematics called **combinatorics**.

1 Find All the Subsets of a Set

We begin by reviewing the ways in which two sets can be compared.

If two sets A and B have precisely the same elements, we say that A and B are **equal** and write $A = B$.

If each element of a set A is also an element of a set B, we say that A is a **subset** of B and write $A \subseteq B$.

If $A \subseteq B$ and $A \neq B$, we say that A is a **proper subset** of B and write $A \subset B$.

If $A \subseteq B$, every element in set A is also in set B, but B may or may not have additional elements. If $A \subset B$, every element in A is also in B, and B has at least one element not found in A.

Finally, we agree that the empty set, $\varnothing$, is a subset of every set; that is,

$$\varnothing \subseteq A \qquad \text{for any set } A$$

EXAMPLE 1 Finding All the Subsets of a Set

Write down all the subsets of the set $\{a, b, c\}$.

Solution To organize the work, write down all the subsets with no elements, then those with one element, then those with two elements, and finally those with three elements. This gives all the subsets. Do you see why?

0 Elements	1 Element	2 Elements	3 Elements
$\varnothing$	$\{a\}, \{b\}, \{c\}$	$\{a, b\}, \{b, c\}, \{a, c\}$	$\{a, b, c\}$

Now Work PROBLEM 9

2 Count the Number of Elements in a Set

As you count the number of students in a classroom or the number of pennies in your pocket, what you are really doing is matching, on a one-to-one basis, each object to be counted with the set of counting numbers, $1, 2, 3, \ldots, n$, for some number n. If a set A matched up in this fashion with the set $\{1, 2, \ldots, 25\}$, you would conclude that there are 25 elements in the set A. The notation $n(A) = 25$ is used to indicate that there are 25 elements in the set A.

Because the empty set has no elements, we write

$$n(\varnothing) = 0$$

If the number of elements in a set is a nonnegative integer, the set is **finite**. Otherwise, it is **infinite**. We shall concern ourselves only with finite sets.

Look again at Example 1. A set with 3 elements has $2^3 = 8$ subsets. This result can be generalized.

In Words
The notation $n(A)$ means "the number of elements in set A."

If A is a set with n elements, then A has 2^n subsets.

For example, the set $\{a, b, c, d, e\}$ has $2^5 = 32$ subsets.

EXAMPLE 2 **Analyzing Survey Data**

In a survey of 100 college students, 35 were registered in College Algebra, 52 were registered in Computer Science I, and 18 were registered in both courses.

(a) How many students were registered in College Algebra or Computer Science I?
(b) How many were registered in neither course?

Solution (a) First, let A = set of students in College Algebra

B = set of students in Computer Science I

Then the given information tells us that

$$n(A) = 35 \qquad n(B) = 52 \qquad n(A \cap B) = 18$$

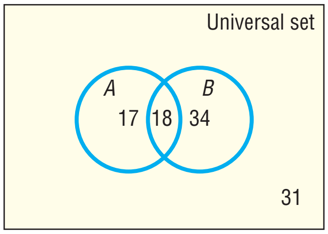


Figure 1

Refer to Figure 1. Since $n(A \cap B) = 18$, the common part of the circles representing set A and set B has 18 elements. In addition, the remaining portion of the circle representing set A will have $35 - 18 = 17$ elements. Similarly, the remaining portion of the circle representing set B has $52 - 18 = 34$ elements. This means that $17 + 18 + 34 = 69$ students were registered in College Algebra or Computer Science I.

(b) Since 100 students were surveyed, it follows that $100 - 69 = 31$ were registered in neither course.

Now Work PROBLEMS 17 AND 27

The solution to Example 2 contains the basis for a general counting formula. If we count the elements in each of two sets A and B, we necessarily count twice any elements that are in both A and B—that is, those elements in $A \cap B$. To count correctly the elements that are in A or B—that is, to find $n(A \cup B)$—we need to subtract those in $A \cap B$ from $n(A) + n(B)$.

THEOREM

Counting Formula

If A and B are finite sets,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) \qquad \textbf{(1)}$$

Refer to Example 2. Using formula (1), we have

$$\begin{aligned} n(A \cup B) &= n(A) + n(B) - n(A \cap B) \\ &= 35 + 52 - 18 \\ &= 69 \end{aligned}$$

There are 69 students registered in College Algebra or Computer Science I.

A special case of the counting formula (1) occurs if A and B have no elements in common. In this case, $A \cap B = \varnothing$, so $n(A \cap B) = 0$.

THEOREM

Addition Principle of Counting

If two sets A and B have no elements in common, that is,

$$\text{if } A \cap B = \varnothing, \text{ then } n(A \cup B) = n(A) + n(B) \qquad \textbf{(2)}$$

Formula (2) can be generalized.

THEOREM

General Addition Principle of Counting

If, for n sets $A_1, A_2, \ldots, A_n$, no two have elements in common, then

$$n(A_1 \cup A_2 \cup \cdots \cup A_n) = n(A_1) + n(A_2) + \cdots + n(A_n) \quad \textbf{(3)}$$

EXAMPLE 3 Counting

Table 1 lists the level of education for all United States residents 25 years of age or older in 2014.

Table 1

Level of Education	Number of U.S. Residents at Least 25 Years Old
Not a high school graduate	24,458,000
High school graduate	62,240,000
Some college, but no degree	34,919,000
Associate's degree	20,790,000
Bachelor's degree	42,256,000
Advanced degree	24,623,000

Source: *U.S. Census Bureau*

(a) How many U.S. residents 25 years of age or older had an associate's degree or a bachelor's degree?

(b) How many U.S. residents 25 years of age or older had an associate's degree, a bachelor's degree, or an advanced degree?

Solution Let A represent the set of associate's degree holders, B represent the set of bachelor's degree holders, and C represent the set of advanced degree holders. No two of the sets A, B, and C have elements in common (although the holder of an advanced degree certainly also holds a bachelor's degree, the individual would be part of the set for which the highest degree has been conferred). Then

$$n(A) = 20{,}790{,}000 \quad n(B) = 42{,}256{,}000 \quad n(C) = 24{,}623{,}000$$

(a) Using formula (2),

$$n(A \cup B) = n(A) + n(B) = 20{,}790{,}000 + 42{,}256{,}000 = 63{,}046{,}000$$

There were 63,046,000 U.S. residents 25 years of age or older who had an associate's degree or a bachelor's degree.

(b) Using formula (3),

$$\begin{aligned} n(A \cup B \cup C) &= n(A) + n(B) + n(C) \\ &= 20{,}790{,}000 + 42{,}256{,}000 + 24{,}623{,}000 \\ &= 87{,}669{,}000 \end{aligned}$$

There were 87,669,000 U.S. residents 25 years of age or older who had an associate's degree, a bachelor's degree, or an advanced degree.

Now Work PROBLEM 31

3 Solve Counting Problems Using the Multiplication Principle

EXAMPLE 4 Counting the Number of Possible Meals

The fixed-price dinner at Mabenka Restaurant provides the following choices:

Appetizer: soup or salad
Entrée: baked chicken, broiled beef patty, beef liver, or roast beef au jus
Dessert: ice cream or cheese cake

How many different meals can be ordered?

Solution Ordering such a meal requires three separate decisions:

Choose an Appetizer	**Choose an Entrée**	**Choose a Dessert**
2 choices	4 choices	2 choices

Look at the **tree diagram** in Figure 2. Note that for each choice of appetizer, there are 4 choices of entrées. And for each of these $2 \cdot 4 = 8$ choices, there are 2 choices for dessert. A total of

$$2 \cdot 4 \cdot 2 = 16$$

different meals can be ordered.

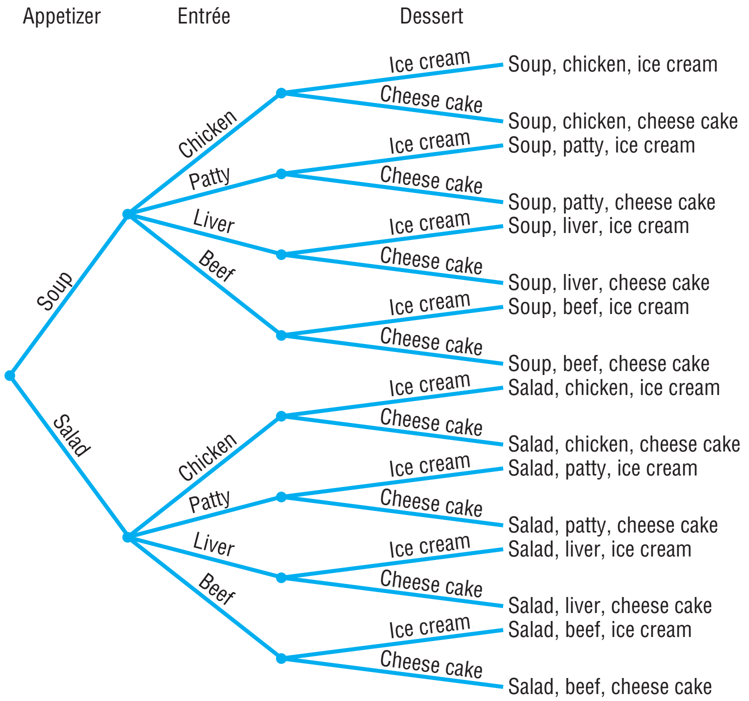


Figure 2

Example 4 demonstrates a general principle of counting.

THEOREM

Multiplication Principle of Counting

If a task consists of a sequence of choices in which there are p selections for the first choice, q selections for the second choice, r selections for the third choice, and so on, the task of making these selections can be done in

$$\boxed{p \cdot q \cdot r \cdot \ldots}$$

different ways.

EXAMPLE 5 Forming Codes

How many two-symbol code words can be formed if the first symbol is an uppercase letter and the second symbol is a digit?

Solution It sometimes helps to begin by listing some of the possibilities. The code consists of an uppercase letter followed by a digit, so some possibilities are A1, A2, B3, X0, and so on. The task consists of making two selections: The first selection requires choosing an uppercase letter (26 choices), and the second task requires choosing a digit (10 choices). By the Multiplication Principle, there are

$$26 \cdot 10 = 260$$

different code words of the type described.

Now Work PROBLEM 23

10.1 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. The ________ of A and B consists of all elements in either A or B or both. (pp. 2–3)
2. The ________ of A with B consists of all elements in both A and B. (pp. 2–3)
3. ***True or False*** The intersection of two sets is always a subset of their union. (pp. 2–3)
4. ***True or False*** If A is a set, the complement of A is the set of all the elements in the universal set that are not in A. (pp. 2–3)

Concepts and Vocabulary

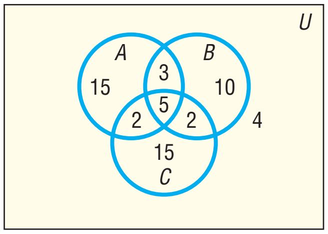
5. If each element of a set A is also an element of a set B, we say that A is a ________ of B and write A ______ B.
6. If the number of elements in a set is a nonnegative integer, we say that the set is ________ .
7. The Counting Formula states that if A and B are finite sets, then $n(A \cup B) =$ ______________________.
8. ***True or False*** If a task consists of a sequence of three choices in which there are p selections for the first choice, q selections for the second choice, and r selections for the third choice, then the task of making these selections can be done in $p \cdot q \cdot r$ different ways.

Skill Building

9. Write down all the subsets of $\{a, b, c, d\}$.
10. Write down all the subsets of $\{a, b, c, d, e\}$.
11. If $n(A) = 15$, $n(B) = 20$, and $n(A \cap B) = 10$, find $n(A \cup B)$.
12. If $n(A) = 30$, $n(B) = 40$, and $n(A \cup B) = 45$, find $n(A \cap B)$.
13. If $n(A \cup B) = 50$, $n(A \cap B) = 10$, and $n(B) = 20$, find $n(A)$.
14. If $n(A \cup B) = 60$, $n(A \cap B) = 40$, and $n(A) = n(B)$, find $n(A)$.

In Problems 15–22, use the information given in the figure.

15. How many are in set A?
16. How many are in set B?
17. How many are in A or B?
18. How many are in A and B?
19. How many are in A but not C?
20. How many are not in A?
21. How many are in A and B and C?
22. How many are in A or B or C?



Applications and Extensions

23. **Shirts and Ties** A man has 5 shirts and 3 ties. How many different shirt-and-tie arrangements can he wear?
24. **Blouses and Skirts** A woman has 5 blouses and 8 skirts. How many different outfits can she wear?
25. **Four-digit Numbers** How many four-digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 if the first digit cannot be 0? Repeated digits are allowed.
26. **Five-digit Numbers** How many five-digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 if the first digit cannot be 0 or 1? Repeated digits are allowed.
27. **Analyzing Survey Data** In a consumer survey of 500 people, 200 indicated that they would be buying a major appliance within the next month, 150 indicated that they would buy a car, and 25 said that they would purchase both a major appliance and a car. How many will purchase neither? How many will purchase only a car?
28. **Analyzing Survey Data** In a student survey, 200 indicated that they would attend Summer Session I, and 150 indicated Summer Session II. If 75 students plan to attend both summer sessions, and 275 indicated that they would attend neither session, how many students participated in the survey?
29. **Analyzing Survey Data** In a survey of 100 investors in the stock market,

    50 owned shares in IBM
    40 owned shares in AT&T
    45 owned shares in GE
    20 owned shares in both IBM and GE
    15 owned shares in both AT&T and GE
    20 owned shares in both IBM and AT&T
    5 owned shares in all three

    (a) How many of the investors surveyed did not have shares in any of the three companies?
    (b) How many owned just IBM shares?
    (c) How many owned just GE shares?
    (d) How many owned neither IBM nor GE?
    (e) How many owned either IBM or AT&T but no GE?
30. **Classifying Blood Types** Human blood is classified as either Rh+ or Rh−. Blood is also classified by type: A, if it contains an A antigen but not a B antigen; B, if it contains a B antigen but not an A antigen; AB, if it contains both A and B antigens; and O, if it contains neither antigen. Draw a Venn diagram illustrating the various blood types. Based on this classification, how many different kinds of blood are there?

31. Demographics The following data represent the marital status of males 18 years old and older in the U.S. in 2014.

Marital Status	Number (in millions)
Married	65.7
Widowed	3.1
Divorced	10.7
Never married	36.3

Source: Current Population Survey

(a) Determine the number of males 18 years old and older who are widowed or divorced.
(b) Determine the number of males 18 years old and older who are married, widowed, or divorced.

32. Demographics The following data represent the marital status of U.S. females 18 years old and older in 2014.

Marital Status	Number (in millions)
Married	66.7
Widowed	11.2
Divorced	14.6
Never married	31.0

Source: Current Population Survey

(a) Determine the number of females 18 years old and older who are widowed or divorced.
(b) Determine the number of females 18 years old and older who are married, widowed, or divorced.

33. Stock Portfolios As a financial planner, you are asked to select one stock each from the following groups: 8 Dow Jones stocks, 15 NASDAQ stocks, and 4 global stocks. How many different portfolios are possible?

Explaining Concepts: Discussion and Writing

34. Make up a problem different from any found in the text that requires the addition principle of counting to solve. Give it to a friend to solve and critique.

35. Investigate the notion of counting as it relates to infinite sets. Write an essay on your findings.

Retain Your Knowledge

Problems 36–39 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

36. Graph $(x-2)^2 + (y+1)^2 = 9$.

37. Given that the point $(3, 8)$ is on the graph of $y = f(x)$, what is the corresponding point on the graph of $y = -2f(x+3) + 5$?

38. Find all the real zeros of the function

$$f(x) = (x-2)(x^2 - 3x - 10)$$

39. Solve: $\log_3 x + \log_3 2 = -2$

'Are You Prepared?' Answers

1. union **2.** intersection **3.** True **4.** True

10.2 Permutations and Combinations

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Factorial (Section 9.1, p. 657)

Now Work the **'Are You Prepared?'** problems on page 711.

OBJECTIVES
1 Solve Counting Problems Using Permutations Involving n Distinct Objects (p. 705)
2 Solve Counting Problems Using Combinations (p. 708)
3 Solve Counting Problems Using Permutations Involving n Nondistinct Objects (p. 710)

1 Solve Counting Problems Using Permutations Involving *n* Distinct Objects

DEFINITION A **permutation** is an ordered arrangement of r objects chosen from n objects.

Three types of permutations are discussed:

1. The n objects are distinct (different), and repetition is allowed in the selection of r of them. [Distinct, with repetition]
2. The n objects are distinct (different), and repetition is not allowed in the selection of r of them, where $r \leq n$. [Distinct, without repetition]
3. The n objects are not distinct, and all of them are used in the arrangement. [Not distinct]

We take up the first two types here and deal with the third type at the end of this section.

The first type of permutation (n distinct objects, repetition allowed) is handled using the Multiplication Principle.

EXAMPLE 1 **Counting Airport Codes [Permutation: Distinct, with Repetition]**

The International Airline Transportation Association (IATA) assigns three-letter codes to represent airport locations. For example, the airport code for Ft. Lauderdale, Florida, is FLL. Notice that repetition is allowed in forming this code. How many airport codes are possible?

Solution An airport code is formed by choosing 3 letters from 26 letters and arranging them in order. In the ordered arrangement, a letter may be repeated. This is an example of a permutation with repetition in which 3 objects are chosen from 26 distinct objects.

The task of counting the number of such arrangements consists of making three selections. Each selection requires choosing a letter of the alphabet (26 choices). By the Multiplication Principle, there are

$$26 \cdot 26 \cdot 26 = 26^3 = 17{,}576$$

possible airport codes. ■

The solution given to Example 1 can be generalized.

THEOREM **Permutations: Distinct Objects with Repetition**

The number of ordered arrangements of r objects chosen from n objects, in which the n objects are distinct and repetition is allowed, is n^r. ■

Now Work PROBLEM 33

Now let's consider permutations in which the objects are distinct and repetition is not allowed.

EXAMPLE 2 **Forming Codes [Permutation: Distinct, without Repetition]**

Suppose that a three-letter code is to be formed using any of the 26 uppercase letters of the alphabet, but no letter is to be used more than once. How many different three-letter codes are there?

Solution Some of the possibilities are ABC, ABD, ABZ, ACB, CBA, and so on. The task consists of making three selections. The first selection requires choosing from 26 letters. Since no letter can be used more than once, the second selection requires choosing from 25 letters. The third selection requires choosing from 24 letters. (Do you see why?) By the Multiplication Principle, there are

$$26 \cdot 25 \cdot 24 = 15{,}600$$

different three-letter codes with no letter repeated. ■

For the second type of permutation, we introduce the following notation.

The notation $\boldsymbol{P(n, r)}$ represents the number of ordered arrangements of r objects chosen from n distinct objects, where $r \leq n$ and repetition is not allowed.

For example, the question posed in Example 2 asks for the number of ways in which the 26 letters of the alphabet can be arranged, in order, using three nonrepeated letters. The answer is

$$P(26, 3) = 26 \cdot 25 \cdot 24 = 15{,}600$$

EXAMPLE 3 Lining People Up

In how many ways can 5 people be lined up?

Solution The 5 people are distinct. Once a person is in line, that person will not be repeated elsewhere in the line; and, in lining people up, order is important. This is a permutation of 5 objects taken 5 at a time, so 5 people can be lined up in

$$P(5, 5) = \underbrace{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}_{\text{5 factors}} = 120 \text{ ways}$$

Now Work PROBLEM 35

To arrive at a formula for $P(n, r)$, note that the task of obtaining an ordered arrangement of n objects in which only $r \leq n$ of them are used, without repeating any of them, requires making r selections. For the first selection, there are n choices; for the second selection, there are $n - 1$ choices; for the third selection, there are $n - 2$ choices; $\ldots$; for the rth selection, there are $n - (r - 1)$ choices. By the Multiplication Principle, this means

$$\begin{aligned} P(n, r) &= \overset{\text{1st}}{n} \cdot \overset{\text{2nd}}{(n-1)} \cdot \overset{\text{3rd}}{(n-2)} \cdot \cdots \cdot \overset{r\text{th}}{[n-(r-1)]} \\ &= n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-r+1) \end{aligned}$$

This formula for $P(n, r)$ can be compactly written using factorial notation.*

$$\begin{aligned} P(n, r) &= n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-r+1) \\ &= n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-r+1) \cdot \frac{(n-r) \cdot \cdots \cdot 3 \cdot 2 \cdot 1}{(n-r) \cdot \cdots \cdot 3 \cdot 2 \cdot 1} = \frac{n!}{(n-r)!} \end{aligned}$$

THEOREM

Permutations of r Objects Chosen from n Distinct Objects without Repetition

The number of arrangements of n objects using $r \leq n$ of them, in which

1. the n objects are distinct,
2. once an object is used it cannot be repeated, and
3. order is important,

is given by the formula

$$P(n, r) = \frac{n!}{(n-r)!} \tag{1}$$

*Recall that $0! = 1,\ 1! = 1,\ 2! = 2 \cdot 1, \ldots, n! = n(n-1) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$.

EXAMPLE 4 **Computing Permutations**

Evaluate: (a) $P(7,3)$ (b) $P(6,1)$ (c) $P(52,5)$

Solution Parts (a) and (b) are each worked two ways.

(a) $P(7,3) = \underbrace{7 \cdot 6 \cdot 5}_{\text{3 factors}} = 210$

or

$$P(7,3) = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4!}}{\cancel{4!}} = 210$$

(b) $P(6,1) = \underbrace{6}_{\text{1 factor}} = 6$

or

$$P(6,1) = \frac{6!}{(6-1)!} = \frac{6!}{5!} = \frac{6 \cdot \cancel{5!}}{\cancel{5!}} = 6$$

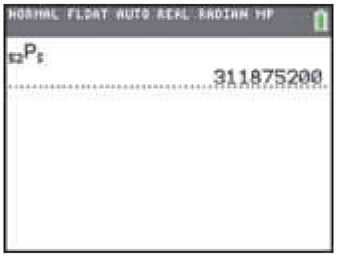


Figure 3 $P(52, 5)$

(c) Figure 3 shows the solution using a TI-84 Plus C graphing calculator. So

$$P(52,5) = 311{,}875{,}200$$

Now Work PROBLEM 7

EXAMPLE 5 **The Birthday Problem**

All we know about Shannon, Patrick, and Ryan is that they have different birthdays. If all the possible ways this could occur were listed, how many would there be? Assume that there are 365 days in a year.

Solution This is an example of a permutation in which 3 birthdays are selected from a possible 365 days, and no birthday may repeat itself. The number of ways this can occur is

$$P(365,3) = \frac{365!}{(365-3)!} = \frac{365 \cdot 364 \cdot 363 \cdot \cancel{362!}}{\cancel{362!}} = 365 \cdot 364 \cdot 363 = 48{,}228{,}180$$

There are 48,228,180 ways in which three people can all have different birthdays.

Now Work PROBLEM 47

2 Solve Counting Problems Using Combinations

In a permutation, order is important. For example, the arrangements *ABC*, *CAB*, *BAC*, . . . are considered different arrangements of the letters *A*, *B*, and *C*. In many situations, though, order is unimportant. For example, in the card game of poker, the order in which the cards are received does not matter; it is the *combination* of the cards that matters.

DEFINITION

A **combination** is an arrangement, without regard to order, of r objects selected from n distinct objects without repetition, where $r \le n$. The notation $\boldsymbol{C(n, r)}$ represents the number of combinations of n distinct objects using r of them.

EXAMPLE 6 **Listing Combinations**

List all the combinations of the 4 objects a, b, c, d taken 2 at a time. What is $C(4,2)$?

Solution One combination of a, b, c, d taken 2 at a time is

$$ab$$

Exclude *ba* from the list because order is not important in a combination (this means that we do not distinguish *ab* from *ba*). The list of all combinations of a, b, c, d taken 2 at a time is

$$ab, \quad ac, \quad ad, \quad bc, \quad bd, \quad cd$$

so

$$C(4, 2) = 6$$

A formula for $C(n, r)$ can be found by noting that the only difference between a permutation of type 2 (distinct, without repetition) and a combination is that order is disregarded in combinations. To determine $C(n, r)$, eliminate from the formula for $P(n, r)$ the number of permutations that are simply rearrangements of a given set of r objects. This can be determined from the formula for $P(n, r)$ by calculating $P(r, r) = r!$. So, dividing $P(n, r)$ by $r!$ gives the desired formula for $C(n, r)$:

$$C(n, r) = \frac{P(n, r)}{r!} \underset{\substack{\uparrow \\ \text{Use formula (1).}}}{=} \frac{\dfrac{n!}{(n-r)!}}{r!} = \frac{n!}{(n-r)!\,r!}$$

We have proved the following result:

THEOREM

Number of Combinations of *n* Distinct Objects Taken *r* at a Time

The number of arrangements of n objects using $r \le n$ of them, in which

1. the n objects are distinct,

2. once an object is used, it cannot be repeated, and

3. order is not important,

is given by the formula

$$\boxed{C(n, r) = \frac{n!}{(n-r)!\,r!} \qquad \textbf{(2)}}$$

Based on formula (2), we discover that the symbol $C(n, r)$ and the symbol $\binom{n}{r}$ for the binomial coefficients are, in fact, the same. The Pascal triangle (see Section 9.5) can be used to find the value of $C(n, r)$. However, because it is more practical and convenient, we will use formula (2) instead.

EXAMPLE 7 Using Formula (2)

Use formula (2) to find the value of each expression.

(a) $C(3, 1)$ (b) $C(6, 3)$ (c) $C(n, n)$ (d) $C(n, 0)$ (e) $C(52, 5)$

Solution

(a) $C(3, 1) = \dfrac{3!}{(3-1)!1!} = \dfrac{3!}{2!1!} = \dfrac{3 \cdot \cancel{2} \cdot 1}{\cancel{2} \cdot 1 \cdot 1} = 3$

(b) $C(6, 3) = \dfrac{6!}{(6-3)!3!} = \dfrac{6 \cdot 5 \cdot 4 \cdot \cancel{3!}}{3! \cdot \cancel{3!}} = \dfrac{\cancel{6} \cdot 5 \cdot 4}{\cancel{6}} = 20$

(c) $C(n, n) = \dfrac{n!}{(n-n)!n!} = \dfrac{\cancel{n!}}{0!\cancel{n!}} = \dfrac{1}{1} = 1$

(d) $C(n, 0) = \dfrac{n!}{(n-0)!0!} = \dfrac{\cancel{n!}}{\cancel{n!}0!} = \dfrac{1}{1} = 1$

(e) Figure 4 shows the solution using a TI-84 Plus C graphing calculator.

$$C(52, 5) = 2{,}598{,}960$$

The value of the expression is 2,598,960. ■

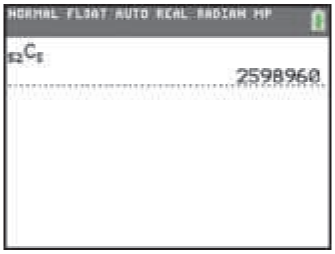


Figure 4 $C(52, 5)$

Now Work PROBLEM 15

EXAMPLE 8 Forming Committees

How many different committees of 3 people can be formed from a pool of 7 people?

Solution The 7 people are distinct. More important, though, is the observation that the order of being selected for a committee is not significant. The problem asks for the number of combinations of 7 objects taken 3 at a time.

$$C(7, 3) = \frac{7!}{4!3!} = \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4!}}{\cancel{4!}3!} = \frac{7 \cdot \cancel{6} \cdot 5}{\cancel{6}} = 35$$

Thirty-five different committees can be formed. ■

EXAMPLE 9 Forming Committees

In how many ways can a committee consisting of 2 faculty members and 3 students be formed if 6 faculty members and 10 students are eligible to serve on the committee?

Solution The problem can be separated into two parts: the number of ways in which the faculty members can be chosen, $C(6, 2)$, and the number of ways in which the student members can be chosen, $C(10, 3)$. By the Multiplication Principle, the committee can be formed in

$$C(6, 2) \cdot C(10, 3) = \frac{6!}{4!2!} \cdot \frac{10!}{7!3!} = \frac{6 \cdot 5 \cdot \cancel{4!}}{\cancel{4!}2!} \cdot \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!}3!}$$

$$= \frac{30}{2} \cdot \frac{720}{6} = 1800 \text{ ways}$$ ■

Now Work PROBLEM 49

3 Solve Counting Problems Using Permutations Involving *n* Nondistinct Objects

EXAMPLE 10 Forming Different Words

How many different words (real or imaginary) can be formed using all the letters in the word REARRANGE?

Solution Each word formed will have 9 letters: 3 R's, 2 A's, 2 E's, 1 N, and 1 G. To construct each word, we need to fill in 9 positions with the 9 letters:

$$\overline{1}\ \ \overline{2}\ \ \overline{3}\ \ \overline{4}\ \ \overline{5}\ \ \overline{6}\ \ \overline{7}\ \ \overline{8}\ \ \overline{9}$$

The process of forming a word consists of five tasks.

Task 1: Choose the positions for the 3 R's.
Task 2: Choose the positions for the 2 A's.
Task 3: Choose the positions for the 2 E's.
Task 4: Choose the position for the 1 N.
Task 5: Choose the position for the 1 G.

Task 1 can be done in $C(9, 3)$ ways. There then remain 6 positions to be filled, so Task 2 can be done in $C(6, 2)$ ways. There remain 4 positions to be filled, so Task 3 can be done in $C(4, 2)$ ways. There remain 2 positions to be filled, so Task 4 can be done in $C(2, 1)$ ways. The last position can be filled in $C(1, 1)$ way. Using the Multiplication Principle, the number of possible words that can be formed is

$$C(9,3)\cdot C(6,2)\cdot C(4,2)\cdot C(2,1)\cdot C(1,1) = \frac{9!}{3!\cdot\cancel{6!}}\cdot\frac{\cancel{6!}}{2!\cdot\cancel{4!}}\cdot\frac{\cancel{4!}}{2!\cdot\cancel{2!}}\cdot\frac{\cancel{2!}}{1!\cdot\cancel{1!}}\cdot\frac{\cancel{1!}}{0!\cdot 1!}$$

$$= \frac{9!}{3!\cdot 2!\cdot 2!\cdot 1!\cdot 1!} = 15{,}120$$

15,120 possible words can be formed. ■

The form of the expression before the answer to Example 10 is suggestive of a general result. Had all the letters in REARRANGE been different, there would have been $P(9, 9) = 9!$ possible words formed. This is the numerator of the answer. The presence of 3 R's, 2 A's, and 2 E's reduces the number of different words, as the entries in the denominator illustrate. This leads to the following result:

THEOREM

Permutations Involving n Objects That Are Not Distinct

The number of permutations of n objects of which n_1 are of one kind, n_2 are of a second kind, . . ., and n_k are of a kth kind is given by

$$\boxed{\frac{n!}{n_1!\cdot n_2!\cdot\cdots\cdot n_k!}} \qquad \textbf{(3)}$$

where $n = n_1 + n_2 + \cdots + n_k$. ■

EXAMPLE 11 **Arranging Flags**

How many different vertical arrangements are there of 8 flags if 4 are white, 3 are blue, and 1 is red?

Solution We seek the number of permutations of 8 objects, of which 4 are of one kind, 3 are of a second kind, and 1 is of a third kind. Using formula (3), we find that there are

$$\frac{8!}{4!\cdot 3!\cdot 1!} = \frac{8\cdot 7\cdot 6\cdot 5\cdot\cancel{4!}}{\cancel{4!}\cdot 3!\cdot 1!} = 280 \text{ different arrangements}$$ ■

Now Work PROBLEM 51

10.2 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. $0! =$ _____; $1! =$ _____. (p. 657)

2. ***True or False*** $n! = \dfrac{(n+1)!}{n}$. (p. 657)

Concepts and Vocabulary

3. A(n) ______________ is an ordered arrangement of r objects chosen from n objects.

4. A(n) ______________ is an arrangement of r objects chosen from n distinct objects, without repetition and without regard to order.

5. $P(n, r) =$ ________.

6. $C(n, r) =$ ________.

Skill Building

In Problems 7–14, find the value of each permutation.

7. $P(6,2)$ 8. $P(7,2)$ 9. $P(4,4)$ 10. $P(8,8)$

11. $P(7,0)$ 12. $P(9,0)$ 13. $P(8,4)$ 14. $P(8,3)$

In Problems 15–22, use formula (2) to find the value of each combination.

15. $C(8,2)$ 16. $C(8,6)$ 17. $C(7,4)$ 18. $C(6,2)$

19. $C(15,15)$ 20. $C(18,1)$ 21. $C(26,13)$ 22. $C(18,9)$

Applications and Extensions

23. List all the ordered arrangements of 5 objects a, b, c, d, and e choosing 3 at a time without repetition. What is $P(5,3)$?

24. List all the ordered arrangements of 5 objects a, b, c, d, and e choosing 2 at a time without repetition. What is $P(5,2)$?

25. List all the ordered arrangements of 4 objects 1, 2, 3, and 4 choosing 3 at a time without repetition. What is $P(4,3)$?

26. List all the ordered arrangements of 6 objects 1, 2, 3, 4, 5, and 6 choosing 3 at a time without repetition. What is $P(6,3)$?

27. List all the combinations of 5 objects a, b, c, d, and e taken 3 at a time. What is $C(5,3)$?

28. List all the combinations of 5 objects a, b, c, d, and e taken 2 at a time. What is $C(5,2)$?

29. List all the combinations of 4 objects 1, 2, 3, and 4 taken 3 at a time. What is $C(4,3)$?

30. List all the combinations of 6 objects 1, 2, 3, 4, 5, and 6 taken 3 at a time. What is $C(6,3)$?

31. Forming Codes How many two-letter codes can be formed using the letters A, B, C, and D? Repeated letters are allowed.

32. Forming Codes How many two-letter codes can be formed using the letters A, B, C, D, and E? Repeated letters are allowed.

33. Forming Numbers How many three-digit numbers can be formed using the digits 0 and 1? Repeated digits are allowed.

34. Forming Numbers How many three-digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9? Repeated digits are allowed.

35. Lining People Up In how many ways can 4 people be lined up?

36. Stacking Boxes In how many ways can 5 different boxes be stacked?

37. Forming Codes How many different three-letter codes are there if only the letters A, B, C, D, and E can be used and no letter can be used more than once?

38. Forming Codes How many different four-letter codes are there if only the letters A, B, C, D, E, and F can be used and no letter can be used more than once?

39. Stocks on the NYSE Companies whose stocks are listed on the New York Stock Exchange (NYSE) have their company name represented by 1, 2, or 3 letters (repetition of letters is allowed). What is the maximum number of companies that can be listed on the NYSE?

40. Stocks on the NASDAQ Companies whose stocks are listed on the NASDAQ stock exchange have their company name represented by either 4 or 5 letters (repetition of letters is allowed). What is the maximum number of companies that can be listed on the NASDAQ?

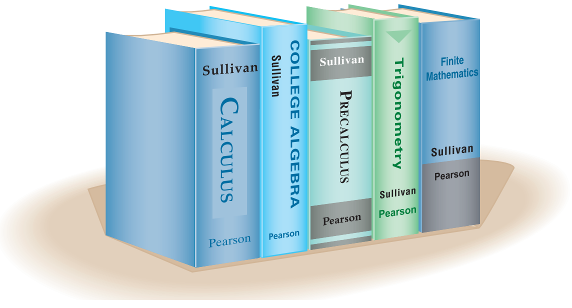
41. Establishing Committees In how many ways can a committee of 4 students be formed from a pool of 7 students?

42. Establishing Committees In how many ways can a committee of 3 professors be formed from a department that has 8 professors?

43. Possible Answers on a True/False Test How many arrangements of answers are possible for a true/false test with 10 questions?

44. Possible Answers on a Multiple-choice Test How many arrangements of answers are possible in a multiple-choice test with 5 questions, each of which has 4 possible answers?

45. Arranging Books Five different mathematics books are to be arranged on a student's desk. How many arrangements are possible?



46. Forming License Plate Numbers How many different license plate numbers can be made using 2 letters followed by 4 digits selected from the digits 0 through 9, if:

(a) Letters and digits may be repeated?
(b) Letters may be repeated, but digits may not be repeated?
(c) Neither letters nor digits may be repeated?

47. Birthday Problem In how many ways can 2 people each have different birthdays? Assume that there are 365 days in a year.

48. Birthday Problem In how many ways can 5 people all have different birthdays? Assume that there are 365 days in a year.

49. Forming a Committee A student dance committee is to be formed consisting of 2 boys and 3 girls. If the membership is to be chosen from 4 boys and 8 girls, how many different committees are possible?

50. **Forming a Committee** The student relations committee of a college consists of 2 administrators, 3 faculty members, and 5 students. Four administrators, 8 faculty members, and 20 students are eligible to serve. How many different committees are possible?

51. **Forming Words** How many different 9-letter words (real or imaginary) can be formed from the letters in the word ECONOMICS?

52. **Forming Words** How many different 11-letter words (real or imaginary) can be formed from the letters in the word MATHEMATICS?

53. **Selecting Objects** An urn contains 7 white balls and 3 red balls. Three balls are selected. In how many ways can the 3 balls be drawn from the total of 10 balls:
    (a) If 2 balls are white and 1 is red?
    (b) If all 3 balls are white?
    (c) If all 3 balls are red?

54. **Selecting Objects** An urn contains 15 red balls and 10 white balls. Five balls are selected. In how many ways can the 5 balls be drawn from the total of 25 balls:
    (a) If all 5 balls are red?
    (b) If 3 balls are red and 2 are white?
    (c) If at least 4 are red balls?

55. **Senate Committees** The U.S. Senate has 100 members. Suppose that it is desired to place each senator on exactly 1 of 7 possible committees. The first committee has 22 members, the second has 13, the third has 10, the fourth has 5, the fifth has 16, and the sixth and seventh have 17 apiece. In how many ways can these committees be formed?

56. **Football Teams** A defensive football squad consists of 25 players. Of these, 10 are linemen, 10 are linebackers, and 5 are safeties. How many different teams of 5 linemen, 3 linebackers, and 3 safeties can be formed?

57. **Baseball** In the American Baseball League, a designated hitter may be used. How many batting orders is it possible for a manager to use? (There are 9 regular players on a team.)

58. **Baseball** In the National Baseball League, the pitcher usually bats ninth. If this is the case, how many batting orders is it possible for a manager to use?

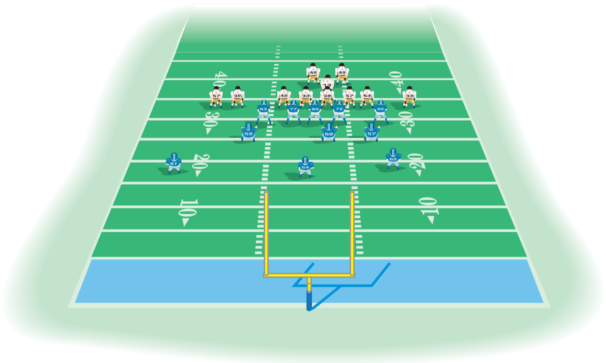
59. **Baseball Teams** A baseball team has 15 members. Four of the players are pitchers, and the remaining 11 members can play any position. How many different teams of 9 players can be formed?

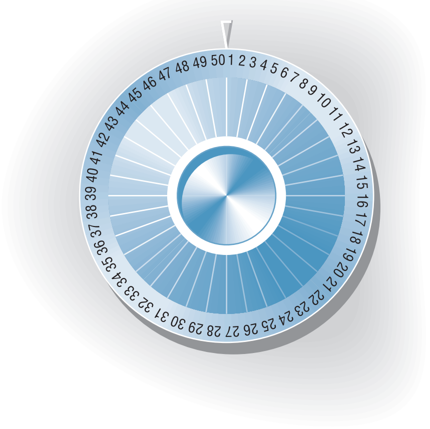
60. **World Series** In the World Series the American League team (A) and the National League team (N) play until one team wins four games. If the sequence of winners is designated by letters (for example, *NAAAA* means that the National League team won the first game and the American League won the next four), how many different sequences are possible?

61. **Basketball Teams** A basketball team has 6 players who play guard (2 of 5 starting positions). How many different teams are possible, assuming that the remaining 3 positions are filled and it is not possible to distinguish a left guard from a right guard?

62. **Basketball Teams** On a basketball team of 12 players, 2 play only center, 3 play only guard, and the rest play forward (5 players on a team: 2 forwards, 2 guards, and 1 center). How many different teams are possible, assuming that it is not possible to distinguish a left guard from a right guard or a left forward from a right forward?

63. **Combination Locks** A combination lock displays 50 numbers. To open it, you turn clockwise to the first number of the "combination," then rotate counterclockwise to the second number, and then rotate clockwise to the third number.
    (a) How many different lock combinations are there?
    (b) Comment on the description of such a lock as a *combination* lock.





Explaining Concepts: Discussion and Writing

64. Create a problem different from any found in the text that requires a permutation to solve. Give it to a friend to solve and critique.

65. Create a problem different from any found in the text that requires a combination to solve. Give it to a friend to solve and critique.

66. Explain the difference between a permutation and a combination. Give an example to illustrate your explanation.

Retain Your Knowledge

Problems 67–70 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

67. Find any asymptotes for the graph of

$$R(x) = \frac{x+3}{x^2 - x - 12}$$

68. If $f(x) = 2x - 1$ and $g(x) = x^2 + x - 2$, find $(g \circ f)(x)$.

69. Solve: $\frac{5}{x-3} \geq 1$

70. Find the 5th term of the geometric sequence with first term $a_1 = 5$ and common ratio $r = -2$.

'Are You Prepared?' Answers

1. 1; 1 **2.** False

10.3 Probability

OBJECTIVES
1 Construct Probability Models (p. 714)
2 Compute Probabilities of Equally Likely Outcomes (p. 716)
3 Find Probabilities of the Union of Two Events (p. 718)
4 Use the Complement Rule to Find Probabilities (p. 719)

Probability is an area of mathematics that deals with experiments that yield random results, yet admit a certain regularity. Such experiments do not always produce the same result or outcome, so the result of any one observation is not predictable. However, the results of the experiment over a long period do produce regular patterns that enable us to make predictions with remarkable accuracy.

EXAMPLE 1 **Tossing a Fair Coin**

If a fair coin is tossed, the outcome is either a head or a tail. On any particular throw, we cannot predict what will happen, but if we toss the coin many times, we observe that the number of times that a head comes up is approximately equal to the number of times that a tail comes up. It seems reasonable, therefore, to assign a probability of $\frac{1}{2}$ that a head comes up and a probability of $\frac{1}{2}$ that a tail comes up. ■

1 Construct Probability Models

The discussion in Example 1 constitutes the construction of a **probability model** for the experiment of tossing a fair coin once. A probability model has two components: a sample space and an assignment of probabilities. A **sample space** S is a set whose elements represent all the possibilities that can occur as a result of the experiment. Each element of S is called an **outcome**. To each outcome a number is assigned, called the **probability** of that outcome, which has two properties:

1. The probability assigned to each outcome is nonnegative.
2. The sum of all the probabilities equals 1.

DEFINITION

A **probability model** with the sample space

$$S = \{e_1, e_2, \ldots, e_n\}$$

where $e_1, e_2, \ldots, e_n$ are the possible outcomes and $P(e_1), P(e_2), \ldots, P(e_n)$ are the respective probabilities of these outcomes, requires that

$$P(e_1) \geq 0, P(e_2) \geq 0, \ldots, P(e_n) \geq 0 \qquad \textbf{(1)}$$

$$\sum_{i=1}^{n} P(e_i) = P(e_1) + P(e_2) + \cdots + P(e_n) = 1 \qquad \textbf{(2)}$$

EXAMPLE 2 Determining Probability Models

In a bag of M&Ms,™ the candies are colored red, green, blue, brown, yellow, and orange. A candy is drawn from the bag and the color is recorded. The sample space of this experiment is {red, green, blue, brown, yellow, orange}. Determine which of the following are probability models.

(a)

Outcome	Probability
red	0.3
green	0.15
blue	0
brown	0.15
yellow	0.2
orange	0.2

(b)

Outcome	Probability
red	0.1
green	0.1
blue	0.1
brown	0.4
yellow	0.2
orange	0.3

(c)

Outcome	Probability
red	0.3
green	−0.3
blue	0.2
brown	0.4
yellow	0.2
orange	0.2

(d)

Outcome	Probability
red	0
green	0
blue	0
brown	0
yellow	1
orange	0

Solution

(a) This model is a probability model because all the outcomes have probabilities that are nonnegative, and the sum of the probabilities is 1.

(b) This model is not a probability model because the sum of the probabilities is not 1.

(c) This model is not a probability model because $P(\text{green})$ is less than 0. Remember that all probabilities must be nonnegative.

(d) This model is a probability model because all the outcomes have probabilities that are nonnegative, and the sum of the probabilities is 1. Notice that $P(\text{yellow}) = 1$, meaning that this outcome will occur with 100% certainty each time that the experiment is repeated. This means that the bag of M&Ms™ contains only yellow candies. ■

Now Work PROBLEM 7

EXAMPLE 3 Constructing a Probability Model

An experiment consists of rolling a fair die once. A die is a cube with each face having 1, 2, 3, 4, 5, or 6 dots on it. See Figure 5. Construct a probability model for this experiment.

Solution A sample space S consists of all the possibilities that can occur. Because rolling the die will result in one of six faces showing, the sample space S consists of

$$S = \{1, 2, 3, 4, 5, 6\}$$

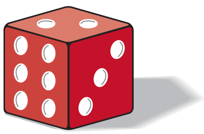


Figure 5 A six-sided die

Because the die is fair, one face is no more likely to occur than another. As a result, our assignment of probabilities is

$$P(1) = \frac{1}{6} \qquad P(2) = \frac{1}{6}$$
$$P(3) = \frac{1}{6} \qquad P(4) = \frac{1}{6}$$
$$P(5) = \frac{1}{6} \qquad P(6) = \frac{1}{6}$$

■

Now suppose that a die is loaded (weighted) so that the probability assignments are

$$P(1) = 0 \quad P(2) = 0 \quad P(3) = \frac{1}{3} \quad P(4) = \frac{2}{3} \quad P(5) = 0 \quad P(6) = 0$$

This assignment would be made if the die were loaded so that only a 3 or 4 could occur and the 4 was twice as likely as the 3 to occur. This assignment is consistent with the definition, since each assignment is nonnegative, and the sum of all the probability assignments equals 1.

Now Work PROBLEM 23

EXAMPLE 4 Constructing a Probability Model

An experiment consists of tossing a coin. The coin is weighted so that heads (H) is three times as likely to occur as tails (T). Construct a probability model for this experiment.

Solution The sample space S is $S = \{H, T\}$. If x denotes the probability that a tail occurs,

$$P(T) = x \quad \text{and} \quad P(H) = 3x$$

The sum of the probabilities of the possible outcomes must equal 1, so

$$\begin{aligned} P(T) + P(H) = x + 3x &= 1 \\ 4x &= 1 \\ x &= \frac{1}{4} \end{aligned}$$

Assign the probabilities

$$P(T) = \frac{1}{4} \qquad P(H) = \frac{3}{4}$$

■

Now Work PROBLEM 27

In Words
$P(S) = 1$ means that one of the outcomes in the sample space must occur in an experiment.

In working with probability models, the term **event** is used to describe a set of possible outcomes of the experiment. An event E is some subset of the sample space S. The **probability of an event** E, $E \neq \varnothing$, denoted by $P(E)$, is defined as the sum of the probabilities of the outcomes in E. We can also think of the probability of an event E as the likelihood that the event E occurs. If $E = \varnothing$, then $P(E) = 0$; if $E = S$, then $P(E) = P(S) = 1$.

2 Compute Probabilities of Equally Likely Outcomes

When the same probability is assigned to each outcome of the sample space, the experiment is said to have **equally likely outcomes**.

THEOREM

Probability for Equally Likely Outcomes

If an experiment has n equally likely outcomes, and if the number of ways in which an event E can occur is m, then the probability of E is

$$P(E) = \frac{\text{Number of ways that } E \text{ can occur}}{\text{Number of possible outcomes}} = \frac{m}{n} \qquad \textbf{(3)}$$

If S is the sample space of this experiment,

$$P(E) = \frac{n(E)}{n(S)} \qquad \textbf{(4)}$$

■

EXAMPLE 5

Calculating Probabilities of Events Involving Equally Likely Outcomes

Calculate the probability that in a 3-child family there are 2 boys and 1 girl. Assume equally likely outcomes.

Solution Begin by constructing a tree diagram to help in listing the possible outcomes of the experiment. See Figure 6, where B stands for "boy" and G for "girl." The sample space S of this experiment is

$$S = \{\text{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG}\}$$

so $n(S) = 8$.

We wish to know the probability of the event E: "having two boys and one girl." From Figure 6, we conclude that $E = \{\text{BBG, BGB, GBB}\}$, so $n(E) = 3$. Since the outcomes are equally likely, the probability of E is

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{8}$$

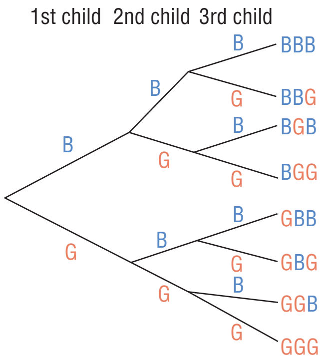


Figure 6

Now Work PROBLEM 37

So far, we have calculated probabilities of single events. Now we compute probabilities of multiple events, which are called **compound probabilities**.

EXAMPLE 6

Computing Compound Probabilities

Consider the experiment of rolling a single fair die. Let E represent the event "roll an odd number," and let F represent the event "roll a 1 or 2."

(a) Write the event E and F. What is $n(E \cap F)$?
(b) Write the event E or F. What is $n(E \cup F)$?
(c) Compute $P(E)$. Compute $P(F)$.
(d) Compute $P(E \cap F)$.
(e) Compute $P(E \cup F)$.

Solution The sample space S of the experiment is $\{1, 2, 3, 4, 5, 6\}$, so $n(S) = 6$. Since the die is fair, the outcomes are equally likely. The event E: "roll an odd number" is $\{1, 3, 5\}$, and the event F: "roll a 1 or 2" is $\{1, 2\}$, so $n(E) = 3$ and $n(F) = 2$.

(a) In probability, the word *and* means the intersection of two events. The event E and F is

$$E \cap F = \{1, 3, 5\} \cap \{1, 2\} = \{1\} \qquad n(E \cap F) = 1$$

(b) In probability, the word *or* means the union of the two events. The event E or F is

$$E \cup F = \{1, 3, 5\} \cup \{1, 2\} = \{1, 2, 3, 5\} \qquad n(E \cup F) = 4$$

(c) Use formula (4). Then

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{6} = \frac{1}{2} \qquad P(F) = \frac{n(F)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

(d) $P(E \cap F) = \dfrac{n(E \cap F)}{n(S)} = \dfrac{1}{6}$

(e) $P(E \cup F) = \dfrac{n(E \cup F)}{n(S)} = \dfrac{4}{6} = \dfrac{2}{3}$

3 Find Probabilities of the Union of Two Events

The next formula can be used to find the probability of the union of two events.

THEOREM

For any two events E and F,

$$P(E \cup F) = P(E) + P(F) - P(E \cap F) \qquad \textbf{(5)}$$

This result is a consequence of the Counting Formula discussed earlier, in Section 10.1.

For example, formula (5) can be used to find $P(E \cup F)$ in Example 6(e). Then

$$P(E \cup F) = P(E) + P(F) - P(E \cap F) = \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{3}{6} + \frac{2}{6} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$$

as before.

EXAMPLE 7 Computing Probabilities of the Union of Two Events

If $P(E) = 0.2$, $P(F) = 0.3$, and $P(E \cap F) = 0.1$, find the probability of E or F. That is, find $P(E \cup F)$.

Solution Use formula (5).

$$\text{Probability of } E \text{ or } F = P(E \cup F) = P(E) + P(F) - P(E \cap F)$$
$$= 0.2 + 0.3 - 0.1 = 0.4$$

A Venn diagram can sometimes be used to obtain probabilities. To construct a Venn diagram representing the information in Example 7, draw two sets E and F. Begin with the fact that $P(E \cap F) = 0.1$. See Figure 7(a). Then, since $P(E) = 0.2$ and $P(F) = 0.3$, fill in E with $0.2 - 0.1 = 0.1$ and fill in F with $0.3 - 0.1 = 0.2$. See Figure 7(b). Since $P(S) = 1$, complete the diagram by inserting $1 - (0.1 + 0.1 + 0.2) = 0.6$ outside the circles. See Figure 7(c). Now it is easy to see, for example, that the probability of F but not E is 0.2. Also, the probability of neither E nor F is 0.6.

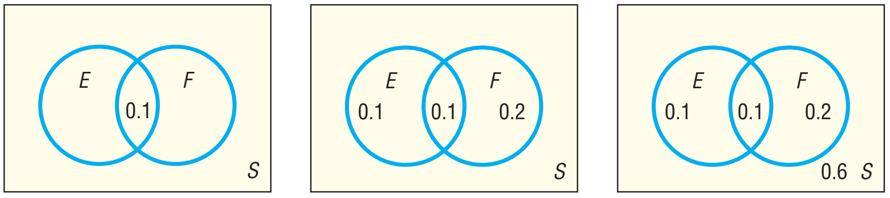


Figure 7 (a) (b) (c)

Now Work PROBLEM 45

If events E and F are disjoint so that $E \cap F = \varnothing$, we say they are **mutually exclusive**. In this case, $P(E \cap F) = 0$, and formula (5) takes the following form:

THEOREM

Mutually Exclusive Events

If E and F are mutually exclusive events,

$$P(E \cup F) = P(E) + P(F) \qquad \textbf{(6)}$$

EXAMPLE 8 **Computing Probabilities of the Union of Two Mutually Exclusive Events**

If $P(E) = 0.4$ and $P(F) = 0.25$, and E and F are mutually exclusive, find $P(E \cup F)$.

Solution Since E and F are mutually exclusive, use formula (6).

$$P(E \cup F) = P(E) + P(F) = 0.4 + 0.25 = 0.65$$

Now Work PROBLEM 47

4 Use the Complement Rule to Find Probabilities

Recall that if A is a set, the complement of A, denoted $\overline{A}$, is the set of all elements in the universal set U that are not in A. We similarly define the complement of an event.

DEFINITION **Complement of an Event**

Let S denote the sample space of an experiment, and let E denote an event. The **complement of E**, denoted $\overline{E}$, is the set of all outcomes in the sample space S that are not outcomes in the event E.

The complement of an event E—that is, $\overline{E}$—in a sample space S has the following two properties:

$$E \cap \overline{E} = \varnothing \qquad E \cup \overline{E} = S$$

Since E and $\overline{E}$ are mutually exclusive, it follows from (6) that

$$P(E \cup \overline{E}) = P(S) = 1 \qquad P(E) + P(\overline{E}) = 1 \qquad P(\overline{E}) = 1 - P(E)$$

We have the following result:

THEOREM **Computing Probabilities of Complementary Events**

If E represents any event and $\overline{E}$ represents the complement of E, then

$$P(\overline{E}) = 1 - P(E) \tag{7}$$

EXAMPLE 9 **Computing Probabilities Using Complements**

On the local news the weather reporter stated that the probability of rain tomorrow is 40%. What is the probability that it will not rain?

Solution The complement of the event "rain" is "no rain."

$$P(\text{no rain}) = 1 - P(\text{rain}) = 1 - 0.4 = 0.6$$

There is a 60% chance of no rain tomorrow.

Now Work PROBLEM 51

EXAMPLE 10 **Birthday Problem**

What is the probability that in a group of 10 people, at least 2 people have the same birthday? Assume that there are 365 days in a year and that a person is as likely to be born on one day as another, so all the outcomes are equally likely.

Solution First determine the number of outcomes in the sample space S. There are 365 possibilities for each person's birthday. Since there are 10 people in the group, there are 365^{10} possibilities for the birthdays. [For one person in the group, there are 365 days on which his or her birthday can fall; for two people, there are $(365)(365) = 365^2$ pairs of days; and, in general, using the Multiplication Principle, for n people there are 365^n possibilities.] So

$$n(S) = 365^{10}$$

We wish to find the probability of the event E: "at least two people have the same birthday." It is difficult to count the elements in this set; it is much easier to count the elements of the complementary event $\overline{E}$: "no two people have the same birthday."

Find $n(\overline{E})$ as follows: Choose one person at random. There are 365 possibilities for his or her birthday. Choose a second person. There are 364 possibilities for this birthday, if no two people are to have the same birthday. Choose a third person. There are 363 possibilities left for this birthday. Finally, arrive at the tenth person. There are 356 possibilities left for this birthday. By the Multiplication Principle, the total number of possibilities is

$$n(\overline{E}) = 365 \cdot 364 \cdot 363 \cdot \cdots \cdot 356$$

The probability of the event $\overline{E}$ is

$$P(\overline{E}) = \frac{n(\overline{E})}{n(S)} = \frac{365 \cdot 364 \cdot 363 \cdot \cdots \cdot 356}{365^{10}} \approx 0.883$$

The probability of two or more people in a group of 10 people having the same birthday is then

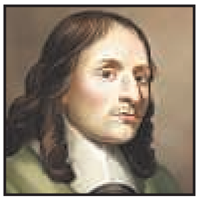
$$P(E) = 1 - P(\overline{E}) \approx 1 - 0.883 = 0.117$$

The birthday problem can be solved for any group size. The following table gives the probabilities for two or more people having the same birthday for various group sizes. Notice that the probability is greater than $\frac{1}{2}$ for any group of 23 or more people.

	Number of People															
	5	10	15	20	21	22	23	24	25	30	40	50	60	70	80	90
Probability That Two or More Have the Same Birthday	0.027	0.117	0.253	0.411	0.444	0.476	0.507	0.538	0.569	0.706	0.891	0.970	0.994	0.99916	0.99991	0.99999

Now Work PROBLEM 71

Historical Feature



Blaise Pascal (1623–1662)

Set theory, counting, and probability first took form as a systematic theory in an exchange of letters (1654) between Pierre de Fermat (1601–1665) and Blaise Pascal (1623–1662). They discussed the problem of how to divide the stakes in a game that is interrupted before completion, knowing how many points each player needs to win. Fermat solved the problem by listing all possibilities and counting the favorable ones, whereas Pascal made use of the triangle that now bears his name. As mentioned in the text, the entries in Pascal's triangle are equivalent to $C(n, r)$. This recognition of the role of $C(n, r)$ in counting is the foundation of all further developments.

The first book on probability, the work of Christiaan Huygens (1629–1695), appeared in 1657. In it, the notion of mathematical expectation is explored. This allows the calculation of the profit or loss that a gambler might expect, knowing the probabilities involved in the game (see the Historical Problem that follows).

Although Girolamo Cardano (1501–1576) wrote a treatise on probability, it was not published until 1663 in Cardano's collected works, and this was too late to have had any effect on the early development of the theory.

In 1713, the posthumously published *Ars Conjectandi* of Jakob Bernoulli (1654–1705) gave the theory the form it would have until 1900. Recently, both combinatorics (counting) and probability have undergone rapid development, thanks to the use of computers.

A final comment about notation. The notations $C(n, r)$ and $P(n, r)$ are variants of a form of notation developed in England after 1830. The notation $\binom{n}{r}$ for $C(n, r)$ goes back to Leonhard Euler (1707–1783) but is now losing ground because it has no clearly related symbolism of the same type for permutations. The set symbols $\cup$ and $\cap$ were introduced by Giuseppe Peano (1858–1932) in 1888 in a slightly different context. The inclusion symbol $\subset$ was introduced by E. Schroeder (1841–1902) about 1890. We owe the treatment of set theory in the text to George Boole (1815–1864), who wrote $A + B$ for $A \cup B$ and AB for $A \cap B$ (statisticians still use AB for $A \cap B$).

Historical Problem

1. *The Problem Discussed by Fermat and Pascal* A game between two equally skilled players, A and B, is interrupted when A needs 2 points to win and B needs 3 points. In what proportion should the stakes be divided?

   (a) *Fermat's solution* List all possible outcomes that can occur as a result of four more plays. Comparing the probabilities for A to win and for B to win then determines how the stakes should be divided.

   (b) *Pascal's solution* Use combinations to determine the number of ways that the 2 points needed for A to win could occur in four plays. Then use combinations to determine the number of ways that the 3 points needed for B to win could occur. This is trickier than it looks, since A can win with 2 points in two plays, in three plays, or in four plays. Compute the probabilities, and compare them with the results in part (a).

10.3 Assess Your Understanding

Concepts and Vocabulary

1. When the same probability is assigned to each outcome of a sample space, the experiment is said to have __________ __________ outcomes.

2. The ______________ of an event E is the set of all outcomes in the sample space S that are not outcomes in the event E.

3. ***True or False*** The probability of an event can never equal 0.

4. ***True or False*** In a probability model, the sum of all probabilities is 1.

Skill Building

5. In a probability model, which of the following numbers could be the probability of an outcome?

$$0 \quad 0.01 \quad 0.35 \quad -0.4 \quad 1 \quad 1.4$$

6. In a probability model, which of the following numbers could be the probability of an outcome?

$$1.5 \quad \frac{1}{2} \quad \frac{3}{4} \quad \frac{2}{3} \quad 0 \quad -\frac{1}{4}$$

7. Determine whether the following is a probability model.

Outcome	Probability
1	0.2
2	0.3
3	0.1
4	0.4

8. Determine whether the following is a probability model.

Outcome	Probability
Steve	0.4
Bob	0.3
Faye	0.1
Patricia	0.2

9. Determine whether the following is a probability model.

Outcome	Probability
Linda	0.3
Jean	0.2
Grant	0.1
Jim	0.3

10. Determine whether the following is a probability model.

Outcome	Probability
Erica	0.3
Joanne	0.2
Laura	0.1
Donna	0.5
Angela	−0.1

In Problems 11–16, construct a probability model for each experiment.

11. Tossing a fair coin twice

12. Tossing two fair coins once

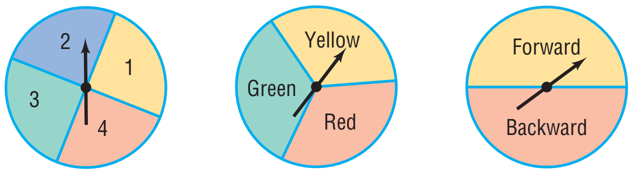
13. Tossing two fair coins and then a fair die

14. Tossing a fair coin, a fair die, and then a fair coin

15. Tossing three fair coins once

16. Tossing one fair coin three times

In Problems 17–22, use the following spinners to construct a probability model for each experiment.



Spinner I (4 equal areas)   Spinner II (3 equal areas)   Spinner III (2 equal areas)

17. Spin spinner I, then spinner II. What is the probability of getting a 2 or a 4, followed by Red?

18. Spin spinner III, then spinner II. What is the probability of getting Forward, followed by Yellow or Green?

19. Spin spinner I, then II, then III. What is the probability of getting a 1, followed by Red or Green, followed by Backward?

20. Spin spinner II, then I, then III. What is the probability of getting Yellow, followed by a 2 or a 4, followed by Forward?

21. Spin spinner I twice, then spinner II. What is the probability of getting a 2, followed by a 2 or a 4, followed by Red or Green?

22. Spin spinner III, then spinner I twice. What is the probability of getting Forward, followed by a 1 or a 3, followed by a 2 or a 4?

In Problems 23–26, consider the experiment of tossing a coin twice. The table lists six possible assignments of probabilities for this experiment. Using this table, answer the following questions.

	Sample Space			
Assignments	**HH**	**HT**	**TH**	**TT**
A	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
B	0	0	0	1
C	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{3}{16}$
D	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
E	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$
F	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{4}{9}$

23. Which of the assignments of probabilities is(are) consistent with the definition of a probability model?

24. Which of the assignments of probabilities should be used if the coin is known to be fair?

25. Which of the assignments of probabilities should be used if the coin is known to always come up tails?

26. Which of the assignments of probabilities should be used if tails is twice as likely as heads to occur?

27. Assigning Probabilities A coin is weighted so that heads is four times as likely as tails to occur. What probability should be assigned to heads? to tails?

28. Assigning Probabilities A coin is weighted so that tails is twice as likely as heads to occur. What probability should be assigned to heads? to tails?

29. Assigning Probabilities A die is weighted so that an odd-numbered face is twice as likely to occur as an even-numbered face. What probability should be assigned to each face?

30. Assigning Probabilities A die is weighted so that a six cannot appear. All the other faces occur with the same probability. What probability should be assigned to each face?

For Problems 31–34, the sample space is

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

Suppose that the outcomes are equally likely.

31. Compute the probability of the event $E = \{1, 2, 3\}$.

32. Compute the probability of the event $F = \{3, 5, 9, 10\}$.

33. Compute the probability of the event E: "an even number."

34. Compute the probability of the event F: "an odd number."

For Problems 35 and 36, an urn contains 5 white marbles, 10 green marbles, 8 yellow marbles, and 7 black marbles.

35. If one marble is selected, determine the probability that it is white.

36. If one marble is selected, determine the probability that it is black.

In Problems 37–40, assume equally likely outcomes.

37. Determine the probability of having 3 boys in a 3-child family.

38. Determine the probability of having 3 girls in a 3-child family.

39. Determine the probability of having 1 girl and 3 boys in a 4-child family.

40. Determine the probability of having 2 girls and 2 boys in a 4-child family.

For Problems 41–44, two fair dice are rolled.

41. Determine the probability that the sum of the faces is 7.

42. Determine the probability that the sum of the faces is 11.

43. Determine the probability that the sum of the faces is 3.

44. Determine the probability that the sum of the faces is 12.

In Problems 45–48, find the probability of the indicated event if $P(A) = 0.25$ and $P(B) = 0.45$.

45. $P(A \cup B)$ if $P(A \cap B) = 0.15$

46. $P(A \cap B)$ if $P(A \cup B) = 0.6$

47. $P(A \cup B)$ if A, B are mutually exclusive

48. $P(A \cap B)$ if A, B are mutually exclusive

49. If $P(A) = 0.60$, $P(A \cup B) = 0.85$, and $P(A \cap B) = 0.05$, find $P(B)$.

50. If $P(B) = 0.30$, $P(A \cup B) = 0.65$, and $P(A \cap B) = 0.15$, find $P(A)$.

51. Automobile Theft According to the Insurance Information Institute, in 2013 there was a 14.2% probability that an automobile theft in the United States would be cleared by arrests. If an automobile theft case from 2013 is randomly selected, what is the probability that it was not cleared by an arrest?

52. Pet Ownership According to the American Pet Products Association's *2015–2016 National Pet Owners Survey,* there is a 65% probability that a U.S. household owns a pet. If a U.S. household is randomly selected, what is the probability that it does not own a pet?

53. Dog Ownership According to the American Pet Products Association's *2015–2016 National Pet Owners Survey,* there is a 44% probability that a U.S. household owns a dog. If a U.S. household is randomly selected, what is the probability that it does not own a dog?

54. Doctorate Degrees According to the National Science Foundation, in 2013 there was a 17.0% probability that a doctoral degree awarded at a U.S. university was awarded in engineering. If a 2013 U.S. doctoral recipient is randomly selected, what is the probability that his or her degree was not in engineering?

55. Online Gambling According to a Casino FYI survey, 6.4% of U.S. adults admitted to having spent money gambling online. If a U.S. adult is selected at random, what is the probability that he or she has never spent any money gambling online?

56. Girl Scout Cookies According to the Girl Scouts of America, 19% of all Girl Scout cookies sold are Samoas/Caramel deLites. If a box of Girl Scout cookies is selected at random, what is the probability that it does not contain Samoas/Caramel deLites?

For Problems 57–60, a golf ball is selected at random from a container. If the container has 9 white balls, 8 green balls, and 3 orange balls, find the probability of each event.

57. The golf ball is white or green.

58. The golf ball is white or orange.

59. The golf ball is not white.

60. The golf ball is not green.

61. On *The Price Is Right,* there is a game in which a bag is filled with 3 strike chips and 5 numbers. Let's say that the numbers in the bag are 0, 1, 3, 6, and 9. What is the probability of selecting a strike chip or the number 1?

62. Another game on *The Price Is Right* requires the contestant to spin a wheel with the numbers 5, 10, 15, 20, . . ., 100. What is the probability that the contestant spins 100 or 30?

Problems 63–66 are based on a survey of annual incomes in 100 U.S. households. The following table gives the data.

Income	\$0–24,999	\$25,000–49,999	\$50,000–74,999	\$75,000–99,999	\$100,000 or more
Number of Households	24	24	18	12	22

63. What is the probability that a household has an annual income of \$75,000 or more?

64. What is the probability that a household has an annual income between \$25,000 and \$74,999, inclusive?

65. What is the probability that a household has an annual income of less than \$50,000?

66. What is the probability that a household has an annual income of \$50,000 or more?

67. Surveys In a survey about the number of motor vehicles per household, the following probability table was constructed:

Number of Motor Vehicles	0	1	2	3	4 or more
Probability	0.09	0.34	0.37	0.14	0.06

Find the probability of a household having:

(a) 1 or 2 motor vehicles
(b) 1 or more motor vehicles
(c) 3 or fewer motor vehicles
(d) 3 or more motor vehicles
(e) Fewer than 2 motor vehicles
(f) Fewer than 1 motor vehicles
(g) 1, 2, or 3 motor vehicles
(h) 2 or more motor vehicles

68. Checkout Lines Through observation, it has been determined that the probability for a given number of people waiting in line at the "5 items or less" checkout register of a supermarket is as follows:

Number Waiting in Line	0	1	2	3	4 or more
Probability	0.10	0.15	0.20	0.24	0.31

Find the probability of:

(a) At most 2 people in line
(b) At least 2 people in line
(c) At least 1 person in line

69. In a certain College Algebra class, there are 18 freshmen and 15 sophomores. Of the 18 freshmen, 10 are male, and of the 15 sophomores, 8 are male. Find the probability that a randomly selected student is:

(a) A freshman or female (b) A sophomore or male

70. The faculty of the mathematics department at Joliet Junior College is composed of 4 females and 9 males. Of the 4 females, 2 are under age 40, and of the 9 males, 3 are under age 40. Find the probability that a randomly selected faculty member is:

(a) Female or under age 40 (b) Male or over age 40

71. Birthday Problem What is the probability that at least 2 people in a group of 12 people have the same birthday? Assume that there are 365 days in a year.

72. Birthday Problem What is the probability that at least 2 people in a group of 35 people have the same birthday? Assume that there are 365 days in a year.

73. Winning a Lottery Powerball is a multistate lottery in which 5 white balls from a drum with 69 balls and 1 red ball from a drum with 26 balls are selected. For a \$2 ticket, players get one chance at winning the jackpot by matching all 6 numbers. What is the probability of selecting the winning numbers on a \$2 play?

Retain Your Knowledge

Problems 74–77 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

74. To graph $g(x) = |x + 2| - 3$, shift the graph of $f(x) = |x|$ $\underset{\text{number}}{\underline{\qquad}}$ units $\underset{\text{left/right}}{\underline{\qquad}}$ and then $\underset{\text{number}}{\underline{\qquad}}$ units $\underset{\text{up/down}}{\underline{\qquad}}$.

75. Simplify: $\sqrt[3]{24x^2y^5}$

76. Solve: $\log_5(x + 3) = 2$

77. Solve the given system using matrices.

$$\begin{cases} 3x + y + 2z = 1 \\ 2x - 2y + 5z = 5 \\ x + 3y + 2z = -9 \end{cases}$$

Chapter Review

Things to Know

Counting formula (p. 701) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Addition Principle of Counting (p. 701) If $A \cap B = \varnothing$, then $n(A \cup B) = n(A) + n(B)$.

Multiplication Principle of Counting (p. 703) If a task consists of a sequence of choices in which there are p selections for the first choice, q selections for the second choice, and so on, the task of making these selections can be done in $p \cdot q \cdot \cdots$ different ways.

Permutation (p. 705) An ordered arrangement of r objects chosen from n objects

Number of permutations: Distinct, with repetition (p. 706) n^r

The n objects are distinct (different), and repetition is allowed in the selection of r of them.

Number of permutations: Distinct, without repetition (p. 707) $P(n, r) = n(n-1) \cdot \cdots \cdot [n - (r-1)] = \dfrac{n!}{(n-r)!}$

The n objects are distinct (different), and repetition is not allowed in the selection of r of them, where $r \le n$.

Combination (p. 708) An arrangement, without regard to order, of r objects selected from n distinct objects, where $r \le n$

Number of combinations (p. 709) $C(n, r) = \dfrac{P(n, r)}{r!} = \dfrac{n!}{(n-r)!\,r!}$

Number of permutations: Not distinct (p. 711) $\dfrac{n!}{n_1!n_2!\cdots n_k!}$

The number of permutations of n objects of which n_1 are of one kind, n_2 are of a second kind, . . . , and n_k are of a kth kind, where $n = n_1 + n_2 + \cdots + n_k$

Sample space (p. 714) Set whose elements represent the possible outcomes that can occur as a result of an experiment

Probability (p. 714) A nonnegative number assigned to each outcome of a sample space; the sum of all the probabilities of the outcomes equals 1.

Probability for equally likely outcomes (p. 716)	$P(E) = \dfrac{n(E)}{n(S)}$ The same probability is assigned to each outcome.
Probability of the union of two events (p. 718)	$P(E \cup F) = P(E) + P(F) - P(E \cap F)$
Probability of the complement of an event (p. 719)	$P(\overline{E}) = 1 - P(E)$

Objectives

Section		You should be able to . . .	Example(s)	Review Exercises
10.1	1	Find all the subsets of a set (p. 700)	1	1
	2	Count the number of elements in a set (p. 700)	2, 3	2–9
	3	Solve counting problems using the Multiplication Principle (p. 702)	4, 5	12, 13, 17, 18
10.2	1	Solve counting problems using permutations involving n distinct objects (p. 705)	1–5	10, 14, 19, 22(a)
	2	Solve counting problems using combinations (p. 708)	6–9	11, 15, 16, 21
	3	Solve counting problems using permutations involving n nondistinct objects (p. 710)	10, 11	20
10.3	1	Construct probability models (p. 714)	2–4	22(b)
	2	Compute probabilities of equally likely outcomes (p. 716)	5, 6	22(b), 23(a), 24, 25
	3	Find probabilities of the union of two events (p. 718)	7, 8	26
	4	Use the Complement Rule to find probabilities (p. 719)	9, 10	22(c), 23(b)

Review Exercises

1. Write down all the subsets of the set {Dave, Joanne, Erica}.

2. If $n(A) = 8$, $n(B) = 12$, and $n(A \cap B) = 3$, find $n(A \cup B)$.

3. If $n(A) = 12$, $n(A \cup B) = 30$, and $n(A \cap B) = 6$, find $n(B)$.

In Problems 4–9, use the information supplied in the figure.

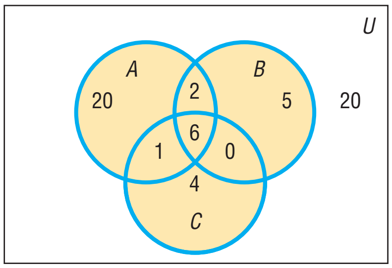
4. How many are in A? **5.** How many are in A or B?

6. How many are in A and C?

7. How many are not in B?

8. How many are in neither A nor C?

9. How many are in B but not in C?



In Problems 10 and 11, compute the value of the given expression.

10. $P(8, 3)$ **11.** $C(8, 3)$

12. Stocking a Store A clothing store sells pure wool and polyester-wool suits. Each suit comes in 3 colors and 10 sizes. How many suits are required for a complete assortment?

13. Baseball On a given day, the American Baseball League schedules 7 games. How many different outcomes are possible, assuming that each game is played to completion?

14. Choosing Seats If 4 people enter a bus that has 9 vacant seats, in how many ways can they be seated?

15. Choosing a Team In how many ways can a squad of 4 relay runners be chosen from a track team of 8 runners?

16. Baseball In how many ways can 2 teams from 14 teams in the American League be chosen without regard to which team is at home?

17. Telephone Numbers Using the digits $0, 1, 2, \ldots, 9$, how many 7-digit numbers can be formed if the first digit cannot be 0 or 9 and if the last digit is greater than or equal to 2 and less than or equal to 3? Repeated digits are allowed.

18. License Plate Possibilities A license plate consists of 1 letter, excluding O and I, followed by a 4-digit number that cannot have a 0 in the lead position. How many different plates are possible?

19. Binary Codes Using the digits 0 and 1, how many different numbers consisting of 8 digits can be formed?

20. Arranging Flags How many different vertical arrangements are there of 10 flags if 4 are white, 3 are blue, 2 are green, and 1 is red?

21. Forming Committees A group of 9 people is going to be formed into committees of 4, 3, and 2 people. How many committees can be formed if:

(a) A person can serve on any number of committees?
(b) No person can serve on more than one committee?

22. Birthday Problem For this problem, assume that a year has 365 days.

(a) In how many ways can 18 people have different birthdays?
(b) What is the probability that no 2 people in a group of 18 people have the same birthday?
(c) What is the probability that at least 2 people in a group of 18 people have the same birthday?

23. Unemployment According to the U.S. Bureau of Labor Statistics, 6.2% of the U.S. labor force was unemployed in 2013.

(a) What is the probability that a randomly selected member of the U.S. labor force was unemployed in 2013?
(b) What is the probability that a randomly selected member of the U.S. labor force was not unemployed in 2013?

24. You have four \$1 bills, three \$5 bills, and two \$10 bills in your wallet. If you pick a bill at random, what is the probability that it will be a \$1 bill?

25. Each of the numbers $1, 2, \ldots, 100$ is written on an index card, and the cards are shuffled. If a card is selected at random, what is the probability that the number on the card is divisible by 5? What is the probability that the card selected either is a 1 or names a prime number?

26. At the Milex tune-up and brake repair shop, the manager has found that a car will require a tune-up with a probability of 0.6, a brake job with a probability of 0.1, and both with a probability of 0.02.

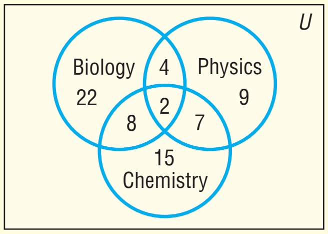
(a) What is the probability that a car requires either a tune-up or a brake job?
(b) What is the probability that a car requires a tune-up but not a brake job?
(c) What is the probability that a car requires neither a tune-up nor a brake job?

Chapter Test

CHAPTER **Test Prep** VIDEOS

The Chapter Test Prep Videos are step-by-step solutions available in MyMathLab®, or on this text's YouTube™ Channel. Flip back to the Resources for Success page for a link to this text's YouTube channel.

In Problems 1–4, a survey of 70 college freshmen asked whether students planned to take biology, chemistry, or physics during their first year. Use the diagram to answer each question.



1. How many of the surveyed students plan to take physics during their first year?

2. How many of the surveyed students do not plan to take biology, chemistry, or physics during their first year?

3. How many of the surveyed students plan to take only biology and chemistry during their first year?

4. How many of the surveyed students plan to take physics or chemistry during their first year?

In Problems 5–7, compute the value of the given expression.

5. 7! **6.** $P(10, 6)$ **7.** $C(11, 5)$

8. M&M's® offers customers the opportunity to create their own color mix of candy. There are 21 colors to choose from, and customers are allowed to select up to 6 different colors. How many different color mixes are possible, assuming that no color is selected more than once and 6 different colors are chosen?

9. How many distinct 8-letter words (real or imaginary) can be formed from the letters in the word REDEEMED?

10. In horse racing, an exacta bet requires the bettor to pick the first two horses in the exact order. If there are 8 horses in a race, in how many ways could you make an exacta bet?

11. On February 20, 2004, the Ohio Bureau of Motor Vehicles unveiled the state's new license plate format. The plate consists of three letters (A–Z) followed by 4 digits (0–9). Assume that all letters and digits may be used, except that the third letter cannot be O, I, or Z. If repetitions are allowed, how many different plates are possible?

12. Kiersten applies for admission to the University of Southern California (USC) and Florida State University (FSU). She estimates that she has a 60% chance of being admitted to USC, a 70% chance of being admitted to FSU, and a 35% chance of being admitted to both universities.

(a) What is the probability that she will be admitted to either USC or FSU?
(b) What is the probability that she will not be admitted to FSU?

13. A cooler contains 8 bottles of Pepsi, 5 bottles of Coke, 4 bottles of Mountain Dew, and 3 bottles of IBC.

(a) What is the probability that a bottle chosen at random is Coke?
(b) What is the probability that a bottle chosen at random is either Pepsi or IBC?

14. A study on the age distribution of students at a community college yielded the following data:

Age	17 and under	18–20	21–24	25–34	35–64	65 and over
Probability	0.03	???	0.23	0.29	0.25	0.01

What must be the probability that a randomly selected student at the college is between 18 and 20 years old?

15. In a certain lottery, there are ten balls numbered 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. Of these, five are drawn in order. If you pick five numbers that match those drawn in the correct order, you win \$1,000,000. What is the probability of winning such a lottery?

16. If you roll a die five times, what is the probability that you obtain exactly 2 fours?

Cumulative Review

1. Solve: $3x^2 - 2x = -1$
2. Graph $f(x) = x^2 + 4x - 5$ by determining whether the graph opens up or down and by finding the vertex, axis of symmetry, and intercepts.
3. Graph $f(x) = 2(x + 1)^2 - 4$ using transformations.
4. Solve: $|x - 4| \le 0.01$
5. Find the complex zeros of

$$f(x) = 5x^4 - 9x^3 - 7x^2 - 31x - 6$$

6. Graph $g(x) = 3^{x-1} + 5$ using transformations. Determine the domain, the range, and the horizontal asymptote of g.
7. What is the exact value of $\log_3 9$?
8. Solve: $\log_2(3x - 2) + \log_2 x = 4$
9. Solve the system: $\begin{cases} x - 2y + z = 15 \\ 3x + y - 3z = -8 \\ -2x + 4y - z = -27 \end{cases}$
10. What is the 33rd term in the sequence $-3, 1, 5, 9, \ldots$? What is the sum of the first 20 terms?

Chapter Projects

I. **The Lottery and Expected Profit** When all of the possible outcomes in a probability model are numeric quantities, useful statistics can be computed for such models. The **expected value**, or **mean**, of such a probability model is found by multiplying each possible numeric outcome by its corresponding probability and then adding these products.

For example, Table 2 provides the probability model for rolling a fair six-sided die. The expected value, $E(x)$, is

$$E(x) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = 3.5$$

When a fair die is rolled repeatedly, the average of the outcomes will approach 3.5.

Mega Millions is a multistate lottery in which a player selects five different "white" numbers from 1 to 75 and one "gold" number from 1 to 15. The probability model shown in Table 3 lists the possible cash prizes and their corresponding probabilities.

1. Verify that Table 3 is a probability model.
2. To win the jackpot, a player must match all six numbers. Verify the probability given in Table 3 of winning the jackpot.

For questions 3–6, assume a single jackpot winner so that the jackpot does not have to be shared.

3. If the jackpot is \$20,000,000, calculate the expected cash prize.
4. If a ticket costs \$1, what is the expected financial result from purchasing one ticket? Interpret (give the meaning of) this result.

Table 2

Outcome	Probability
1	$\frac{1}{6}$
2	$\frac{1}{6}$
3	$\frac{1}{6}$
4	$\frac{1}{6}$
5	$\frac{1}{6}$
6	$\frac{1}{6}$

Table 3

Cash Prize	Probability
Jackpot	0.00000000386
\$1,000,000	0.00000005408
\$5000	0.00000135192
\$500	0.00001892689
\$50	0.00009328256
\$5	0.00342036036
\$2	0.01770813839
\$1	0.04674948535
\$0	0.93200839659

5. If the jackpot is $100,000,000, what is the expected cash prize? What is the expected financial result from purchasing one $1 ticket? Interpret this result.
6. What amount must the jackpot be so that a profit from one $1 ticket is expected?
7. Research the Powerball lottery, and create a probability model similar to Table 3 for it. Repeat questions 3–6 for Powerball. Be sure to adjust the price for a Powerball ticket. Based on what you have learned, which lottery would you prefer to play? Justify your decision.

The following projects are available at the Instructor's Resource Center (IRC):

II. Project at Motorola ***Probability of Error in Digital Wireless Communications*** Transmission errors in digital communications can often be detected by adding an extra digit of code to each transmitted signal. Investigate the probability of identifying an erroneous code using this simple coding method.

III. Surveys Polling (or taking a survey) is big business in the United States. Take and analyze a survey; then consider why different pollsters might get different results.

IV. Law of Large Numbers The probability that an event occurs, such as a head in a coin toss, is the proportion of heads you expect in the long run. A simulation is used to show that as a coin is flipped more and more times, the proportion of heads gets close to 0.5.

V. Simulation Electronic simulation of an experiment is often an economical way to investigate a theoretical probability. Develop a theory without leaving your desk.

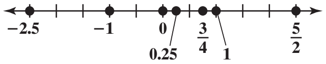
Answers

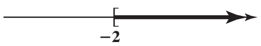
CHAPTER R Review

R.1 Assess Your Understanding *(page 16)*

1. rational **2.** 31 **3.** Distributive **4.** $5(x+3)=6$ **5.** a **6.** b **7.** T **8.** F **9.** F **10.** T **11.** $\{1,2,3,4,5,6,7,8,9\}$ **13.** $\{4\}$ **15.** $\{1,3,4,6\}$ **17.** $\{0,2,6,7,8\}$ **19.** $\{0,1,2,3,5,6,7,8,9\}$ **21.** $\{0,1,2,3,5,6,7,8,9\}$ **23. (a)** $\{2,5\}$ **(b)** $\{-6,2,5\}$ **(c)** $\left\{-6,\frac{1}{2},-1.333\ldots,2,5\right\}$ **(d)** $\{\pi\}$ **(e)** $\left\{-6,\frac{1}{2},-1.333\ldots,\pi,2,5\right\}$ **25. (a)** $\{1\}$ **(b)** $\{0,1\}$ **(c)** $\left\{0,1,\frac{1}{2},\frac{1}{3},\frac{1}{4}\right\}$ **(d)** None **(e)** $\left\{0,1,\frac{1}{2},\frac{1}{3},\frac{1}{4}\right\}$ **27. (a)** None **(b)** None **(c)** None **(d)** $\left\{\sqrt{2},\pi,\sqrt{2}+1,\pi+\frac{1}{2}\right\}$ **(e)** $\left\{\sqrt{2},\pi,\sqrt{2}+1,\pi+\frac{1}{2}\right\}$ **29. (a)** 18.953 **(b)** 18.952 **31. (a)** 28.653 **(b)** 28.653 **33. (a)** 0.063 **(b)** 0.062 **35. (a)** 9.999 **(b)** 9.998 **37. (a)** 0.429 **(b)** 0.428 **39. (a)** 34.733 **(b)** 34.733 **41.** $3+2=5$ **43.** $x+2=3\cdot 4$ **45.** $3y=1+2$ **47.** $x-2=6$ **49.** $\frac{x}{2}=6$ **51.** 7 **53.** 6 **55.** 1 **57.** $\frac{13}{3}$ **59.** -11 **61.** 11 **63.** -4 **65.** 1 **67.** 6 **69.** $\frac{2}{7}$ **71.** $\frac{4}{45}$ **73.** $\frac{23}{20}$ **75.** $\frac{79}{30}$ **77.** $\frac{13}{36}$ **79.** $-\frac{16}{45}$ **81.** $\frac{1}{60}$ **83.** $\frac{15}{22}$ **85.** 1 **87.** $\frac{15}{8}$ **89.** $6x+24$ **91.** x^2-4x **93.** $\frac{3}{2}x-1$ **95.** x^2+6x+8 **97.** x^2-x-2 **99.** $x^2-10x+16$ **101.** $2x+3x=(2+3)x=5x$ **103.** $2(3\cdot 4)=2\cdot 12=24$; $(2\cdot 3)\cdot(2\cdot 4)=6\cdot 8=48$ **105.** No; $2-3\neq 3-2$ **107.** No; $2\div 3\neq 3\div 2$ **109.** Symmetric Property **111.** No; no

R.2 Assess Your Understanding *(page 27)*

1. variable **2.** origin **3.** strict **4.** base; exponent, or power **5.** 1.2345678×10^3 **6.** T **7.** T **8.** F **9.** F **10.** F **11.** d **12.** b **13.** 

15. $>$ **17.** $>$ **19.** $>$ **21.** $=$ **23.** $<$ **25.** $x>0$ **27.** $x<2$ **29.** $x\leq 1$ **31.** 

33. 

35. 1 **37.** 2 **39.** 6 **41.** 4 **43.** -28 **45.** $\frac{4}{5}$ **47.** 0 **49.** 1 **51.** 5 **53.** 1 **55.** 22 **57.** 2 **59.** $x=0$ **61.** $x=3$ **63.** None **65.** $x=0, x=1, x=-1$ **67.** $\{x\mid x\neq 5\}$ **69.** $\{x\mid x\neq -4\}$ **71.** 0°C **73.** 25°C **75.** 16 **77.** $\frac{1}{16}$ **79.** $\frac{1}{9}$ **81.** 9 **83.** 5 **85.** 4 **87.** $64x^6$ **89.** $\frac{x^4}{y^2}$ **91.** $\frac{x}{y}$ **93.** $-\frac{8x^3z}{9y}$ **95.** $\frac{16x^2}{9y^2}$ **97.** -4 **99.** 5 **101.** 4 **103.** 2 **105.** $\sqrt{5}$ **107.** $\frac{1}{2}$ **109.** 10; 0 **111.** 81 **113.** 304,006.671 **115.** 0.004 **117.** 481.890 **119.** 0.000 **121.** 4.542×10^2 **123.** 1.3×10^{-2} **125.** 3.2155×10^4 **127.** 4.23×10^{-4} **129.** 61,500 **131.** 0.001214 **133.** 110,000,000 **135.** 0.081 **137.** $A=lw$ **139.** $C=\pi d$ **141.** $A=\frac{\sqrt{3}}{4}x^2$ **143.** $V=\frac{4}{3}\pi r^3$ **145.** $V=x^3$ **147. (a)** \$6000 **(b)** \$8000 **149.** $|x-4|\geq 6$ **151. (a)** $2\leq 5$ **(b)** $6>5$ **153. (a)** Yes **(b)** No **155.** 400,000,000 m **157.** 0.0000005 m **159.** 5×10^{-4} in. **161.** 5.865696×10^{12} mi **163.** No; $\frac{1}{3}$ is larger; 0.000333… **165.** No

R.3 Assess Your Understanding *(page 37)*

1. right; hypotenuse **2.** $A=\frac{1}{2}bh$ **3.** $C=2\pi r$ **4.** similar **5.** c **6.** b **7.** T **8.** T **9.** F **10.** T **11.** T **12.** F **13.** 13 **15.** 26 **17.** 25 **19.** Right triangle; 5 **21.** Not a right triangle **23.** Right triangle; 25 **25.** Not a right triangle **27.** 8 in.^{2} **29.** 4 in.^{2} **31.** $A=25\pi$ m^2; $C=10\pi$ m **33.** $V=224$ ft^3; $S=232$ ft^2 **35.** $V=\frac{256}{3}\pi$ cm^3; $S=64\pi$ cm^2 **37.** $V=648\pi$ in.^{3}; $S=306\pi$ in.^{2} **39.** π square units **41.** 2π square units **43.** $x=4$ units; $A=90°$; $B=60°$; $C=30°$ **45.** $x=67.5$ units; $A=60°$; $B=95°$; $C=25°$ **47.** About 16.8 ft **49.** 64 ft^2 **51.** $24+2\pi\approx 30.28$ ft^2; $16+2\pi\approx 22.28$ ft **53.** 160 paces **55.** About 5.477 mi **57.** From 100 ft: 12.2 mi; From 150 ft: 15.0 mi

R.4 Assess Your Understanding *(page 48)*

1. 4; 3 **2.** x^4-16 **3.** x^3-8 **4.** a **5.** c **6.** F **7.** T **8.** F **9.** Monomial; variable: x; coefficient: 2; degree: 3 **11.** Not a monomial; the exponent of the variable is not a nonnegative integer **13.** Monomial; variables: x, y; coefficient: -2; degree: 3 **15.** Not a monomial; the exponent of one of the variables is not a nonnegative integer **17.** Not a monomial; it has more than one term **19.** Yes; 2 **21.** Yes; 0 **23.** No; the exponent of the variable of one of the terms is not a nonnegative integer **25.** Yes; 3 **27.** No; the polynomial of the denominator has a degree greater than 0 **29.** x^2+7x+2 **31.** x^3-4x^2+9x+7 **33.** $6x^5+5x^4+3x^2+x$ **35.** $7x^2-x-7$ **37.** $-2x^3+18x^2-18$ **39.** $2x^2-4x+6$ **41.** $15y^2-27y+30$ **43.** x^3+x^2-4x **45.** $-8x^5-10x^2$ **47.** x^3+3x^2-2x-4 **49.** x^2+6x+8 **51.** $2x^2+9x+10$ **53.** x^2-2x-8 **55.** x^2-5x+6 **57.** $2x^2-x-6$ **59.** $-2x^2+11x-12$ **61.** $2x^2+8x+8$ **63.** $x^2-xy-2y^2$ **65.** $-6x^2-13xy-6y^2$ **67.** x^2-49 **69.** $4x^2-9$ **71.** $x^2+8x+16$ **73.** $x^2-8x+16$ **75.** $9x^2-16$ **77.** $4x^2-12x+9$ **79.** x^2-y^2 **81.** $9x^2-y^2$ **83.** $x^2+2xy+y^2$ **85.** $x^2-4xy+4y^2$ **87.** $x^3-6x^2+12x-8$ **89.** $8x^3+12x^2+6x+1$ **91.** $4x^2-11x+23$; remainder -45 **93.** $4x-3$; remainder $x+1$ **95.** $5x^2-13$; remainder $x+27$ **97.** $2x^2$; remainder $-x^2+x+1$ **99.** $x^2-2x+\frac{1}{2}$; remainder $\frac{5}{2}x+\frac{1}{2}$ **101.** $-4x^2-3x-3$; remainder -7 **103.** x^2-x-1; remainder $2x+2$ **105.** x^2+ax+a^2; remainder 0

R.5 Assess Your Understanding *(page 58)*

1. $3x(x-2)(x+2)$ **2.** prime **3.** c **4.** b **5.** d **6.** c **7.** T **8.** F **9.** $3(x+2)$ **11.** $a(x^2+1)$ **13.** $x(x^2+x+1)$ **15.** $2x(x-1)$

17. $3xy(x-2y+4)$ **19.** $(x+1)(x-1)$ **21.** $(2x+1)(2x-1)$ **23.** $(x+4)(x-4)$ **25.** $(5x+2)(5x-2)$ **27.** $(x+1)^2$ **29.** $(x+2)^2$ **31.** $(x-5)^2$ **33.** $(2x+1)^2$ **35.** $(4x+1)^2$ **37.** $(x-3)(x^2+3x+9)$ **39.** $(x+3)(x^2-3x+9)$ **41.** $(2x+3)(4x^2-6x+9)$ **43.** $(x+2)(x+3)$ **45.** $(x+6)(x+1)$ **47.** $(x+5)(x+2)$ **49.** $(x-8)(x-2)$ **51.** $(x-8)(x+1)$ **53.** $(x+8)(x-1)$ **55.** $(x+2)(2x+3)$ **57.** $(x-2)(2x+1)$ **59.** $3(2x+3)(3x+2)$ **61.** $(3x+1)(x+1)$ **63.** $(z+1)(2z+3)$ **65.** $(x+2)(3x-4)$ **67.** $(x-2)(3x+4)$ **69.** $4x^2(x+4)(3x+2)$ **71.** $(x+4)(3x-2)$ **73.** $25; (x+5)^2$ **75.** $9; (y-3)^2$ **77.** $\frac{1}{16}; \left(x-\frac{1}{4}\right)^2$ **79.** $(x+6)(x-6)$ **81.** $2(1+2x)(1-2x)$ **83.** $8(x+1)(x+10)$ **85.** $(x-7)(x-3)$ **87.** $4(x^2-2x+8)$ **89.** Prime **91.** $-(x-5)(x+3)$ **93.** $3(x+2)(x-6)$ **95.** $y^2(y+5)(y+6)$ **97.** $2x^3(2x+3)^2$ **99.** $2(3x+1)(x+1)$ **101.** $(x-3)(x+3)(x^2+9)$ **103.** $(x-1)^2(x^2+x+1)^2$ **105.** $x^5(x-1)(x+1)$ **107.** $(4x+3)^2$ **109.** $-(4x-5)(4x+1)$ **111.** $(2y-5)(2y-3)$ **113.** $-(3x-1)(3x+1)(x^2+1)$ **115.** $(x+3)(x-6)$ **117.** $(x+2)(x-3)$ **119.** $(3x-5)(9x^2-3x+7)$ **121.** $(x+5)(3x+11)$ **123.** $(x-1)(x+1)(x+2)$ **125.** $(x-1)(x+1)(x^2-x+1)$ **127.** $2(3x+4)(9x+13)$ **129.** $2x(3x+5)$ **131.** $5(x+3)(x-2)^2(x+1)$ **133.** $3(4x-3)(4x-1)$ **135.** $6(3x-5)(2x+1)^2(5x-4)$ **137.** The possibilities are $(x \pm 1)(x \pm 4) = x^2 \pm 5x + 4$ or $(x \pm 2)(x \pm 2) = x^2 \pm 4x + 4$, none of which equals $x^2 + 4$.

R.6 Assess Your Understanding *(page 62)*

1. quotient; divisor; remainder **2.** $-3\overline{)2\ 0\ -5\ 1}$ **3.** d **4.** a **5.** T **6.** T **7.** x^2+x+4; remainder 12 **9.** $3x^2+11x+32$; remainder 99 **11.** $x^4-3x^3+5x^2-15x+46$; remainder -138 **13.** $4x^5+4x^4+x^3+x^2+2x+2$; remainder 7 **15.** $0.1x^2-0.11x+0.321$; remainder -0.3531 **17.** $x^4+x^3+x^2+x+1$; remainder 0 **19.** No **21.** Yes **23.** Yes **25.** No **27.** Yes **29.** -9

R.7 Assess Your Understanding *(page 71)*

1. lowest terms **2.** least common multiple **3.** d **4.** a **5.** T **6.** F **7.** $\frac{3}{x-3}$ **9.** $\frac{x}{3}$ **11.** $\frac{4x}{2x-1}$ **13.** $\frac{y+5}{2(y+1)}$ **15.** $\frac{x+5}{x-1}$ **17.** $-(x+7)$ **19.** $\frac{3}{5x(x-2)}$ **21.** $\frac{2x(x^2+4x+16)}{x+4}$ **23.** $\frac{8}{3x}$ **25.** $\frac{x-3}{x+7}$ **27.** $\frac{4x}{(x-2)(x-3)}$ **29.** $\frac{4}{5(x-1)}$ **31.** $-\frac{(x-4)^2}{4x}$ **33.** $\frac{(x+3)^2}{(x-3)^2}$ **35.** $\frac{(x-4)(x+3)}{(x-1)(2x+1)}$ **37.** $\frac{x+5}{2}$ **39.** $\frac{(x-2)(x+2)}{2x-3}$ **41.** $\frac{3x-2}{x-3}$ **43.** $\frac{x+9}{2x-1}$ **45.** $\frac{4-x}{x-2}$ **47.** $\frac{2(x+5)}{(x-1)(x+2)}$ **49.** $\frac{3x^2-2x-3}{(x+1)(x-1)}$ **51.** $\frac{-(11x+2)}{(x+2)(x-2)}$ **53.** $\frac{2(x^2-2)}{x(x-2)(x+2)}$ **55.** $(x-2)(x+2)(x+1)$ **57.** $x(x-1)(x+1)$ **59.** $x^3(2x-1)^2$ **61.** $x(x-1)^2(x+1)(x^2+x+1)$ **63.** $\frac{5x}{(x-6)(x-1)(x+4)}$ **65.** $\frac{2(2x^2+5x-2)}{(x-2)(x+2)(x+3)}$ **67.** $\frac{5x+1}{(x-1)^2(x+1)^2}$ **69.** $\frac{-x^2+3x+13}{(x-2)(x+1)(x+4)}$ **71.** $\frac{x^3-2x^2+4x+3}{x^2(x+1)(x-1)}$ **73.** $\frac{-1}{x(x+h)}$ **75.** $\frac{x+1}{x-1}$ **77.** $\frac{(x-1)(x+1)}{2x(2x+1)}$ **79.** $\frac{2(5x-1)}{(x-2)(x+1)^2}$ **81.** $\frac{-2x(x^2-2)}{(x+2)(x^2-x-3)}$ **83.** $\frac{-1}{x-1}$ **85.** $\frac{3x-1}{2x+1}$ **87.** $\frac{19}{(3x-5)^2}$ **89.** $\frac{(x+1)(x-1)}{(x^2+1)^2}$ **91.** $\frac{x(3x+2)}{(3x+1)^2}$ **93.** $-\frac{(x+3)(3x-1)}{(x^2+1)^2}$ **95.** $f = \frac{R_1 \cdot R_2}{(n-1)(R_1+R_2)}; \frac{2}{15}$ m

R.8 Assess Your Understanding *(page 79)*

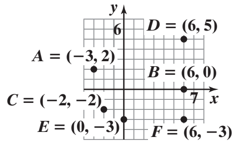
3. index **4.** cube root **5.** b **6.** d **7.** c **8.** c **9.** T **10.** F **11.** 3 **13.** -2 **15.** $2\sqrt{2}$ **17.** $10\sqrt{7}$ **19.** $2\sqrt[3]{4}$ **21.** $-2x\sqrt[3]{x}$ **23.** $3\sqrt[4]{3}$ **25.** x^3y^2 **27.** x^2y **29.** $6\sqrt{x}$ **31.** $3x^2y^3\sqrt[4]{2x}$ **33.** $6x\sqrt{x}$ **35.** $15\sqrt[3]{3}$ **37.** $12\sqrt{3}$ **39.** $7\sqrt{2}$ **41.** $\sqrt{2}$ **43.** $2\sqrt{3}$ **45.** $-\sqrt[3]{2}$ **47.** $x-2\sqrt{x}+1$ **49.** $(2x-1)\sqrt[3]{2x}$ **51.** $(2x-15)\sqrt{2x}$ **53.** $-(x+5y)\sqrt[3]{2xy}$ **55.** $\frac{\sqrt{2}}{2}$ **57.** $-\frac{\sqrt{15}}{5}$ **59.** $\frac{(5+\sqrt{2})\sqrt{3}}{23}$ **61.** $\frac{8\sqrt{5}-19}{41}$ **63.** $5\sqrt{2}+5$ **65.** $\frac{5\sqrt[3]{4}}{2}$ **67.** $\frac{2x+h-2\sqrt{x^2+xh}}{h}$ **69.** 4 **71.** -3 **73.** 64 **75.** $\frac{1}{27}$ **77.** $\frac{27\sqrt{2}}{32}$ **79.** $\frac{27\sqrt{2}}{32}$ **81.** $-\frac{1}{10}$ **83.** $\frac{25}{16}$ **85.** $x^{7/12}$ **87.** xy^2 **89.** $x^{2/3}y$ **91.** $\frac{8x^{5/4}}{y^{3/4}}$ **93.** $\frac{3x+2}{(1+x)^{1/2}}$ **95.** $\frac{x(3x^2+2)}{(x^2+1)^{1/2}}$ **97.** $\frac{22x+5}{10\sqrt{(x-5)(4x+3)}}$ **99.** $\frac{2+x}{2(1+x)^{3/2}}$ **101.** $\frac{4-x}{(x+4)^{3/2}}$ **103.** $\frac{1}{x^2(x^2-1)^{1/2}}$ **105.** $\frac{1-3x^2}{2\sqrt{x}(1+x^2)^2}$ **107.** $\frac{1}{2}(5x+2)(x+1)^{1/2}$ **109.** $2x^{1/2}(3x-4)(x+1)$ **111.** $(x^2+4)^{1/3}(11x^2+12)$ **113.** $(3x+5)^{1/3}(2x+3)^{1/2}(17x+27)$ **115.** $\frac{3(x+2)}{2x^{1/2}}$ **117.** 1.41 **119.** 1.59 **121.** 4.89 **123.** 2.15 **125. (a)** 15,660.4 gal **(b)** 390.7 gal **127.** $2\sqrt{2}\pi \approx 8.89$ sec

CHAPTER 1 Graphs, Equations, and Inequalities

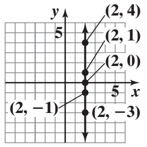
1.1 Assess Your Understanding *(page 94)*

7. x-coordinate or abscissa; y-coordinate or ordinate **8.** quadrants **9.** midpoint **10.** F **11.** F **12.** T **13.** d **14.** c

15. (a) quadrant II **(b)** x-axis **(c)** quadrant III **(d)** quadrant I **(e)** y-axis **(f)** quadrant IV



17. The points will be on a vertical line that is 2 units to the right of the y-axis.

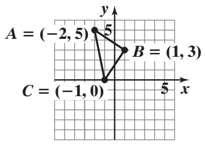


19. $(-1, 4)$; quadrant II
21. $(3, 1)$; quadrant I
23. $X\text{min} = -11$, $X\text{max} = 5$, $X\text{scl} = 1$, $Y\text{min} = -3$, $Y\text{max} = 6$, $Y\text{scl} = 1$
25. $X\text{min} = -30$, $X\text{max} = 50$, $X\text{scl} = 10$, $Y\text{min} = -90$, $Y\text{max} = 50$, $Y\text{scl} = 10$
27. $X\text{min} = -10$, $X\text{max} = 110$, $X\text{scl} = 10$, $Y\text{min} = -10$, $Y\text{max} = 160$, $Y\text{scl} = 10$
29. $X\text{min} = -6$, $X\text{max} = 6$, $X\text{scl} = 2$, $Y\text{min} = -4$, $Y\text{max} = 4$, $Y\text{scl} = 2$

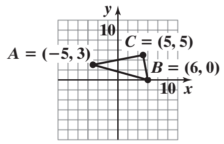
31. $X\text{min} = -6$, $X\text{max} = 6$, $X\text{scl} = 2$, $Y\text{min} = -1$, $Y\text{max} = 3$, $Y\text{scl} = 1$ **33.** $X\text{min} = 3$, $X\text{max} = 9$, $X\text{scl} = 1$, $Y\text{min} = 2$, $Y\text{max} = 10$, $Y\text{scl} = 2$

35. $\sqrt{5}$ **37.** $\sqrt{10}$ **39.** $2\sqrt{17}$ **41.** 20 **43.** $\sqrt{53}$ **45.** $\sqrt{a^2 + b^2}$ **47.** $4\sqrt{10}$ **49.** $2\sqrt{65}$

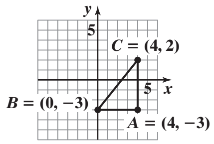
51. $d(A, B) = \sqrt{13}$
$d(B, C) = \sqrt{13}$
$d(A, C) = \sqrt{26}$
$(\sqrt{13})^2 + (\sqrt{13})^2 = (\sqrt{26})^2$
$\text{Area} = \dfrac{13}{2}$ square units



53. $d(A, B) = \sqrt{130}$
$d(B, C) = \sqrt{26}$
$d(A, C) = 2\sqrt{26}$
$(\sqrt{26})^2 + (2\sqrt{26})^2 = (\sqrt{130})^2$
Area = 26 square units

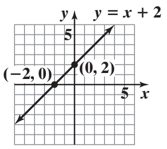


55. $d(A, B) = 4$
$d(B, C) = \sqrt{41}$
$d(A, C) = 5$
$4^2 + 5^2 = (\sqrt{41})^2$
Area = 10 square units

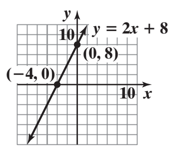


57. $(4, 0)$ **59.** $(3, 3)$ **61.** $(5, -1)$ **63.** $\left(\dfrac{a}{2}, \dfrac{b}{2}\right)$ **65.** $(0, 0)$ is on the graph.
67. $(0, 3)$ is on the graph. **69.** $(0, 2)$ and $(\sqrt{2}, \sqrt{2})$ are on the graph. **71.** $(-1, 0), (1, 0)$
73. $\left(-\dfrac{\pi}{2}, 0\right), (0, 1), \left(\dfrac{\pi}{2}, 0\right)$ **75.** $(0, 2), (1, 0), (0, -2)$ **77.** $(-4, 0), (-1, 0), (0, -3), (4, 0)$

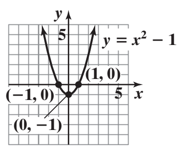
79.



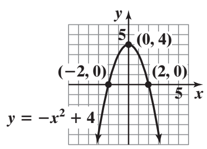
81.



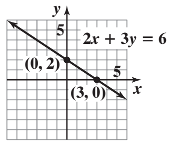
83.



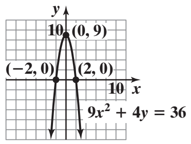
85.



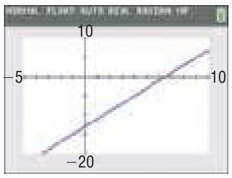
87.



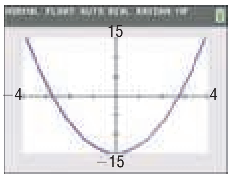
89.



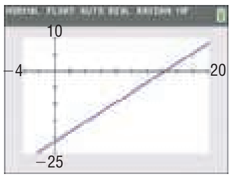
91. x-intercept: 6.5
y-intercept: -13



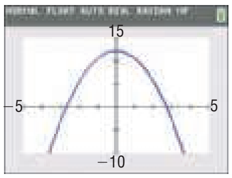
93. x-intercepts: $-2.74, 2.74$
y-intercept: -15



95. x-intercept: 14.33
y-intercept: -21.5



97. x-intercepts: $-2.72, 2.72$
y-intercept: 12.33



99. $(5, 3)$ **101.** $\sqrt{17}; 2\sqrt{5}; \sqrt{29}$ **103.** $d(P_1, P_2) = 6$; $d(P_2, P_3) = 4$; $d(P_1, P_3) = 2\sqrt{13}$; right triangle
105. $d(P_1, P_2) = 2\sqrt{17}$; $d(P_2, P_3) = \sqrt{34}$; $d(P_1, P_3) = \sqrt{34}$; isosceles right triangle **107.** $(5, -2)$ **109.** $90\sqrt{2} \approx 127.28$ ft
111. (a) $(90, 0), (90, 90), (0, 90)$ **(b)** $5\sqrt{2161} \approx 232.43$ ft **(c)** $30\sqrt{149} \approx 366.20$ ft **113.** $d = 50t$ mi **115. (a)** $(2.65, 1.6)$ **(b)** ≈ 1.285 units
117. \$21,582.50

1.2 Assess Your Understanding *(page 107)*

5. b **6.** identity **7.** linear; first **8.** d **9.** T **10.** F **11.** $\{3\}$ **13.** $\{-5\}$ **15.** $\left\{\dfrac{3}{2}\right\}$ **17.** $\left\{\dfrac{5}{4}\right\}$ **19.** $\{-2.21, 0.54, 1.68\}$ **21.** $\{-1.55, 1.15\}$
23. $\{-1.12, 0.36\}$ **25.** $\{-2.69, -0.49, 1.51\}$ **27.** $\{-2.86, -1.34, 0.20, 1.00\}$ **29.** No real solutions **31.** $\{-2\}$ **33.** $\{3\}$ **35.** $\{-1\}$ **37.** $\{-18\}$

39. $\{-4\}$ **41.** $\left\{-\frac{5}{4}\right\}$ **43.** $\left\{-\frac{3}{4}\right\}$ **45.** $\{-2\}$ **47.** $\{0.5\}$ **49.** $\left\{\frac{29}{10}\right\}$ **51.** $\{2\}$ **53.** $\{8\}$ **55.** $\{2\}$ **57.** $\{-1\}$ **59.** $\{3\}$ **61.** No solution **63.** No solution **65.** $\{-6\}$ **67.** $\{34\}$ **69.** $\left\{-\frac{20}{39}\right\}$ **71.** $\{-1\}$ **73.** $\left\{-\frac{11}{6}\right\}$ **75.** $\{-6\}$ **77.** $-\frac{2}{5}$ **79.** $2a + 3b = 6$ **81.** $x = \frac{b+c}{a}$ **83.** $x = \frac{abc}{a+b}$ **85.** $x = a^2$ **87.** $a = 3$ **89.** $R = \frac{R_1R_2}{R_1 + R_2}$ **91.** $R = \frac{mv^2}{F}$ **93.** $r = \frac{S-a}{S}$ **95.** \$11,500 will be invested in bonds and \$8500 in CDs. **97.** Scott will receive \$400,000, Alice \$300,000, and Tricia \$200,000. **99.** The regular hourly rate is \$11.50. **101.** Brooke needs a score of 85. **103.** The original price was \$200,000; purchasing the model saves \$30,000. **105.** The theater paid \$0.80 for each box. **107.** There were 400 adults. **109.** The length is 19 ft; the width is 11 ft. **111.** Approximately 184 million people **113.** In obtaining step (7) we divided by $x - 2$. Since $x = 2$ from step (1), we actually divided by 0.

Historical Problems *(page 117)*

1. The area of each shaded square is 9, so the larger square will have area $85 + 4(9) = 121$. The area of the larger square is also given by the expression $(x+6)^2$, so $(x+6)^2 = 121$. Taking the positive square root of each side, $x + 6 = 11$ or $x = 5$.
2. Let $z = -6$, so $z^2 + 12z - 85 = -121$. We get the equation $u^2 - 121 = 0$ or $u^2 = 121$. Thus $u = \pm 11$, so $x = \pm 11 - 6$. $x = -17$ or $x = 5$.

3.
$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2$$
$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2 = 0$$
$$\left(x + \frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}\right)\left(x + \frac{b}{2a} + \frac{\sqrt{b^2-4ac}}{2a}\right) = 0$$
$$\left(x + \frac{b - \sqrt{b^2-4ac}}{2a}\right)\left(x + \frac{b + \sqrt{b^2-4ac}}{2a}\right) = 0$$
$$x = \frac{-b + \sqrt{b^2-4ac}}{2a} \text{ or } x = \frac{-b - \sqrt{b^2-4ac}}{2a}$$

1.3 Assess Your Understanding *(page 117)*

6. repeated; multiplicity 2 **7.** discriminant; negative **8.** F **9.** b **10.** d **11.** $\{0, 9\}$ **13.** $\{-5, 5\}$ **15.** $\{-3, 2\}$ **17.** $\left\{-\frac{1}{2}, 3\right\}$ **19.** $\{-4, 4\}$ **21.** $\{2, 6\}$ **23.** $\left\{\frac{3}{2}\right\}$ **25.** $\left\{-\frac{2}{3}, \frac{3}{2}\right\}$ **27.** $\left\{-\frac{2}{3}, \frac{3}{2}\right\}$ **29.** $\left\{-\frac{3}{4}, 2\right\}$ **31.** $\{-5, 5\}$ **33.** $\{-1, 3\}$ **35.** $\{-3, 0\}$ **37.** $\{-7, 3\}$ **39.** $\left\{-\frac{1}{4}, \frac{3}{4}\right\}$ **41.** $\left\{\frac{-1-\sqrt{7}}{6}, \frac{-1+\sqrt{7}}{6}\right\}$ **43.** $\{2 - \sqrt{2}, 2 + \sqrt{2}\}$ **45.** $\{2 - \sqrt{5}, 2 + \sqrt{5}\}$ **47.** $\left\{1, \frac{3}{2}\right\}$ **49.** No real solution **51.** $\left\{\frac{-1-\sqrt{5}}{4}, \frac{-1+\sqrt{5}}{4}\right\}$ **53.** $\left\{0, \frac{9}{4}\right\}$ **55.** $\left\{\frac{1}{3}\right\}$ **57.** $\left\{-\frac{2}{3}, 1\right\}$ **59.** $\left\{\frac{3-\sqrt{29}}{10}, \frac{3+\sqrt{29}}{10}\right\}$ **61.** $\left\{\frac{-2-\sqrt{10}}{2}, \frac{-2+\sqrt{10}}{2}\right\}$ **63.** $\left\{\frac{1-\sqrt{33}}{8}, \frac{1+\sqrt{33}}{8}\right\}$ **65.** $\left\{\frac{9-\sqrt{73}}{2}, \frac{9+\sqrt{73}}{2}\right\}$ **67.** No real solution **69.** Repeated real solution **71.** Two unequal real solutions **73.** $\{-\sqrt{5}, \sqrt{5}\}$ **75.** $\left\{\frac{1}{4}\right\}$ **77.** $\left\{-\frac{3}{5}, \frac{5}{2}\right\}$ **79.** $\left\{-\frac{1}{2}, \frac{2}{3}\right\}$ **81.** $\left\{\frac{-\sqrt{2}-2}{2}, \frac{-\sqrt{2}+2}{2}\right\}$ **83.** $\left\{\frac{-1-\sqrt{17}}{2}, \frac{-1+\sqrt{17}}{2}\right\}$ **85.** $\left\{-\frac{1}{4}, \frac{2}{3}\right\}$ **87.** $\{5\}$ **89.** 2; 5 meters, 12 meters, 13 meters; 20 meters, 21 meters, 29 meters **91.** The dimensions are 11 ft by 13 ft. **93.** The dimensions are 5 m by 8 m. **95.** The dimensions should be 4 ft by 4 ft. **97. (a)** The ball strikes the ground after 6 sec. **(b)** The ball passes the top of the building on its way down after 5 sec. **99.** The dimensions should be 11.55 cm by 6.55 cm by 3 cm. **101.** The border will be 2.71 ft wide. **103.** The border will be 2.56 ft wide. **105.** The screen of an iPad Air in 4:3 format has an area of 45.16 square inches; the screen of the Google Nexus in 16:10 format has an area of 44.94 square inches. The iPad Air has a larger screen. **107.** 1.1 ft **109.** 29 hours **111.** 36 consecutive integers must be added.
113. $\frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a} = \frac{-2b}{2a} = -\frac{b}{a}$ **115.** $k = \frac{1}{2}$ or $k = -\frac{1}{2}$ **117.** $ax^2 + bx + c = 0, x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$;
$ax^2 - bx + c = 0, x = \frac{b \pm \sqrt{(-b)^2-4ac}}{2a} = \frac{b \pm \sqrt{b^2-4ac}}{2a} = -\frac{-b \pm \sqrt{b^2-4ac}}{2a}$ **119.** b

1.4 Assess Your Understanding *(page 128)*

4. real; imaginary; imaginary unit **5.** F **6.** T **7.** F **8.** b **9.** a **10.** c **11.** $8 + 5i$ **13.** $-7 + 6i$ **15.** $-6 - 11i$ **17.** $6 - 18i$ **19.** $6 + 4i$ **21.** $10 - 5i$ **23.** 37 **25.** $\frac{6}{5} + \frac{8}{5}i$ **27.** $1 - 2i$ **29.** $\frac{5}{2} - \frac{7}{2}i$ **31.** $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ **33.** $2i$ **35.** $-i$ **37.** i **39.** -6 **41.** $-10i$ **43.** $-2 + 2i$ **45.** 0 **47.** 0 **49.** $2i$ **51.** $5i$ **53.** $2\sqrt{3}i$ **55.** $10\sqrt{2}i$ **57.** $5i$ **59.** $\{-2i, 2i\}$ **61.** $\{-4, 4\}$ **63.** $\{3 - 2i, 3 + 2i\}$ **65.** $\{3 - i, 3 + i\}$ **67.** $\left\{\frac{1}{4} - \frac{1}{4}i, \frac{1}{4} + \frac{1}{4}i\right\}$ **69.** $\left\{\frac{1}{5} - \frac{2}{5}i, \frac{1}{5} + \frac{2}{5}i\right\}$ **71.** $\left\{-\frac{1}{2} - \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i\right\}$ **73.** $\{2, -1 - \sqrt{3}i, -1 + \sqrt{3}i\}$ **75.** $\{-2, 2, -2i, 2i\}$ **77.** $\{-3i, -2i, 2i, 3i\}$ **79.** Two complex solutions that are conjugates of each other **81.** Two unequal real solutions **83.** A repeated real solution **85.** $2 - 3i$ **87.** 6 **89.** 25 **91.** $2 + 3i$ ohms **93.** $z + \bar{z} = (a + bi) + (a - bi) = 2a$; $z - \bar{z} = (a + bi) - (a - bi) = 2bi$
95. $\overline{z + w} = \overline{(a + bi) + (c + di)} = \overline{(a + c) + (b + d)i} = (a + c) - (b + d)i = (a - bi) + (c - di) = \bar{z} + \bar{w}$

1.5 Assess Your Understanding *(page 135)*

6. F **7.** quadratic in form **8.** T **9.** a **10.** c **11.** $\{22\}$ **13.** $\{1\}$ **15.** No real solution **17.** $\{-13\}$ **19.** $\{4\}$ **21.** $\{-1\}$ **23.** $\{0, 64\}$ **25.** $\{3\}$ **27.** $\{2\}$ **29.** $\left\{-\frac{8}{5}\right\}$ **31.** $\{8\}$ **33.** $\{-1, 3\}$ **35.** $\{1, 5\}$ **37.** $\{1\}$ **39.** $\{5\}$ **41.** $\{2\}$ **43.** $\{-4, 4\}$ **45.** $\{-2, 2\}$ **47.** $\{-2, -1, 1, 2\}$ **49.** $\{-1, 1\}$ **51.** $\{-2, 1\}$ **53.** $\{-6, -5\}$ **55.** $\left\{-\frac{3}{2}, 2\right\}$ **57.** $\{0, 16\}$ **59.** $\{16\}$ **61.** $\{1\}$ **63.** $\left\{\left(\frac{9-\sqrt{17}}{8}\right)^4, \left(\frac{9+\sqrt{17}}{8}\right)^4\right\}$ **65.** $\left\{-2, -\frac{1}{2}\right\}$ **67.** $\left\{-\frac{3}{2}, \frac{1}{3}\right\}$ **69.** $\left\{-\frac{1}{8}, 27\right\}$ **71.** $\{-4, 1\}$ **73.** $\left\{-1, \frac{3}{2}\right\}$ **75.** $\{-4, 4\}$ **77.** $\left\{-\frac{1}{2}, \frac{1}{2}\right\}$ **79.** $\left\{-\frac{27}{2}, \frac{27}{2}\right\}$ **81.** $\left\{-\frac{36}{5}, \frac{24}{5}\right\}$ **83.** No real solution **85.** $\{-3, 3\}$ **87.** $\{-1, 3\}$ **89.** $\{-3, 0, 3\}$ **91.** $\{-5, 0, 4\}$ **93.** $\{-1, 1\}$ **95.** $\{-2, 2, 3\}$ **97.** $\left\{-2, \frac{1}{2}, 2\right\}$ **99.** $\{0.34, 11.66\}$ **101.** $\{-1.03, 1.03\}$ **103.** $\left\{-4, \frac{5}{3}\right\}$ **105.** $\left\{-3, -\frac{2}{5}, 3\right\}$ **107.** $\left\{-\frac{1}{5}, 1\right\}$ **109.** $\{5\}$ **111.** $\left\{-2, -\frac{4}{5}\right\}$ **113.** $\{-2, 6\}$ **115.** $\{2\}$ **117.** $\left\{\frac{-3-\sqrt{6}}{3}, \frac{-3+\sqrt{6}}{3}\right\}$ **119.** $\{-2, -1, 0, 1\}$ **121.** $\{\sqrt{2}, \sqrt{3}\}$ **123.** $\{-2, 2, -2i, 2i\}$ **125.** $\left\{1, 2, -1-\sqrt{3}i, -1+\sqrt{3}i, -\frac{1}{2}-\frac{\sqrt{3}}{2}i, -\frac{1}{2}+\frac{\sqrt{3}}{2}i\right\}$ **127.** $\left\{\frac{3}{2}, 5\right\}$ **129.** $(2, 2)$; $(2, -4)$ **131.** $(0, 0)$; $(8, 0)$ **133.** $\{1, 3\}$ **135.** The distance is approximately 229.94 ft. **137.** approx. 221 feet

1.6 Assess Your Understanding *(page 143)*

1. mathematical modeling **2.** interest **3.** uniform motion **4.** F **5.** T **6.** a **7.** b **8.** c **9.** $A = \pi r^2$; r = radius, A = area **11.** $A = s^2$; A = area, s = length of a side **13.** $F = ma$; F = force, m = mass, a = acceleration **15.** $W = Fd$; W = work, F = force, d = distance **17.** $C = 150x$; C = total variable cost, x = number of dishwashers **19.** Invest \$31,250 in bonds and \$18,750 in CDs. **21.** \$11,600 was loaned out at 8%. **23.** Mix 75 lb of Earl Grey tea with 25 lb of Orange Pekoe tea. **25.** Mix 160 lb of cashews with the almonds. **27.** The speed of the current is 2.286 mi/h. **29.** The speed of the current is 5 mi/h. **31.** Karen walked at 4.05 ft/sec. **33.** A doubles tennis court is 78 feet long and 36 feet wide. **35.** Working together, it takes 12 min. **37. (a)** The dimensions are 10 ft by 5 ft. **(b)** The area is 50 sq ft. **(c)** The dimensions would be 7.5 ft by 7.5 ft. **(d)** The area would be 56.25 sq ft. **39.** The defensive back catches up to the tight end at the tight end's 45-yd line. **41.** Add $\frac{2}{3}$ gal of water. **43.** Evaporate 10.67 oz of water. **45.** 40 g of 12-karat gold should be mixed with 20 g of pure gold. **47.** Mike passes Dan $\frac{1}{3}$ mile from the start, 2 min from the time Mike started to run. **49.** Start the auxiliary pump at 9:45 AM. **51.** The tub will fill in 1 h. **53.** Run: 12 miles; bicycle: 75 miles **55.** Bolt would beat Burke by 19.25 m. **57.** Set the original price at \$40. At 50% off, there will be no profit. **61.** The tail wind was 91.47 knots.

1.7 Assess Your Understanding *(page 154)*

4. negative **5.** closed interval **6.** $-a \le u \le a$ **7.** $(-\infty, a]$ **8.** T **9.** T **10.** F **11.** a **12.** c **13.** $[0, 2]$; $0 \le x \le 2$ **15.** $[2, \infty)$; $x \ge 2$ **17.** $[0, 3)$; $0 \le x < 3$ **19. (a)** $6 < 8$ **(b)** $-2 < 0$ **(c)** $9 < 15$ **(d)** $-6 > -10$ **21. (a)** $2x + 4 < 5$ **(b)** $2x - 4 < -3$ **(c)** $6x + 3 < 6$ **(d)** $-4x - 2 > -4$

23. $[0, 4]$

25. $[4, 6)$

27. $[4, \infty)$

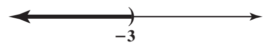
29. $(-\infty, -4)$

31. $2 \le x \le 5$

33. $-3 < x < -2$

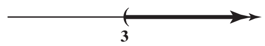
35. $x \ge 4$

37. $x < -3$

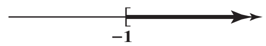


39. $<$ **41.** $>$ **43.** $\ge$ **45.** $<$ **47.** $<$ **49.** $\ge$

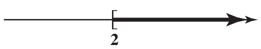
51. $\{x \mid x > 3\}$ or $(3, \infty)$



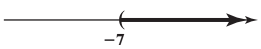
53. $\{x \mid x \ge -1\}$ or $[-1, \infty)$



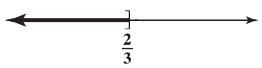
55. $\{x \mid x \ge 2\}$ or $[2, \infty)$



57. $\{x \mid x > -7\}$ or $(-7, \infty)$



59. $\left\{x \middle| x \le \frac{2}{3}\right\}$ or $\left(-\infty, \frac{2}{3}\right]$



61. $\{x \mid x < -20\}$ or $(-\infty, -20)$

63. $\{x \mid 3 \le x \le 5\}$ or $[3, 5]$

65. $\left\{x \middle| \frac{2}{3} \le x \le 3\right\}$ or $\left[\frac{2}{3}, 3\right]$

67. $\left\{x \middle| -\frac{11}{2} < x < \frac{1}{2}\right\}$ or $\left(-\frac{11}{2}, \frac{1}{2}\right)$

69. $\{x \mid -6 < x < 0\}$ or $(-6, 0)$

71. $\{x \mid x < -5\}$ or $(-\infty, -5)$

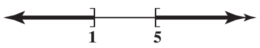
73. $\{x \mid -4 < x < 4\}$; $(-4, 4)$

75. $\{x \mid x < -4 \text{ or } x > 4\}$; $(-\infty, -4) \cup (4, \infty)$

77. $\left\{t \middle| -\frac{2}{3} \le t \le 2\right\}$; $\left[-\frac{2}{3}, 2\right]$

79. $\{x \mid 1 < x < 3\}$; $(1, 3)$

81. $\{x \mid x \le 1 \text{ or } x \ge 5\}$; $(-\infty, 1] \cup [5, \infty)$



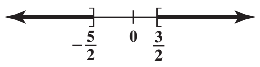
83. $\{x \mid x < -1 \text{ or } x > 2\}$; $(-\infty, -1)$ or $(2, \infty)$

85. $\left\{x \,\middle|\, -1 < x < \frac{3}{2}\right\}$; $\left(-1, \frac{3}{2}\right)$

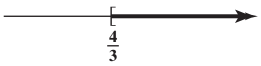
87. No solution

89. $\{x \mid x > -2\}$; $(-2, \infty)$

91. $\left\{x \,\middle|\, x \le -\frac{5}{2} \text{ or } x \ge \frac{3}{2}\right\}$; $\left(-\infty, -\frac{5}{2}\right] \cup \left[\frac{3}{2}, \infty\right)$

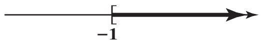


93. $\left\{x \,\middle|\, x \ge \frac{4}{3}\right\}$; $\left[\frac{4}{3}, \infty\right)$



95. $\{x \mid -3 \le x < 7\}$; $[-3, 7)$

97. $\{x \mid x \ge -1\}$; $[-1, \infty)$;



99. $\left\{x \,\middle|\, \frac{17}{6} < x < \frac{19}{6}\right\}$; $\left(\frac{17}{6}, \frac{19}{6}\right)$

101. $\{x \mid -3 \le x < 4\}$; $[-3, 4)$

103. $\{x \mid -2 < x < 4\}$; $(-2, 4)$

105. $\{x \mid x > 5\}$; $(5, \infty)$

107. $\{x \mid x > 3\}$; $(3, \infty)$

109. $|x - 2| < \frac{1}{2}$; $\left\{x \,\middle|\, \frac{3}{2} < x < \frac{5}{2}\right\}$

111. $|x + 3| > 2$; $\{x \mid x < -5 \text{ or } x > -1\}$ **113.** $21 < \text{age} < 30$ **115.** $|x - 98.6| \ge 1.5$; $\{x \mid x \le 97.1 \text{ or } x \ge 100.1\}$
117. **(a)** Male ≥ 81.9 **(b)** Female ≥ 85.6 **(c)** A female can expect to live at least 3.7 years longer. **119.** The agent's commission ranges from \$45,000 to \$95,000, inclusive. As a percent of selling price, the commission ranges from 5% to approximately 8.6%, inclusive.
121. The amount withheld varies from \$134.50 to \$184.50, inclusive. **123.** The usage varies from approximately 700 to 2700 kilowatt-hours, inclusive.
125. The dealer's cost varies from \$7457.63 to \$7857.14, inclusive. **127.** **(a)** You need at least a 74 on the last test.
(b) You need at least a 77 on the last test. **129.** $|x - 13.6| < 1.8$; between 11.8 and 15.4 books per year are read, on average.

131. $\frac{a+b}{2} - a = \frac{a+b-2a}{2} = \frac{b-a}{2} > 0$; therefore, $a < \frac{a+b}{2}$

$b - \frac{a+b}{2} = \frac{2b-a-b}{2} = \frac{b-a}{2} > 0$; therefore, $b > \frac{a+b}{2}$.

133. $(\sqrt{ab})^2 - a^2 = ab - a^2 = a(b - a) > 0$; thus, $(\sqrt{ab})^2 > a^2$ and $\sqrt{ab} > a$.

$b^2 - (\sqrt{ab})^2 = b^2 - ab = b(b - a) > 0$; thus, $b^2 > (\sqrt{ab})^2$ and $b > \sqrt{ab}$.

135. $\frac{1}{h} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right) = \frac{1}{2}\left(\frac{b+a}{ab}\right)$; $h = \frac{2ab}{b+a}$

$$h - a = \frac{2ab}{b+a} - a = \frac{2ab - a(b+a)}{b+a} = \frac{2ab - ab - a^2}{b+a} = \frac{ab - a^2}{b+a} = \frac{a(b-a)}{b+a} > 0, \text{ since } 0 < a < b.$$

$$b - h = b - \frac{2ab}{b+a} = \frac{b(b+a) - 2ab}{b+a} = \frac{b^2 + ab - 2ab}{b+a} = \frac{b^2 - ab}{b+a} = \frac{b(b-a)}{b+a} > 0, \text{ since } 0 < a < b.$$

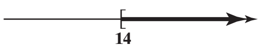
Review Exercises *(page 159)*

1. $\{-18\}$ **2.** $\{6\}$ **3.** $\left\{\frac{1}{5}\right\}$ **4.** $\{6\}$ **5.** No real solution **6.** $\left\{-\frac{27}{13}\right\}$ **7.** $\left\{-2, \frac{3}{2}\right\}$ **8.** $\left\{\frac{1-\sqrt{13}}{4}, \frac{1+\sqrt{13}}{4}\right\}$ **9.** $\{-3, 3\}$ **10.** No real solution

11. No real solution **12.** $\{-2, -1, 1, 2\}$ **13.** $\{2\}$ **14.** $\left\{\frac{13}{2}\right\}$ **15.** $\left\{\frac{\sqrt{5}}{2}\right\}$ **16.** $\left\{\frac{9}{4}\right\}$ **17.** $\left\{-1, \frac{1}{2}\right\}$ **18.** $\left\{\frac{m}{1-n}, \frac{m}{1+n}\right\}$ **19.** $\left\{-\frac{9b}{5a}, \frac{2b}{a}\right\}$

20. $\left\{-\frac{9}{5}\right\}$ **21.** $\{-5, 2\}$ **22.** $\left\{-\frac{5}{3}, 3\right\}$ **23.** $\left\{0, \frac{3}{2}\right\}$ **24.** $\left\{-\frac{5}{2}, -2, 2\right\}$ **25.** $\{3\}$ **26.** $\{-1, 5\}$ **27.** $\{-2.49, 0.66, 1.83\}$ **28.** $\{-1.14, 1.64\}$

29. $\{x \mid x \ge 14\}$; $[14, \infty)$



30. $\left\{x \,\middle|\, -\frac{31}{2} \le x \le \frac{33}{2}\right\}$; $\left[-\frac{31}{2}, \frac{33}{2}\right]$

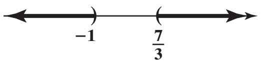
31. $\{x \mid -23 < x < -7\}$; $(-23, -7)$

32. $\left\{x \,\middle|\, -\frac{3}{2} < x < -\frac{7}{6}\right\}$; $\left(-\frac{3}{2}, -\frac{7}{6}\right)$

33. $\{x \mid x \le -2 \text{ or } x \ge 7\}$; $(-\infty, -2] \cup [7, \infty)$

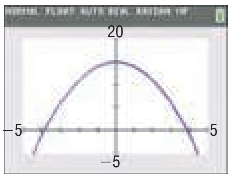
34. $\left\{x \,\middle|\, 0 \le x \le \frac{4}{3}\right\}$; $\left[0, \frac{4}{3}\right]$

35. $\left\{x \middle| x < -1 \text{ or } x > \frac{7}{3}\right\}; (-\infty, -1) \cup \left(\frac{7}{3}, \infty\right)$ **36.** $4 + 7i$ **37.** $-3 + 2i$ **38.** $\frac{9}{10} - \frac{3}{10}i$ **39.** -1 **40.** $-46 + 9i$ **41.** $\left\{-\frac{1}{2} - \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i\right\}$

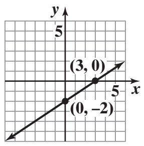


42. $\left\{\frac{-1 - \sqrt{17}}{4}, \frac{-1 + \sqrt{17}}{4}\right\}$ **43.** $\left\{\frac{1}{2} - \frac{\sqrt{11}}{2}i, \frac{1}{2} + \frac{\sqrt{11}}{2}i\right\}$ **44.** $\{-2, 1\}$ **45. (a)** $2\sqrt{5}$ **(b)** $(2, 1)$ **46. (a)** 5 **(b)** $\left(-\frac{1}{2}, 1\right)$ **47. (a)** 12 **(b)** $(4, 2)$

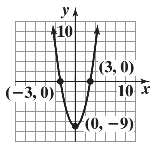
48.



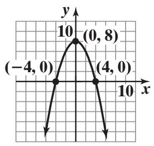
49. $(-4, 0), (0, 2), (0, 0), (0, -2), (2, 0)$ **50.**



51.



52.



53. $d(A, B) = \sqrt{13}; d(B, C) = \sqrt{13}$ **54.** -4 and 8 **55.** $p = 2l + 2w$ **56.** The interest is \$630. **57.** He should invest \$50,000 in A-rated bonds and \$20,000 in the CD. **58.** The storm is 3300 ft away. **59.** The range of distances is from 0.5 to 0.75 m, $0.5 \le x \le 0.75$. **60.** The search plane can go as far as 616 mi. **61.** The helicopter will reach the life raft in a little less than 1 hr 35 min. **62.** The bees meet for the first time in 18.75 seconds. The bees meet for the second time 37.5 seconds later. **63. (a)** The object will strike the ground in 8 seconds. **(b)** The height is 896 ft. **64.** It takes Clarissa 10 days by herself.

65. Add $6\frac{2}{3}$ lb of \$8/lb coffee to get $26\frac{2}{3}$ lb of \$5/lb coffee. **66.** Evaporate 51.2 oz of water. **67.** 5 cm and 12 cm **68.** The freight train is 190.67 ft long.

69. (a) 6.5 in. by 6.5 in.; 12.5 in. by 12.5 in. **(b)** $8\frac{2}{3}$ in. by $4\frac{1}{3}$ in.; $14\frac{2}{3}$ in. by $10\frac{1}{3}$ in. **70.** It will take the smaller pump 2 h.

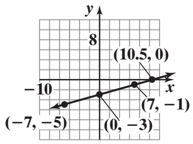
71. The length should be approximately 6.47 ft. **72.** 36 seniors went on the trip; each paid \$13.40.

73. (a) No **(b)** Todd wins again. **(c)** Todd wins by $\frac{1}{4}$ m. **(d)** Todd should line up 5.26 m behind the start line. **(e)** Yes

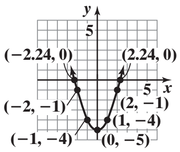
Chapter Test *(page 162)*

1. (a) $d = 10$ **(b)** $M = (1, 1)$ **2.** $\left\{-\frac{3}{2}, -1\right\}$ **3.** $\{2\}$ **4.** $\left\{-\frac{3}{2}, 2\right\}$ **5.** $\{2\}$ **6.** $\{0, 3\}$ **7.** $\{-2, 2\}$ **8.** $\{2 - \sqrt{2}, 2 + \sqrt{2}\}$ **9.** $\{3 - 2\sqrt{3}, 3 + 2\sqrt{3}\}$

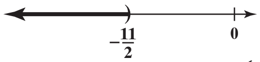
10.



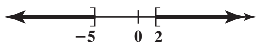
11.



12. $\{-1, 0.5, 1\}$
13. $\{-2.50, 2.50\}$
14. $\{-2.46, -0.24, 1.70\}$
15. $\left\{x \middle| x < -\frac{11}{2}\right\}; \left(-\infty, -\frac{11}{2}\right)$



16. $\{x|x \le -5 \text{ or } x \ge 2\}; (\infty, -5] \cup [2, \infty)$



17. $\{x|-1 \le x < 2\}; [-1, 2)$

18. $\left\{x \middle| -4 < x < \frac{4}{3}\right\}; \left(-4, \frac{4}{3}\right)$

19. $2 - 25i$ **20.** $14 + 83i$ **21.** $\frac{7}{34} + \frac{11}{34}i$ **22.** $\left\{\frac{1}{2} - i, \frac{1}{2} + i\right\}$ **23.** About 204.63 min (3.41 h) **24.** 23.75 lb of banana chips

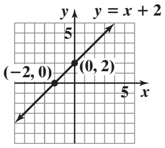
25. The sale price is \$159.50. **26.** Glenn will earn \$100 in interest after 3 months.

CHAPTER 2 Graphs

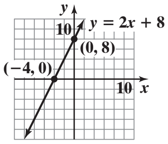
2.1 Assess Your Understanding *(page 170)*

3. intercepts **4.** y-axis **5.** 4 **6.** $(-3, 4)$ **7.** T **8.** F **9.** a **10.** c

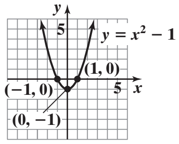
11. $(-2, 0), (0, 2)$



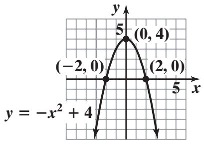
13. $(-4, 0), (0, 8)$



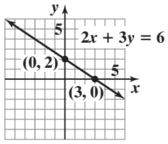
15. $(-1, 0), (1, 0), (0, -1)$



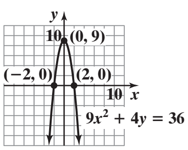
17. $(-2, 0), (2, 0), (0, 4)$



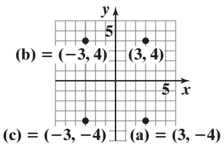
19. $(3, 0), (0, 2)$



21. $(-2, 0), (2, 0), (0, 9)$



23.



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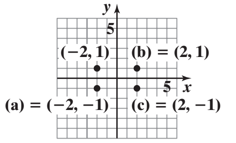

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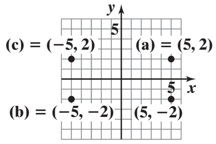

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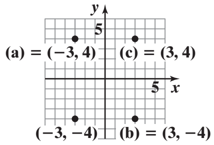
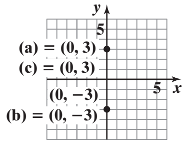
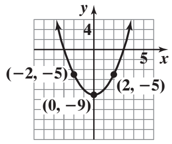


31.



33. (a) $(-1, 0), (1, 0)$ **(b)** Symmetric with respect to the x-axis, the y-axis, and the origin **35. (a)** $\left(-\frac{\pi}{2}, 0\right), (0, 1), \left(\frac{\pi}{2}, 0\right)$ **(b)** Symmetric with respect to the y-axis **37. (a)** $(0, 0)$ **(b)** Symmetric with respect to the x-axis **39. (a)** $(-2, 0), (0, 0), (2, 0)$ **(b)** Symmetric with respect to the origin **41. (a)** $(x, 0), -2 \le x \le 1$ **(b)** No Symmetry **43. (a)** No intercepts **(b)** Symmetric with respect to the origin

45.



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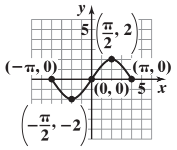
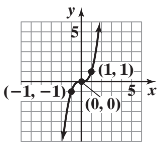
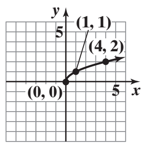


49. $(-4, 0), (0, -2), (0, 2)$; symmetric with respect to the x-axis **51.** $(0, 0)$; symmetric with respect to the origin **53.** $(0, -9), (3, 0), (-3, 0)$; symmetric with respect to the y-axis **55.** $(-2, 0), (2, 0), (0, -3), (0, 3)$; symmetric with respect to the x-axis, y-axis, and origin **57.** $(0, -27), (3, 0)$; no symmetry **59.** $(0, -4), (4, 0), (-1, 0)$; no symmetry **61.** $(0, 0)$; symmetric with respect to the origin **63.** $(0, 0)$; symmetric with respect to the origin

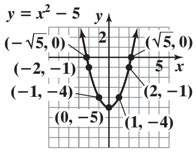
65.



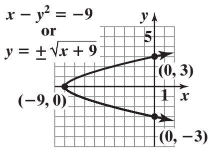
67.



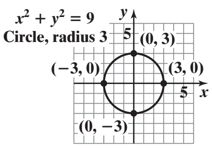
69. $b = 13$ **71.** $a = -4$ or $a = 1$
73. (a) $(0, -5), (-\sqrt{5}, 0), (\sqrt{5}, 0)$ **(b)** Symmetric with respect to the y-axis **(c)**



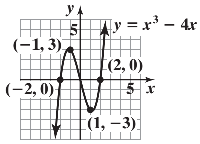
75. (a) $(-9, 0), (0, -3), (0, 3)$ **(b)** Symmetric with respect to the x-axis **(c)**



77. (a) $(0, 3), (0, -3), (-3, 0), (3, 0)$ **(b)** Symmetric with respect to the x-axis, y-axis, and origin **(c)**



79. (a) $(0, 0), (-2, 0), (2, 0)$ **(b)** Symmetric with respect to the origin **(c)**

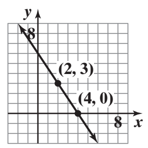


81. $(-1, -2)$ **83.** 4 **85. (a)** $(0, 0), (2, 0), (0, 1), (0, -1)$ **(b)** x-axis symmetry **87. (a)** $y = \sqrt{x^2}$ and $y = |x|$ have the same graph. **(b)** $\sqrt{x^2} = |x|$ **(c)** $x \ge 0$ for $y = (\sqrt{x})^2$, while x can be any real number for $y = x$. **(d)** $y \ge 0$ for $y = \sqrt{x^2}$
94. $\frac{1}{2}$ **95.** $3(x - 5)^2$ **96.** $14i$ **97.** $\{4 - 2\sqrt{3}, 4 + 2\sqrt{3}\}$

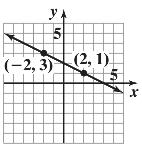
2.2 Assess Your Understanding *(page 184)*

1. undefined; 0 **2.** 3; 2 **3.** T **4.** F **5.** T **6.** $m_1 = m_2$; y-intercepts; $m_1 m_2 = -1$ **7.** 2 **8.** $-\frac{1}{2}$ **9.** F **10.** d **11.** c **12.** b
13. (a) Slope $= \frac{1}{2}$ **(b)** If x increases by 2 units, y will increase by 1 unit. **15. (a)** Slope $= -\frac{1}{3}$ **(b)** If x increases by 3 units, y will decrease by 1 unit.

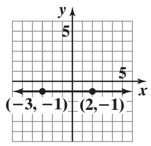
17. Slope $= -\frac{3}{2}$



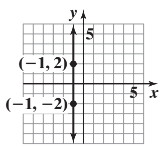
19. Slope $= -\frac{1}{2}$



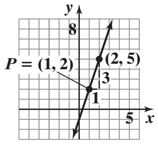
21. Slope $= 0$



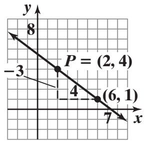
23. Slope undefined



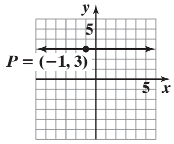
25.



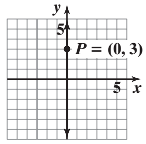
27.



29.

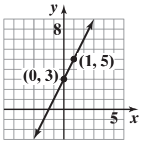


31.

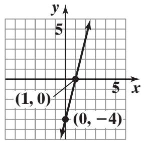


33. $(2, 6); (3, 10); (4, 14)$ **35.** $(4, -7); (6, -10); (8, -13)$ **37.** $(-1, -5); (0, -7); (1, -9)$ **39.** $x - 2y = 0$ or $y = \frac{1}{2}x$
41. $x + y = 2$ or $y = -x + 2$ **43.** $2x - y = 3$ or $y = 2x - 3$ **45.** $x + 2y = 5$ or $y = -\frac{1}{2}x + \frac{5}{2}$ **47.** $3x - y = -9$ or $y = 3x + 9$
49. $2x + 3y = -1$ or $y = -\frac{2}{3}x - \frac{1}{3}$ **51.** $3x + y = 3$ or $y = -3x + 3$ **53.** $x - 2y = -5$ or $y = \frac{1}{2}x + \frac{5}{2}$ **55.** $x - 2y = 2$ or $y = \frac{1}{2}x - 1$
57. $x = 2$; no slope–intercept form **59.** $y = 2$ **61.** $4x - y = -6$ or $y = 4x + 6$ **63.** $5x - y = 0$ or $y = 5x$ **65.** $x = 4$; no slope–intercept form
67. $6x + y = 0$ or $y = -6x$ **69.** $5x - 2y = -3$ or $y = \frac{5}{2}x + \frac{3}{2}$ **71.** $y = 4$

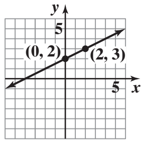
73. Slope $= 2$; y-intercept $= 3$



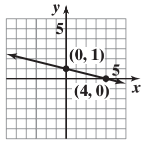
75. Slope $= 4$; y-intercept $= -4$



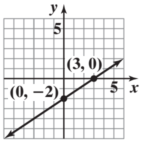
77. Slope $= \frac{1}{2}$; y-intercept $= 2$



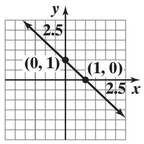
79. Slope $= -\frac{1}{4}$; y-intercept $= 1$



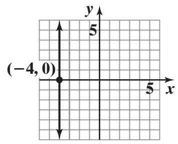
81. Slope $= \frac{2}{3}$; y-intercept $= -2$



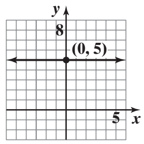
83. Slope $= -1$; y-intercept $= 1$



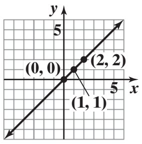
85. Slope undefined; no y-intercept



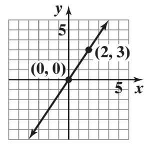
87. Slope $= 0$; y-intercept $= 5$



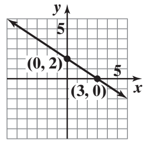
89. Slope $= 1$; y-intercept $= 0$



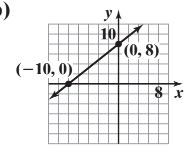
91. Slope $= \frac{3}{2}$; y-intercept $= 0$



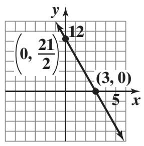
93. (a) x-intercept: 3; y-intercept: 2 **(b)**



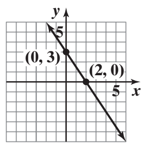
95. (a) x-intercept: -10; y-intercept: 8 **(b)**



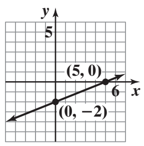
97. (a) x-intercept: 3; y-intercept: $\frac{21}{2}$ **(b)**



99. (a) x-intercept: 2; y-intercept: 3 **(b)**



101. (a) x-intercept: 5; y-intercept: -2 **(b)**



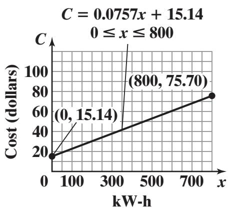
103. $y = 0$
105. Parallel
107. Neither
109. $x - y = -2$ or $y = x + 2$
111. $x + 3y = 3$ or $y = -\frac{1}{3}x + 1$

113. $P_1 = (-2, 5)$, $P_2 = (1, 3)$, $m_1 = -\frac{2}{3}$; $P_2 = (1, 3)$, $P_3 = (-1, 0)$, $m_2 = \frac{3}{2}$; because $m_1 m_2 = -1$, the lines are perpendicular and the points $(-2, 5)$, $(1, 3)$, and $(-1, 0)$ are the vertices of a right triangle.

115. $P_1 = (-1, 0)$, $P_2 = (2, 3)$, $m = 1$; $P_3 = (1, -2)$, $P_4 = (4, 1)$, $m = 1$; $P_1 = (-1, 0)$, $P_3 = (1, -2)$, $m = -1$; $P_2 = (2, 3)$, $P_4 = (4, 1)$, $m = -1$; opposite sides are parallel, and adjacent sides are perpendicular; the points are the vertices of a rectangle.

117. $C = 0.60x + 39$; \$105; \$177 **119.** $C = 0.16x + 1461$

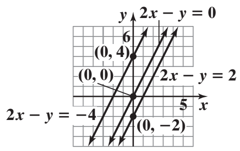
121. (a) $C = 0.0757x + 15.14$, $0 \le x \le 800$ **(b)**



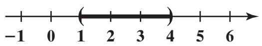
(c) \$30.28 **(d)** \$52.99
(e) Each additional kW-h used adds \$0.0757 to the bill.

123. $°C = \frac{5}{9}(°F - 32)$; approximately 21.1°C **125. (a)** $y = -\frac{2}{25}x + 30$ **(b)** x-intercept: 375; The ramp meets the floor 375 in. (31.25 ft) from the base of the platform. **(c)** The ramp does not meet design requirements. It has a run of 31.25 ft. **(d)** The only slope possible for the ramp to comply with the requirement is for it to drop 1 in. for every 12-in. run.

127. (a) $A = \frac{1}{5}x + 20{,}000$ **(b)** \$80,000 **(c)** Each additional box sold requires an additional \$0.20 in advertising. **129.** All have the same slope, 2; the lines are parallel.



131. (b), (c), (e), (g) **133.** (c) **139.** No; no **141.** They are the same line. **143.** Yes, if the y-intercept is 0. **146.** x^4y^{16} **147.** 17
148. $\{3 - 2\sqrt{6}, 3 + 2\sqrt{6}\}$ **149.** $\{x \mid 1 < x < 4\}$ or $(1, 4)$

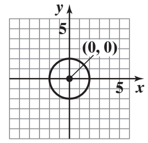


2.3 Assess Your Understanding *(page 193)*

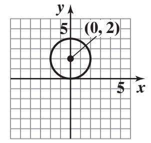
3. F **4.** radius **5.** T **6.** F **7.** d **8.** a **9.** Center $(2, 1)$; radius $= 2$; $(x - 2)^2 + (y - 1)^2 = 4$

11. Center $\left(\frac{5}{2}, 2\right)$; radius $= \frac{3}{2}$; $\left(x - \frac{5}{2}\right)^2 + (y - 2)^2 = \frac{9}{4}$

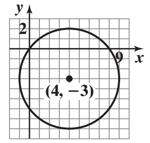
13. $x^2 + y^2 = 4$; $x^2 + y^2 - 4 = 0$



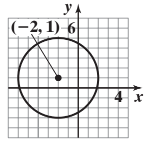
15. $x^2 + (y - 2)^2 = 4$; $x^2 + y^2 - 4y = 0$



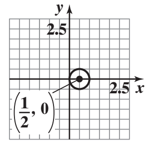
17. $(x - 4)^2 + (y + 3)^2 = 25$; $x^2 + y^2 - 8x + 6y = 0$



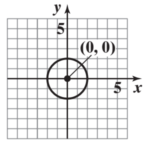
19. $(x + 2)^2 + (y - 1)^2 = 16$; $x^2 + y^2 + 4x - 2y - 11 = 0$



21. $\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$; $x^2 + y^2 - x = 0$

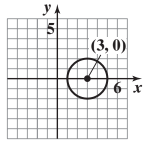


23. (a) $(h, k) = (0, 0); r = 2$ **(b)**



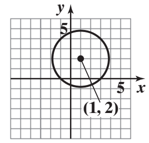
(c) $(\pm 2, 0); (0, \pm 2)$

25. (a) $(h, k) = (3, 0); r = 2$ **(b)**



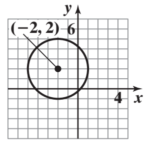
(c) $(1, 0); (5, 0)$

27. (a) $(h, k) = (1, 2); r = 3$ **(b)**



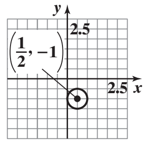
(c) $(1 \pm \sqrt{5}, 0); (0, 2 \pm 2\sqrt{2})$

29. (a) $(h, k) = (-2, 2); r = 3$ **(b)**



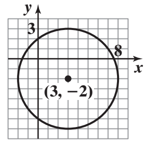
(c) $(-2 \pm \sqrt{5}, 0)$; $(0, 2 \pm \sqrt{5})$

31. (a) $(h, k) = \left(\frac{1}{2}, -1\right); r = \frac{1}{2}$ **(b)**



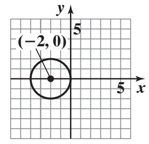
(c) $(0, -1)$

33. (a) $(h, k) = (3, -2); r = 5$ **(b)**



(c) $(3 \pm \sqrt{21}, 0)$; $(0, -6), (0, 2)$

35. (a) $(h, k) = (-2, 0); r = 2$ **(b)**



(c) $(0, 0), (-4, 0)$

37. $x^2 + y^2 = 13$ **39.** $(x - 2)^2 + (y - 3)^2 = 9$ **41.** $(x + 1)^2 + (y - 3)^2 = 5$ **43.** $(x + 1)^2 + (y - 3)^2 = 1$ **45.** (c) **47.** (b) **49.** 18 units^2 **51.** $x^2 + (y - 139)^2 = 15{,}625$ **53.** $x^2 + y^2 + 2x + 4y - 4168.16 = 0$ **55.** $\sqrt{2}x + 4y - 9\sqrt{2} = 0$ **57.** $(1, 0)$ **59.** $y = 2$ **61.** (b), (c), (e), (g) **65.** $A = 169\pi \text{cm}^2$; $C = 26\pi \text{cm}$ **66.** $3x^3 - 8x^2 + 13x - 6$ **67.** {1} **68.** 12.32 min

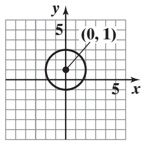
2.4 Assess Your Understanding *(page 199)*

1. $y = kx$ **2.** F **3.** b **4.** c **5.** $y = \frac{1}{5}x$ **7.** $A = \pi x^2$ **9.** $F = \frac{250}{d^2}$ **11.** $z = \frac{1}{5}(x^2 + y^2)$ **13.** $M = \frac{9d^2}{2\sqrt{x}}$ **15.** $T^2 = \frac{8a^3}{d^2}$ **17.** $V = \frac{4\pi}{3}r^3$ **19.** $A = \frac{1}{2}bh$ **21.** $F = 6.67 \times 10^{-11}\left(\frac{mM}{d^2}\right)$ **23.** $p = 0.00649B$; \$941.05 **25.** 144 ft; 2 sec **27.** 2.25 **29.** $R = 3.95g$; \$41.48 **31. (a)** $D = \frac{429}{p}$ **(b)** 143 bags **33.** 450 cm^3 **35.** 124.76 lb **37.** $V = \pi r^2 h$ **39.** 0.012 foot-candle **41.** $\sqrt[3]{6} \approx 1.82$ in. **43.** 2812.5 joules **45.** 384 psi **51.** $(3x + 25)(x + 2)(x - 2)$ **52.** $\frac{6}{x + 4}$ **53.** $\frac{8}{125}$ **54.** $\sqrt{7} + 2$

Review Exercises *(page 203)*

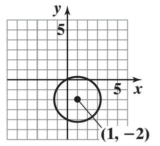
1. (a) $\frac{1}{2}$ **(b)** For each run of 2, there is a rise of 1. **2. (a)** $-\frac{4}{3}$ **(b)** For each run of 3, there is a rise of -4. **3. (a)** undefined **(b)** no change in x **4. (a)** 0 **(b)** no change in y **5.** $(0, 0)$; symmetric with respect to the x-axis **6.** $(\pm 4, 0), (0, \pm 2)$; symmetric with respect to the x-axis, y-axis, and origin **7.** $(\pm 2, 0), (0, -4)$; symmetric with respect to the y-axis **8.** $(0, 0), (\pm 1, 0)$; symmetric with respect to the origin **9.** $(0, 0), (-1, 0), (0, -2)$; no symmetry **10.** $(x + 2)^2 + (y - 3)^2 = 16$ **11.** $(x + 1)^2 + (y + 2)^2 = 1$

12. Center $(0, 1)$; radius $= 2$



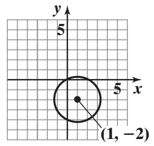
Intercepts: $(-\sqrt{3}, 0), (\sqrt{3}, 0)$, $(0, -1), (0, 3)$

13. Center $(1, -2)$; radius $= 3$



Intercepts: $(1 - \sqrt{5}, 0), (1 + \sqrt{5}, 0)$, $(0, -2 - 2\sqrt{2}), (0, -2 + 2\sqrt{2})$

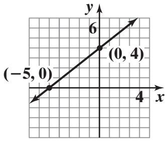
14. Center $(1, -2)$; radius $= \sqrt{5}$



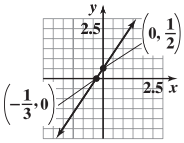
Intercepts: $(0, 0), (2, 0), (0, -4)$

15. $2x + y = 5$ or $y = -2x + 5$ **16.** $y = 4$ **17.** $x = -3$; no slope–intercept form **18.** $5x + 2y = 10$ or $y = -\frac{5}{2}x + 5$ **19.** $x + 5y = -10$ or $y = -\frac{1}{5}x - 2$ **20.** $5x + y = 11$ or $y = -5x + 11$ **21.** $2x - 3y = -19$ or $y = \frac{2}{3}x + \frac{19}{3}$ **22.** $x + 3y = 10$ or $y = -\frac{1}{3}x + \frac{10}{3}$

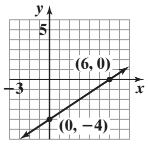
23. Slope $= \frac{4}{5}$; y-intercept $= 4$



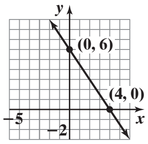
24. Slope $= \frac{3}{2}$; y-intercept $= \frac{1}{2}$



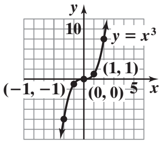
25. Intercepts: $(6, 0)$, $(0, -4)$



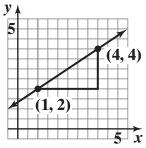
26. Intercepts: $(4, 0)$, $(0, 6)$



27.



28.

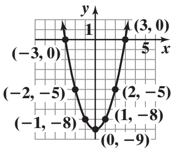


29. Slope from A to B is -2; slope from A to C is $\frac{1}{2}$. Since $(-2)\left(\frac{1}{2}\right) = -1$, the lines are perpendicular. **30.** Center: $(1, -2)$; radius: $4\sqrt{2}$; $(x-1)^2 + (y+2)^2 = 32$ **31.** Slope from A to B is -1; slope from A to C is -1. **32.** $p = \frac{854}{130{,}000}B$; \$1083.92 **33.** 199.9 lb **34.** 189 Btu

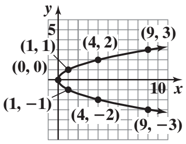
Chapter Test *(page 204)*

1. (a) $m = -\frac{2}{3}$ **(b)** For every 3-unit change in x, y will change by -2 units.

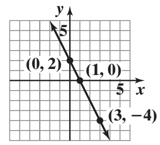
2.



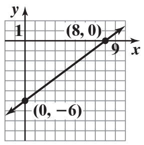
3.



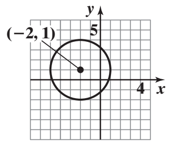
4. Intercepts: $(-3, 0)$, $(3, 0)$, $(0, 9)$; symmetric with respect to the y-axis **5.** $y = -2x + 2$



6. Slope $= -\frac{2}{3}$; y-intercept $= 3$ **7.**



8. $x^2 + y^2 - 8x + 6y = 0$ **9.** Center: $(-2, 1)$; radius: 3

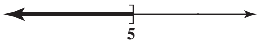


10. Parallel line: $y = -\frac{2}{3}x - \frac{1}{3}$; perpendicular line: $y = \frac{3}{2}x + 3$ **11.** 14.69 ohms

Cumulative Review *(page 204)*

1. $\left\{\frac{5}{3}\right\}$ **2.** $\{-3, 4\}$ **3.** $\left\{-\frac{1}{2}, 3\right\}$ **4.** $\{1 - \sqrt{3}, 1 + \sqrt{3}\}$ **5.** No real solution

6. $\{4\}$ **7.** $\{1, 3\}$ **8.** $\{-2 - 2\sqrt{2}, -2 + 2\sqrt{2}\}$ **9.** $\{-3i, 3i\}$ **10.** $\{1 - 2i, 1 + 2i\}$ **11.** $\{x \mid x \le 5\}$ or $(\infty, 5]$;



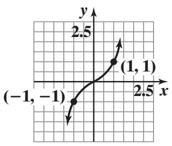
12. $\{x \mid -5 < x < 1\}$ or $(-5, 1)$;

13. $\{x \mid 1 \le x \le 3\}$ or $[1, 3]$;

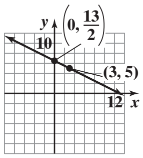
14. $\{x \mid x < -5$ or $x > 1\}$ or $(\infty, -5) \cup (1, \infty)$;

15. $5\sqrt{2}$; $\left(\frac{3}{2}, \frac{1}{2}\right)$ **16.** (a), (b)

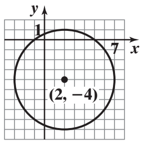
17.



18. $y = -2x + 2$ **19.** $y = -\frac{1}{2}x + \frac{13}{2}$;



20.



CHAPTER 3 Functions and Their Graphs

3.1 Assess Your Understanding *(page 218)*

7. independent; dependent **8.** $[0, 5]$ **9.** $\neq$; f; g **10.** $(g - f)(x)$ **11.** F **12.** T **13.** F **14.** F **15.** a **16.** c **17.** d **18.** a
19. Function; Domain: {Elvis, Colleen, Kaleigh, Marissa}; Range: {January 8, March 15, September 17} **21.** Not a function; Domain: {20, 30, 40}; Range: {200, 300, 350, 425} **23.** Not a function; Domain: $\{-3, 2, 4\}$; Range: {6, 9, 10} **25.** Function; Domain: {1, 2, 3, 4}; Range: {3}
27. Not a function; Domain: $\{-2, 0, 3\}$; Range: {3, 4, 6, 7} **29.** Function; Domain: $\{-2, -1, 0, 1\}$; Range: {0, 1, 4} **31.** Function **33.** Function

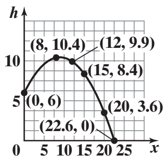
35. Not a function **37.** Not a function **39.** Function **41.** Not a function **43. (a)** -4 **(b)** 1 **(c)** -3 **(d)** $3x^2-2x-4$ **(e)** $-3x^2-2x+4$ **(f)** $3x^2+8x+1$ **(g)** $12x^2+4x-4$ **(h)** $3x^2+6xh+3h^2+2x+2h-4$ **45. (a)** 0 **(b)** $\frac{1}{2}$ **(c)** $-\frac{1}{2}$ **(d)** $\frac{-x}{x^2+1}$ **(e)** $\frac{-x}{x^2+1}$ **(f)** $\frac{x+1}{x^2+2x+2}$ **(g)** $\frac{2x}{4x^2+1}$ **(h)** $\frac{x+h}{x^2+2xh+h^2+1}$ **47. (a)** 4 **(b)** 5 **(c)** 5 **(d)** $|x|+4$ **(e)** $-|x|-4$ **(f)** $|x+1|+4$ **(g)** $2|x|+4$ **(h)** $|x+h|+4$ **49. (a)** $-\frac{1}{5}$ **(b)** $-\frac{3}{2}$ **(c)** $\frac{1}{8}$ **(d)** $\frac{2x-1}{3x+5}$ **(e)** $\frac{-2x-1}{3x-5}$ **(f)** $\frac{2x+3}{3x-2}$ **(g)** $\frac{4x+1}{6x-5}$ **(h)** $\frac{2x+2h+1}{3x+3h-5}$ **51.** All real numbers **53.** All real numbers **55.** $\{x \mid x \neq -4, x \neq 4\}$ **57.** $\{x \mid x \neq 0\}$ **59.** $\{x \mid x \geq 4\}$ **61.** $\{x \mid x > 1\}$ **63.** $\{x \mid x > 4\}$ **65.** $\{t \mid t \geq 4, t \neq 7\}$ **67. (a)** $(f+g)(x)=5x+1$; All real numbers **(b)** $(f-g)(x)=x+7$; All real numbers **(c)** $(f \cdot g)(x)=6x^2-x-12$; All real numbers **(d)** $\left(\frac{f}{g}\right)(x)=\frac{3x+4}{2x-3}$; $\left\{x \,\middle|\, x \neq \frac{3}{2}\right\}$ **(e)** 16 **(f)** 11 **(g)** 10 **(h)** -7 **69. (a)** $(f+g)(x)=2x^2+x-1$; All real numbers **(b)** $(f-g)(x)=-2x^2+x-1$; All real numbers **(c)** $(f \cdot g)(x)=2x^3-2x^2$; All real numbers **(d)** $\left(\frac{f}{g}\right)(x)=\frac{x-1}{2x^2}$; $\{x \mid x \neq 0\}$ **(e)** 20 **(f)** -29 **(g)** 8 **(h)** 0 **71. (a)** $(f+g)(x)=\sqrt{x}+3x-5$; $\{x \mid x \geq 0\}$ **(b)** $(f-g)(x)=\sqrt{x}-3x+5$; $\{x \mid x \geq 0\}$ **(c)** $(f \cdot g)(x)=3x\sqrt{x}-5\sqrt{x}$; $\{x \mid x \geq 0\}$ **(d)** $\left(\frac{f}{g}\right)(x)=\frac{\sqrt{x}}{3x-5}$; $\left\{x \,\middle|\, x \geq 0, x \neq \frac{5}{3}\right\}$ **(e)** $\sqrt{3}+4$ **(f)** -5 **(g)** $\sqrt{2}$ **(h)** $-\frac{1}{2}$ **73. (a)** $(f+g)(x)=1+\frac{2}{x}$; $\{x \mid x \neq 0\}$ **(b)** $(f-g)(x)=1$; $\{x \mid x \neq 0\}$ **(c)** $(f \cdot g)(x)=\frac{1}{x}+\frac{1}{x^2}$; $\{x \mid x \neq 0\}$ **(d)** $\left(\frac{f}{g}\right)(x)=x+1$; $\{x \mid x \neq 0\}$ **(e)** $\frac{5}{3}$ **(f)** 1 **(g)** $\frac{3}{4}$ **(h)** 2 **75. (a)** $(f+g)(x)=\frac{6x+3}{3x-2}$; $\left\{x \,\middle|\, x \neq \frac{2}{3}\right\}$ **(b)** $(f-g)(x)=\frac{-2x+3}{3x-2}$; $\left\{x \,\middle|\, x \neq \frac{2}{3}\right\}$ **(c)** $(f \cdot g)(x)=\frac{8x^2+12x}{(3x-2)^2}$; $\left\{x \,\middle|\, x \neq \frac{2}{3}\right\}$ **(d)** $\left(\frac{f}{g}\right)(x)=\frac{2x+3}{4x}$; $\left\{x \,\middle|\, x \neq 0, x \neq \frac{2}{3}\right\}$ **(e)** 3 **(f)** $-\frac{1}{2}$ **(g)** $\frac{7}{2}$ **(h)** $\frac{5}{4}$ **77.** $g(x)=5-\frac{7}{2}x$ **79.** 4 **81.** $2x+h$ **83.** $2x+h-1$ **85.** $\frac{-(2x+h)}{x^2(x+h)^2}$ **87.** $\frac{6}{(x+3)(x+h+3)}$ **89.** $\frac{1}{\sqrt{x+h-2}+\sqrt{x-2}}$ **91.** $\{-2, 4\}$ **93.** $A=-\frac{7}{2}$ **95.** $A=-4$ **97.** $A(x)=\frac{1}{2}x^2$ **99.** $G(x)=14x$ **101. (a)** P is the dependent variable; a is the independent variable. **(b)** $P(20)=231.427$ million; In 2012, there were 231.427 million people 20 years of age or older. **(c)** $P(0)=327.287$ million; In 2012, there were 327.287 million people. **103. (a)** 15.1 m, 14.071 m, 12.944 m, 11.719 m **(b)** 1.01 sec, 1.43 sec, 1.75 sec **(c)** 2.02 sec **105. (a)** \$222 **(b)** \$225 **(c)** \$220 **(d)** \$230 **107.** $R(x)=\frac{L(x)}{P(x)}$ **109.** $H(x)=P(x) \cdot I(x)$ **111. (a)** $P(x)=-0.05x^3+0.8x^2+155x-500$ **(b)** $P(15)=\$1836.25$ **(c)** When 15 hundred cellphones are sold, the profit is \$1836.25. **113. (a)** $D(v)=0.05v^2+2.6v-15$ **(b)** 321 feet **(c)** The car will need 321 feet to stop once the impediment is observed. **115.** No; domain of f is all real numbers; domain of g is $\{x \mid x \neq -1\}$ **117.** $H(x)=\frac{3x-x^3}{\text{age}}$ **118.** Intercepts: $(-16, 0)$, $(-8, 0)$; x-axis symmetry **119.** $(4, 32)$ **120.** 7.5 lb **121.** $\{-3, 2, 3\}$

3.2 Assess Your Understanding *(page 226)*

3. vertical **4.** 5; -3 **5.** $a=-2$ **6.** F **7.** F **8.** T **9.** c **10.** a **11. (a)** $f(0)=3$; $f(-6)=-3$ **(b)** $f(6)=0$; $f(11)=1$ **(c)** Positive **(d)** Negative **(e)** -3, 6, and 10 **(f)** $-3<x<6$; $10<x\leq 11$ **(g)** $\{x \mid -6 \leq x \leq 11\}$ **(h)** $\{y \mid -3 \leq y \leq 4\}$ **(i)** $-3, 6, 10$ **(j)** 3 **(k)** 3 times **(l)** Once **(m)** 0, 4 **(n)** -5, 8 **13.** Not a function **15.** Function **(a)** Domain: $\{x \mid -\pi \leq x \leq \pi\}$; Range: $\{y \mid -1 \leq y \leq 1\}$ **(b)** $\left(-\frac{\pi}{2}, 0\right)$, $\left(\frac{\pi}{2}, 0\right)$, $(0, 1)$ **(c)** y-axis **17.** Not a function **19.** Function **(a)** Domain: $\{x \mid 0<x<3\}$; Range: $\{y \mid y<2\}$ **(b)** $(1, 0)$ **(c)** None **21.** Function **(a)** Domain: all real numbers; Range: $\{y \mid y \leq 2\}$ **(b)** $(-3, 0)$, $(3, 0)$, $(0, 2)$ **(c)** y-axis **23.** Function **(a)** Domain: all real numbers; Range: $\{y \mid y \geq -3\}$ **(b)** $(1, 0)$, $(3, 0)$, $(0, 9)$ **(c)** None **25. (a)** Yes **(b)** $f(-2)=9$; $(-2, 9)$ **(c)** $0, \frac{1}{2}$; $(0, -1)$, $\left(\frac{1}{2}, -1\right)$ **(d)** All real numbers **(e)** $-\frac{1}{2}, 1$ **(f)** -1 **27. (a)** No **(b)** $f(4)=-3$; $(4, -3)$ **(c)** 14; $(14, 2)$ **(d)** $\{x \mid x \neq 6\}$ **(e)** -2 **(f)** $-\frac{1}{3}$ **29. (a)** Yes **(b)** $f(2)=\frac{8}{17}$; $\left(2, \frac{8}{17}\right)$ **(c)** $-1, 1$; $(-1, 1)$, $(1, 1)$ **(d)** All real numbers **(e)** 0 **(f)** 0

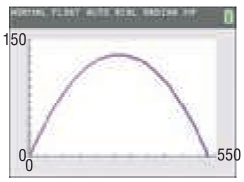
31. (a) 3 **(b)** -2 **(c)** -1 **(d)** 1 **(e)** 2 **(f)** $-\frac{1}{3}$

33. (a) Approximately 10.4 ft high
(b) Approximately 9.9 ft high
(c)



(d) The ball will not go through the hoop; $h(15) \approx 8.4$ ft. If $v=30$ ft/sec, $h(15)=10$ ft.

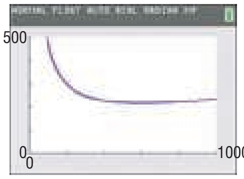
35. (a) About 81.07 ft **(b)** About 129.59 ft **(c)** About 26.63 ft
(d) About 528.13 ft
(e)



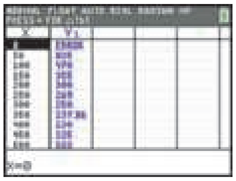
(f) About 115.07 ft and 413.05 ft
(g) 275 ft; maximum height shown in the table is 131.8 ft **(h)** 264 ft

37. (a) \$223; \$220 **(b)** $\{x \mid x > 0\}$

(c)



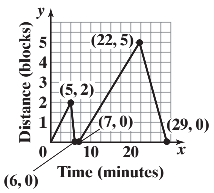
(d)



(e) 600 mi/h

39. (a) \$30; It costs \$30 if you use 0 gigabytes. **(b)** \$30; It costs \$30 if you use 5 gigabytes. **(c)** \$90; It costs \$90 if you use 15 gigabytes. **(d)** $\{g \mid 0 \le g \le 60\}$. There are at most 60 gigabytes in a month. **41.** The x-intercepts can number anywhere from 0 to infinitely many. There is at most one y-intercept. **43. (a)** III **(b)** IV **(c)** I **(d)** V **(e)** II

45.

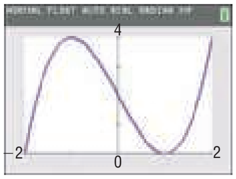


47. (a) 2 hr elapsed during which Kevin was between 0 and 3 mi from home **(b)** 0.5 hr elapsed during which Kevin was 3 mi from home **(c)** 0.3 hr elapsed during which Kevin was between 0 and 3 mi from home **(d)** 0.2 hr elapsed during which Kevin was 0 mi from home **(e)** 0.9 hr elapsed during which Kevin was between 0 and 2.8 mi from home **(f)** 0.3 hr elapsed during which Kevin was 2.8 mi from home **(g)** 1.1 hr elapsed during which Kevin was between 0 and 2.8 mi from home **(h)** 3 mi **(i)** Twice
49. No points whose x-coordinate is 5 or whose y-coordinate is 0 can be on the graph.
52. $2(2x+3)^2$ **53.** $2\sqrt{10}$ **54.** $y = \frac{2}{3}x + 8$ **55.** $4x^3 - 8x^2 - 5x + 4$

3.3 Assess Your Understanding *(page 240)*

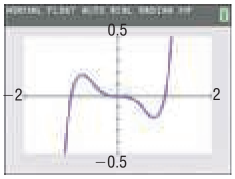
6. increasing **7.** even; odd **8.** T **9.** T **10.** F **11.** c **12.** d **13.** Yes **15.** No **17.** $[-8,-2]$; $[0,2]$; $[5,7]$ **19.** Yes; 10 **21.** $-2, 2$; 6, 10
23. $f(-8) = -4$ **25. (a)** $(-2,0)$, $(0,3)$, $(2,0)$ **(b)** Domain: $\{x \mid -4 \le x \le 4\}$ or $[-4,4]$; Range: $\{y \mid 0 \le y \le 3\}$ or $[0,3]$
(c) Increasing on $[-2,0]$ and $[2,4]$; Decreasing on $[-4,-2]$ and $[0,2]$ **(d)** Even **27. (a)** $(0,1)$
(b) Domain: all real numbers; Range: $\{y \mid y > 0\}$ or $(0,\infty)$ **(c)** Increasing on $(-\infty,\infty)$ **(d)** Neither
29. (a) $(-\pi,0)$, $(0,0)$, $(\pi,0)$ **(b)** Domain: $\{x \mid -\pi \le x \le \pi\}$ or $[-\pi,\pi]$; Range: $\{y \mid -1 \le y \le 1\}$ or $[-1,1]$
(c) Increasing on $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$; Decreasing on $\left[-\pi,-\frac{\pi}{2}\right]$ and $\left[\frac{\pi}{2},\pi\right]$ **(d)** Odd **31. (a)** $\left(0,\frac{1}{2}\right)$, $\left(\frac{1}{3},0\right)$, $\left(\frac{5}{2},0\right)$
(b) Domain: $\{x \mid -3 \le x \le 3\}$ or $[-3,3]$; Range: $\{y \mid -1 \le y \le 2\}$ or $[-1,2]$ **(c)** Increasing on $[2,3]$; Decreasing on $[-1,1]$; Constant on $[-3,-1]$ and $[1,2]$ **(d)** Neither **33. (a)** 0; 3 **(b)** $-2, 2$; 0, 0
35. (a) $\frac{\pi}{2}$; 1 **(b)** $-\frac{\pi}{2}$; -1 **37.** Odd **39.** Even **41.** Odd **43.** Neither **45.** Even **47.** Odd
49. Absolute maximum: $f(1) = 4$; absolute minimum: $f(5) = 1$; local maximum: $f(3) = 3$; local minimum: $f(2) = 2$
51. Absolute maximum: $f(3) = 4$; absolute minimum: $f(1) = 1$; local maximum: $f(3) = 4$; local minimum: $f(1) = 1$
53. Absolute maximum: none; absolute minimum: $f(0) = 0$: local maximum: $f(2) = 3$; local minimum: $f(0) = 0$ and $f(3) = 2$
55. Absolute maximum: none; absolute minimum: none: local maximum: none; local minimum: none

57.



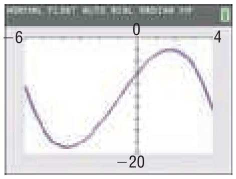
Increasing: $[-2,-1]$, $[1,2]$
Decreasing: $[-1,1]$
Local maximum: $f(-1) = 4$
Local minimum: $f(1) = 0$

59.



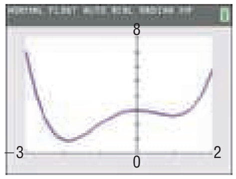
Increasing: $[-2,-0.77]$, $[0.77,2]$
Decreasing: $[-0.77,0.77]$
Local maximum: $f(-0.77) = 0.19$
Local minimum: $f(0.77) = -0.19$

61.



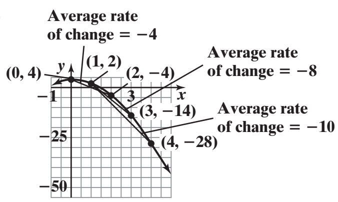
Increasing: $[-3.77,1.77]$
Decreasing: $[-6,-3.77]$, $[1.77,4]$
Local maximum: $f(1.77) = -1.91$
Local minimum: $f(-3.77) = -18.89$

63.

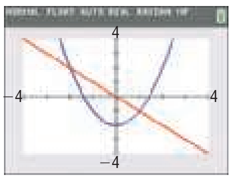


Increasing: $[-1.87,0]$, $[0.97,2]$
Decreasing: $[-3,-1.87]$, $[0,0.97]$
Local maximum: $f(0) = 3$
Local minima: $f(-1.87) = 0.95$, $f(0.97) = 2.65$

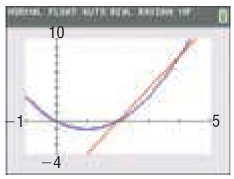
65. (a) -4 **(b)** -8 **(c)** -10
(d)



67. (a) 17 **(b)** -1 **(c)** 11 **69. (a)** 5 **(b)** $y = 5x - 2$
71. (a) -1 **(b)** $y = -x$
(c)

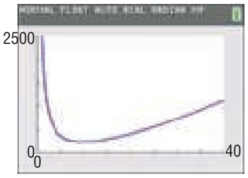


73. (a) 4 **(b)** $y = 4x - 8$
(c)



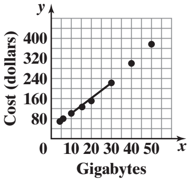
75. (a) Odd **(b)** Local maximum value: 54 at $x = -3$ **77. (a)** Even **(b)** Local maximum value: 25 at $x = -2$ **(c)** 50.4 sq. units

79. (a)



(b) 10 riding lawn mowers
(c) \$239/mower

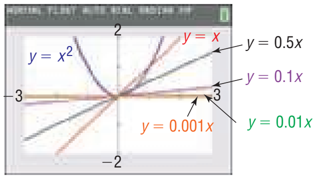
81. (a), (b)



(c) \$5/gigabyte
(d) \$6.25/gigabyte
(e) \$7.50/gigabyte
(f) The average rate of change is increasing as the number of gigabytes increases.

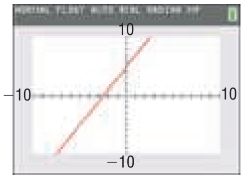
83. (a) On average, the population is increasing at a rate of 0.036 g/h from 0 to 2.5 h. **(b)** On average, from 4.5 to 6 h, the population is increasing at a rate of 0.1 g/h. **(c)** The average rate of change is increasing over time.

85. (a) 1 **(b)** 0.5 **(c)** 0.1 **(d)** 0.01
(e).0001
(f)

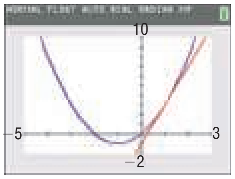


(g) They are getting closer to the tangent line at $(0, 0)$.
(h) They are getting closer to 0.

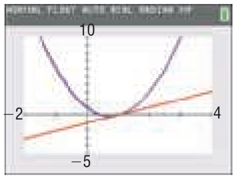
87. (a) 2
(b) 2; 2; 2; 2
(c) $y = 2x + 5$
(d)



89. (a) $2x + h + 2$
(b) 4.5; 4.1; 4.01; 4
(c) $y = 4.01x - 1.01$
(d)



91. (a) $4x + 2h - 3$
(b) 2; 1.2; 1.02; 1
(c) $y = 1.02x - 1.02$
(d)

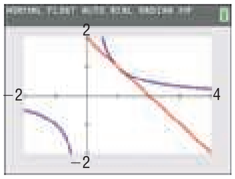


93. (a) $-\dfrac{1}{(x+h)x}$

(b) $-\dfrac{2}{3}; -\dfrac{10}{11}; -\dfrac{100}{101}; -1$

(c) $y = -\dfrac{100}{101}x + \dfrac{201}{101}$

(d)

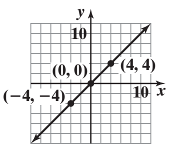


97. At most one **99.** Yes; the function $f(x) = 0$ is both even and odd. **101.** Not necessarily. It just means $f(5) > f(2)$.
103. (a) 7.01×10^{-6} **(b)** 2.305×10^{9} **104.** $6\sqrt{15}$ **105.** $[-8, -3)$ **106.** 8.25 days

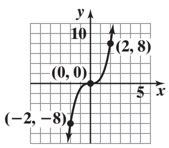
3.4 Assess Your Understanding *(page 252)*

4. $(-\infty, 0]$ **5.** piecewise-defined **6.** T **7.** F **8.** F **9.** b **10.** a **11.** C **13.** E **15.** B **17.** F

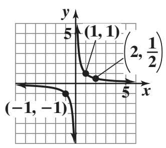
19.



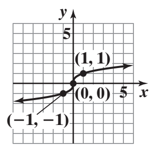
21.



23.

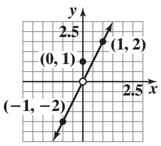


25.



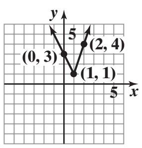
27. (a) 4 **(b)** 2 **(c)** 5 **29. (a)** −4 **(b)** −2 **(c)** 0 **(d)** 25

31. (a) All real numbers
(b) $(0, 1)$
(c)



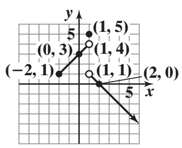
(d) $\{y|y \neq 0\}$; $(-\infty, 0) \cup (0, \infty)$

33. (a) All real numbers
(b) $(0, 3)$
(c)



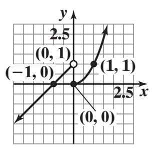
(d) $\{y|y \geq 1\}$; $[1, \infty)$

35. (a) $\{x|x \geq -2\}$; $[-2, \infty)$
(b) $(0, 3)$, $(2, 0)$
(c)



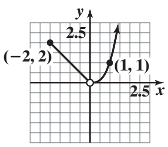
(d) $\{y|y < 4, y = 5\}$; $(-\infty, 4) \cup \{5\}$

37. (a) All real numbers
(b) $(-1, 0)$, $(0, 0)$
(c)



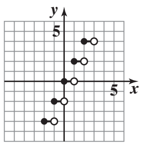
(d) All real numbers

39. (a) $\{x|x \geq -2, x \neq 0\}$; $[-2, 0) \cup (0, \infty)$
(b) No intercepts
(c)



(d) $\{y|y > 0\}$; $(0, \infty)$

41. (a) All real numbers
(b) $(x, 0)$ for $0 \leq x < 1$
(c)



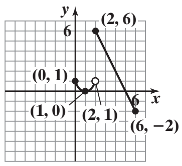
(d) Set of even integers

43. $f(x) = \begin{cases} -x & \text{if } -1 \leq x \leq 0 \\ \frac{1}{2}x & \text{if } 0 < x \leq 2 \end{cases}$ (Other answers are possible.)

45. $f(x) = \begin{cases} -x & \text{if } x \leq 0 \\ -x + 2 & \text{if } 0 < x \leq 2 \end{cases}$ (Other answers are possible.)

47. (a) 2 **(b)** 3 **(c)** −4

49. (a)



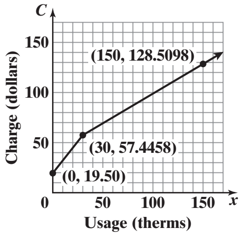
(b) $[0, 6]$
(c) Absolute maximum: $f(2) = 6$; absolute minimum: $f(6) = -2$
(d) Local maximum: $f(2) = 6$; local minimum: $f(1) = 0$

51. (a) \$34.99 **(b)** \$64.99 **(c)** \$184.99

53. (a) \$44.80 **(b)** \$128.51

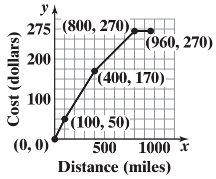
(c) $C(x) = \begin{cases} 1.26486x + 19.50 & \text{if } 0 \le x \le 30 \\ 0.5922x + 39.6798 & \text{if } x > 30 \end{cases}$

(d)



55. $f(x) = \begin{cases} 0.10x & \text{if} & 0 < x \le 9{,}225 \\ 922.50 + 0.15(x - 9{,}225) & \text{if} & 9{,}225 < x \le 37{,}450 \\ 5{,}156.25 + 0.25(x - 37{,}450) & \text{if} & 37{,}450 < x \le 90{,}750 \\ 18{,}481.25 + 0.28(x - 90{,}750) & \text{if} & 90{,}750 < x \le 189{,}300 \\ 46{,}075.25 + 0.33(x - 189{,}300) & \text{if} & 189{,}300 < x \le 411{,}500 \\ 119{,}401.25 + 0.35(x - 411{,}500) & \text{if} & 411{,}500 < x \le 413{,}200 \\ 119{,}996.25 + 0.396(x - 413{,}200) & \text{if} & x > 413{,}200 \end{cases}$

57. (a)

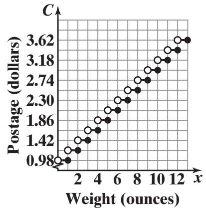


(b) $C(x) = 10 + 0.4x$ **(c)** $C(x) = 70 + 0.25x$

59. (a) $C(s) = \begin{cases} 9000 & \text{if} & s \le 659 \\ 7500 & \text{if} & 660 \le s \le 679 \\ 5250 & \text{if} & 680 \le s \le 699 \\ 3000 & \text{if} & 700 \le s \le 719 \\ 1500 & \text{if} & 720 \le s \le 739 \\ 750 & \text{if} & s \ge 740 \end{cases}$ **(b)** \$1500 **(c)** \$7500

61. (a) 10°C **(b)** 4°C **(c)** −3°C **(d)** −4°C
(e) The wind chill is equal to the air temperature.
(f) At wind speed greater than 20 m/sec, the wind chill factor depends only on the air temperature.

63. $C(x) = \begin{cases} 0.98 & \text{if} & 0 < x \le 1 \\ 1.20 & \text{if} & 1 < x \le 2 \\ 1.42 & \text{if} & 2 < x \le 3 \\ 1.64 & \text{if} & 3 < x \le 4 \\ 1.86 & \text{if} & 4 < x \le 5 \\ 2.08 & \text{if} & 5 < x \le 6 \\ 2.30 & \text{if} & 6 < x \le 7 \\ 2.52 & \text{if} & 7 < x \le 8 \\ 2.74 & \text{if} & 8 < x \le 9 \\ 2.96 & \text{if} & 9 < x \le 10 \\ 3.18 & \text{if} & 10 < x \le 11 \\ 3.40 & \text{if} & 11 < x \le 12 \\ 3.62 & \text{if} & 12 < x \le 13 \end{cases}$

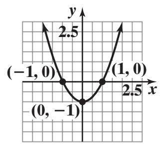


65. Each graph is that of $y = x^2$, but shifted horizontally. If $y = (x - k)^2$, $k > 0$, the shift is right k units; if $y = (x + k)^2$, $k > 0$, the shift is left k units. **67.** The graph of $y = -f(x)$ is the reflection about the x-axis of the graph of $y = f(x)$. **69.** Yes. The graph of $y = (x - 1)^3 + 2$ is the graph of $y = x^3$ shifted right 1 unit and up 2 units. **71.** They all have the same general shape. All three go through the points $(-1, -1)$, $(0, 0)$, and $(1, 1)$. As the exponent increases, the steepness of the curve increases (except near $x = 0$). **74.** $22 - 7i$ **75.** $(h, k) = (0, 3)$; $r = 5$
76. $\{-8\}$ **77.** CD: \$22,000; Mutual fund: \$38,000

3.5 Assess Your Understanding *(page 264)*

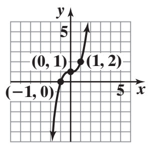
1. horizontal; right **2.** y **3.** F **4.** T **5.** d **6.** a **7.** B **9.** H **11.** I **13.** L **15.** F **17.** G **19.** $y = (x - 4)^3$ **21.** $y = x^3 + 4$ **23.** $y = -x^3$ **25.** $y = 4x^3$ **27.** $y = -(\sqrt{-x} + 2)$ **29.** $y = -\sqrt{x + 3} + 2$ **31.** c **33.** c **35. (a)** −7 and 1 **(b)** −3 and 5 **(c)** −5 and 3 **(d)** −3 and 5 **37. (a)** $[-3, 3]$ **(b)** $[4, 10]$ **(c)** Decreasing on $[-1, 5]$ **(d)** Decreasing on $[-5, 1]$

39.



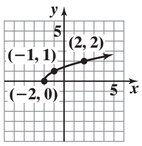
Domain: $(-\infty, \infty)$; Range: $[-1, \infty)$

41.



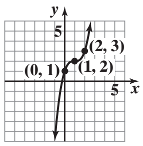
Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

43.



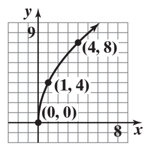
Domain: $[-2, \infty)$; Range: $[0, \infty)$

45.



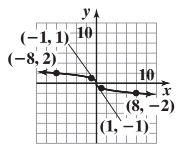
Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

47.



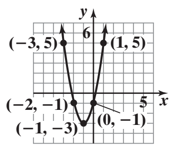
Domain: $[0, \infty)$; Range: $[0, \infty)$

49.



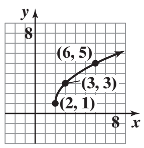
Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

51.



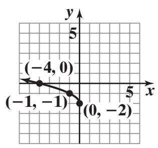
Domain: $(-\infty, \infty)$; Range: $[-3, \infty)$

53.



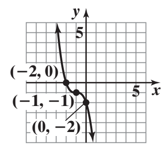
Domain: $[2, \infty)$; Range: $[1, \infty)$

55.



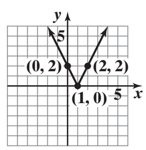
Domain: $(-\infty, 0]$; Range: $[-2, \infty)$

57.



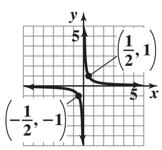
Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

59.



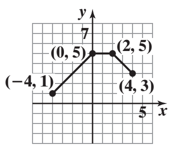
Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

61.

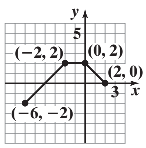


Domain: $(-\infty, 0) \cup (0, \infty)$ Range: $(-\infty, 0) \cup (0, \infty)$

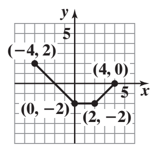
63. (a) $F(x) = f(x) + 3$



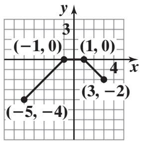
(b) $G(x) = f(x + 2)$



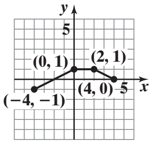
(c) $P(x) = -f(x)$



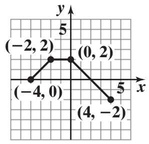
(d) $H(x) = f(x+1) - 2$



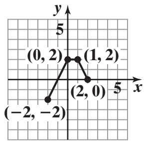
(e) $Q(x) = \frac{1}{2}f(x)$



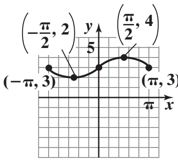
(f) $g(x) = f(-x)$



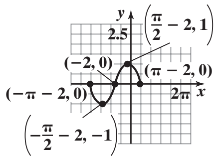
(g) $h(x) = f(2x)$



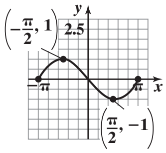
65. (a) $F(x) = f(x) + 3$



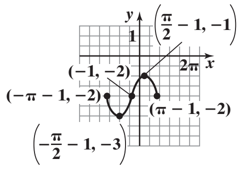
(b) $G(x) = f(x+2)$



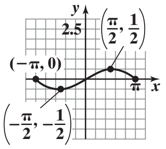
(c) $P(x) = -f(x)$



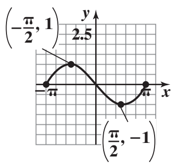
(d) $H(x) = f(x+1) - 2$



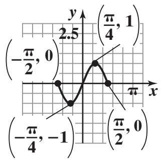
(e) $Q(x) = \frac{1}{2}f(x)$



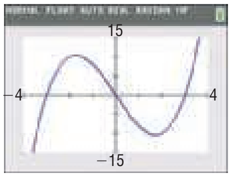
(f) $g(x) = f(-x)$



(g) $h(x) = f(2x)$

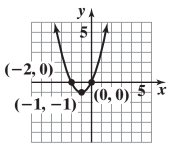


67. (a)

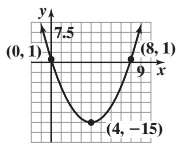


(b) $-3, 0, 3$ **(c)** Local maximum: 10.39 at $x = -1.73$; local minimum: -10.39 at $x = 1.73$ **(d)** Increasing: $[-4, -1.73]$, $[1.73, 4]$; decreasing: $[-1.73, 1.73]$ **(e)** Intercepts: $-5, -2, 1$; local maximum: 10.39 at $x = -3.73$; local minimum: -10.39 at $x = -0.27$; increasing: $[-6, -3.73]$, $[-0.27, 2]$; decreasing: $[-3.73, -0.27]$ **(f)** Intercepts: $-3, 0, 3$; Local maximum: 20.78 at $x = -1.73$; local minimum: -20.78 at $x = 1.73$; increasing: $[-4, -1.73]$, $[1.73, 4]$; decreasing: $[-1.73, 1.73]$ **(g)** Intercepts: $-3, 0, 3$; local maximum: 10.39 at $x = 1.73$; local minimum: -10.39 at $x = -1.73$; increasing: $[-1.73, 1.73]$; decreasing: $[-4, -1.73]$, $[1.73, 4]$

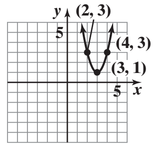
69. $f(x) = (x+1)^2 - 1$



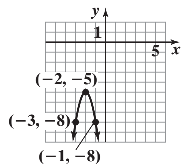
71. $f(x) = (x-4)^2 - 15$



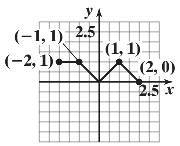
73. $f(x) = 2(x-3)^2 + 1$



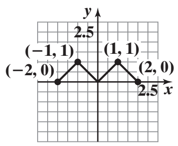
75. $f(x) = -3(x+2)^2 - 5$



77. (a)

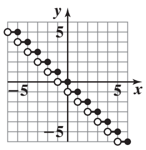


(b)

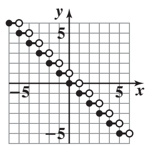


79. (a) $(-2, 2)$ **(b)** $(3, -5)$ **(c)** $(-1, 3)$

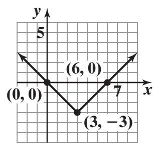
81. (a)



(b)

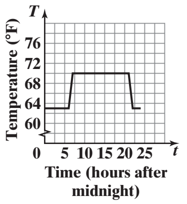


83. (a)

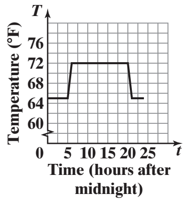


(b) 9 square units

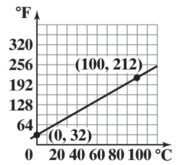
85. (a) 72°F; 65°F
(b) The temperature decreases by 2° to 70°F during the day and 63°F overnight.

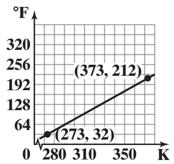


(c) The time at which the temperature adjusts between the daytime and overnight settings is moved to 1 hr sooner. It begins warming up at 5:00 AM instead of 6:00 AM, and it begins cooling down at 8:00 PM instead of 9:00 PM.

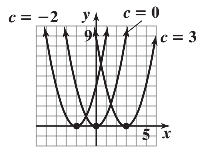


87.





89.

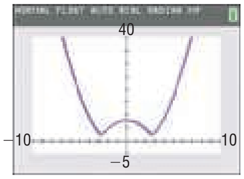


91. The graph of $y = 4f(x)$ is a vertical stretch by a factor of 4. The graph of $y = f(4x)$ is a horizontal compression by a factor of $\frac{1}{4}$.

93. $\frac{16}{3}$ sq. units **95.** The domain of $g(x) = \sqrt{x}$ is $[0, \infty)$. The graph of $g(x-k)$ is the graph of g shifted k units to the right, so the domain of $g(x-k)$ is $[k, \infty)$. **96.** $m = \frac{3}{5}$; $b = -6$ **97.** $\frac{y^2}{x^4}$ **98.** 15.75 gal **99.** Intercepts: $(0, -2)$, $(0, 2)$, $(-4, 0)$; x-axis symmetry

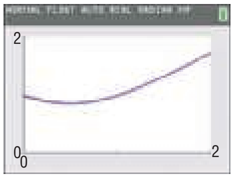
3.6 Assess Your Understanding *(page 270)*

1. (a) $d(x) = \sqrt{x^4 - 15x^2 + 64}$ **(b)** $d(0) = 8$ **(c)** $d(1) = \sqrt{50} \approx 7.07$ **(d)**



(e) d is smallest when $x \approx -2.74$ or $x \approx 2.74$

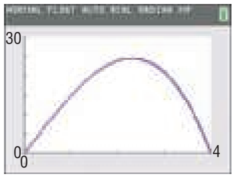
3. (a) $d(x) = \sqrt{x^2 - x + 1}$ **(b)**



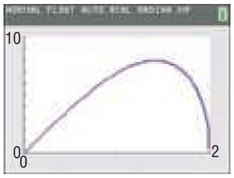
(c) d is smallest when $x = 0.5$.

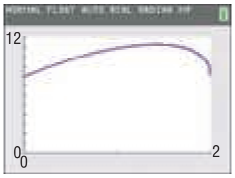
5. $A(x) = \dfrac{1}{2}x^4$

7. (a) $A(x) = x(16 - x^2)$ **(b)** Domain: $\{x \mid 0 < x < 4\}$ **(c)** The area is largest when $x \approx 2.31$.

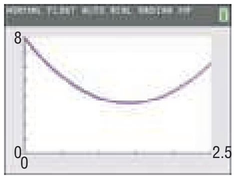


9. (a) $A(x) = 4x\sqrt{4 - x^2}$ **(b)** $p(x) = 4x + 4\sqrt{4 - x^2}$ **(c)** A is largest when $x \approx 1.41$. **(d)** p is largest when $x \approx 1.41$.



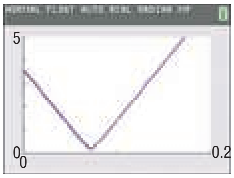


11. (a) $A(x) = x^2 + \dfrac{25 - 20x + 4x^2}{\pi}$ **(b)** Domain: $\{x \mid 0 < x < 2.5\}$ **(c)** A is smallest when $x \approx 1.40$ m.



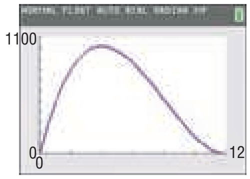
13. (a) $C(x) = x$ **(b)** $A(x) = \dfrac{x^2}{4\pi}$ **15. (a)** $A(r) = 2r^2$ **(b)** $p(r) = 6r$ **17.** $A(x) = \left(\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{4}\right)x^2$

19. (a) $d(t) = \sqrt{2500t^2 - 360t + 13}$ **(b)** d is smallest when $t \approx 0.07$ hr.



21. $V(r) = \dfrac{\pi H(R - r)r^2}{R}$ **23. (a)** $T(x) = \dfrac{12 - x}{5} + \dfrac{\sqrt{x^2 + 4}}{3}$ **(b)** $\{x \mid 0 \le x \le 12\}$ **(c)** 3.09 hr **(d)** 3.55 hr

25. (a) $V(x) = x(24 - 2x)^2$ **(b)** 972 in.^{3} **(c)** 160 in.^{3} **(d)** V is largest when $x = 4$.



27. $\{0, 3\}$ **28.** 64 mi/h **29.** $m = -4$ **30.** 5.6

Review Exercises *(page 275)*

1. Function; domain $\{-1, 2, 4\}$, range $\{0, 3\}$ **2.** Not a function **3. (a)** 2 **(b)** -2 **(c)** $-\dfrac{3x}{x^2 - 1}$ **(d)** $-\dfrac{3x}{x^2 - 1}$ **(e)** $\dfrac{3(x - 2)}{x^2 - 4x + 3}$ **(f)** $\dfrac{6x}{4x^2 - 1}$

4. (a) 0 **(b)** 0 **(c)** $\sqrt{x^2 - 4}$ **(d)** $-\sqrt{x^2 - 4}$ **(e)** $\sqrt{x^2 - 4x}$ **(f)** $2\sqrt{x^2 - 1}$ **5. (a)** 0 **(b)** 0 **(c)** $\dfrac{x^2 - 4}{x^2}$ **(d)** $-\dfrac{x^2 - 4}{x^2}$ **(e)** $\dfrac{x(x - 4)}{(x - 2)^2}$ **(f)** $\dfrac{x^2 - 1}{x^2}$

6. $\{x \mid x \ne -3, x \ne 3\}$ **7.** $\{x \mid x \le 2\}$ **8.** $\{x \mid x \ne 0\}$ **9.** $\{x \mid x \ne -3, x \ne 1\}$ **10.** $[-1, 2) \cup (2, \infty)$ **11.** $\{x \mid x > -8\}$

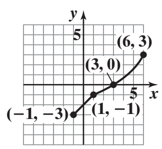
12. $(f + g)(x) = 2x + 3$; Domain: all real numbers
$(f - g)(x) = -4x + 1$; Domain: all real numbers
$(f \cdot g)(x) = -3x^2 + 5x + 2$; Domain: all real numbers
$\left(\dfrac{f}{g}\right)(x) = \dfrac{2 - x}{3x + 1}$; Domain: $\left\{x \,\middle|\, x \ne -\dfrac{1}{3}\right\}$

13. $(f + g)(x) = 3x^2 + 4x + 1$; Domain: all real numbers
$(f - g)(x) = 3x^2 - 2x + 1$; Domain: all real numbers
$(f \cdot g)(x) = 9x^3 + 3x^2 + 3x$; Domain: all real numbers
$\left(\dfrac{f}{g}\right)(x) = \dfrac{3x^2 + x + 1}{3x}$; Domain: $\{x \mid x \ne 0\}$

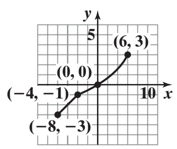
14. $(f + g)(x) = \dfrac{x^2 + 2x - 1}{x(x - 1)}$; Domain: $\{x \mid x \ne 0, x \ne 1\}$
$(f - g)(x) = \dfrac{x^2 + 1}{x(x - 1)}$; Domain: $\{x \mid x \ne 0, x \ne 1\}$
$(f \cdot g)(x) = \dfrac{x + 1}{x(x - 1)}$; Domain: $\{x \mid x \ne 0, x \ne 1\}$
$\left(\dfrac{f}{g}\right)(x) = \dfrac{x(x + 1)}{x - 1}$; Domain: $\{x \mid x \ne 0, x \ne 1\}$

15. $-4x + 1 - 2h$ **16. (a)** Domain: $\{x \mid -4 \le x \le 3\}$; Range: $\{y \mid -3 \le y \le 3\}$ **(b)** $(0, 0)$ **(c)** -1 **(d)** -4 **(e)** $\{x \mid 0 < x \le 3\}$

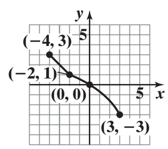
(f)



(g)



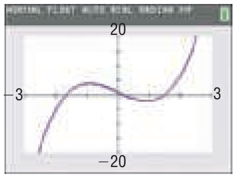
(h)



17. (a) Domain: $\{x \mid x \le 4\}$ or $(-\infty, 4]$
Range: $\{y \mid y \le 3\}$ or $(-\infty, 3]$
(b) Increasing on $(-\infty, -2]$ and $[2, 4]$; Decreasing on $[-2, 2]$
(c) Local maximum value is 1 and occurs at $x = -2$.
Local minimum value is -1 and occurs at $x = 2$.
(d) Absolute maximum: $f(4) = 3$
Absolute minimum: none
(e) No symmetry
(f) Neither
(g) x-intercepts: $-3, 0, 3$ y-intercept: 0

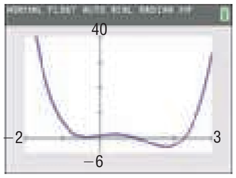
18. Odd **19.** Even **20.** Neither **21.** Odd

22.



Local maximum value: 4.04 at $x = -0.91$
Local minimum value: -2.04 at $x = 0.91$
Increasing: $[-3, -0.91]$; $[0.91, 3]$
Decreasing: $[-0.91, 0.91]$

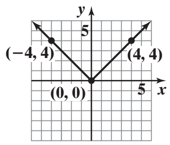
23.



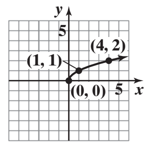
Local maximum value: 1.53 at $x = 0.41$
Local minima values: 0.54 at $x = -0.34$ and -3.56 at $x = 1.80$
Increasing: $[-0.34, 0.41]$; $[1.80, 3]$
Decreasing: $[-2, -0.34]$; $[0.41, 1.80]$

24. **(a)** 23 **(b)** 7 **(c)** 47 **25.** -5 **26.** -17 **27.** No **28.** Yes

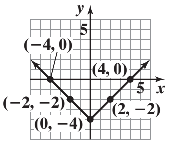
29.



30.

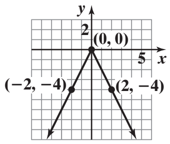


31.



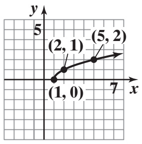
Intercepts: $(-4, 0)$, $(4, 0)$, $(0, -4)$
Domain: all real numbers
Range: $\{y|y \geq -4\}$ or $[-4, \infty)$

32.



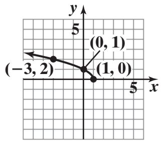
Intercept: $(0, 0)$
Domain: all real numbers
Range: $\{y|y \leq 0\}$ or $(-\infty, 0]$

33.



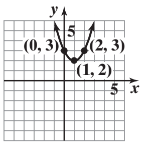
Intercept: $(1, 0)$
Domain: $\{x|x \geq 1\}$ or $[1, \infty)$
Range: $\{y|y \geq 0\}$ or $[0, \infty)$

34.



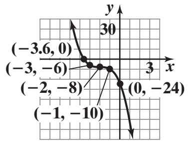
Intercepts: $(0, 1)$, $(1, 0)$
Domain: $\{x|x \leq 1\}$ or $(-\infty, 1]$
Range: $\{y|y \geq 0\}$ or $[0, \infty)$

35.



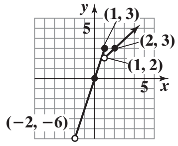
Intercept: $(0, 3)$
Domain: all real numbers
Range: $\{y|y \geq 2\}$ or $[2, \infty)$

36.



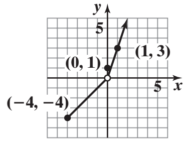
Intercepts: $(0, -24)$, $(-2 - \sqrt[3]{4}, 0)$ or about $(-3.6, 0)$
Domain: all real numbers
Range: all real numbers

37. **(a)** $\{x|x > -2\}$ or $(-2, \infty)$
(b) $(0, 0)$
(c)



(d) $\{y|y > -6\}$ or $(-6, \infty)$

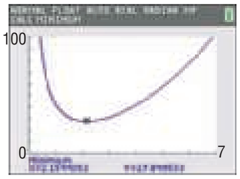
38. **(a)** $\{x|x \geq -4\}$ or $[-4, \infty)$
(b) $(0, 1)$
(c)



(d) $\{y|-4 \leq y < 0 \text{ or } y > 0\}$ or $[-4, 0) \cup (0, \infty)$

39. $A = 11$

40. **(a)** $A(x) = 2x^2 + \dfrac{40}{x}$
(b) 42 ft^2
(c) 28 ft^2
(d)



A is smallest when $x \approx 2.15$ ft.

41. **(a)** $A(x) = 10x - x^3$
(b) The largest area that can be enclosed by the rectangle is approximately 12.17 square units.

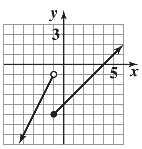
Chapter Test *(page 277)*

1. **(a)** Function; Domain: $\{2, 4, 6, 8\}$; Range: $\{5, 6, 7, 8\}$ **(b)** Not a function **(c)** Not a function **(d)** Function; Domain: all real numbers; Range: $\{y|y \geq 2\}$

2. Domain: $\left\{x \middle| x \leq \dfrac{4}{5}\right\}$; $f(-1) = 3$ **3.** Domain: $\{x|x \neq -2\}$; $g(-1) = 1$ **4.** Domain: $\{x|x \neq -9, x \neq 4\}$; $h(-1) = \dfrac{1}{8}$

5. **(a)** Domain: $\{x|-5 \leq x \leq 5\}$; Range: $\{y|-3 \leq y \leq 3\}$ **(b)** $(0, 2)$, $(-2, 0)$, and $(2, 0)$ **(c)** $f(1) = 3$ **(d)** $x = -5$ and $x = 3$ **(e)** $\{x|-5 \leq x < -2 \text{ or } 2 < x \leq 5\}$ or $[-5, -2) \cup (2, 5]$ **6.** Local maxima values: $f(-0.85) \approx -0.86$; $f(2.35) \approx 15.55$; local minimum value: $f(0) = -2$; the function is increasing on the intervals $[-5, -0.85]$ and $[0, 2.35]$ and decreasing on the intervals $[-0.85, 0]$ and $[2.35, 5]$.

7. **(a)**

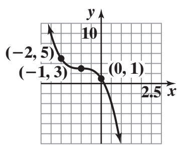


(b) $(0, -4)$, $(4, 0)$
(c) $g(-5) = -9$
(d) $g(2) = -2$

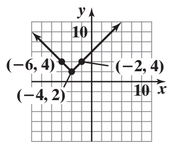
8. 19

9. **(a)** $(f - g)(x) = 2x^2 - 3x + 3$
(b) $(f \cdot g)(x) = 6x^3 - 4x^2 + 3x - 2$
(c) $f(x + h) - f(x) = 4xh + 2h^2$

10. **(a)**



(b)



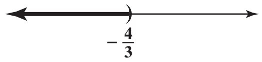
11. **(a)** 8.67% occurring in 1997 $(x \approx 5)$ **(b)** The model predicts that the interest rate will be -10.343%. This is not reasonable.

12. **(a)** $V(x) = \dfrac{x^2}{8} - \dfrac{5x}{4} + \dfrac{\pi x^2}{64}$ **(b)** 1297.61 ft^3

Cumulative Review *(page 278)*

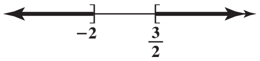
1. $\{6\}$ **2.** $\left\{0, \frac{1}{3}\right\}$ **3.** $\{-1, 9\}$ **4.** $\left\{\frac{1}{3}, \frac{1}{2}\right\}$ **5.** $\left\{-\frac{7}{2}, \frac{1}{2}\right\}$ **6.** $\left\{\frac{1}{2}\right\}$

7. $\left\{x \middle| x < -\frac{4}{3}\right\}; \left(-\infty, -\frac{4}{3}\right)$



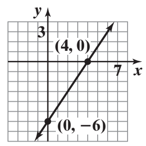
8. $\{x \mid 1 < x < 4\}$; $(1, 4)$

9. $\left\{x \middle| x \le -2 \text{ or } x \ge \frac{3}{2}\right\}; (-\infty, -2] \cup \left[\frac{3}{2}, \infty\right)$

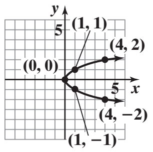


10. (a) distance: $\sqrt{29}$ **(b)** midpoint: $\left(\frac{1}{2}, -4\right)$ **(c)** slope: $-\frac{2}{5}$

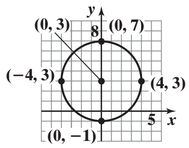
11.



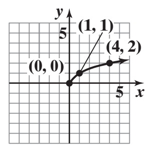
12.



13.



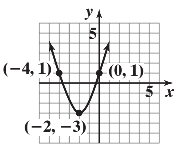
14.



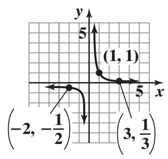
15. Intercepts: $(0, -3)$, $(-2, 0)$, $(2, 0)$; symmetry with respect to the y-axis

16. $y = \frac{1}{2}x + 5$

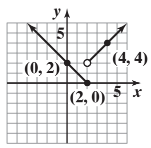
17.



18.



19.

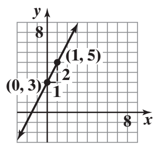


CHAPTER 4 Linear and Quadratic Functions

4.1 Assess Your Understanding *(page 287)*

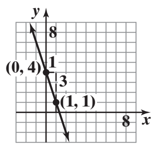
7. slope; y-intercept **8.** positive **9.** T **10.** F **11.** a **12.** d

13. (a) $m = 2; b = 3$
(b)



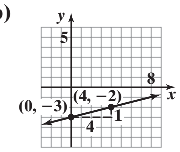
(c) 2 **(d)** Increasing

15. (a) $m = -3; b = 4$
(b)



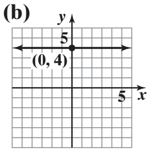
(c) -3 **(d)** Decreasing

17. (a) $m = \frac{1}{4}; b = -3$
(b)



(c) $\frac{1}{4}$ **(d)** Increasing

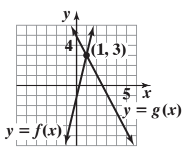
19. (a) $m = 0; b = 4$
(b)



(c) 0 **(d)** Constant

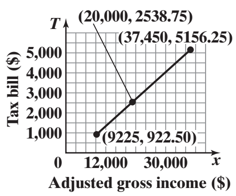
21. Linear; $f(x) = -3x - 2$
23. Nonlinear
25. Nonlinear
27. Linear; $f(x) = 8$

29. (a) $\frac{1}{4}$ **(b)** $\left\{x \middle| x > \frac{1}{4}\right\}$ or $\left(\frac{1}{4}, \infty\right)$
(c) 1 **(d)** $\{x \mid x \le 1\}$ or $(-\infty, 1]$
(e)



31. (a) 40 **(b)** 88 **(c)** -40 **(d)** $\{x \mid x > 40\}$ or $(40, \infty)$ **(e)** $\{x \mid x \le 88\}$ or $(-\infty, 88]$ **(f)** $\{x \mid -40 < x < 88\}$ or $(-40, 88)$ **33. (a)** -4 **(b)** $\{x \mid x < -4\}$ or $(-\infty, -4)$ **35. (a)** -6 **(b)** $\{x \mid -6 \le x < 5\}$ or $[-6, 5)$ **37. (a)** \$59 **(b)** 180 mi **(c)** 300 mi **(d)** $\{x \mid x \ge 0\}$ or $[0, \infty)$ **(e)** The cost of renting the car for a day increases \$0.35 for each mile driven, or there is a charge of \$0.35 per mile to rent the car in addition to a fixed charge of \$45. **(f)** It costs \$45 to rent the car if 0 miles are driven, or there is a fixed charge of \$45 to rent the car in addition to a charge that depends on mileage. **39. (a)** \$24; 600 T-shirts **(b)** $\$0 \le p < \$24$ **(c)** The price will increase.

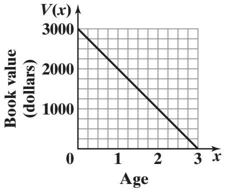
41. (a) $\{x \mid 9225 \le x \le 37{,}450\}$ or $[9225, 37{,}450]$
(b) \$2538.75
(c) The independent variable is adjusted gross income, x. The dependent variable is the tax bill, T.
(d)



(e) \$27,500
(f) For each additional dollar of taxable income between \$9225 and \$37,450, the tax bill of a single person in 2015 increased by \$0.15.

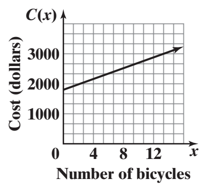
43. (a) $x = 5000$
(b) $x > 5000$

45. (a) $V(x) = -1000x + 3000$
(b) $\{x \mid 0 \le x \le 3\}$ or $[0, 3]$
(c)



(d) \$1000
(e) After 1 year

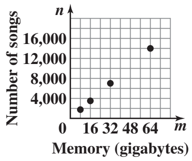
47. (a) $C(x) = 90x + 1800$
(b)



(c) \$3060 **(d)** 22 bicycles

49. (a) $C(x) = 0.89x + 39.95$ **(b)** \$137.85; \$244.65

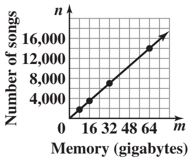
51. (a)



(b) Since each input (memory) corresponds to a single output (number of songs), we know that number of songs is a function of memory. Also, because the average rate of change is a constant 218.75 songs per gigabyte, the function is linear.

(c) $n(m) = 218.75\,m$ **(d)** $\{m \mid m \geq 0\}$ or $[0, \infty)$

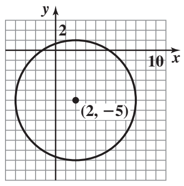
(e)



(f) If memory increases by 1 GB, then the number of songs increases by 218.75.

53. (d), (e) **55.** $b = 0$; yes, $f(x) = b$

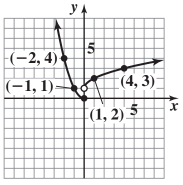
57.



58. 6

59. 7

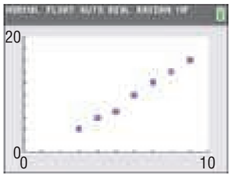
60.



4.2 Assess Your Understanding *(page 294)*

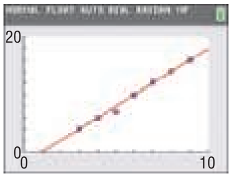
3. scatter diagram **4.** decrease; 0.008 **5.** Linear relation, $m > 0$ **7.** Linear relation, $m < 0$ **9.** Nonlinear relation

11. (a)



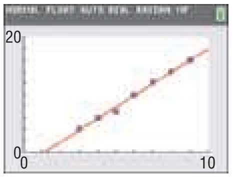
(b) Answers will vary. Using (4, 6) and (8, 14), $y = 2x - 2$.

(c)

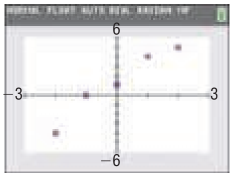


(d) $y = 2.0357x - 2.3571$

(e)

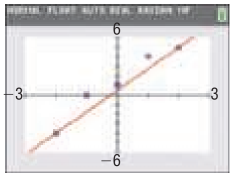


13. (a)



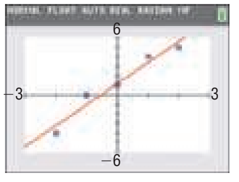
(b) Answers will vary. Using $(-2, -4)$ and $(2, 5)$, $y = \frac{9}{4}x + \frac{1}{2}$.

(c)

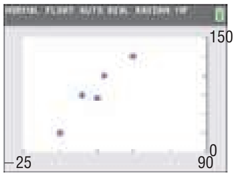


(d) $y = 2.2x + 1.2$

(e)

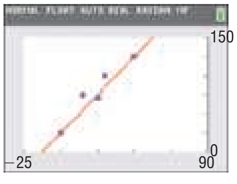


15. (a)



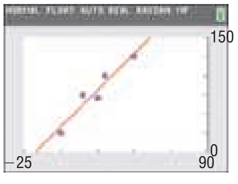
(b) Answers will vary. Using $(-20, 100)$ and $(-10, 140)$, $y = 4x + 180$.

(c)

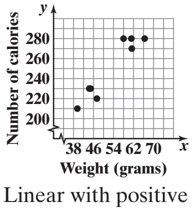


(d) $y = 3.8613x + 180.2920$

(e)



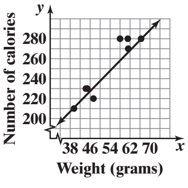
17. (a)



(b) Linear with positive slope

(c) Answers will vary. Using the points (39.52, 210) and (66.45, 280), $y = 2.599x + 107.288$.

(d)

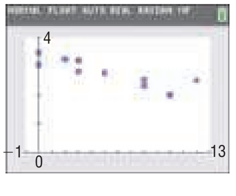


(e) 269 calories

(f) If the weight of a candy bar is increased by 1 gram, the number of calories will increase by 2.599, on average.

19. (a) The independent variable is the number of hours spent playing video games, and cumulative grade-point average is the dependent variable, because we are using number of hours playing video games to predict (or explain) cumulative grade-point average.

(b)



(c) $G(h) = -0.0942h + 3.2763$

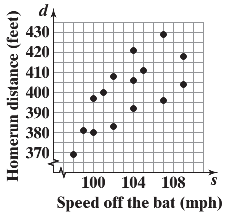
(d) If the number of hours playing video games in a week increases by 1 hour, the cumulative grade-point average decreases 0.09, on average.

(e) 2.52

(f) Approximately 9.3 hours

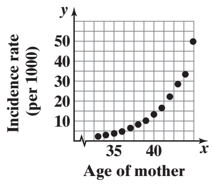
21. (a) No

(b)



(c) $d = 3.3641s + 51.8233$
(d) If the speed off the bat increases by 1 mile per hour, the homerun distance increases by 3.3641 feet, on average.
(e) $d(s) = 3.3641s + 51.8233$
(f) $\{s \mid s > 0\}$ or $(0, \infty)$
(g) Approximately 398 feet

23.



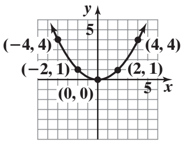
No, the data do not follow a linear pattern.

25. No linear relation **27.** 34.8 hours; A student whose GPA is 0 spends 34.8 hours each week playing video games. $G(0) = 3.28$. The average GPA of a student who does not play video games is 3.28. **28.** $2x + y = 3$ or $y = -2x + 3$ **29.** $\{x \mid x \neq -5, x \neq 5\}$ **30.** $(g - f)(x) = x^2 - 8x + 12$
31. $y = (x + 3)^2 - 4$

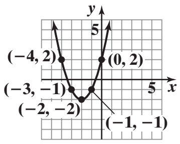
4.3 Assess Your Understanding *(page 306)*

5. parabola **6.** axis or axis of symmetry **7.** $-\dfrac{b}{2a}$ **8.** T **9.** T **10.** T **11.** a **12.** d **13.** C **15.** F **17.** G **19.** H

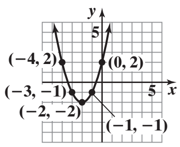
21.



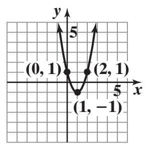
23.



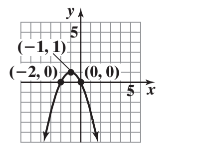
25. $f(x) = (x + 2)^2 - 2$



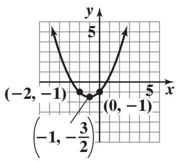
27. $f(x) = 2(x - 1)^2 - 1$



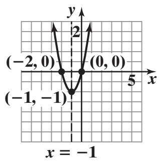
29. $f(x) = -(x + 1)^2 + 1$



31. $f(x) = \dfrac{1}{2}(x + 1)^2 - \dfrac{3}{2}$

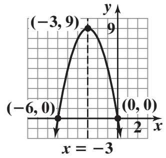


33. (a)



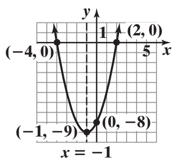
(b) Domain: $(-\infty, \infty)$
Range: $[-1, \infty)$
(c) Decreasing: $(-\infty, -1]$
Increasing: $[-1, \infty)$

35. (a)



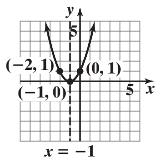
(b) Domain: $(-\infty, \infty)$
Range: $(-\infty, 9]$
(c) Increasing: $(-\infty, -3]$
Decreasing: $[-3, \infty)$

37. (a)



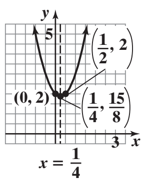
(b) Domain: $(-\infty, \infty)$
Range: $[-9, \infty)$
(c) Decreasing: $(-\infty, -1]$
Increasing: $[-1, \infty)$

39. (a)



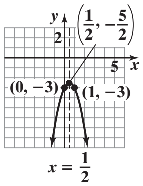
(b) Domain: $(-\infty, \infty)$
Range: $[0, \infty)$
(c) Decreasing: $(-\infty, -1]$
Increasing: $[-1, \infty)$

41. (a)



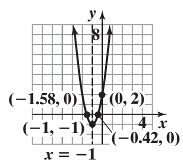
(b) Domain: $(-\infty, \infty)$
Range: $\left[\dfrac{15}{8}, \infty\right)$
(c) Decreasing: $\left(-\infty, \dfrac{1}{4}\right]$
Increasing: $\left[\dfrac{1}{4}, \infty\right)$

43. (a)



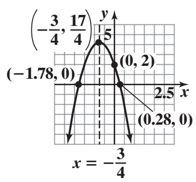
(b) Domain: $(-\infty, \infty)$
Range: $\left(-\infty, -\dfrac{5}{2}\right]$
(c) Increasing: $\left(-\infty, \dfrac{1}{2}\right]$
Decreasing: $\left[\dfrac{1}{2}, \infty\right)$

45. (a)



(b) Domain: $(-\infty, \infty)$
Range: $[-1, \infty)$
(c) Decreasing: $(-\infty, -1]$
Increasing: $[-1, \infty)$

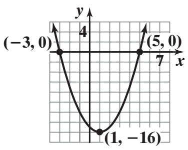
47. (a)



(b) Domain: $(-\infty, \infty)$
Range: $\left(-\infty, \dfrac{17}{4}\right]$
(c) Increasing: $\left(-\infty, -\dfrac{3}{4}\right]$
Decreasing: $\left[-\dfrac{3}{4}, \infty\right)$

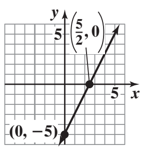
49. $f(x) = (x + 1)^2 - 2 = x^2 + 2x - 1$ **51.** $f(x) = -(x + 3)^2 + 5 = -x^2 - 6x - 4$ **53.** $f(x) = 2(x - 1)^2 - 3 = 2x^2 - 4x - 1$
55. Minimum value; -18 **57.** Minimum value; -21 **59.** Maximum value; 21 **61.** Maximum value; 13

63. (a)



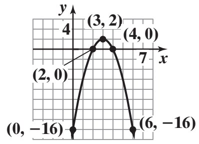
(b) Domain: $(-\infty, \infty)$
Range: $[-16, \infty)$
(c) Decreasing: $(-\infty, 1]$
Increasing: $[1, \infty)$

65. (a)



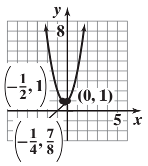
(b) Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$
(c) Increasing: $(-\infty, \infty)$

67. (a)



(b) Domain: $(-\infty, \infty)$
Range: $(-\infty, 2]$
(c) Increasing: $(-\infty, 3]$
Decreasing: $[3, \infty)$

69. (a)



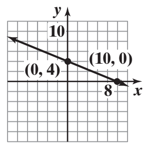
(b) Domain: $(-\infty, \infty)$

Range: $\left[\frac{7}{8}, \infty\right)$

(c) Decreasing: $\left(-\infty, -\frac{1}{4}\right]$

Increasing: $\left[-\frac{1}{4}, \infty\right)$

71. (a)

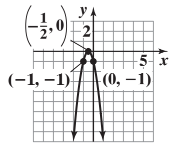


(b) Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

(c) Decreasing: $(-\infty, \infty)$

73. (a)



(b) Domain: $(-\infty, \infty)$

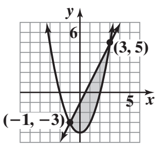
Range: $(-\infty, 0]$

(c) Increasing: $\left(-\infty, -\frac{1}{2}\right]$

Decreasing: $\left[-\frac{1}{2}, \infty\right)$

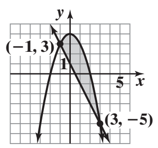
75. $a = 6, b = 0, c = 2$

77. (a), (c), (d)



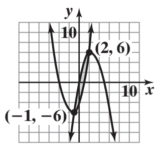
(b) $\{-1, 3\}$

79. (a), (c), (d)



(b) $\{-1, 3\}$

81. (a), (c), (d)



(b) $\{-1, 2\}$

83. (a) $a = 1: f(x) = (x+3)(x-1) = x^2 + 2x - 3$
$a = 2: f(x) = 2(x+3)(x-1) = 2x^2 + 4x - 6$
$a = -2: f(x) = -2(x+3)(x-1) = -2x^2 - 4x + 6$
$a = 5: f(x) = 5(x+3)(x-1) = 5x^2 + 10x - 15$

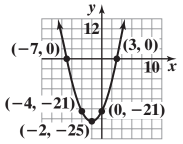
(b) The value of a does not affect the x-intercepts, but it changes the y-intercept by a factor of a.

(c) The value of a does not affect the axis of symmetry. It is $x = -1$ for all values of a.

(d) The value of a does not affect the x-coordinate of the vertex. However, the y-coordinate of the vertex is multiplied by a.

(e) The mean of the x-intercepts is the x-coordinate of the vertex.

85. (a) $(-2, -25)$
(b) $-7, 3$
(c) $-4, 0; (-4, -21), (0, -21)$
(d)

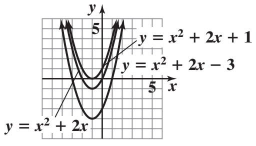


87. $(2, 2)$ **89.** \$500; \$1,000,000 **91. (a)** 70,000 digital music players **(b)** \$2500

93. (a) 187 or 188 watches; \$7031.20 **(b)** $P(x) = -0.2x^2 + 43x - 1750$
(c) 107 or 108 watches; \$561.20 **95. (a)** 171 ft **(b)** 49 mph **(c)** Reaction time

97. If x is even, then ax^2 and bx are even and $ax^2 + bx$ is even, which means that $ax^2 + bx + c$ is odd. If x is odd, then ax^2 and bx are odd and $ax^2 + bx$ is even, which means that $ax^2 + bx + c$ is odd. In either case, $f(x)$ is odd.

99.



101. $b^2 - 4ac < 0$ **103.** No

105. Symmetric with respect to the x-axis, the y-axis, and the origin.

106. $\{x \mid x \le 4\}$ or $(-\infty, 4]$ **107.** Center $(5, -2)$; radius $= 3$

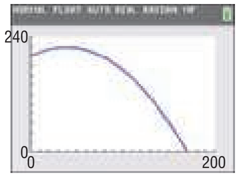
108. $y = \sqrt{-x}$

4.4 Assess Your Understanding *(page 315)*

3. (a) $R(x) = -\frac{1}{6}x^2 + 100x$ **(b)** $\{x \mid 0 \le x \le 600\}$ **(c)** \$13,333.33 **(d)** 300; \$15,000 **(e)** \$50 **5. (a)** $R(x) = -\frac{1}{5}x^2 + 20x$ **(b)** \$255 **(c)** 50; \$500 **(d)** \$10 **(e)** Between \$8 and \$12 **7. (a)** $A(w) = -w^2 + 200w$ **(b)** A is largest when $w = 100$ yd. **(c)** 10,000 yd^2 **9.** 2,000,000 m^2

11. (a) $\frac{625}{16} \approx 39$ ft **(b)** $\frac{7025}{32} \approx 219.5$ ft **(c)** About 170 ft

(d)



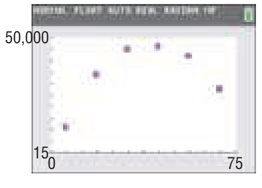
(f) When the height is 100 ft, the projectile is about 135.7 ft from the cliff.

13. 18.75 m **15. (a)** 3 in. **(b)** Between 2 in. and 4 in.

17. $\frac{750}{\pi} \approx 238.73$ m by 375 m

19. $x = \frac{a}{2}$ **21.** $\frac{38}{3}$ **23.** $\frac{248}{3}$

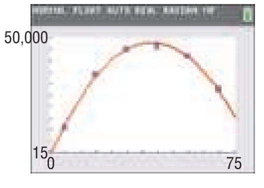
25. (a)



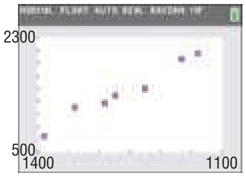
The data appear to follow a quadratic relation with $a < 0$.

(b) $I(x) = -44.759x^2 + 4295.356x - 55{,}045.418$
(c) About 48.0 years of age
(d) Approximately \$48,007

(e)

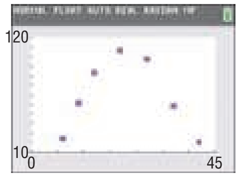


27. (a)



The data appear to be linearly related with positive slope.
(b) $R(x) = 1.229x + 917.385$ **(c)** \$1993

29. (a)



The data appear to follow a quadratic relation with $a < 0$.
(b) $B(a) = -0.547a^2 + 31.190a - 342.218$ **(c)** 79.357

32. $15i$ **33.** 13 **34.** $(x + 6)^2 + y^2 = 7$ **35.** $\left\{\dfrac{-4 - \sqrt{31}}{5}, \dfrac{-4 + \sqrt{31}}{5}\right\}$

4.5 Assess Your Understanding *(page 322)*

3. (a) $\{x \mid x < -2 \text{ or } x > 2\}$; $(-\infty, -2) \cup (2, \infty)$ **(b)** $\{x \mid -2 \le x \le 2\}$; $[-2, 2]$

5. (a) $\{x \mid -2 \le x \le 1\}$; $[-2, 1]$ **(b)** $\{x \mid x < -2 \text{ or } x > 1\}$; $(-\infty, -2) \cup (1, \infty)$ **7.** $\{x \mid -2 < x < 5\}$; $(-2, 5)$

9. $\{x \mid x < 0 \text{ or } x > 4\}$; $(-\infty, 0) \cup (4, \infty)$ **11.** $\{x \mid -3 < x < 3\}$; $(-3, 3)$ **13.** $\{x \mid x < -4 \text{ or } x > 3\}$; $(-\infty, -4) \cup (3, \infty)$

15. $\left\{x \,\middle|\, -\dfrac{1}{2} < x < 3\right\}$; $\left(-\dfrac{1}{2}, 3\right)$ **17.** No real solution **19.** No real solution **21.** $\left\{x \,\middle|\, x < -\dfrac{2}{3} \text{ or } x > \dfrac{3}{2}\right\}$; $\left(-\infty, -\dfrac{2}{3}\right) \cup \left(\dfrac{3}{2}, \infty\right)$

23. $\{x \mid x \le -4 \text{ or } x \ge 4\}$; $(-\infty, -4] \cup [4, \infty)$

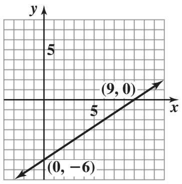
25. (a) $\{-1, 1\}$ **(b)** $\{-1\}$ **(c)** $\{-1, 4\}$ **(d)** $\{x \mid x < -1 \text{ or } x > 1\}$; $(-\infty, -1) \cup (1, \infty)$ **(e)** $\{x \mid x \le -1\}$; $(-\infty, -1]$
(f) $\{x \mid x < -1 \text{ or } x > 4\}$; $(-\infty, -1) \cup (4, \infty)$ **(g)** $\{x \mid x \le -\sqrt{2} \text{ or } x \ge \sqrt{2}\}$; $(-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)$

27. (a) $\{-1, 1\}$ **(b)** $\left\{-\dfrac{1}{4}\right\}$ **(c)** $\{-4, 0\}$ **(d)** $\{x \mid -1 < x < 1\}$; $(-1, 1)$ **(e)** $\left\{x \,\middle|\, x \le -\dfrac{1}{4}\right\}$; $\left(-\infty, -\dfrac{1}{4}\right]$ **(f)** $\{x \mid -4 < x < 0\}$; $(-4, 0)$ **(g)** $\{0\}$

29. (a) $\{-2, 2\}$ **(b)** $\{-2, 2\}$ **(c)** $\{-2, 2\}$ **(d)** $\{x \mid x < -2 \text{ or } x > 2\}$; $(-\infty, -2) \cup (2, \infty)$ **(e)** $\{x \mid x \le -2 \text{ or } x \ge 2\}$; $(-\infty, -2] \cup [2, \infty)$
(f) $\{x \mid x < -2 \text{ or } x > 2\}$; $(-\infty, -2) \cup (2, \infty)$ **(g)** $\{x \mid x \le -\sqrt{5} \text{ or } x \ge \sqrt{5}\}$; $(-\infty, -\sqrt{5}] \cup [\sqrt{5}, \infty)$

31. (a) $\{-1, 2\}$ **(b)** $\{-2, 1\}$ **(c)** $\{0\}$ **(d)** $\{x \mid x < -1 \text{ or } x > 2\}$; $(-\infty, -1) \cup (2, \infty)$ **(e)** $\{x \mid -2 \le x \le 1\}$; $[-2, 1]$ **(f)** $\{x \mid x < 0\}$; $(-\infty, 0)$
(g) $\left\{x \,\middle|\, x \le \dfrac{1 - \sqrt{13}}{2} \text{ or } x \ge \dfrac{1 + \sqrt{13}}{2}\right\}$; $\left(-\infty, \dfrac{1 - \sqrt{13}}{2}\right] \cup \left[\dfrac{1 + \sqrt{13}}{2}, \infty\right)$

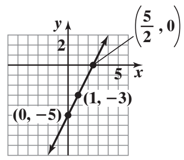
33. (a) 5 sec **(b)** The ball is more than 96 ft above the ground for time t between 2 and 3 sec, $2 < t < 3$.
35. (a) \$0, \$1000 **(b)** The revenue is more than \$800,000 for prices between \$276.39 and \$723.61, $\$276.39 < p < \$723.61$.
37. (a) $\{c \mid 0.112 < c < 81.907\}$; $(0.112, 81.907)$ **(b)** It is possible to hit a target 75 km away if $c = 0.651$ or $c = 1.536$. **44.** $\{x \mid x \le 5\}$
45. (a) $(9, 0)$, $(0, -6)$
(b)



46. Odd
47. $19 - 26i$

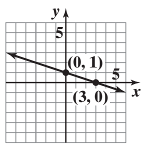
Review Exercises *(page 325)*

1. (a) $m = 2$; $b = -5$ **(b)** 2
(c)



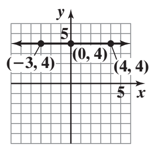
(d) Increasing

2. (a) $m = -\dfrac{1}{3}$; $b = 1$ **(b)** $-\dfrac{1}{3}$
(c)



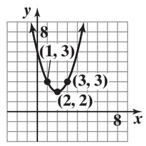
(d) Decreasing

3. (a) $m = 0$; $b = 4$ **(b)** 0
(c)

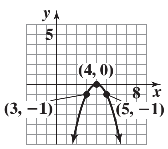


(d) Constant
4. Linear; $f(x) = 5x + 3$
5. Nonlinear

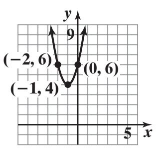
6.



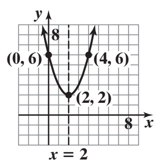
7.



8.

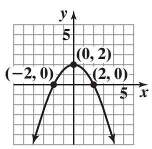


9. (a)



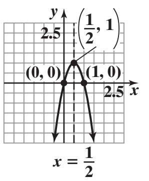
(b) Domain: $(-\infty, \infty)$
Range: $[2, \infty)$
(c) Decreasing: $(-\infty, 2]$
Increasing: $[2, \infty)$

10. (a)



(b) Domain: $(-\infty, \infty)$
Range: $(-\infty, 2]$
(c) Increasing: $(-\infty, 0]$
Decreasing: $[0, \infty)$

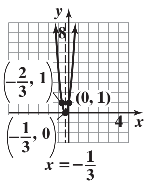
11. (a)



(b) Domain: $(-\infty, \infty)$
Range: $(-\infty, 1]$

(c) Increasing: $\left(-\infty, \frac{1}{2}\right]$
Decreasing: $\left[\frac{1}{2}, \infty\right)$

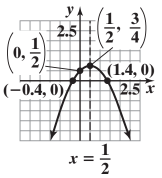
12. (a)



(b) Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

(c) Decreasing: $\left(-\infty, -\frac{1}{3}\right]$
Increasing: $\left[-\frac{1}{3}, \infty\right)$

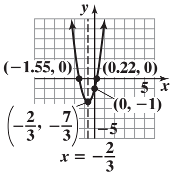
13. (a)



(b) Domain: $(-\infty, \infty)$
Range: $\left(-\infty, \frac{3}{4}\right]$

(c) Increasing: $\left(-\infty, \frac{1}{2}\right]$
Decreasing: $\left[\frac{1}{2}, \infty\right)$

14. (a)



(b) Domain: $(-\infty, \infty)$
Range: $\left[-\frac{7}{3}, \infty\right)$

(c) Decreasing: $\left(-\infty, -\frac{2}{3}\right]$
Increasing: $\left[-\frac{2}{3}, \infty\right)$

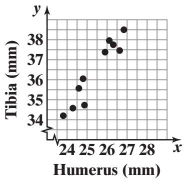
15. Minimum value; 1 **16.** Maximum value; 12 **17.** Maximum value; 4 **18.** $\{x \mid -8 < x < 2\}$; $(-8, -2)$

19. $\left\{x \middle| x \le -\frac{1}{3} \text{ or } x \ge 5\right\}$; $\left(-\infty, -\frac{1}{3}\right] \cup [5, \infty)$ **20.** $y = x^2 + 2x + 3$ **21.** $y = x^2 - 4x + 5$

22. (a) $S(x) = 0.01x + 25{,}000$ **(b)** \$35,000 **(c)** \$7,500,000 **(d)** $x > \$12{,}500{,}000$ **23. (a)** $R(x) = -\frac{1}{10}x^2 + 150x$ **(b)** \$14,000 **(c)** 750; \$56,250 **(d)** \$75

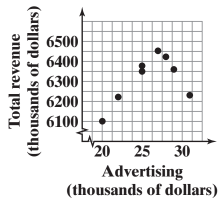
24. 4,166,666.7 m^2 **25. (a)** 63 clubs **(b)** \$151.90 **26. (a)** $A(x) = -x^2 + 10x$; 25 units^2 **27.** 3.6 ft

28. (a)



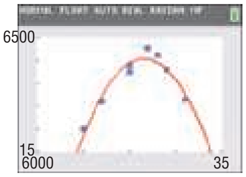
(b) Yes
(c) $y = 1.3092x + 1.1140$
(d) 37.95 mm

29. (a) Quadratic, $a < 0$



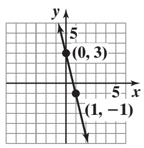
(b) About \$26.5 thousand
(c) \$6408 thousand

(e)



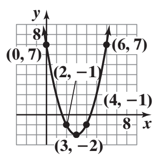
Chapter Test *(page 327)*

1. (a) Slope: −4; y-intercept: 3
(b) Decreasing
(c)



2. Linear; $y = -5x + 2$

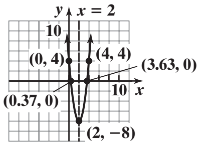
3.



4. (a) Opens up **(b)** $(2, -8)$ **(c)** $x = 2$

(d) x-intercepts: $\frac{6 - 2\sqrt{6}}{3}, \frac{6 + 2\sqrt{6}}{3}$;
y-intercept: 4

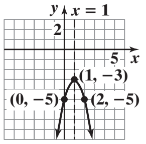
(e)



(f) Domain: All real numbers; $(-\infty, \infty)$
Range: $\{y \mid y \ge -8\}$; $[-8, \infty)$
(g) Decreasing: $(-\infty, 2]$; Increasing: $[2, \infty)$

5. (a) Opens down **(b)** $(1, -3)$ **(c)** $x = 1$
(d) No x-intercepts; y-intercept: −5

(e)

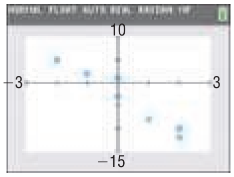


(f) Domain: All real numbers; $(-\infty, \infty)$
Range: $\{y \mid y \le -3\}$; $(-\infty, -3]$
(g) Increasing: $(-\infty, 1]$; Decreasing: $[1, \infty)$

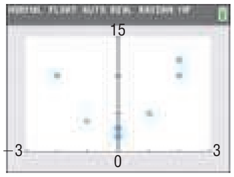
6. $f(x) = 2x^2 - 4x - 30$ **7.** Maximum value; 21 **8.** $\{x \mid x \le 4 \text{ or } x \ge 6\}$; $(-\infty, 4] \cup [6, \infty)$ **9. (a)** $C(m) = 0.15m + 129.50$ **(b)** \$258.50

(c) 562 miles **10. (a)** $R(x) = -\frac{1}{10}x^2 + 1000x$ **(b)** \$384,000 **(c)** 5000 units; \$2,500,000 **(d)** \$500

11. (a)



Linear with negative slope



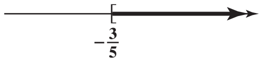
Quadratic that opens up

(b) $y = -4.234x - 2.362$ **(c)** $y = 1.993x^2 + 0.289x + 2.503$

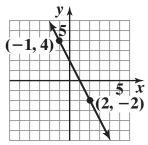
Cumulative Review *(page 328)*

1. $5\sqrt{2}$; $\left(\frac{3}{2}, \frac{1}{2}\right)$ **2.** $(-2, -1)$ and $(2, 3)$ are on the graph.

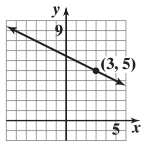
3. $\left\{x \middle| x \geq -\frac{3}{5}\right\}$ or $\left[-\frac{3}{5}, \infty\right)$



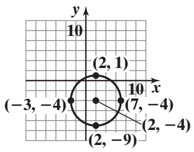
4. $y = -2x + 2$



5. $y = -\frac{1}{2}x + \frac{13}{2}$



6. $(x - 2)^2 + (y + 4)^2 = 25$

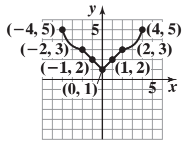


7. Yes **8. (a)** -3 **(b)** $x^2 - 4x - 2$ **(c)** $x^2 + 4x + 1$ **(d)** $-x^2 + 4x - 1$ **(e)** $x^2 - 3$ **(f)** $2x + h - 4$ **9.** $\left\{z \middle| z \neq \frac{7}{6}\right\}$ **10.** Yes

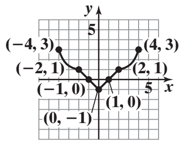
11. (a) No **(b)** -1; $(-2, -1)$ is on the graph. **(c)** -8; $(-8, 2)$ is on the graph. **12.** Neither **13.** Local maximum value is 5.30 and occurs at $x = -1.29$. Local minimum value is -3.30 and occurs at $x = 1.29$. Increasing: $[-4, -1.29]$ and $[1.29, 4]$; Decreasing: $[-1.29, 1.29]$

14. (a) -4 **(b)** $\{x | x > -4\}$ or $(-4, \infty)$ **15. (a)** Domain: $\{x | -4 \leq x \leq 4\}$; Range: $\{y | -1 \leq y \leq 3\}$ **(b)** $(-1, 0)$, $(0, -1)$, $(1, 0)$ **(c)** y-axis **(d)** 1 **(e)** -4 and 4 **(f)** $\{x | -1 < x < 1\}$

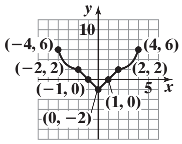
(g)



(h)



(i)



(j) Even **(k)** $[0, 4]$

CHAPTER 5 Polynomial and Rational Functions

5.1 Assess Your Understanding *(page 346)*

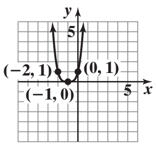
7. smooth; continuous **8.** touches **9.** $(-1, 1)$; $(0, 0)$; $(1, 1)$ **10.** r is a real zero of f; r is an x-intercept of the graph of f; $x - r$ is a factor of f. **11.** turning points **12.** ∞; ∞ **13.** ∞; $-\infty$ **14.** As x increases in the positive direction, $f(x)$ decreases without bound. **15.** b **16.** d

17. Yes; degree 3; $f(x) = x^3 + 4x$; leading term: x^3; constant term: 0 **19.** Yes; degree 2; $g(x) = -\frac{1}{2}x^2 + \frac{1}{2}$; leading term: $-\frac{1}{2}x^2$; constant term: $\frac{1}{2}$

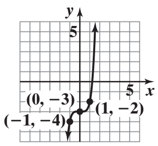
21. No; x is raised to the -1 power **23.** No; x is raised to the $\frac{3}{2}$ power **25.** Yes; degree 4; $F(x) = 5x^4 - \pi x^3 + \frac{1}{2}$; leading term: $5x^4$; constant term: $\frac{1}{2}$

27. Yes; degree 4; $G(x) = 2x^4 - 4x^3 + 4x^2 - 4x + 2$; leading term: $2x^4$; constant term: 2

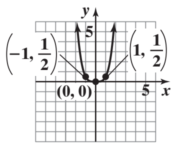
29.



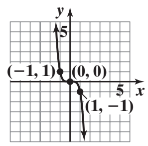
31.



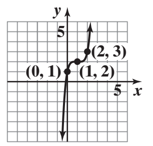
33.



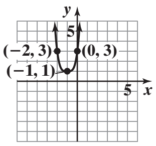
35.



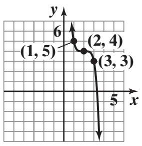
37.



39.



41.



43. $f(x) = (x + 1)(x - 1)(x - 3)$
$= x^3 - 3x^2 - x + 3$ for $a = 1$

45. $f(x) = x(x + 3)(x - 4)$
$= x^3 - x^2 - 12x$ for $a = 1$

47. $f(x) = (x + 4)(x + 1)(x - 2)(x - 3)$
$= x^4 - 15x^2 + 10x + 24$ for $a = 1$

49. $f(x) = (x + 1)(x - 3)^2$
$= x^3 - 5x^2 + 3x + 9$ for $a = 1$

51. $f(x) = 2(x + 3)(x - 1)(x - 4)$
$= 2x^3 - 4x^2 - 22x + 24$

53. $f(x) = 16x(x + 1)(x - 2)(x - 4)$
$= 16x^4 - 80x^3 + 32x^2 + 128x$

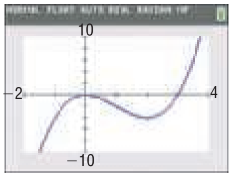
55. $f(x) = 5(x + 1)^2(x - 1)^2$
$= 5x^4 - 10x^2 + 5$

57. (a) 7, multiplicity 1; -3, multiplicity 2 **(b)** Graph touches the x-axis at -3 and crosses it at 7. **(c)** 2 **(d)** $y = 3x^3$

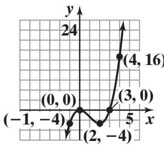
59. (a) 2, multiplicity 3 **(b)** Graph crosses the x-axis at 2. **(c)** 4 **(d)** $y = 4x^5$ **61. (a)** $-\frac{1}{2}$, multiplicity 2; -4, multiplicity 3 **(b)** Graph touches the x-axis at $-\frac{1}{2}$ and crosses at -4. **(c)** 4 **(d)** $y = -2x^5$ **63. (a)** 5, multiplicity 3; -4, multiplicity 2 **(b)** Graph touches the x-axis at -4 and crosses it at 5. **(c)** 4 **(d)** $y = x^5$ **65. (a)** No real zeros **(b)** Graph neither crosses nor touches the x-axis. **(c)** 5 **(d)** $y = 3x^6$ **67. (a)** 0, multiplicity 2; $-\sqrt{2}$, $\sqrt{2}$, multiplicity 1 **(b)** Graph touches the x-axis at 0 and crosses at $-\sqrt{2}$ and $\sqrt{2}$. **(c)** 3 **(d)** $y = -2x^4$ **69.** Could be; zeros: $-1, 1, 2$; Least degree is 3. **71.** Cannot be the graph of a polynomial; gap at $x = -1$ **73.** $f(x) = x(x - 1)(x - 2)$

75. $f(x) = -\frac{1}{2}(x + 1)(x - 1)^2(x - 2)$ **77.** $f(x) = 0.2(x + 4)(x + 1)^2(x - 3)$ **79.** $f(x) = -x(x + 3)^2(x - 3)^2$

81. Step 1: $y = x^3$
Step 2: x-intercepts: 0, 3; y-intercept: 0
Step 3: 0: multiplicity 2; touches; 3: multiplicity 1; crosses
Step 4:

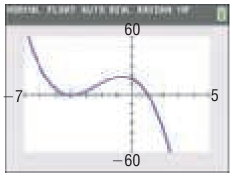


Step 5: $(2, -4)$; $(0, 0)$
Step 6:

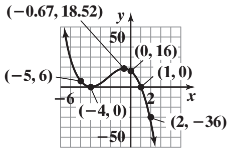


Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$
Step 8: Increasing on $(-\infty, 0]$ and $[2, \infty)$
Decreasing on $[0, 2]$

83. Step 1: $y = -x^3$
Step 2: x-intercepts: -4, 1; y-intercept: 16
Step 3: -4: multiplicity 2, touches; 1: multiplicity 1, crosses
Step 4:

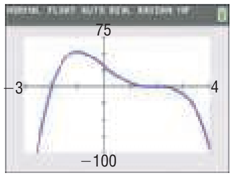


Step 5: $(-4, 0)$; $(-0.67, 18.52)$
Step 6:

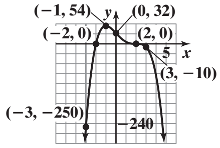


Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$
Step 8: Increasing on $[-4, -0.67]$
Decreasing on $(-\infty, -4]$ and $[-0.67, \infty)$

85. Step 1: $y = -2x^4$
Step 2: x-intercepts: -2, 2; y-intercept: 32
Step 3: -2: multiplicity 1, crosses; 2: multiplicity 3, crosses
Step 4:

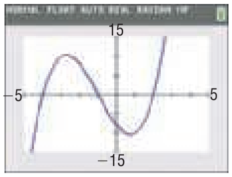


Step 5: $(-1, 54)$
Step 6:

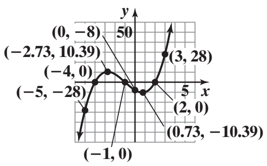


Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, 54]$
Step 8: Increasing on $(-\infty, -1]$
Decreasing on $[-1, \infty)$

87. Step 1: $y = x^3$
Step 2: x-intercepts: $-4, -1, 2$; y-intercept: -8
Step 3: $-4, -1, 2$: multiplicity 1, crosses
Step 4:

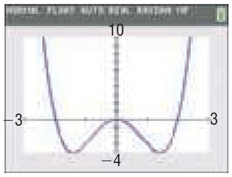


Step 5: $(-2.73, 10.39)$; $(0.73, -10.39)$
Step 6:

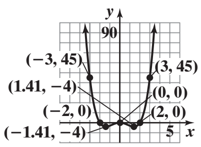


Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$
Step 8: Increasing on $(-\infty, -2.73]$ and $[0.73, \infty)$
Decreasing on $[-2.73, 0.73]$

89. Step 1: $y = x^4$
Step 2: x-intercepts: $-2, 0, 2$; y-intercept: 0
Step 3: $-2, 2$: multiplicity 1, crosses; 0: multiplicity 2, touches
Step 4:

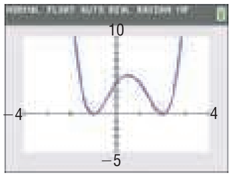


Step 5: $(-1.41, -4)$; $(1.41, -4)$; $(0, 0)$
Step 6:

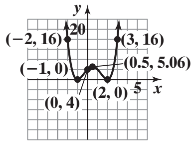


Step 7: Domain: $(-\infty, \infty)$; Range: $[-4, \infty)$
Step 8: Increasing on $[-1.41, 0]$ and $[1.41, \infty)$
Decreasing on $(-\infty, -1.41]$, and $[0, 1.41]$

91. Step 1: $y = x^4$
Step 2: x-intercepts: -1, 2; y-intercept: 4
Step 3: -1, 2: multiplicity 2, touches
Step 4:

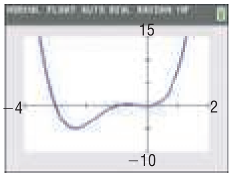


Step 5: $(-1, 0)$; $(2, 0)$; $(0.5, 5.06)$
Step 6:

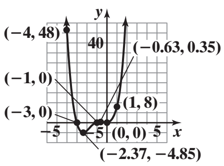


Step 7: Domain: $(-\infty, \infty)$; Range: $[0, \infty)$
Step 8: Increasing on $[-1, 0.5]$ and $[2, \infty)$
Decreasing on $(-\infty, -1]$ and $[0.5, 2]$

93. Step 1: $y = x^4$
Step 2: x-intercepts: $-1, 0, -3$; y-intercept: 0
Step 3: $-1, -3$; multiplicity 1, crosses; 0: multiplicity 2, touches
Step 4:

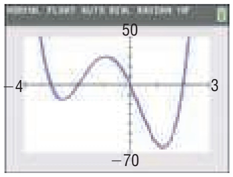


Step 5: $(-2.37, -4.85)$; $(-0.63, 0.35)$; $(0, 0)$
Step 6:

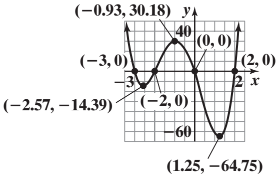


Step 7: Domain: $(-\infty, \infty)$; Range: $[-4.85, \infty)$
Step 8: Increasing on $[-2.37, -0.63]$ and $[0, \infty)$
Decreasing on $(-\infty, -2.37]$ and $[-0.63, 0]$

95. Step 1: $y = 5x^4$
Step 2: x-intercepts: $-3, -2, 0, 2$; y-intercept: 0
Step 3: $-3, -2, 0, 2$: multiplicity 1, crosses
Step 4:

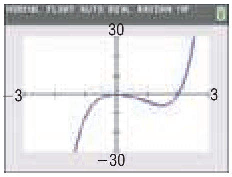


Step 5: $(-2.57, -14.39)$; $(-0.93, 30.18)$; $(1.25, -64.75)$
Step 6:

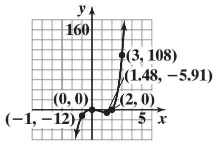


Step 7: Domain: $(-\infty, \infty)$; Range: $[-64.75, \infty)$
Step 8: Increasing on $[-2.57, -0.93]$ and $[1.25, \infty)$
Decreasing on $(-\infty, -2.57]$ and $[-0.93, 1.25]$

97. Step 1: $y = x^5$
Step 2: x-intercepts: 0, 2; y-intercept: 0
Step 3: 0: multiplicity 2, touches; 2: multiplicity 1, crosses
Step 4:

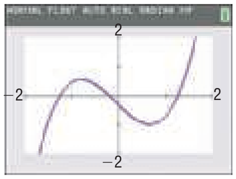


Step 5: $(0, 0)$; $(1.48, -5.91)$
Step 6:



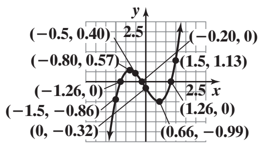
Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$
Step 8: Increasing on $(-\infty, 0]$ and $[1.48, \infty)$
Decreasing on $[0, 1.48]$

99. Step 1: $y = x^3$
Step 2:



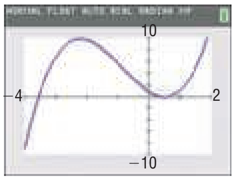
Step 3: x-intercepts: $-1.26, -0.20, 1.26$; y-intercept: -0.31752
Step 4:

Step 5: $(-0.80, 0.57)$; $(0.66, -0.99)$
Step 6:

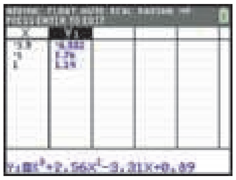


Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$
Step 8: Increasing on $(-\infty, -0.80]$ and $[0.66, \infty)$
Decreasing on $[-0.80, 0.66]$

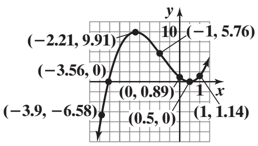
101. Step 1: $y = x^3$
Step 2:



Step 3: x-intercepts: $-3.56, 0.50$; y-intercept: 0.89
Step 4:

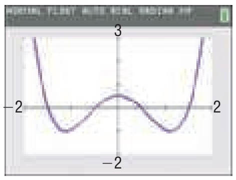


Step 5: $(-2.21, 9.91)$; $(0.50, 0)$
Step 6:



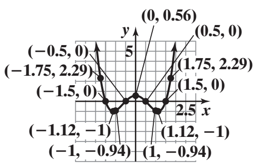
Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$
Step 8: Increasing on $(-\infty, -2.21]$ and $[0.50, \infty)$
Decreasing on $[-2.21, 0.50]$

103. Step 1: $y = x^4$
Step 2:



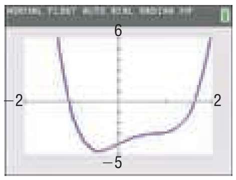
Step 3: x-intercepts: $-1.5, -0.5, 0.5, 1.5$; y-intercept: 0.5625
Step 4:

Step 5: $(-1.12, -1)$; $(1.12, -1)$, $(0, 0.56)$
Step 6:

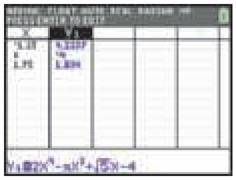


Step 7: Domain: $(-\infty, \infty)$; Range: $[-1, \infty)$
Step 8: Increasing on $[-1.12, 0]$ and $[1.12, \infty)$
Decreasing on $(-\infty, -1.12]$ and $[0, 1.12]$

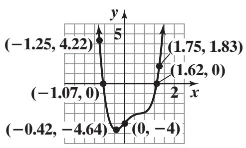
105. Step 1: $y = 2x^4$
Step 2:



Step 3: x-intercepts: $-1.07, 1.62$; y-intercept: -4
Step 4:

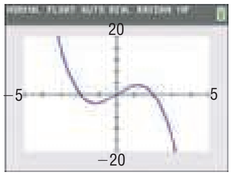


Step 5: $(-0.42, -4.64)$
Step 6:

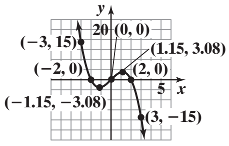


Step 7: Domain: $(-\infty, \infty)$; Range: $[-4.64, \infty)$
Step 8: Increasing on $[-0.42, \infty)$
Decreasing on $(-\infty, -0.42]$

107. $f(x) = -x(x+2)(x-2)$
Step 1: $y = -x^3$
Step 2: x-intercepts: $-2, 0, 2$; y-intercept: 0
Step 3: $-2, 0, 2$: multiplicity 1, crosses
Step 4:

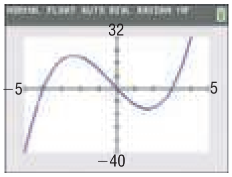


Step 5: $(-1.15, -3.08)$; $(1.15, 3.08)$
Step 6:

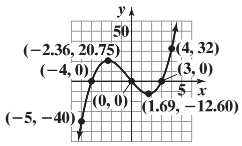


Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$
Step 8: Increasing on $[-1.15, 1.15]$
Decreasing on $(-\infty, -1.15]$ and $[1.15, \infty)$

109. $f(x) = x(x+4)(x-3)$
Step 1: $y = x^3$
Step 2: x-intercepts: $-4, 0, 3$; y-intercept: 0
Step 3: $-4, 0, 3$: multiplicity 1, crosses
Step 4:

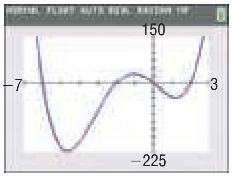


Step 5: $(-2.36, 20.75)$; $(1.69, -12.60)$
Step 6:

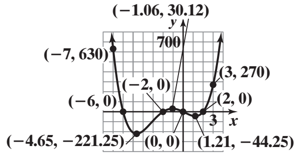


Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$
Step 8: Increasing on $(-\infty, -2.36]$ and $[1.69, \infty)$
Decreasing on $[-2.36, 1.69]$

111. $f(x) = 2x(x+6)(x-2)(x+2)$
Step 1: $y = 2x^4$
Step 2: x-intercepts: $-6, -2, 0, 2$; y-intercept: 0
Step 3: $-6, -2, 0, 2$: multiplicity 1, crosses
Step 4:

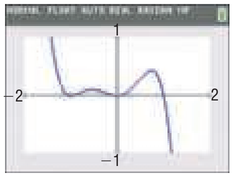


Step 5: $(-4.65, -221.25)$; $(-1.06, 30.12)$; $(1.21, -44.25)$
Step 6:

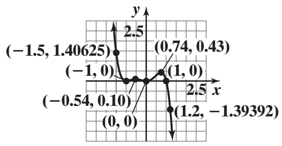


Step 7: Domain: $(-\infty, \infty)$; Range: $[-221.25, \infty)$
Step 8: Decreasing on $(-\infty, -4.65]$ and $[-1.06, 1.21]$
Increasing on $[-4.65, -1.06]$ and $[1.21, \infty)$

113. $f(x) = -x^2(x+1)^2(x-1)$
Step 1: $y = -x^5$
Step 2: x-intercepts: $-1, 0, 1$; y-intercept: 0
Step 3: 1: multiplicity 1, crosses; -1, 0: multiplicity 2, touches
Step 4:



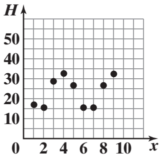
Step 5: $(-1, 0)$; $(-0.54, 0.10)$; $(0, 0)$; $(0.74, 0.43)$
Step 6:



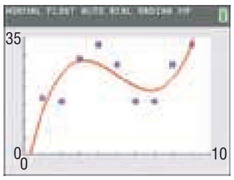
Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$
Step 8: Increasing on $[-1, -0.54]$ and $[0, 0.74]$
Decreasing on $(-\infty, -1]$, $[-0.54, 0]$, and $[0.74, \infty)$

115. $f(x) = 3(x+3)(x-1)(x-4)$ **117.** $f(x) = -2(x+5)^2(x-2)(x-4)$ **119. (a)** $-3, 2$ **(b)** $-6, -1$

121. (a)

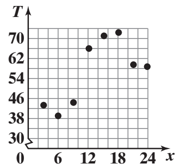


The relation appears to be cubic.
(b) $H(x) = 0.3948x^3 - 5.9563x^2 + 26.1965x - 7.4127$ **(c)** ≈ 24
(d)

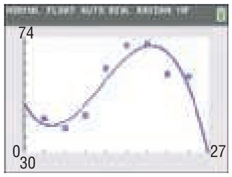


(e) ≈ 54; no. The end behavior of the model indicates that as time goes on, the number of major hurricanes will continue to increase each decade without limit. This is unrealistic. End behavior should not be used to make predictions too far outside the data used to create the model.

123. (a)



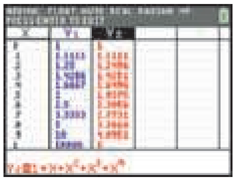
The relation appears to be cubic.
(b) 6°/h **(c)** 0.17°/h
(d) $T(x) = -0.01992x^2 + 0.6745x^2 - 4.4360x + 48.4643$; 70.1°F
(e)

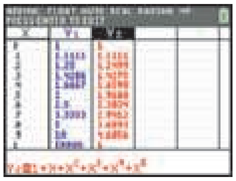


(f) The predicted temperature at midnight is 48.5°F.

125. (a)

(b)



(c)  

(d) As more terms are added, the values of the polynomial function get closer to the values of f. The approximations near 0 are better than those near -1 or 1.

131. (a)–(d) **135.** $y = -\frac{2}{5}x - \frac{11}{5}$ **136.** $\{x \mid x \neq -5\}$ **137.** $\frac{-2-\sqrt{7}}{2}, \frac{-2+\sqrt{7}}{2}$ **138.** $\left\{-\frac{4}{5}, 2\right\}$

Historical Problems *(page 363)*

1.
$$\left(x - \frac{b}{3}\right)^3 + b\left(x - \frac{b}{3}\right)^2 + c\left(x - \frac{b}{3}\right) + d = 0$$
$$x^3 - bx^2 + \frac{b^2x}{3} - \frac{b^3}{27} + bx^2 - \frac{2b^2x}{3} + \frac{b^3}{9} + cx - \frac{bc}{3} + d = 0$$
$$x^3 + \left(c - \frac{b^2}{3}\right)x + \left(\frac{2b^3}{27} - \frac{bc}{3} + d\right) = 0$$
Let $p = c - \frac{b^2}{3}$ and $q = \frac{2b^3}{27} - \frac{bc}{3} + d$. Then $x^3 + px + q = 0$.

2.
$$(H + K)^3 + p(H + K) + q = 0$$
$$H^3 + 3H^2K + 3HK^2 + K^3 + pH + pK + q = 0$$
Let $3HK = -p$.
$$H^3 - pH - pK + K^3 + pH + pK + q = 0, \quad H^3 + K^3 = -q$$

3.
$$3HK = -p$$
$$K = -\frac{p}{3H}$$
$$H^3 + \left(-\frac{p}{3H}\right)^3 = -q$$
$$H^3 - \frac{p^3}{27H^3} = -q$$
$$27H^6 - p^3 = -27qH^3$$
$$27H^6 + 27qH^3 - p^3 = 0$$
$$H^3 = \frac{-27q \pm \sqrt{(27q)^2 - 4(27)(-p^3)}}{2 \cdot 27}$$
$$H^3 = \frac{-q}{2} \pm \sqrt{\frac{27^2q^2}{2^2(27^2)} + \frac{4(27)p^3}{2^2(27^2)}}$$
$$H^3 = \frac{-q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$
$$H = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$
Choose the positive root for now.

4.
$$H^3 + K^3 = -q$$
$$K^3 = -q - H^3$$
$$K^3 = -q - \left[\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right]$$
$$K^3 = \frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$
$$K = \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

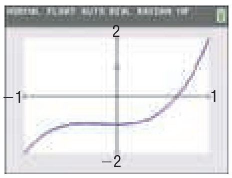
5. $x = H + K$
$$x = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$
(Note that had we used the negative root in 3 the result would be the same.)

6. $x = 3$ **7.** $x = 2$ **8.** $x = 2$

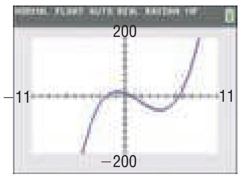
5.2 Assess Your Understanding *(page 363)*

5. a **6.** $f(c)$ **7.** b **8.** F **9.** 0 **10.** T **11.** $R = f(2) = 8$; no **13.** $R = f(2) = 0$; yes **15.** $R = f(-3) = 0$; yes **17.** $R = f(-4) = 1$; no **19.** $R = f\left(\frac{1}{2}\right) = 0$; yes **21.** 7; 3 or 1 positive; 2 or 0 negative **23.** 6; 2 or 0 positive; 2 or 0 negative **25.** 3; 2 or 0 positive; 1 negative **27.** 4; 2 or 0 positive; 2 or 0 negative **29.** 5; 0 positive; 3 or 1 negative **31.** 6; 1 positive; 1 negative **33.** 4; $\pm 1, \pm\frac{1}{3}$ **35.** 5; $\pm 1, \pm 3$ **37.** 3; $\pm 1, \pm 2, \pm\frac{1}{4}, \pm\frac{1}{2}$ **39.** 4; $\pm 1, \pm 3, \pm 9, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm\frac{3}{2}, \pm\frac{9}{2}$ **41.** 5; $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm\frac{1}{2}, \pm\frac{3}{2}$ **43.** 4; $\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20, \pm\frac{1}{2}, \pm\frac{5}{2}, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{5}{3}, \pm\frac{10}{3}, \pm\frac{20}{3}, \pm\frac{1}{6}, \pm\frac{5}{6}$

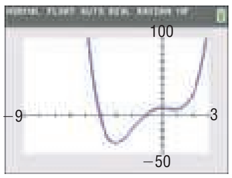
45. -1 and 1



47. -11 and 11

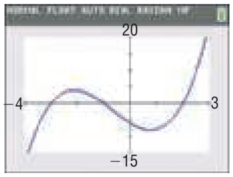


49. -9 and 3

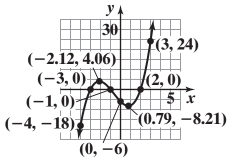


51. $-3, -1, 2; f(x) = (x+3)(x+1)(x-2)$ **53.** $\frac{1}{2}, 3, 3; f(x) = (2x-1)(x-3)^2$ **55.** $-\frac{1}{3}; f(x) = (3x+1)(x^2+x+1)$
57. $4, 3-\sqrt{5}, 3+\sqrt{5}; f(x) = (x-4)(x-3+\sqrt{5})(x-3-\sqrt{5})$ **59.** $-2, -1, 1, 1; f(x) = (x+2)(x+1)(x-1)^2$
61. $-\frac{7}{3}, -1, \frac{2}{7}, 2; f(x) = (x-2)(x+1)(3x+7)(7x-2)$ **63.** $3, \frac{5+\sqrt{17}}{2}, \frac{5-\sqrt{17}}{2}; f(x) = (x-3)\left(x - \frac{5+\sqrt{17}}{2}\right)\left(x - \frac{5-\sqrt{17}}{2}\right)$
65. $-\frac{1}{2}, \frac{1}{2}; f(x) = (2x+1)(2x-1)(x^2+2)$ **67.** $\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 2; f(x) = (x-2)(\sqrt{2}x-1)(\sqrt{2}x+1)(2x^2+1)$ **69.** $-5.9, -0.3, 3$
71. $-3.8, 4.5$ **73.** $-43.5, 1, 23$ **75.** $\{-1, 2\}$ **77.** $\left\{\frac{2}{3}, -1+\sqrt{2}, -1-\sqrt{2}\right\}$ **79.** $\left\{\frac{1}{3}, \sqrt{5}, -\sqrt{5}\right\}$ **81.** $\{-3, -2\}$ **83.** $\left\{-\frac{1}{3}\right\}$
85. $f(0) = -1; f(1) = 10$; Zero: 0.21 **87.** $f(-5) = -58; f(-4) = 2$; Zero: -4.04 **89.** $f(1.4) = -0.17536; f(1.5) = 1.40625$; Zero: 1.41

91. Step 1: $y = x^3$
Step 2: x-intercepts: $-3, -1, 2$; y-intercept: -6
Step 3: $-3, -1, 2$: multiplicity 1, crosses

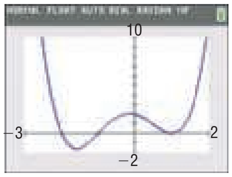
Step 4:


Step 5: $(-2.12, 4.06)$; $(0.79, -8.21)$

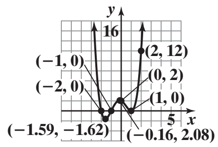
Step 6:


Step 7: Domain and range: $(-\infty, \infty)$
Step 8: Increasing: $(-\infty, -2.12]$, $[0.79, \infty)$
Decreasing: $[-2.12, 0.79]$

93. Step 1: $y = x^4$
Step 2: x-intercepts: $-2, -1, 1$; y-intercept: 2
Step 3: $-2, -1$: multiplicity 1, crosses; 1: multiplicity 2, touches

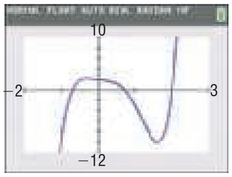
Step 4:


Step 5: $(-1.59, -1.62)$, $(-0.16, 2.08)$, $(1, 0)$

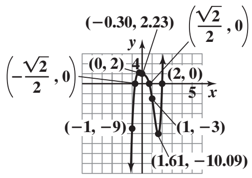
Step 6:


Step 7: Domain: $(-\infty, \infty)$;
Range: $[-1.62, \infty)$
Step 8: Increasing: $[-1.59, -0.16]$, $[1, \infty)$
Decreasing: $(-\infty, -1.59]$, $[-0.16, 1]$

95. Step 1: $y = 4x^5$
Step 2: x-intercepts: $-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 2$; y-intercept: 2
Step 3: $-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 2$: multiplicity 1, crosses

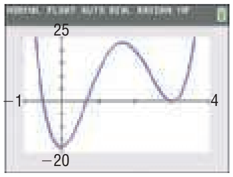
Step 4:


Step 5: $(-0.30, 2.23)$; $(1.61, -10.09)$

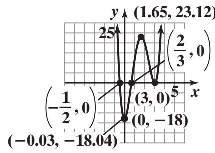
Step 6:


Step 7: Domain and range: $(-\infty, \infty)$
Step 8: Increasing: $(-\infty, -0.30]$, $[1.61, \infty)$
Decreasing: $[-0.30, 1.61]$

97. Step 1: $y = 6x^4$
Step 2: x-intercepts: $-\frac{1}{2}, \frac{2}{3}, 3$; y-intercept: -18
Step 3: $-\frac{1}{2}, \frac{2}{3}$: multiplicity 1, crosses; 3: multiplicity 2, touches

Step 4:


Step 5: $(-0.03, -18.04)$, $(1.65, 23.12)$, $(3, 0)$

Step 6:


Step 7: Domain: $(-\infty, \infty)$;
Range: $[-18.04, \infty)$
Step 8: Increasing: $[-0.03, 1.65]$, $[3, \infty)$;
Decreasing: $(-\infty, -0.03]$, $[1.65, 3]$

99. $k = 5$ **101.** -7 **103.** If $f(x) = x^n - c^n$, then $f(c) = c^n - c^n = 0$; so $x - c$ is a factor of f. **105.** 5 **107.** 7 in.
109. All the potential rational zeros are integers, so r is either an integer or is not a rational zero (and is, therefore, irrational). **111.** 0.215
113. No; by the Rational Zeros Theorem, $\frac{1}{3}$ is not a potential rational zero. **115.** No; by the Rational Zeros Theorem, $\frac{2}{3}$ is not a potential rational zero.
116. $y = \frac{2}{5}x - \frac{3}{5}$ **117.** $[3, 8)$ **118.** $(0, -2\sqrt{3})$, $(0, 2\sqrt{3})$, $(4, 0)$ **119.** $[-3, 2]$ and $[5, \infty)$

5.3 Assess Your Understanding *(page 370)*

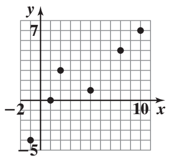
3. one **4.** $3 - 4i$ **5.** T **6.** F **7.** $4 + i$ **9.** $-i, 1 - i$ **11.** $-i, -2i$ **13.** $-i$ **15.** $2 - i, -3 + i$ **17.** $f(x) = x^4 - 14x^3 + 77x^2 - 200x + 208; a = 1$
19. $f(x) = x^5 - 4x^4 + 7x^3 - 8x^2 + 6x - 4; a = 1$ **21.** $f(x) = x^4 - 6x^3 + 10x^2 - 6x + 9; a = 1$ **23.** $-2i, 4$ **25.** $2i, -3, \frac{1}{2}$ **27.** $3 + 2i, -2, 5$
29. $4i, -\sqrt{11}, \sqrt{11}, -\frac{2}{3}$ **31.** $1, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i; f(x) = (x-1)\left(x + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(x + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$
33. $2, 3 - 2i, 3 + 2i; f(x) = (x-2)(x-3+2i)(x-3-2i)$ **35.** $-i, i, -2i, 2i; f(x) = (x+i)(x-i)(x+2i)(x-2i)$

37. $-5i, 5i, -3, 1; f(x) = (x + 5i)(x - 5i)(x + 3)(x - 1)$ **39.** $-4, \frac{1}{3}, 2 - 3i, 2 + 3i; f(x) = 3(x + 4)\left(x - \frac{1}{3}\right)(x - 2 + 3i)(x - 2 - 3i)$

41. 130 **43. (a)** $f(x) = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1)$ **(b)** $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

45. Zeros that are complex numbers must occur in conjugate pairs; or a polynomial with real coefficients of odd degree must have at least one real zero.
47. If the remaining zero were a complex number, its conjugate would also be a zero, creating a polynomial of degree 5.

49.



50. $\{-22\}$ **51.** $6x^3 - 13x^2 - 13x + 20$ **52.** $A = 9\pi \text{ ft}^2$ (about 28.274 ft^2); $C = 6\pi$ ft (about 18.850 ft)

5.4 Assess Your Understanding *(page 379)*

5. F **6.** horizontal asymptote **7.** vertical asymptote **8.** proper **9.** T **10.** F **11.** $y = 0$ **12.** T **13.** d **14.** a

15. All real numbers except 3; $\{x | x \neq 3\}$ **17.** All real numbers except 2 and -4; $\{x | x \neq 2, x \neq -4\}$

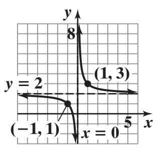
19. All real numbers except $-\frac{1}{2}$ and 3; $\left\{x \middle| x \neq -\frac{1}{2}, x \neq 3\right\}$ **21.** All real numbers except 2; $\{x | x \neq 2\}$ **23.** All real numbers

25. All real numbers except -3 and 3; $\{x | x \neq -3, x \neq 3\}$
27. (a) Domain: $\{x | x \neq 2\}$; range: $\{y | y \neq 1\}$ **(b)** $(0, 0)$ **(c)** $y = 1$ **(d)** $x = 2$ **(e)** None

29. (a) Domain: $\{x | x \neq 0\}$; range: all real numbers **(b)** $(-1, 0), (1, 0)$ **(c)** None **(d)** $x = 0$ **(e)** $y = 2x$

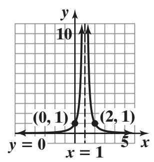
31. (a) Domain: $\{x | x \neq -2, x \neq 2\}$; range: $\{y | y \leq 0, y > 1\}$ **(b)** $(0, 0)$ **(c)** $y = 1$ **(d)** $x = -2, x = 2$ **(e)** None

33. (a)



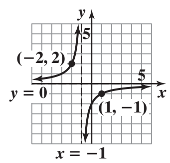
(b) Domain: $\{x | x \neq 0\}$; range: $\{y | y \neq 2\}$
(c) Vertical asymptote: $x = 0$;
horizontal asymptote: $y = 2$

35. (a)



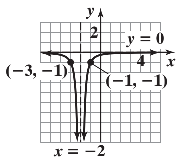
(b) Domain: $\{x | x \neq 1\}$; range: $\{y | y > 0\}$
(c) Vertical asymptote: $x = 1$;
horizontal asymptote: $y = 0$

37. (a)



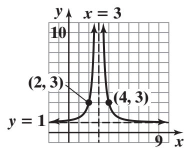
(b) Domain: $\{x | x \neq -1\}$; range: $\{y | y \neq 0\}$
(c) Vertical asymptote: $x = -1$;
horizontal asymptote: $y = 0$

39. (a)



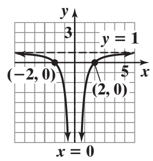
(b) Domain: $\{x | x \neq -2\}$; range: $\{y | y < 0\}$
(c) Vertical asymptote: $x = -2$;
horizontal asymptote: $y = 0$

41. (a)



(b) Domain: $\{x | x \neq 3\}$; range: $\{y | y > 1\}$
(c) Vertical asymptote: $x = 3$;
horizontal asymptote: $y = 1$

43. (a)



(b) Domain: $\{x | x \neq 0\}$; range: $\{y | y < 1\}$
(c) Vertical asymptote: $x = 0$;
horizontal asymptote: $y = 1$

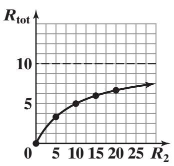
45. Vertical asymptote: $x = -4$; horizontal asymptote: $y = 3$ **47.** Vertical asymptote: $x = 3$; oblique asymptote: $y = x + 5$

49. Vertical asymptotes: $x = 1, x = -1$; horizontal asymptote: $y = 0$ **51.** Vertical asymptote: $x = -\frac{1}{3}$; horizontal asymptote: $y = \frac{2}{3}$

53. No asymptotes **55.** Vertical asymptote: $x = 0$; no horizontal or oblique asymptote

57. (a) 9.8208 m/sec^2 **(b)** 9.8195 m/sec^2 **(c)** 9.7936 m/sec^2 **(d)** $y = 0$ **(e)** $\varnothing$

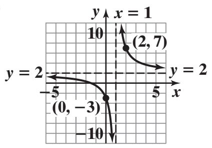
59. (a)



(b) Horizontal: $R_{tot} = 10$; as the resistance of R_2 increases without bound, the total resistance approaches 10 ohms, the resistance R_1.
(c) $R_1 \approx 103.5$ ohms

61. (a) $R(x) = 2 + \frac{5}{x - 1} = 5\left(\frac{1}{x - 1}\right) + 2$

(b)



(c) Vertical asymptote: $x = 1$;
horizontal asymptote: $y = 2$

67. $x = 5$ **68.** $\left\{-\frac{4}{19}\right\}$

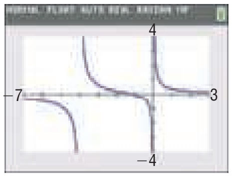
69. x-axis symmetry

70. $(-3, 11), (2, -4)$

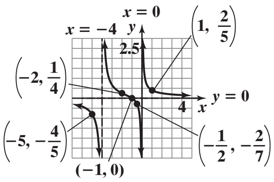
5.5 Assess Your Understanding *(page 390)*

2. False **3.** c **4.** a **5.** **(a)** $\{x \mid x \neq 2\}$ **(b)** 0 **6.** True

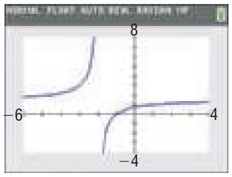
7. Step 1: Domain: $\{x \mid x \neq 0,\ x \neq -4\}$
Step 2: R is in lowest terms
Step 3: no y-intercept; x-intercept: -1
Step 4: R is in lowest terms; vertical asymptotes: $x = 0,\ x = -4$
Step 5: Horizontal asymptote: $y = 0$, intersected at $(-1, 0)$
Step 6:



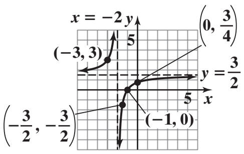
Step 7:



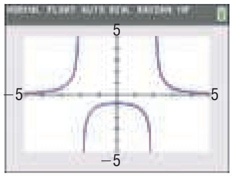
9. Step 1: $R(x) = \dfrac{3(x+1)}{2(x+2)}$; domain: $\{x \mid x \neq -2\}$
Step 2: R is in lowest terms
Step 3: y-intercept: $\dfrac{3}{4}$; x-intercept: -1
Step 4: R is in lowest terms; vertical asymptote: $x = -2$
Step 5: Horizontal asymptote: $y = \dfrac{3}{2}$, not intersected
Step 6:



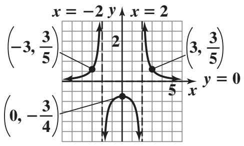
Step 7:



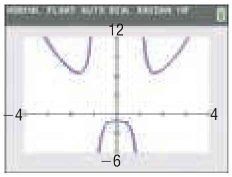
11. Step 1: $R(x) = \dfrac{3}{(x+2)(x-2)}$; domain: $\{x \mid x \neq -2, x \neq 2\}$
Step 2: R is in lowest terms
Step 3: y-intercept: $-\dfrac{3}{4}$; no x-intercept
Step 4: R is in lowest terms; vertical asymptotes: $x = 2, x = -2$
Step 5: Horizontal asymptote: $y = 0$, not intersected
Step 6:



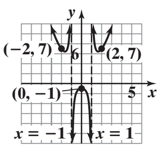
Step 7:



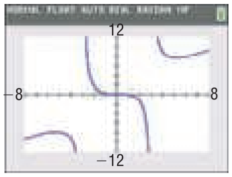
13. Step 1: $P(x) = \dfrac{(x^2 + x + 1)(x^2 - x + 1)}{(x+1)(x-1)}$; domain: $\{x \mid x \neq -1, x \neq 1\}$
Step 2: P is in lowest terms
Step 3: y-intercept: -1; no x-intercept
Step 4: P is in lowest terms; vertical asymptotes: $x = -1, x = 1$
Step 5: No horizontal or oblique asymptote
Step 6:



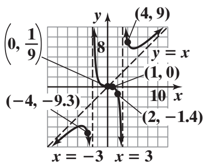
Step 7:



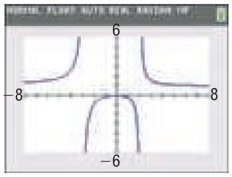
15. Step 1: $H(x) = \dfrac{(x-1)(x^2 + x + 1)}{(x+3)(x-3)}$; domain: $\{x \mid x \neq -3, x \neq 3\}$
Step 2: H is in lowest terms
Step 3: y-intercept: $\dfrac{1}{9}$; x-intercept: 1
Step 4: H is in lowest terms; vertical asymptotes: $x = 3, x = -3$
Step 5: Oblique asymptote: $y = x$, intersected at $\left(\dfrac{1}{9}, \dfrac{1}{9}\right)$
Step 6:



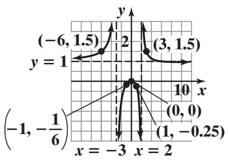
Step 7:



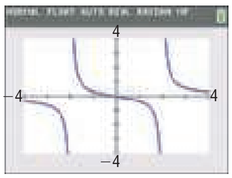
17. Step 1: $R(x) = \dfrac{x^2}{(x+3)(x-2)}$; domain: $\{x \neq -3, x \neq 2\}$
Step 2: R is in lowest terms
Step 3: y-intercept: 0; x-intercept: 0
Step 4: R is in lowest terms; vertical asymptotes: $x = 2, x = -3$
Step 5: Horizontal asymptote: $y = 1$, intersected at $(6, 1)$
Step 6:



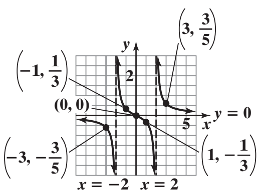
Step 7:



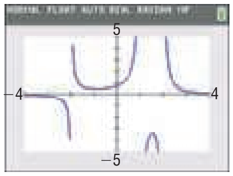
19. Step 1: $G(x) = \dfrac{x}{(x+2)(x-2)}$; domain: $\{x \mid x \neq -2, x \neq 2\}$
Step 2: G is in lowest terms
Step 3: y-intercept: 0; x-intercept: 0
Step 4: G is in lowest terms; vertical asymptotes: $x = -2, x = 2$
Step 5: Horizontal asymptote: $y = 0$, intersected at $(0, 0)$
Step 6:



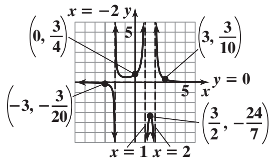
Step 7:



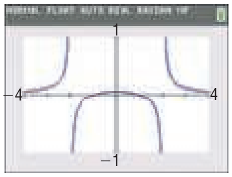
21. Step 1: $R(x) = \dfrac{3}{(x-1)(x+2)(x-2)}$; domain: $\{x \mid x \neq 1, x \neq -2, x \neq 2\}$
Step 2: R is in lowest terms
Step 3: y-intercept: $\dfrac{3}{4}$; no x-intercept
Step 4: R is in lowest terms; vertical asymptotes: $x = -2, x = 1, x = 2$
Step 5: Horizontal asymptote: $y = 0$, not intersected
Step 6:



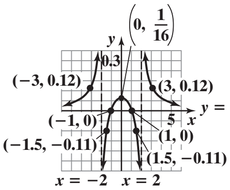
Step 7:



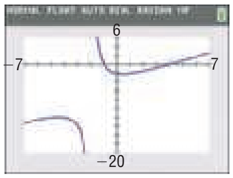
23. Step 1: $H(x) = \dfrac{(x+1)(x-1)}{(x^2+4)(x+2)(x-2)}$; domain: $\{x \mid x \neq -2, x \neq 2\}$
Step 2: H is in lowest terms
Step 3: y-intercept: $\dfrac{1}{16}$; x-intercepts: $-1, 1$
Step 4: H is in lowest terms; vertical asymptotes: $x = -2, x = 2$
Step 5: Horizontal asymptote: $y = 0$, intersected at $(-1, 0)$ and $(1, 0)$
Step 6:



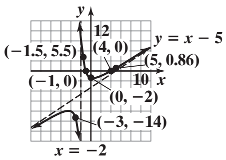
Step 7:



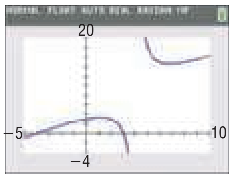
25. Step 1: $F(x) = \dfrac{(x+1)(x-4)}{x+2}$; domain: $\{x \mid x \neq -2\}$
Step 2: F is in lowest terms
Step 3: y-intercept: -2; x-intercepts: $-1, 4$
Step 4: F is in lowest terms; vertical asymptote: $x = -2$
Step 5: Oblique asymptote: $y = x - 5$, not intersected
Step 6:



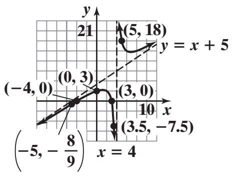
Step 7:



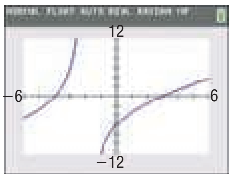
27. Step 1: $R(x) = \dfrac{(x+4)(x-3)}{x-4}$; domain: $\{x \mid x \neq 4\}$
Step 2: R is in lowest terms
Step 3: y-intercept: 3; x-intercepts: $-4, 3$
Step 4: R is in lowest terms; vertical asymptote: $x = 4$
Step 5: Oblique asymptote: $y = x + 5$, not intersected
Step 6:



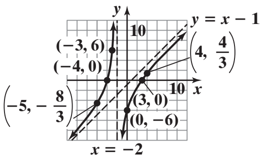
Step 7:



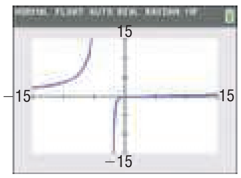
29. Step 1: $F(x) = \dfrac{(x+4)(x-3)}{x+2}$; domain: $\{x \mid x \neq -2\}$
Step 2: F is in lowest terms
Step 3: y-intercept: -6; x-intercepts: $-4, 3$
Step 4: F is in lowest terms; vertical asymptote: $x = -2$
Step 5: Oblique asymptote: $y = x - 1$, not intersected
Step 6:

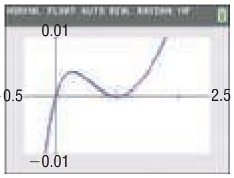


Step 7:

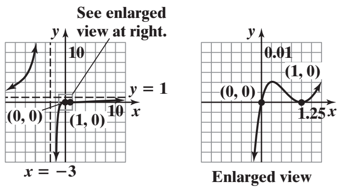


31. Step 1: Domain: $\{x \mid x \neq -3\}$
Step 2: R is in lowest terms
Step 3: y-intercept: 0; x-intercepts: 0, 1
Step 4: Vertical asymptote: $x = -3$
Step 5: Horizontal asymptote: $y = 1$, not intersected
Step 6:

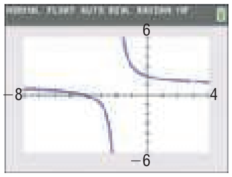




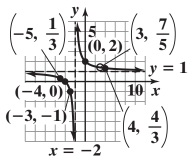
Step 7:



33. Step 1: $R(x) = \dfrac{(x+4)(x-3)}{(x+2)(x-3)}$; domain: $\{x \mid x \neq -2, x \neq 3\}$
Step 2: In lowest terms, $R(x) = \dfrac{x+4}{x+2}$
Step 3: y-intercept: 2; x-intercept: -4
Step 4: Vertical asymptote: $x = -2$; hole at $\left(3, \dfrac{7}{5}\right)$
Step 5: Horizontal asymptote: $y = 1$, not intersected
Step 6:



Step 7:



35. Step 1: $R(x) = \dfrac{(3x+1)(2x-3)}{(x-2)(2x-3)}$; domain: $\left\{x \middle| x \neq \dfrac{3}{2}, x \neq 2\right\}$

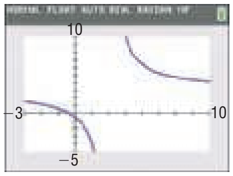
Step 2: In lowest terms, $R(x) = \dfrac{3x+1}{x-2}$

Step 3: y-intercept: $-\dfrac{1}{2}$; x-intercept: $-\dfrac{1}{3}$

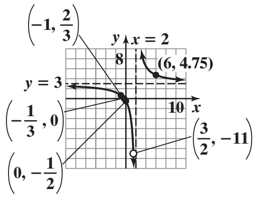
Step 4: Vertical asymptote: $x = 2$; hole at $\left(\dfrac{3}{2}, -11\right)$

Step 5: Horizontal asymptote: $y = 3$, not intersected

Step 6:



Step 7:



37. Step 1: $R(x) = \dfrac{(x+3)(x+2)}{x+3}$; domain: $\{x | x \neq -3\}$

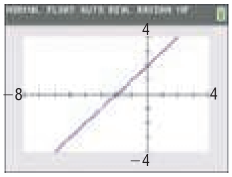
Step 2: In lowest terms, $R(x) = x + 2$

Step 3: y-intercept: 2; x-intercept: -2

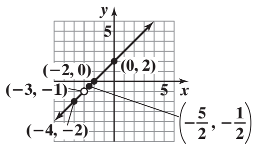
Step 4: Vertical asymptote: none; hole at $(-3, -1)$

Step 5: No horizontal or oblique asymptote

Step 6:



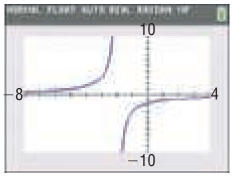
Step 7:



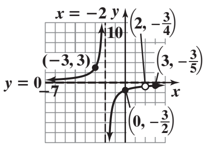
39. 1. $H(x) = \dfrac{-3(x-2)}{(x-2)(x+2)}$; domain: $\{x | x \neq -2, x \neq 2\}$ **2.** In lowest terms, $H(x) = \dfrac{-3}{x+2}$ **3.** y-intercept: $-\dfrac{3}{2}$; no x-intercept

4. Vertical asymptote: $x = -2$; hole at $\left(2, -\dfrac{3}{4}\right)$ **5.** Horizontal asymptote: $y = 0$; not intersected

6.



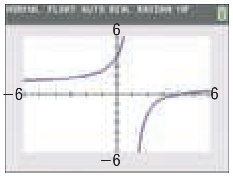
7.



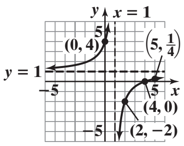
41. 1. $F(x) = \dfrac{(x-1)(x-4)}{(x-1)^2}$; domain: $\{x | x \neq 1\}$ **2.** In lowest terms, $F(x) = \dfrac{x-4}{x-1}$ **3.** y-intercept: 4; x-intercept: 4

4. Vertical asymptote: $x = 1$ **5.** Horizontal asymptote: $y = 1$; not intersected

6.



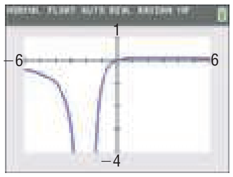
7.



43. 1. $G(x) = \dfrac{x}{(x+2)^2}$; domain: $\{x | x \neq -2\}$ **2.** G is in lowest terms **3.** y-intercept: 0; x-intercept: 0 **4.** Vertical asymptote: $x = -2$

5. Horizontal asymptote: $y = 0$; intersected at $(0, 0)$

6.



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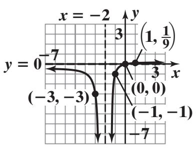


45. Step 1: $f(x) = \dfrac{x^2+1}{x}$; domain: $\{x \mid x \neq 0\}$

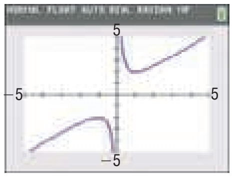
Step 2: f is in lowest terms

Step 3: no y-intercept; no x-intercepts

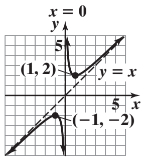
Step 4: f is in lowest terms; vertical asymptote: $x = 0$

Step 5: Oblique asymptote: $y = x$, not intersected

Step 6:



Step 7:



47. Step 1: $f(x) = \dfrac{x^3+1}{x} = \dfrac{(x+1)(x^2-x+1)}{x}$; domain: $\{x \mid x \neq 0\}$

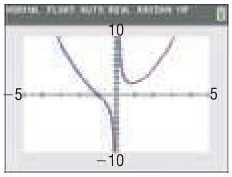
Step 2: f is in lowest terms

Step 3: no y-intercept; x-intercept: -1

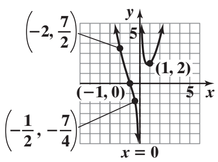
Step 4: f is in lowest terms; vertical asymptote: $x = 0$

Step 5: No horizontal or oblique asymptote

Step 6:



Step 7:



49. Step 1: $f(x) = \dfrac{x^4+1}{x^3}$; domain: $\{x \mid x \neq 0\}$

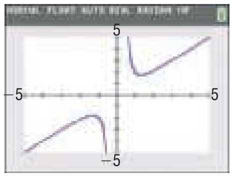
Step 2: f is in lowest terms

Step 3: no y-intercept; no x-intercepts

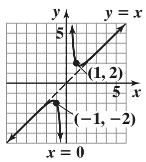
Step 4: f is in lowest terms; vertical asymptote: $x = 0$

Step 5: Oblique asymptote: $y = x$, not intersected

Step 6:

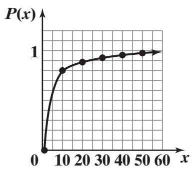


Step 7:



51. One possibility: $R(x) = \dfrac{x^2}{x^2-4}$ **53.** One possibility: $R(x) = \dfrac{(x-1)(x-3)\left(x^2+\frac{4}{3}\right)}{(x+1)^2(x-2)^2}$

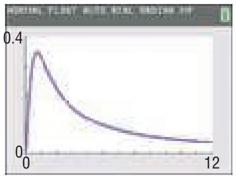
55.



The likelihood of your ball being chosen decreases very quickly and approaches 0 as the number of attendees, x, increases.

57. (a) t-axis; $C(t) \to 0$

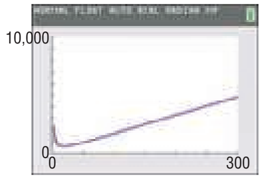
(b)



(c) 0.71 h after injection

59. (a) $C(x) = 16x + \dfrac{5000}{x} + 100$ **(b)** $x > 0$

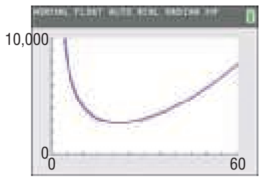
(c)



(d) Approximately 17.7 ft by 56.6 ft (longer side parallel to river)

61. (a) $S(x) = 2x^2 + \dfrac{40{,}000}{x}$

(b)



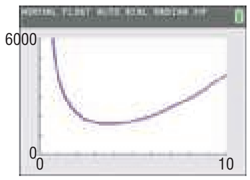
(c) 2784.95 in.^{2}

(d) 21.54 in. × 21.54 in. × 21.54 in.

(e) To minimize the cost of materials needed for construction

63. (a) $C(r) = 12\pi r^2 + \dfrac{4000}{r}$

(b)



The cost is smallest when $r = 3.76$ cm.

65. No. Each function is a quotient of polynomials, but it is not written in lowest terms. Each function is undefined for $x = 1$; each graph has a hole at $x = 1$. **71.** If there is a common factor between the numerator and the denominator, and the factor yields a real zero, then the graph will have a hole.

72. $4x^3 - 5x^2 + 2x + 2$ **73.** $\left\{-\dfrac{1}{10}\right\}$ **74.** $\dfrac{17}{2}$ **75.** ≈ -0.164

5.6 Assess Your Understanding *(page 398)*

3. c **4.** F **5. (a)** $\{x \mid 0 < x < 1 \text{ or } x > 2\}$; $(0,1) \cup (2,\infty)$ **(b)** $\{x \mid x \le 0 \text{ or } 1 \le x \le 2\}$; $(-\infty, 0] \cup [1,2]$

7. (a) $\{x \mid -1 < x < 0 \text{ or } x > 1\}$; $(-1,0) \cup (1,\infty)$ **(b)** $\{x \mid x < -1 \text{ or } 0 \le x < 1\}$; $(-\infty,-1) \cup [0,1)$

9. $\{x \mid x < 0 \text{ or } 0 < x < 3\}$; $(-\infty,0) \cup (0,3)$ **11.** $\{x \mid x \le 1\}$; $(-\infty,1]$ **13.** $\{x \mid x \le -2 \text{ or } x \ge 2\}$; $(-\infty,-2] \cup [2,\infty)$

15. $\{x \mid -4 < x < -1 \text{ or } x > 0\}$; $(-4,-1) \cup (0,\infty)$ **17.** $\{x \mid -2 < x \le -1\}$; $(-2,-1]$ **19.** $\{x \mid x < -2\}$; $(-\infty,-2)$ **21.** $\{x \mid x > 4\}$; $(4,\infty)$

23. $\{x \mid -4 < x < 0 \text{ or } x > 0\}$; $(-4,0) \cup (0,\infty)$ **25.** $\{x \mid x \le 1 \text{ or } 2 \le x \le 3\}$; $(-\infty,1] \cup [2,3]$ **27.** $\{x \mid -1 < x < 0 \text{ or } x > 3\}$; $(-1,0) \cup (3,\infty)$

29. $\{x \mid x < -1 \text{ or } x > 1\}$; $(-\infty,-1) \cup (1,\infty)$ **31.** $\{x \mid x < -1 \text{ or } x > 1\}$; $(-\infty,-1) \cup (1,\infty)$ **33.** $\{x \mid x < -1 \text{ or } x > 1\}$; $(-\infty,-1) \cup (1,\infty)$

35. $\{x \mid x \le -1 \text{ or } 0 < x \le 1\}$; $(-\infty,-1] \cup (0,1]$ **37.** $\{x \mid x < -1 \text{ or } x > 1\}$; $(-\infty,-1) \cup (1,\infty)$ **39.** $\{x \mid x < 2\}$; $(-\infty,2)$

41. $\{x \mid -2 < x \le 9\}$; $(-2,9]$ **43.** $\{x \mid x < 2 \text{ or } 3 < x < 5\}$; $(-\infty,2) \cup (3,5)$

45. $\{x|x < -5 \text{ or } -4 \le x \le -3 \text{ or } x = 0 \text{ or } x > 1\}$; $(-\infty, -5) \cup [-4, -3] \cup \{0\} \cup (1, \infty)$

47. $\left\{x\left|-\frac{1}{2} < x < 1 \text{ or } x > 3\right.\right\}$; $\left(-\frac{1}{2}, 1\right) \cup (3, \infty)$ **49.** $\{x|-1 < x < 3 \text{ or } x > 5\}$; $(-1, 3) \cup (5, \infty)$

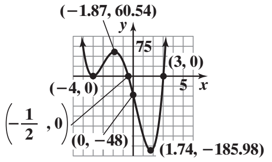
51. $\left\{x\left|x \le -4 \text{ or } x \ge \frac{1}{2}\right.\right\}$; $(-\infty, -4] \cup \left[\frac{1}{2}, \infty\right)$ **53.** $\{x|x < 3 \text{ or } x \ge 7\}$; $(-\infty, 3) \cup [7, \infty)$ **55.** $\{x|x < 2\}$; $(-\infty, 2)$

57. $\left\{x\left|x < -\frac{2}{3} \text{ or } 0 < x < \frac{3}{2}\right.\right\}$; $\left(-\infty, -\frac{2}{3}\right) \cup \left(0, \frac{3}{2}\right)$ **59.** $\{x|x \le -3 \text{ or } 0 \le x \le 3\}$; $(-\infty, -3] \cup [0, 3]$

61. (a) $-4, -\frac{1}{2}, 3$

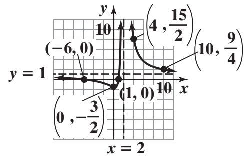
(b) $f(x) = (x + 4)^2(2x + 1)(x - 3)$

(c)



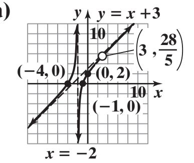
(d) $\left(-\frac{1}{2}, 3\right)$

63. (a)



(b) $(-\infty, -6] \cup [1, 2) \cup (2, \infty)$

65. (a)



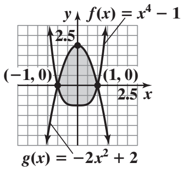
(b) $[-4, -2) \cup [-1, 3) \cup (3, \infty)$

67. $\{x|x > 4\}$; $(4, \infty)$

69. $\{x|x \le -2 \text{ or } x \ge 2\}$; $(-\infty, -2] \cup [2, \infty)$

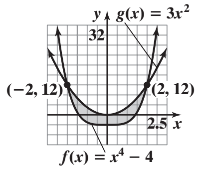
71. $\{x|x < -4 \text{ or } x \ge 2\}$; $(-\infty, -4) \cup [2, \infty)$

73.



$f(x) \le g(x)$ if $-1 \le x \le 1$

75.



$f(x) \le g(x)$ if $-2 \le x \le 2$

77. Produce at least 250 bicycles

79. (a) The stretch is less than 39 ft.

(b) The ledge should be at least 84 ft above the ground for a 150-lb jumper.

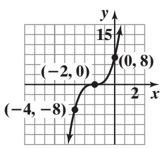
81. At least 50 students must attend. **86.** $\left[\frac{4}{3}, \infty\right)$

87. $x^2 - x - 4$ **88.** $3x^2y^4(x + 2y)(2x - 3y)$ **89.** $y = \frac{2}{3}\sqrt{x}$

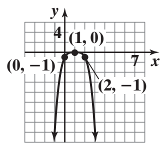
Review Exercises *(page 402)*

1. Polynomial of degree 5 **2.** Rational **3.** Neither **4.** Polynomial of degree 0

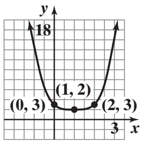
5.



6.



7.

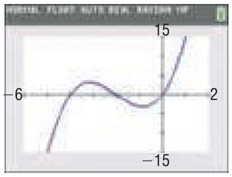


8. Step 1: $y = x^3$

Step 2: x-intercepts: $-4, -2, 0$; y-intercept: 0

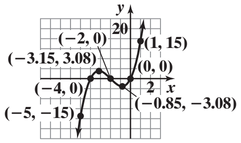
Step 3: $-4, -2, 0$: multiplicity 1; crosses

Step 4:



Step 5: $(-3.15, 3.08)$, $(-0.85, -3.08)$

Step 6:



Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

Step 8: Increasing on $(-\infty, -3.15]$ and $[-0.85, \infty)$

Decreasing on $[-3.15, -0.85]$

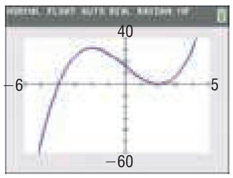
9. Step 1: $y = x^3$

Step 2: x-intercepts: $-4, 2$; y-intercept: 16

Step 3: -4: multiplicity 1; crosses;

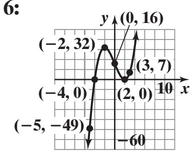
2: multiplicity 2; touches

Step 4:



Step 5: $(-2, 32)$, $(2, 0)$

Step 6:



Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

Step 8: Increasing on $(-\infty, -2]$ and $[2, \infty)$

Decreasing on $[-2, 2]$

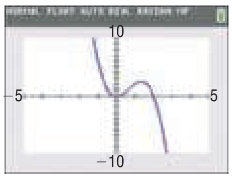
10. Step 1: $y = -2x^3$

Step 2: x-intercepts: 0, 2; y-intercept: 0

Step 3: 0: multiplicity 2; touches;

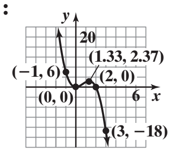
2: multiplicity 1; crosses

Step 4:



Step 5: $(0, 0)$, $(1.33, 2.37)$

Step 6 :



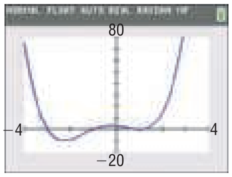
Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

Step 8: Increasing on $[0, 1.33]$

Decreasing on $(-\infty, 0]$ and $[1.33, \infty)$

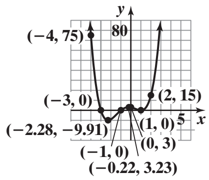
11. Step 1: $y = x^4$
Step 2: x-intercepts: $-3, -1, 1$; y-intercept: 3
Step 3: $-3, -1$: multiplicity 1; crosses; 1: multiplicity 2; touches

Step 4:



Step 5: $(-2.28, -9.91)$, $(-0.22, 3.23)$, $(1, 0)$

Step 6:



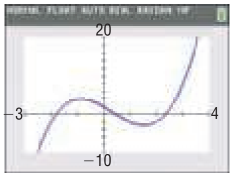
Step 7: Domain: $(-\infty, \infty)$; Range: $[-9.91, \infty)$
Step 8: Increasing on $[-2.28, -0.22]$ and $[1, \infty)$
Decreasing on $(-\infty, -2.28]$ and $[-0.22, 1]$

12. $R = 10$; g is not a factor of f. **13.** $R = 0$; g is a factor of f. **14.** $f(4) = 47{,}105$ **15.** 4, 2, or 0 positive; 2 or 0 negative **16.** 1 positive; 2 or 0 negative

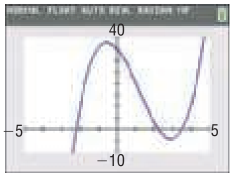
17. $\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{1}{6}, \pm\frac{1}{12}$ **18.** $-2, 1, 4$; $f(x) = (x + 2)(x - 1)(x - 4)$ **19.** $\frac{1}{2}$, multiplicity 2; -2; $f(x) = 4\left(x - \frac{1}{2}\right)^2(x + 2)$

20. 2, multiplicity 2; $f(x) = (x - 2)^2(x^2 + 5)$ **21.** $\{-3, 2\}$ **22.** $\left\{-3, -1, -\frac{1}{2}, 1\right\}$

23. -2 and 3



24. -5 and 5

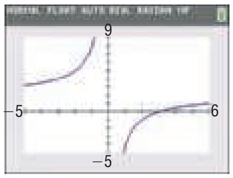


25. $f(0) = -1$; $f(1) = 1$ **26.** $f(0) = -1$; $f(1) = 1$ **27.** 1.52 **28.** 0.93
29. $4 - i$; $f(x) = x^3 - 14x^2 + 65x - 102$ **30.** $-i, 1 - i$; $f(x) = x^4 - 2x^3 + 3x^2 - 2x + 2$
31. $-2, 1, 4$; $f(x) = (x + 2)(x - 1)(x - 4)$
32. $-2, \frac{1}{2}$ (multiplicity 2); $f(x) = 4(x + 2)\left(x - \frac{1}{2}\right)^2$

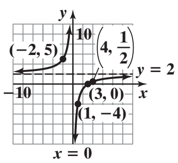
33. 2 (multiplicity 2), $-\sqrt{5}i, \sqrt{5}i$; $f(x) = (x + \sqrt{5}i)(x - \sqrt{5}i)(x - 2)^2$ **34.** $-3, 2, -\frac{\sqrt{2}}{2}i, \frac{\sqrt{2}}{2}i$; $f(x) = 2(x + 3)(x - 2)\left(x + \frac{\sqrt{2}}{2}i\right)\left(x - \frac{\sqrt{2}}{2}i\right)$

35. Domain: $\{x \mid x \neq -3, x \neq 3\}$: horizontal asymptote: $y = 0$; vertical asymptotes: $x = -3, x = 3$
36. Domain: $\{x \mid x \neq -2\}$; horizontal asymptote: $y = 1$; vertical asymptote: $x = -2$

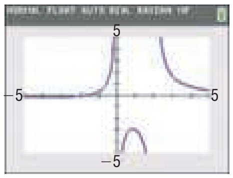
37. Step 1: $R(x) = \dfrac{2(x - 3)}{x}$; domain: $\{x \mid x \neq 0\}$
Step 2: R is in lowest terms
Step 3: no y-intercept; x-intercept: 3
Step 4: R is in lowest terms; vertical asymptote: $x = 0$
Step 5: Horizontal asymptote: $y = 2$; not intersected
Step 6:



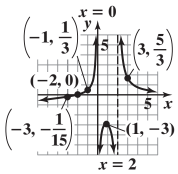
Step 7:



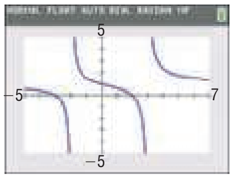
38. Step 1: Domain: $\{x \mid x \neq 0, x \neq 2\}$
Step 2: H is in lowest terms
Step 3: no y-intercept; x-intercept: -2
Step 4: H is in lowest terms; vertical asymptote: $x = 0, x = 2$
Step 5: Horizontal asymptote: $y = 0$; intersected at $(-2, 0)$
Step 6:



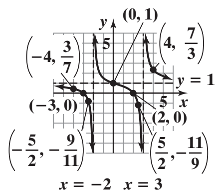
Step 7:



39. Step 1: $R(x) = \dfrac{(x + 3)(x - 2)}{(x - 3)(x + 2)}$; domain: $\{x \mid x \neq -2, x \neq 3\}$
Step 2: R is in lowest terms
Step 3: y-intercept: 1; x-intercepts: $-3, 2$
Step 4: R is in lowest terms; vertical asymptotes: $x = -2, x = 3$
Step 5: Horizontal asymptote: $y = 1$; intersected at $(0, 1)$
Step 6:

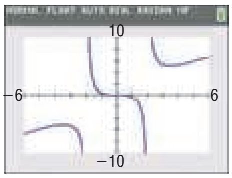


Step 7:

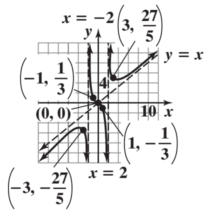


40. Step 1: $F(x) = \dfrac{x^3}{(x + 2)(x - 2)}$; domain: $\{x \mid x \neq -2, x \neq 2\}$
Step 2: F is in lowest terms
Step 3: y-intercept: 0; x-intercept: 0
Step 4: F is in lowest terms; vertical asymptotes: $x = -2, x = 2$
Step 5: Oblique asymptote: $y = x$; intersected at $(0, 0)$

Step 6:

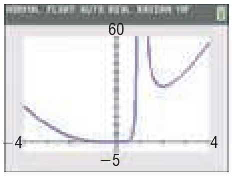


Step 7:

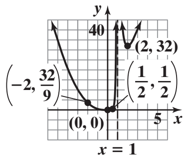


41. Step 1: Domain: $\{x \mid x \neq 1\}$
Step 2: R is in lowest terms
Step 3: y-intercept: 0; x-intercept: 0
Step 4: R is in lowest terms;
vertical asymptote: $x = 1$
Step 5: No oblique or horizontal asymptote

Step 6:

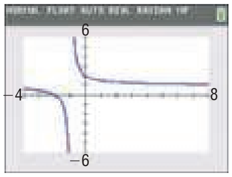


Step 7:

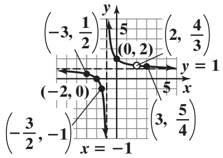


42. Step 1: $G(x) = \dfrac{(x+2)(x-2)}{(x+1)(x-2)}$;
domain: $\{x \mid x \neq -1, x \neq 2\}$
Step 2: In lowest terms, $G(x) = \dfrac{x+2}{x+1}$
Step 3: y-intercept: 2; x-intercept: -2
Step 4: Vertical asymptote: $x = -1$;
hole at $\left(2, \dfrac{4}{3}\right)$
Step 5: Horizontal asymptote: $y = 1$,
not intersected

Step 6:



Step 7:

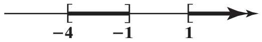


43. (a) $\{-3, 2\}$ **(b)** $(-3, 2) \cup (2, \infty)$ **(c)** $(-\infty, -3] \cup \{2\}$ **(d)** $f(x) = (x-2)^2(x+3)$

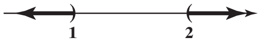
44. (a) $y = 0.25$ **(b)** $x = -2$, $x = 2$ **(c)** $(-3, -2) \cup (-1, 2)$ **(d)** $(-\infty, -3] \cup (-2, -1] \cup (2, \infty)$ **(e)** $R(x) = \dfrac{x^2 + 4x + 3}{4x^2 - 16}$

45. $\{x \mid x < -2 \text{ or } -1 < x < 2\}$; $(-\infty, -2) \cup (-1, 2)$

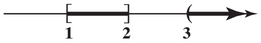
46. $\{x \mid -4 \leq x \leq -1 \text{ or } x \geq 1\}$; $[-4, -1] \cup [1, \infty)$



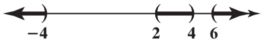
47. $\{x \mid x < 1 \text{ or } x > 2\}$; $(-\infty, 1) \cup (2, \infty)$



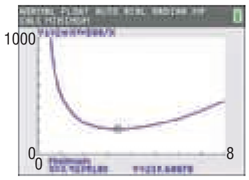
48. $\{x \mid 1 \leq x \leq 2 \text{ or } x > 3\}$; $[1, 2] \cup (3, \infty)$



49. $\{x \mid x < -4 \text{ or } 2 < x < 4 \text{ or } x > 6\}$; $(-\infty, -4) \cup (2, 4) \cup (6, \infty)$

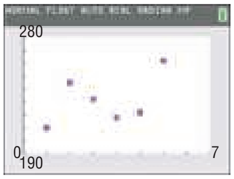


50. (a) $A(r) = 2\pi r^2 + \dfrac{500}{r}$
(b) 223.22 cm^2
(c) 257.08 cm^2
(d)

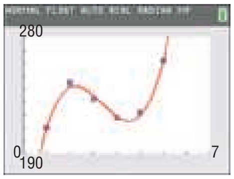


A is smallest when $r \approx 3.41$ cm.

51. (a)



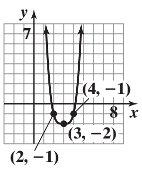
The relation appears to be cubic.
(b) $P(t) = 4.4926t^3 - 45.5294t^2 + 136.1209t + 115.4667$; $\approx \$928{,}000$
(c)



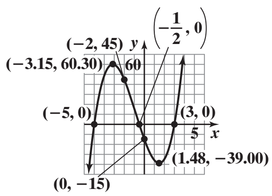
52. (a) Even **(b)** Positive **(c)** Even **(d)** The graph touches the x-axis at $x = 0$, but does not cross it there. **(e)** 8

Chapter Test *(page 404)*

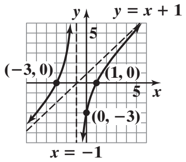
1.



2. (a) 3 **(b)** $\dfrac{p}{q}$: $\pm\dfrac{1}{2}, \pm 1, \pm\dfrac{3}{2}, \pm\dfrac{5}{2}, \pm 3, \pm 5, \pm\dfrac{15}{2}, \pm 15$ **(c)** $-5, -\dfrac{1}{2}, 3$; $g(x) = (x+5)(2x+1)(x-3)$
(d) y-intercept: -15; x-intercepts: $-5, -\dfrac{1}{2}, 3$ **(e)** Crosses at $-5, -\dfrac{1}{2}, 3$ **(f)** $y = 2x^3$
(g) $(-3.15, 60.30)$, $(1.48, -39.00)$ **(h)**



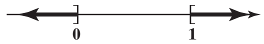
3. $4, -5i, 5i$ **4.** $\left\{1, \dfrac{5-\sqrt{61}}{6}, \dfrac{5+\sqrt{61}}{6}\right\}$ **5.** Domain: $\{x \mid x \neq -10, x \neq 4\}$; asymptotes: $x = -10$, $y = 2$
6. Domain: $\{x \mid x \neq -1\}$; asymptotes: $x = -1$, $y = x + 1$
7.



8. Answers may vary. One possibility is $f(x) = x^4 - 4x^3 - 2x^2 + 20x$.
9. Answers may vary. One possibility is $r(x) = \dfrac{2(x-9)(x-1)}{(x-4)(x-9)}$.
10. $f(0) = 8$; $f(4) = -36$; Since $f(0) = 8 > 0$ and $f(4) = -36 < 0$, the Intermediate Value Theorem guarantees that there is at least one real zero between 0 and 4. **11.** $\{x \mid x < 3 \text{ or } x > 8\}$; $(-\infty, 3) \cup (8, \infty)$

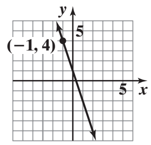
Cumulative Review *(page 404)*

1. $\sqrt{26}$ **2.** $\{x \mid x \leq 0 \text{ or } x \geq 1\}$; $(-\infty, 0]$ or $[1, \infty)$

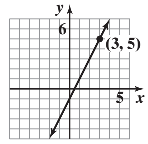


3. $\{x \mid -1 < x < 4\}$; $(-1, 4)$

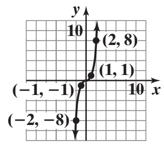
4. $f(x) = -3x + 1$



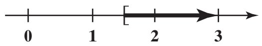
5. $y = 2x - 1$



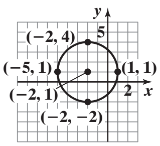
6.



7. Not a functions; 3 has two images. **8.** $\{0, 2, 4\}$ **9.** $\left\{x \,\middle|\, x \geq \dfrac{3}{2}\right\}$; $\left[\dfrac{3}{2}, \infty\right)$

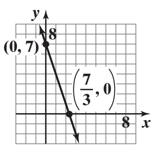


10. Center: $(-2, 1)$; radius: 3

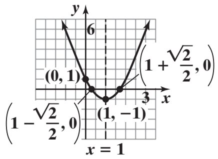


11. x-intercepts: $-3, 0, 3$; y-intercept: 0; symmetric with respect to the origin **12.** $y = -\dfrac{2}{3}x + \dfrac{17}{3}$
13. Not a function; it fails the Vertical Line Test. **14. (a)** 22 **(b)** $x^2 - 5x - 2$ **(c)** $-x^2 - 5x + 2$
(d) $9x^2 + 15x - 2$ **(e)** $2x + h + 5$ **15. (a)** $\{x \mid x \neq 1\}$ **(b)** No; (2,7) is on the graph. **(c)** 4; (3, 4) is on the graph.
(d) $\dfrac{7}{4}$; $\left(\dfrac{7}{4}, 9\right)$ is on the graph. **(e)** Rational

16.

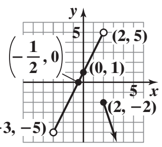


17.



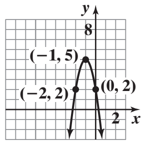
18. 6; $y = 6x - 1$ **19. (a)** x-intercepts: $-5, -1, 5$; y-intercept: -3 **(b)** No symmetry
(c) Neither **(d)** Increasing: $(-\infty, -3]$ and $[2, \infty)$; decreasing: $[-3, 2]$
(e) Local maximum value is 5 and occurs at $x = -3$.
(f) Local minimum value is -6 and occurs at $x = 2$. **20.** Odd

21. (a) Domain: $\{x \mid x > -3\}$ or $(-3, \infty)$
(b) x-intercept: $-\dfrac{1}{2}$; y-intercept: 1
(c)



(d) Range: $\{y \mid y < 5\}$ or $(-\infty, 5)$

22.



23. (a) $(f + g)(x) = x^2 - 9x - 6$; domain: all real numbers
(b) $\left(\dfrac{f}{g}\right)(x) = \dfrac{x^2 - 5x + 1}{-4x - 7}$; domain: $\left\{x \,\middle|\, x \neq -\dfrac{7}{4}\right\}$
24. (a) $R(x) = -\dfrac{1}{10}x^2 + 150x$ **(b)** \$14,000 **(c)** 750; \$56,250 **(d)** \$75

CHAPTER 6 Exponential and Logarithmic Functions

6.1 Assess Your Understanding *(page 413)*

4. composite function; $f(g(x))$ **5.** F **6.** c **7.** a **8.** F **9. (a)** -1 **(b)** -1 **(c)** 8 **(d)** 0 **(e)** 8 **(f)** -7 **11. (a)** 4 **(b)** 5 **(c)** -1 **(d)** -2
13. (a) 98 **(b)** 49 **(c)** 4 **(d)** 4 **15. (a)** 97 **(b)** $-\dfrac{163}{2}$ **(c)** 1 **(d)** $-\dfrac{3}{2}$ **17. (a)** $2\sqrt{2}$ **(b)** $2\sqrt{2}$ **(c)** 1 **(d)** 0 **19. (a)** $\dfrac{1}{17}$ **(b)** $\dfrac{1}{5}$ **(c)** 1 **(d)** $\dfrac{1}{2}$
21. (a) $\dfrac{3}{\sqrt[3]{4} + 1}$ **(b)** 1 **(c)** $\dfrac{6}{5}$ **(d)** 0 **23. (a)** $(f \circ g)(x) = 6x + 3$; all real numbers **(b)** $(g \circ f)(x) = 6x + 9$; all real numbers
(c) $(f \circ f)(x) = 4x + 9$; all real numbers **(d)** $(g \circ g)(x) = 9x$; all real numbers **25. (a)** $(f \circ g)(x) = 3x^2 + 1$; all real numbers
(b) $(g \circ f)(x) = 9x^2 + 6x + 1$; all real numbers **(c)** $(f \circ f)(x) = 9x + 4$; all real numbers **(d)** $(g \circ g)(x) = x^4$; all real numbers
27. (a) $(f \circ g)(x) = x^4 + 8x^2 + 16$; all real numbers **(b)** $(g \circ f)(x) = x^4 + 4$; all real numbers **(c)** $(f \circ f)(x) = x^4$; all real numbers
(d) $(g \circ g)(x) = x^4 + 8x^2 + 20$; all real numbers **29. (a)** $(f \circ g)(x) = \dfrac{3x}{2 - x}$; $\{x \mid x \neq 0, x \neq 2\}$ **(b)** $(g \circ f)(x) = \dfrac{2(x-1)}{3}$; $\{x \mid x \neq 1\}$
(c) $(f \circ f)(x) = \dfrac{3(x-1)}{4 - x}$; $\{x \mid x \neq 1, x \neq 4\}$ **(d)** $(g \circ g)(x) = x$; $\{x \mid x \neq 0\}$ **31. (a)** $(f \circ g)(x) = \dfrac{4}{4 + x}$; $\{x \mid x \neq -4, x \neq 0\}$
(b) $(g \circ f)(x) = \dfrac{-4(x-1)}{x}$; $\{x \mid x \neq 0, x \neq 1\}$ **(c)** $(f \circ f)(x) = x$; $\{x \mid x \neq 1\}$ **(d)** $(g \circ g)(x) = x$; $\{x \mid x \neq 0\}$

33. **(a)** $(f \circ g)(x) = \sqrt{2x+3}; \left\{x \middle| x \geq -\frac{3}{2}\right\}$ **(b)** $(g \circ f)(x) = 2\sqrt{x} + 3; \{x|x \geq 0\}$ **(c)** $(f \circ f)(x) = \sqrt[4]{x}; \{x|x \geq 0\}$
(d) $(g \circ g)(x) = 4x + 9$; all real numbers **35.** **(a)** $(f \circ g)(x) = x; \{x|x \geq 1\}$ **(b)** $(g \circ f)(x) = |x|$; all real numbers
(c) $(f \circ f)(x) = x^4 + 2x^2 + 2$; all real numbers **(d)** $(g \circ g)(x) = \sqrt{\sqrt{x-1} - 1}; \{x|x \geq 2\}$ **37.** **(a)** $(f \circ g)(x) = -\frac{4x-17}{2x-1}; \left\{x \middle| x \neq 3; x \neq \frac{1}{2}\right\}$
(b) $(g \circ f)(x) = -\frac{3x-3}{2x+8}; \{x|x \neq -4; x \neq -1\}$ **(c)** $(f \circ f)(x) = -\frac{2x+5}{x-2}; \{x|x \neq -1; x \neq 2\}$ **(d)** $(g \circ g)(x) = -\frac{3x-4}{2x-11}; \left\{x \middle| x \neq \frac{11}{2}; x \neq 3\right\}$

39. $(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{2}x\right) = 2\left(\frac{1}{2}x\right) = x$; $(g \circ f)(x) = g(f(x)) = g(2x) = \frac{1}{2}(2x) = x$

41. $(f \circ g)(x) = f(g(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x$; $(g \circ f)(x) = g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x$

43. $(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{2}(x+6)\right) = 2\left[\frac{1}{2}(x+6)\right] - 6 = x + 6 - 6 = x$; $(g \circ f)(x) = g(f(x)) = g(2x-6) = \frac{1}{2}(2x - 6 + 6) = \frac{1}{2}(2x) = x$

45. $(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{a}(x-b)\right) = a\left[\frac{1}{a}(x-b)\right] + b = x$; $(g \circ f)(x) = g(f(x)) = g(ax+b) = \frac{1}{a}(ax + b - b) = x$

47. $f(x) = x^4; g(x) = 2x + 3$ (Other answers are possible.) **49.** $f(x) = \sqrt{x}; g(x) = x^2 + 1$ (Other answers are possible.)
51. $f(x) = |x|; g(x) = 2x + 1$ (Other answers are possible.) **53.** $(f \circ g)(x) = 11$; $(g \circ f)(x) = 2$ **55.** $-3, 3$ **57.** **(a)** $(f \circ g)(x) = acx + ad + b$
(b) $(g \circ f)(x) = acx + bc + d$ **(c)** The domains of both $f \circ g$ and $g \circ f$ are all real numbers. **(d)** $f \circ g = g \circ f$ when $ad + b = bc + d$

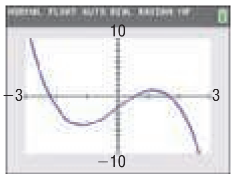
59. $S(t) = \frac{16}{9}\pi t^6$ **61.** $C(t) = 15{,}000 + 800{,}000t - 40{,}000t^2$ **63.** $C(p) = \frac{2\sqrt{100-p}}{25} + 600, 0 \leq p \leq 100$ **65.** $V(r) = 2\pi r^3$

67. **(a)** $f(x) = 0.9428x$ **(b)** $g(x) = 126.457x$ **(c)** $g(f(x)) = g(0.9428x) = 119.2236596x$ **(d)** 119,223.6596 yen **69.** **(a)** $f(p) = p - 200$
(b) $g(p) = 0.8p$ **(c)** $(f \circ g)(p) = 0.8p - 200$; $(g \circ f)(p) = 0.8p - 160$; The 20% discount followed by the \$200 rebate is the better deal. **71.** 15
73. f is an odd function, so $f(-x) = -f(x)$. g is an even function, so $g(-x) = g(x)$. Then $(f \circ g)(-x) = f(g(-x)) = f(g(x)) = (f \circ g)(x)$. So $f \circ g$ is even. Also, $(g \circ f)(-x) = g(f(-x)) = g(-f(x)) = g(f(x)) = (g \circ f)(x)$, so $g \circ f$ is even.

74. $(f + g)(x) = 4x + 3$; Domain: all real numbers
$(f - g)(x) = 2x + 13$; Domain: all real numbers
$(f \cdot g)(x) = 3x^2 - 7x - 40$; Domain: all real numbers
$\left(\frac{f}{g}\right)(x) = \frac{3x+8}{x-5}$; Domain: $\{x|x \neq 5\}$

75. $\frac{1}{4}, 4$

76.



Local minimum: -5.08 at $x = -1.15$
Local maximum: 1.08 at $x = 1.15$
Decreasing: $[-3, -1.15]$; $[1.15, 3]$
Increasing: $[-1.15, 1.15]$

77. Domain: $\{x|x \neq 3\}$
Vertical asymptote: $x = 3$
Oblique asymptote: $y = x + 9$

6.2 Assess Your Understanding *(page 424)*

5. $f(x_1) \neq f(x_2)$ **6.** one-to-one **7.** 3 **8.** $y = x$ **9.** $[4, \infty)$ **10.** T **11.** a **12.** d **13.** one-to-one **15.** not one-to-one
17. not one-to-one **19.** one-to-one **21.** one-to-one **23.** not one-to-one **25.** one-to-one

27.

Annual Rainfall (inches)		Location
49.7	→	Atlanta, Georgia
43.8	→	Boston, Massachusetts
4.2	→	Las Vegas, Nevada
61.9	→	Miami, Florida
12.8	→	Los Angeles, California

Domain: $\{49.7, 43.8, 4.2, 61.9, 12.8\}$
Range: {Atlanta, Boston, Las Vegas, Miami, Los Angeles}

29.

Monthly Cost of Life Insurance		Age
\$10.59	→	30
\$12.52	→	40
\$15.94	→	45

Domain: {\$10.59, \$12.52, \$15.94}
Range: $\{30, 40, 45\}$

31. $\{(5, -3), (9, -2), (2, -1), (11, 0), (-5, 1)\}$
Domain: $\{5, 9, 2, 11, -5\}$
Range: $\{-3, -2, -1, 0, 1\}$

33. $\{(1, -2), (2, -3), (0, -10), (9, 1), (4, 2)\}$
Domain: $\{1, 2, 0, 9, 4\}$
Range: $\{-2, -3, -10, 1, 2\}$

35. $f(g(x)) = f\left(\frac{1}{3}(x-4)\right) = 3\left[\frac{1}{3}(x-4)\right] + 4 = (x-4) + 4 = x$
$g(f(x)) = g(3x+4) = \frac{1}{3}[(3x+4) - 4] = \frac{1}{3}(3x) = x$

37. $f(g(x)) = f\left(\frac{x}{4} + 2\right) = 4\left[\frac{x}{4} + 2\right] - 8 = (x + 8) - 8 = x$
$g(f(x)) = g(4x - 8) = \frac{4x-8}{4} + 2 = (x - 2) + 2 = x$

39. $f(g(x)) = f(\sqrt[3]{x+8}) = (\sqrt[3]{x+8})^3 - 8 = (x+8) - 8 = x$
$g(f(x)) = g(x^3 - 8) = \sqrt[3]{(x^3 - 8) + 8} = \sqrt[3]{x^3} = x$

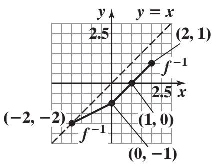
41. $f(g(x)) = f\left(\frac{1}{x}\right) = \frac{1}{\left(\frac{1}{x}\right)} = x; x \neq 0$, $g(f(x)) = g\left(\frac{1}{x}\right) = \frac{1}{\left(\frac{1}{x}\right)} = x,\ x \neq 0$

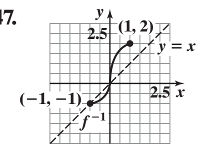
43. $f(g(x)) = f\left(\dfrac{4x-3}{2-x}\right) = \dfrac{2\left(\dfrac{4x-3}{2-x}\right)+3}{\dfrac{4x-3}{2-x}+4}$

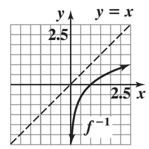
$= \dfrac{2(4x-3)+3(2-x)}{4x-3+4(2-x)} = \dfrac{5x}{5} = x, x \neq 2$

$g(f(x)) = g\left(\dfrac{2x+3}{x+4}\right) = \dfrac{4\left(\dfrac{2x+3}{x+4}\right)-3}{2-\dfrac{2x+3}{x+4}}$

$= \dfrac{4(2x+3)-3(x+4)}{2(x+4)-(2x+3)} = \dfrac{5x}{5} = x, x \neq -4$

45.


47.


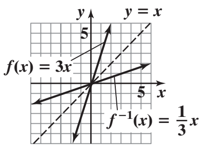
49.


51. (a) $f^{-1}(x) = \dfrac{1}{3}x$

$f(f^{-1}(x)) = f\left(\dfrac{1}{3}x\right) = 3\left(\dfrac{1}{3}x\right) = x$

$f^{-1}(f(x)) = f^{-1}(3x) = \dfrac{1}{3}(3x) = x$

(b) Domain of f = Range of f^{-1} = All real numbers; Range of f = Domain of f^{-1} = All real numbers

(c)


53. (a) $f^{-1}(x) = \dfrac{x}{4} - \dfrac{1}{2}$

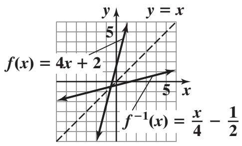
$f(f^{-1}(x)) = f\left(\dfrac{x}{4}-\dfrac{1}{2}\right) = 4\left(\dfrac{x}{4}-\dfrac{1}{2}\right) + 2$

$= (x-2) + 2 = x$

$f^{-1}(f(x)) = f^{-1}(4x+2) = \dfrac{4x+2}{4} - \dfrac{1}{2}$

$= \left(x + \dfrac{1}{2}\right) - \dfrac{1}{2} = x$

(b) Domain of f = Range of f^{-1} = All real numbers; Range of f = Domain of f^{-1} = All real numbers

(c)


55. (a) $f^{-1}(x) = \sqrt[3]{x+1}$

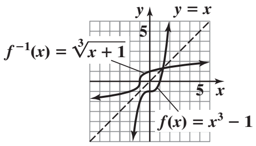
$f(f^{-1}(x)) = f(\sqrt[3]{x+1})$

$= (\sqrt[3]{x+1})^3 - 1 = x$

$f^{-1}(f(x)) = f^{-1}(x^3 - 1)$

$= \sqrt[3]{(x^3-1)+1} = x$

(b) Domain of f = Range of f^{-1} = All real numbers; Range of f = Domain of f^{-1} = All real numbers

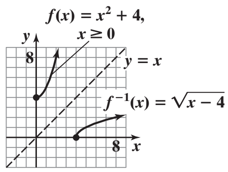
(c)


57. (a) $f^{-1}(x) = \sqrt{x-4}, x \geq 4$

$f(f^{-1}(x)) = f(\sqrt{x-4}) = (\sqrt{x-4})^2 + 4 = x$

$f^{-1}(f(x)) = f^{-1}(x^2+4) = \sqrt{(x^2+4)-4} = \sqrt{x^2} = x, x \geq 0$

(b) Domain of f = Range of f^{-1} = $\{x \mid x \geq 0\}$; Range of f = Domain of f^{-1} = $\{x \mid x \geq 4\}$

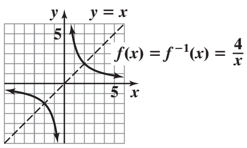
(c)


59. (a) $f^{-1}(x) = \dfrac{4}{x}$

$f(f^{-1}(x)) = f\left(\dfrac{4}{x}\right) = \dfrac{4}{\left(\dfrac{4}{x}\right)} = x$

$f^{-1}(f(x)) = f^{-1}\left(\dfrac{4}{x}\right) = \dfrac{4}{\left(\dfrac{4}{x}\right)} = x$

(b) Domain of f = Range of f^{-1} = $\{x \mid x \neq 0\}$; Range of f = Domain of f^{-1} = $\{x \mid x \neq 0\}$

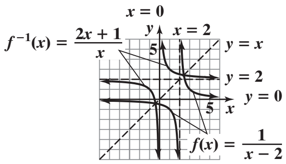
(c)


61. (a) $f^{-1}(x) = \dfrac{2x+1}{x}$

$f(f^{-1}(x)) = f\left(\dfrac{2x+1}{x}\right) = \dfrac{1}{\dfrac{2x+1}{x} - 2} = \dfrac{x}{(2x+1)-2x} = x$

$f^{-1}(f(x)) = f^{-1}\left(\dfrac{1}{x-2}\right) = \dfrac{2\left(\dfrac{1}{x-2}\right)+1}{\dfrac{1}{x-2}} = \dfrac{2+(x-2)}{1} = x$

(b) Domain of f = Range of f^{-1} = $\{x \mid x \neq 2\}$; Range of f = Domain of f^{-1} = $\{x \mid x \neq 0\}$

(c)


63. (a) $f^{-1}(x) = \dfrac{2-3x}{x}$

$$f(f^{-1}(x)) = f\left(\frac{2-3x}{x}\right) = \frac{2}{3+\dfrac{2-3x}{x}} = \frac{2x}{3x+2-3x} = \frac{2x}{2} = x$$

$$f^{-1}(f(x)) = f^{-1}\left(\frac{2}{3+x}\right) = \frac{2-3\left(\dfrac{2}{3+x}\right)}{\dfrac{2}{3+x}} = \frac{2(3+x)-3\cdot 2}{2} = \frac{2x}{2} = x$$

(b) Domain of f = Range of $f^{-1} = \{x \mid x \neq -3\}$; Range of f = Domain of $f^{-1} = \{x \mid x \neq 0\}$

65. (a) $f^{-1}(x) = \dfrac{-2x}{x-3}$

$$f(f^{-1}(x)) = f\left(\frac{-2x}{x-3}\right) = \frac{3\left(\dfrac{-2x}{x-3}\right)}{\dfrac{-2x}{x-3}+2} = \frac{3(-2x)}{-2x+2(x-3)} = \frac{-6x}{-6} = x$$

$$f^{-1}(f(x)) = f^{-1}\left(\frac{3x}{x+2}\right) = \frac{-2\left(\dfrac{3x}{x+2}\right)}{\dfrac{3x}{x+2}-3} = \frac{-2(3x)}{3x-3(x+2)} = \frac{-6x}{-6} = x$$

(b) Domain of f = Range of $f^{-1} = \{x \mid x \neq -2\}$; Range of f = Domain of $f^{-1} = \{x \mid x \neq 3\}$

67. (a) $f^{-1}(x) = \dfrac{x}{3x-2}$

$$f(f^{-1}(x)) = f\left(\frac{x}{3x-2}\right) = \frac{2\left(\dfrac{x}{3x-2}\right)}{3\left(\dfrac{x}{3x-2}\right)-1} = \frac{2x}{3x-(3x-2)} = \frac{2x}{2} = x$$

$$f^{-1}(f(x)) = f^{-1}\left(\frac{2x}{3x-1}\right) = \frac{\dfrac{2x}{3x-1}}{3\left(\dfrac{2x}{3x-1}\right)-2} = \frac{2x}{6x-2(3x-1)} = \frac{2x}{2} = x$$

(b) Domain of f = Range of $f^{-1} = \left\{x \,\middle|\, x \neq \frac{1}{3}\right\}$; Range of f = Domain of $f^{-1} = \left\{x \,\middle|\, x \neq \frac{2}{3}\right\}$

69. (a) $f^{-1}(x) = \dfrac{3x+4}{2x-3}$

$$f(f^{-1}(x)) = f\left(\frac{3x+4}{2x-3}\right) = \frac{3\left(\dfrac{3x+4}{2x-3}\right)+4}{2\left(\dfrac{3x+4}{2x-3}\right)-3} = \frac{3(3x+4)+4(2x-3)}{2(3x+4)-3(2x-3)} = \frac{17x}{17} = x$$

$$f^{-1}(f(x)) = f^{-1}\left(\frac{3x+4}{2x-3}\right) = \frac{3\left(\dfrac{3x+4}{2x-3}\right)+4}{2\left(\dfrac{3x+4}{2x-3}\right)-3} = \frac{3(3x+4)+4(2x-3)}{2(3x+4)-3(2x-3)} = \frac{17x}{17} = x$$

(b) Domain of f = Range of $f^{-1} = \left\{x \,\middle|\, x \neq \frac{3}{2}\right\}$; Range of f = Domain of $f^{-1} = \left\{x \,\middle|\, x \neq \frac{3}{2}\right\}$

71. (a) $f^{-1}(x) = \dfrac{-2x+3}{x-2}$

$$f(f^{-1}(x)) = f\left(\frac{-2x+3}{x-2}\right) = \frac{2\left(\dfrac{-2x+3}{x-2}\right)+3}{\dfrac{-2x+3}{x-2}+2} = \frac{2(-2x+3)+3(x-2)}{-2x+3+2(x-2)} = \frac{-x}{-1} = x$$

$$f^{-1}(f(x)) = f^{-1}\left(\frac{2x+3}{x+2}\right) = \frac{-2\left(\dfrac{2x+3}{x+2}\right)+3}{\dfrac{2x+3}{x+2}-2} = \frac{-2(2x+3)+3(x+2)}{2x+3-2(x+2)} = \frac{-x}{-1} = x$$

(b) Domain of f = Range of $f^{-1} = \{x \mid x \neq -2\}$; Range of f = Domain of $f^{-1} = \{x \mid x \neq 2\}$

73. (a) $f^{-1}(x) = \dfrac{2}{\sqrt{1-2x}}$

$$f(f^{-1}(x)) = f\left(\frac{2}{\sqrt{1-2x}}\right) = \frac{\dfrac{4}{1-2x} - 4}{2\cdot\dfrac{4}{1-2x}} = \frac{4-4(1-2x)}{2\cdot 4} = \frac{8x}{8} = x$$

$$f^{-1}(f(x)) = f^{-1}\left(\frac{x^2-4}{2x^2}\right) = \frac{2}{\sqrt{1-2\left(\dfrac{x^2-4}{2x^2}\right)}} = \frac{2}{\sqrt{\dfrac{4}{x^2}}} = \sqrt{x^2} = x, \text{ since } x > 0$$

(b) Domain of f = Range of $f^{-1} = \{x \mid x > 0\}$; Range of f = Domain of $f^{-1} = \left\{x \,\middle|\, x < \dfrac{1}{2}\right\}$

75. (a) 0 **(b)** 2 **(c)** 0 **(d)** 1 **77.** 7 **79.** Domain of f^{-1}: $[-2, \infty)$; range of f^{-1}: $[5, \infty)$ **81.** Domain of g^{-1}: $[0, \infty)$; range of g^{-1}: $(-\infty, 0]$

83. Increasing on the interval $[f(0), f(5)]$ **85.** $f^{-1}(x) = \dfrac{1}{m}(x-b), m \neq 0$ **87.** Quadrant I

89. Possible answer: $f(x) = |x|, x \geq 0$, is one-to-one; $f^{-1}(x) = x, x \geq 0$

91. (a) $r(d) = \dfrac{d + 90.39}{6.97}$

(b) $r(d(r)) = \dfrac{6.97r - 90.39 + 90.39}{6.97} = \dfrac{6.97r}{6.97} = r$

$d(r(d)) = 6.97\left(\dfrac{d+90.39}{6.97}\right) - 90.39 = d + 90.39 - 90.39 = d$

(c) 56 miles per hour

93. (a) 77.6 kg

(b) $h(W) = \dfrac{W-50}{2.3} + 60 = \dfrac{W+88}{2.3}$

(c) $h(W(h)) = \dfrac{50 + 2.3(h-60) + 88}{2.3} = \dfrac{2.3h}{2.3} = h$

$W(h(W)) = 50 + 2.3\left(\dfrac{W+88}{2.3} - 60\right)$

$= 50 + W + 88 - 138 = W$

(d) 73 inches

95. (a) $\{g \mid 37{,}450 \leq g \leq 90{,}750\}$

(b) $\{T \mid 5156.25 \leq T \leq 18{,}481.25\}$

(c) $g(T) = \dfrac{T - 5156.25}{0.25} + 37{,}450$

Domain: $\{T \mid 5156.25 \leq T \leq 18{,}481.25\}$

Range: $\{g \mid 37{,}450 \leq g \leq 90{,}750\}$

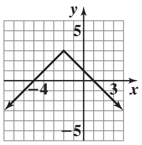
97. (a) t represents time, so $t \geq 0$.

(b) $t(H) = \sqrt{\dfrac{H-100}{-4.9}} = \sqrt{\dfrac{100-H}{4.9}}$

(c) 2.02 seconds

99. $f^{-1}(x) = \dfrac{-dx+b}{cx-a}$; $f = f^{-1}$ if $a = -d$ **103.** No

107.



108. Zeros: $\dfrac{-5-\sqrt{13}}{6}, \dfrac{-5+\sqrt{13}}{6}$, x-intercepts: $\dfrac{-5-\sqrt{13}}{6}, \dfrac{-5+\sqrt{13}}{6}$

109. Domain: $\left\{x \,\middle|\, x \neq -\dfrac{3}{2}, x \neq 2\right\}$; Vertical asymptote: $x = -\dfrac{3}{2}$,

Horizontal asymptote: $y = 3$

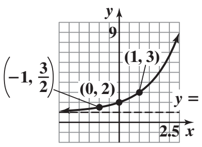
110. $6xh + 3h^2 - 7h$

6.3 Assess Your Understanding *(page 439)*

6. Exponential function; growth factor; initial value **7.** a **8.** T **9.** T **10.** $\left(-1, \dfrac{1}{a}\right)$; $(0, 1)$; $(1, a)$ **11.** 4 **12.** F **13.** b **14.** c

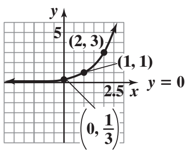
15. (a) 8.815 **(b)** 8.821 **(c)** 8.824 **(d)** 8.825 **17. (a)** 21.217 **(b)** 22.217 **(c)** 22.440 **(d)** 22.459 **19.** 1.265 **21.** 0.347 **23.** 3.320 **25.** 149.952 **27.** Neither **29.** Exponential; $H(x) = 4^x$ **31.** Exponential; $f(x) = 3(2^x)$ **33.** Linear; $H(x) = 2x + 4$ **35.** B **37.** D **39.** A **41.** E

43.



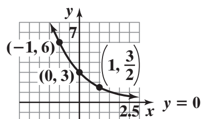
Domain: All real numbers
Range: $\{y \mid y > 1\}$ or $(1, \infty)$
Horizontal asymptote: $y = 1$

45.



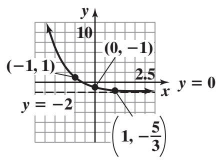
Domain: All real numbers
Range: $\{y \mid y > 0\}$ or $(0, \infty)$
Horizontal asymptote: $y = 0$

47.



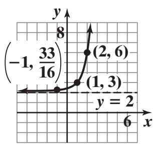
Domain: All real numbers
Range: $\{y \mid y > 0\}$ or $(0, \infty)$
Horizontal asymptote: $y = 0$

49.



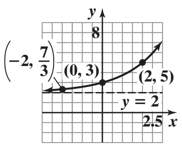
Domain: All real numbers
Range: $\{y \mid y > -2\}$ or $(-2, \infty)$
Horizontal asymptote: $y = -2$

51.



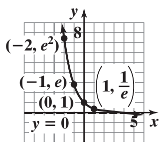
Domain: All real numbers
Range: $\{y \mid y > 2\}$ or $(2, \infty)$
Horizontal asymptote: $y = 2$

53.



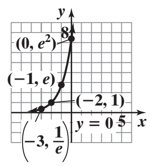
Domain: All real numbers
Range: $\{y \mid y > 2\}$ or $(2, \infty)$
Horizontal asymptote: $y = 2$

55.



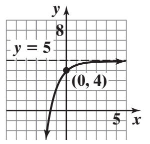
Domain: All real numbers
Range: $\{y \mid y > 0\}$ or $(0, \infty)$
Horizontal asymptote: $y = 0$

57.



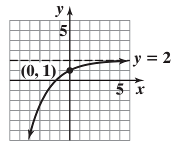
Domain: All real numbers
Range: $\{y \mid y > 0\}$ or $(0, \infty)$
Horizontal asymptote: $y = 0$

59.



Domain: All real numbers
Range: $\{y \mid y < 5\}$ or $(-\infty, 5)$
Horizontal asymptote: $y = 5$

61.



Domain: All real numbers
Range: $\{y \mid y < 2\}$ or $(-\infty, 2)$
Horizontal asymptote: $y = 2$

63. $\{3\}$ **65.** $\{-4\}$ **67.** $\{2\}$ **69.** $\left\{\frac{3}{2}\right\}$ **71.** $\{-\sqrt{2}, 0, \sqrt{2}\}$

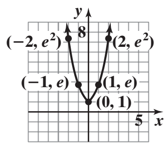
73. $\{6\}$ **75.** $\{-1, 7\}$ **77.** $\{-4, 2\}$ **79.** $\{-4\}$ **81.** $\{1, 2\}$ **83.** $\frac{1}{49}$

85. $\frac{1}{4}$ **87.** 5 **89.** $f(x) = 3^x$ **91.** $f(x) = -6^x$ **93.** $f(x) = 3^x + 2$

95. (a) 16; $(4, 16)$ **(b)** -4; $\left(-4, \frac{1}{16}\right)$ **97. (a)** $\frac{9}{4}$; $\left(-1, \frac{9}{4}\right)$ **(b)** 3; $(3, 66)$

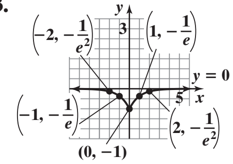
99. (a) 60; $(-6, 60)$ **(b)** -4; $(-4, 12)$ **(c)** -2

101.



Domain: $(-\infty, \infty)$
Range: $[1, \infty)$
Intercept: $(0, 1)$

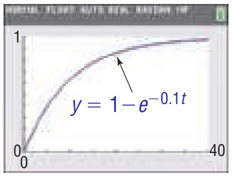
103.



Domain: $(-\infty, \infty)$
Range: $[-1, 0)$
Intercept: $(0, -1)$

105. (a) 74% **(b)** 47% **(c)** Each pane allows only 97% of light to pass through. **107. (a)** \$16,231 **(b)** \$8626 **(c)** As each year passes, the sedan is worth 90% of its value the previous year. **109. (a)** 30% **(b)** 9% **(c)** Each year only 30% of the previous survivors survive again. **111.** 3.35 mg; 0.45 mg

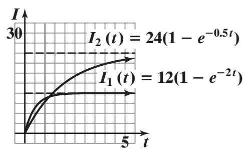
113. (a) 0.632 **(b)** 0.982 **(c)** 1 **(d)**



(e) About 7 min

115. (a) 0.0516 **(b)** 0.0888 **117. (a)** 70.95% **(b)** 72.62% **(c)** 100%

119. (a) 5.41 amp, 7.59 amp, 10.38 amp **(b)** 12 amp **(c), (f)**



(d) 3.34 amp, 5.31 amp, 9.44 amp
(e) 24 amp

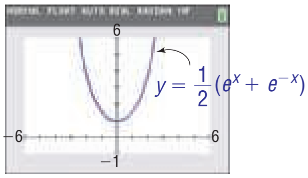
121. 36 **123.**

Final Denominator	Value of Expression	Compare Value to $e \approx$ 2.718281828
1 + 1	2.5	$2.5 < e$
2 + 2	2.8	$2.8 > e$
3 + 3	2.7	$2.7 < e$
4 + 4	2.721649485	$2.721649485 > e$
5 + 5	2.717770035	$2.717770035 < e$
6 + 6	2.718348855	$2.718348855 > e$

125. $f(A + B) = a^{A+B} = a^A \cdot a^B = f(A) \cdot f(B)$ **127.** $f(\alpha x) = a^{\alpha x} = (a^x)^\alpha = [f(x)]^\alpha$

129. (a) $f(-x) = \frac{1}{2}(e^{-x} + e^{-(-x)})$
$= \frac{1}{2}(e^{-x} + e^x)$
$= \frac{1}{2}(e^x + e^{-x})$
$= f(x)$

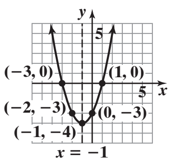
(b)



(c) $(\cosh x)^2 - (\sinh x)^2$
$= \left[\frac{1}{2}(e^x + e^{-x})\right]^2 - \left[\frac{1}{2}(e^x - e^{-x})\right]^2$
$= \frac{1}{4}[e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}]$
$= \frac{1}{4}(4) = 1$

131. 59 minutes **135.** $a^{-x} = (a^{-1})^x = \left(\frac{1}{a}\right)^x$ **136.** $(-\infty, -5] \cup [-2, 2]$ **137.** $(2, \infty)$ **138.** $f(x) = -2x^2 + 12x - 13$

139. (a)



(b) Domain: $(-\infty, \infty)$; Range: $[-4, \infty)$
(c) Decreasing: $(-\infty, -1]$; Increasing: $[-1, \infty)$

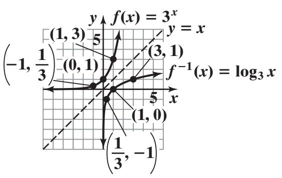
6.4 Assess Your Understanding *(page 454)*

4. $\{x \mid x > 0\}$ or $(0, \infty)$ **5.** $\left(\frac{1}{a}, -1\right)$, $(1, 0)$, $(a, 1)$ **6.** 1 **7.** F **8.** T **9.** a **10.** c **11.** $\log_3 9 = 2$ **13.** $\log_a 1.6 = 2$ **15.** $\log_2 7.2 = x$ **17.** $\ln 8 = x$

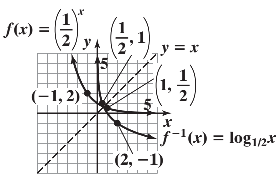
19. $2^3 = 8$ **21.** $a^6 = 3$ **23.** $3^x = 2$ **25.** $e^x = 4$ **27.** 0 **29.** 2 **31.** -4 **33.** $\frac{1}{2}$ **35.** 4 **37.** $\frac{1}{2}$ **39.** $\{x \mid x > 3\}$; $(3, \infty)$

41. All real numbers except 0; $\{x \mid x \neq 0\}$; $(-\infty, 0) \cup (0, \infty)$ **43.** $\{x \mid x > 10\}$; $(10, \infty)$ **45.** $\{x \mid x > -1\}$; $(-1, \infty)$ **47.** $\{x \mid x < -1 \text{ or } x > 0\}$; $(-\infty, -1) \cup (0, \infty)$ **49.** $\{x \mid x \geq 1\}$; $[1, \infty)$ **51.** 0.511 **53.** 30.099 **55.** 2.303 **57.** -53.991 **59.** $\sqrt{2}$

61.



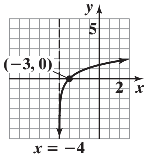
63.



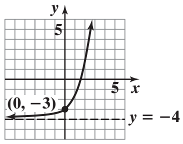
65. B **67.** D **69.** A **71.** E

73. (a) Domain: $(-4, \infty)$

(b)

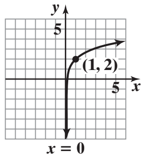


(c) Range: $(-\infty, \infty)$
Vertical asymptote: $x = -4$
(d) $f^{-1}(x) = e^x - 4$
(e) Domain of f^{-1}: $(-\infty, \infty)$
Range of f^{-1}: $(-4, \infty)$
(f)

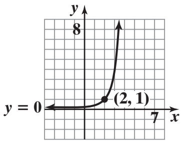


75. (a) Domain: $(0, \infty)$

(b)

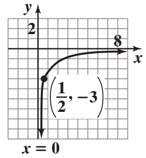


(c) Range: $(-\infty, \infty)$
Vertical asymptote: $x = 0$
(d) $f^{-1}(x) = e^{x-2}$
(e) Domain of f^{-1}: $(-\infty, \infty)$
Range of f^{-1}: $(0, \infty)$
(f)

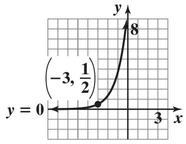


77. (a) Domain: $(0, \infty)$

(b)

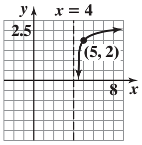


(c) Range: $(-\infty, \infty)$
Vertical asymptote: $x = 0$
(d) $f^{-1}(x) = \frac{1}{2}e^{x+3}$
(e) Domain of f^{-1}: $(-\infty, \infty)$
Range of f^{-1}: $(0, \infty)$
(f)

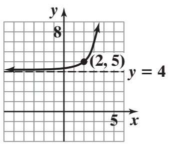


79. (a) Domain: $(4, \infty)$

(b)

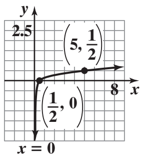


(c) Range: $(-\infty, \infty)$
Vertical asymptote: $x = 4$
(d) $f^{-1}(x) = 10^{x-2} + 4$
(e) Domain of f^{-1}: $(-\infty, \infty)$
Range of f^{-1}: $(4, \infty)$
(f)

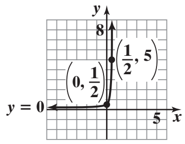


81. (a) Domain: $(0, \infty)$

(b)

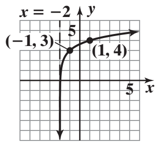


(c) Range: $(-\infty, \infty)$
Vertical asymptote: $x = 0$
(d) $f^{-1}(x) = \frac{1}{2} \cdot 10^{2x}$
(e) Domain of f^{-1}: $(-\infty, \infty)$
Range of f^{-1}: $(0, \infty)$
(f)

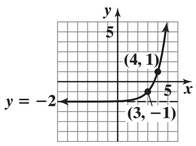


83. (a) Domain: $(-2, \infty)$

(b)

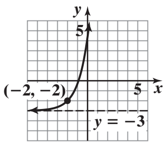


(c) Range: $(-\infty, \infty)$
Vertical asymptote: $x = -2$
(d) $f^{-1}(x) = 3^{x-3} - 2$
(e) Domain of f^{-1}: $(-\infty, \infty)$
Range of f^{-1}: $(-2, \infty)$
(f)

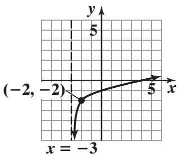


85. (a) Domain: $(-\infty, \infty)$

(b)

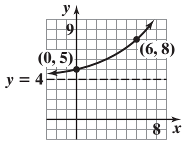


(c) Range: $(-3, \infty)$
Horizontal asymptote: $y = -3$
(d) $f^{-1}(x) = \ln(x + 3) - 2$
(e) Domain of f^{-1}: $(-3, \infty)$
Range of f^{-1}: $(-\infty, \infty)$
(f)

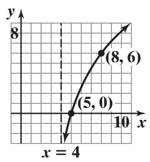


87. (a) Domain: $(-\infty, \infty)$

(b)



(c) Range: $(4, \infty)$
Horizontal asymptote: $y = 4$
(d) $f^{-1}(x) = 3\log_2(x - 4)$
(e) Domain of f^{-1}: $(4, \infty)$
Range of f^{-1}: $(-\infty, \infty)$
(f)



89. $\{9\}$ **91.** $\left\{\frac{7}{2}\right\}$ **93.** $\{2\}$ **95.** $\{5\}$ **97.** $\{3\}$

99. $\{2\}$ **101.** $\left\{\frac{\ln 10}{3}\right\}$ **103.** $\left\{\frac{\ln 8 - 5}{2}\right\}$

105. $\{-2\sqrt{2}, 2\sqrt{2}\}$

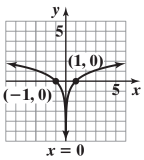
107. $\{-1\}$

109. $\left\{5 \ln \frac{7}{5}\right\}$

111. $\left\{2 - \log \frac{5}{2}\right\}$

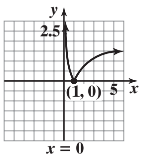
113. (a) $\left\{x \middle| x > -\frac{1}{2}\right\}; \left(-\frac{1}{2}, \infty\right)$
(b) 2; (40, 2) **(c)** 121; (121, 3) **(d)** 4

115.



Domain: $\{x \mid x \neq 0\}$
Range: $(-\infty, \infty)$
Intercepts: $(-1, 0)$, $(1, 0)$

117.



Domain: $\{x \mid x > 0\}$
Range: $\{y \mid y \geq 0\}$
Intercept: $(1, 0)$

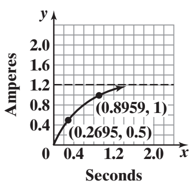
119. (a) 1 **(b)** 2 **(c)** 3
(d) It increases. **(e)** 0.000316
(f) 3.981×10^{-8}

121. (a) 5.97 km **(b)** 0.90 km

123. (a) 6.93 min **(b)** 16.09 min

125. $h \approx 2.29$, so the time between injections is about 2 h, 17 min.

127. 0.2695 s
0.8959 s



129. 50 decibels (dB) **131.** 90 dB **133.** 8.1 **135. (a)** $k \approx 11.216$ **(b)** 6.73 **(c)** 0.41% **(d)** 0.14%

137. Because $y = \log_1 x$ means $1^y = 1 = x$, which cannot be true for $x \neq 1$ **139.** Zeros: $-3, -\frac{1}{2}, \frac{1}{2}, 3$; x-intercepts: $-3, -\frac{1}{2}, \frac{1}{2}, 3$
140. 12 **141.** $f(1) = -5; f(2) = 17$ **142.** $3 + i; f(x) = x^4 - 7x^3 + 14x^2 + 2x - 20; a = 1$

6.5 Assess Your Understanding *(page 465)*

1. 0 **2.** M **3.** r **4.** $\log_a M; \log_a N$ **5.** $\log_a M; \log_a N$ **6.** $r \log_a M$ **7.** 7 **8.** F **9.** F **10.** F **11.** b **12.** b **13.** 71 **15.** -4 **17.** 7 **19.** 1 **21.** 1

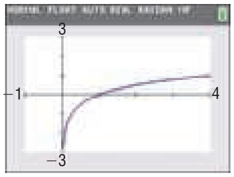
23. 3 **25.** $\frac{5}{4}$ **27.** 4 **29.** $a + b$ **31.** $b - a$ **33.** $3a$ **35.** $\frac{1}{5}(a + b)$ **37.** $2 + \log_5 x$ **39.** $3 \log_2 z$ **41.** $1 + \ln x$ **43.** $\ln x - x$ **45.** $2 \log_a u + 3 \log_a v$

47. $2 \ln x + \frac{1}{2}\ln(1 - x)$ **49.** $3 \log_2 x - \log_2(x - 3)$ **51.** $\log x + \log(x + 2) - 2\log(x + 3)$ **53.** $\frac{1}{3}\ln(x - 2) + \frac{1}{3}\ln(x + 1) - \frac{2}{3}\ln(x + 4)$

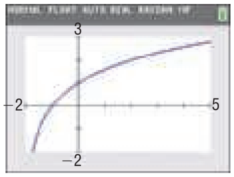
55. $\ln 5 + \ln x + \frac{1}{2}\ln(1 + 3x) - 3\ln(x - 4)$ **57.** $\log_5(u^3v^4)$ **59.** $\log_3\left(\frac{1}{x^{5/2}}\right)$ **61.** $\log_4\left[\frac{x - 1}{(x + 1)^4}\right]$ **63.** $-2\ln(x - 1)$ **65.** $\log_2[x(3x - 2)^4]$

67. $\log_a\left(\frac{25x^6}{\sqrt{2x + 3}}\right)$ **69.** $\log_2\left[\frac{(x + 1)^2}{(x + 3)(x - 1)}\right]$ **71.** 2.771 **73.** -3.880 **75.** 5.615 **77.** 0.874

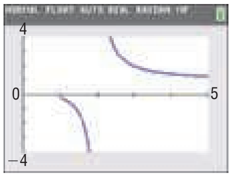
79. $y = \frac{\log x}{\log 4}$



81. $y = \frac{\log(x + 2)}{\log 2}$



83. $y = \frac{\log(x + 1)}{\log(x - 1)}$



85. (a) $(f \circ g)(x) = x$; $\{x \mid x \text{ is any real number}\}$ or $(-\infty, \infty)$
(b) $(g \circ f)(x) = x$; $\{x \mid x > 0\}$ or $(0, \infty)$ **(c)** 5
(d) $(f \circ h)(x) = \ln x^2$; $\{x \mid x \neq 0\}$ or $(-\infty, 0) \cup (0, \infty)$ **(e)** 2

87. $y = Cx$ **89.** $y = Cx(x + 1)$ **91.** $y = Ce^{3x}$ **93.** $y = Ce^{-4x} + 3$

95. $y = \frac{\sqrt[3]{C}(2x + 1)^{1/6}}{(x + 4)^{1/9}}$ **97.** 3 **99.** 1

101. $\log_a(x + \sqrt{x^2 - 1}) + \log_a(x - \sqrt{x^2 - 1}) = \log_a[(x + \sqrt{x^2 - 1})(x - \sqrt{x^2 - 1})] = \log_a[x^2 - (x^2 - 1)] = \log_a 1 = 0$

103. $\ln(1 + e^{2x}) = \ln[e^{2x}(e^{-2x} + 1)] = \ln e^{2x} + \ln(e^{-2x} + 1) = 2x + \ln(1 + e^{-2x})$

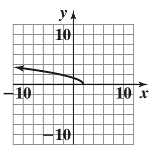
105. $y = f(x) = \log_a x$; $a^y = x$ implies $a^y = \left(\frac{1}{a}\right)^{-y} = x$, so $-y = \log_{1/a} x = -f(x)$.

107. $f(x) = \log_a x$; $f\left(\frac{1}{x}\right) = \log_a \frac{1}{x} = \log_a 1 - \log_a x = -f(x)$

109. $\log_a \frac{M}{N} = \log_a(M \cdot N^{-1}) = \log_a M + \log_a N^{-1} = \log_a M - \log_a N$, since $a^{\log_a N^{-1}} = N^{-1}$ implies $a^{-\log_a N^{-1}} = N$; that is, $\log_a N = -\log_a N^{-1}$.

115. $\{-1.78, 1.29, 3.49\}$ **116.** A repeated real solution (double root) **117.** $-2, \frac{1}{5}, \frac{-5 - \sqrt{21}}{2}, \frac{-5 + \sqrt{21}}{2}$

118.



Domain: $\{x \mid x \leq 2\}$ or $(-\infty, 2]$
Range: $\{y \mid y \geq 0\}$ or $[0, \infty)$

6.6 Assess Your Understanding *(page 472)*

5. $\{16\}$ **7.** $\left\{\frac{16}{5}\right\}$ **9.** $\{6\}$ **11.** $\{16\}$ **13.** $\left\{\frac{1}{3}\right\}$ **15.** $\{3\}$ **17.** $\{5\}$ **19.** $\left\{\frac{21}{8}\right\}$ **21.** $\{-6\}$ **23.** $\{-2\}$ **25.** $\{-1 + \sqrt{1 + e^4}\} \approx \{6.456\}$

27. $\left\{\frac{-5 + 3\sqrt{5}}{2}\right\} \approx \{0.854\}$ **29.** $\{2\}$ **31.** $\left\{\frac{9}{2}\right\}$ **33.** $\{7\}$ **35.** $\{-2 + 4\sqrt{2}\}$ **37.** $\{-\sqrt{3}, \sqrt{3}\}$ **39.** $\left\{\frac{1}{3}, 729\right\}$ **41.** $\{8\}$

43. $\{\log_2 10\} = \left\{\frac{\ln 10}{\ln 2}\right\} \approx \{3.322\}$ **45.** $\{-\log_8 1.2\} = \left\{-\frac{\ln 1.2}{\ln 8}\right\} \approx \{-0.088\}$ **47.** $\left\{\frac{1}{3}\log_2 \frac{8}{5}\right\} = \left\{\frac{\ln \frac{8}{5}}{3 \ln 2}\right\} \approx \{0.226\}$

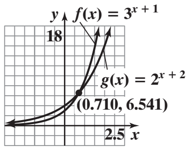
49. $\left\{\frac{\ln 3}{2\ln 3 + \ln 4}\right\} \approx \{0.307\}$ **51.** $\left\{\frac{\ln 7}{\ln 0.6 + \ln 7}\right\} \approx \{1.356\}$ **53.** $\{0\}$ **55.** $\left\{\frac{\ln \pi}{1 + \ln \pi}\right\} \approx \{0.534\}$ **57.** $\left\{\frac{\ln 3}{\ln 2}\right\} \approx \{1.585\}$

59. $\{0\}$ **61.** $\left\{\log_4(-2 + \sqrt{7})\right\} \approx \{-0.315\}$ **63.** $\{\log_5 4\} \approx \{0.861\}$ **65.** No real solution **67.** $\{\log_4 5\} \approx \{1.161\}$ **69.** $\{2.79\}$

71. $\{-0.57\}$ **73.** $\{-0.70\}$ **75.** $\{0.57\}$ **77.** $\{0.39, 1.00\}$ **79.** $\{1.32\}$ **81.** $\{1.31\}$ **83.** $\{1\}$ **85.** $\{16\}$ **87.** $\left\{-1, \frac{2}{3}\right\}$ **89.** $\{0\}$

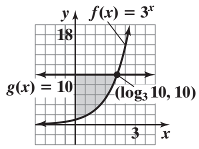
91. $\{\ln(2+\sqrt{5})\} \approx \{1.444\}$ **93.** $\left\{e^{\frac{\ln 5 \cdot \ln 3}{\ln 15}}\right\} \approx \{1.921\}$ **95.(a)** $\{5\}$; $(5, 3)$ **(b)** $\{5\}$; $(5, 4)$ **(c)** $\{1\}$; yes, at $(1, 2)$ **(d)** $\{5\}$ **(e)** $\left\{-\frac{1}{11}\right\}$

97. (a), (b)

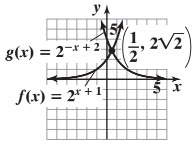


(c) $\{x \mid x > 0.710\}$ or $(0.710, \infty)$

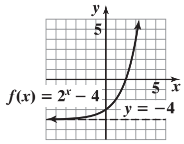
99. (a), (b), (c)



101. (a), (b), (c)



103. (a)



(b) 2 **(c)** $\{x \mid x < 2\}$ or $(-\infty, 2)$

105. (a) 2047 **(b)** 2059
107. (a) After 4.2 yr
(b) After 6.5 yr
(c) After 12.8 yr

110. $\left\{-3, \frac{1}{4}, 2\right\}$
111. one-to-one
112. $(f \circ g)(x) = \frac{x+5}{-x+11}$; $\{x \mid x \neq 3, x \neq 11\}$
113. $\{x \mid x \geq 1\}$, or $[1, \infty)$

6.7 Assess Your Understanding *(page 481)*

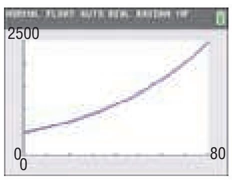
3. principal **4.** *I*; *Prt*; simple interest **5.** 4 **6.** effective rate of interest **7.** \$108.29 **9.** \$609.50 **11.** \$697.09 **13.** \$1246.08 **15.** \$88.72 **17.** \$860.72 **19.** \$554.09 **21.** \$59.71 **23.** 5.095% **25.** 5.127% **27.** $6\frac{1}{4}$% compounded annually **29.** 9% compounded monthly **31.** 25.992% **33.** 24.573% **35. (a)** About 8.69 yr **(b)** About 8.66 yr **37.** 6.823% **39.** 10.15 yr; 10.14 yr **41.** 15.27 yr or 15 yr, 3 mo **43.** \$104,335 **45.** \$12,910.62 **47.** About \$30.17 per share or \$3017 **49.** Not quite. Jim will have \$1057.60. The second bank gives a better deal, since Jim will have \$1060.62 after 1 yr. **51.** Will has \$11,632.73; Henry has \$10,947.89. **53.(a)** \$64,589 **(b)** \$45,062 **55.** About \$1020 billion; about \$233 billion **57.** \$940.90 **59.** 2.53% **61.** 34.31 yr **63. (a)** \$3686.45 **(b)** \$3678.79 **65.** \$6439.28

67. (a) 11.90 yr **(b)** 22.11 yr **(c)**
$$mP = P\left(1+\frac{r}{n}\right)^{nt}$$
$$m = \left(1+\frac{r}{n}\right)^{nt}$$
$$\ln m = \ln\left(1+\frac{r}{n}\right)^{nt} = nt\ln\left(1+\frac{r}{n}\right)$$
$$t = \frac{\ln m}{n\ln\left(1+\frac{r}{n}\right)}$$

69. (a) 1.99% **(b)** In 2026 or after 17 yr **71.** 22.7 yr **76.** $R = 0$; yes
77. $f^{-1}(x) = \frac{2x}{x-1}$ **78.** $-2, 5$; $f(x) = (x+2)^2(x-5)(x^2+1)$ **79.** $\{6\}$

6.8 Assess Your Understanding *(page 492)*

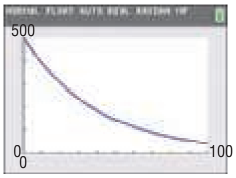
1. (a) 500 insects **(b)** $0.02 = 2\%$ per day **(c)**



(d) About 611 insects **(e)** After about 23.5 days **(f)** After about 34.7 days

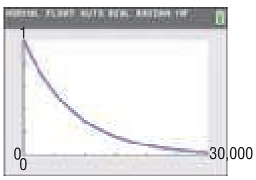
3. (a) $-0.0244 = -2.44\%$ per year
(c) About 391.7 g **(d)** After about 9.1 yr
(e) 28.4 yr

(b)



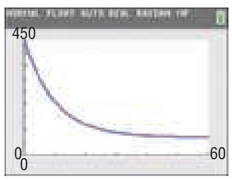
5. (a) $N(t) = N_0e^{kt}$ **(b)** 5832 **(c)** 3.9 days

7. (a) $N(t) = N_0e^{kt}$ **(b)** 25,198 **9.** 9.797 g
11. (a) 9953 years ago
(b)



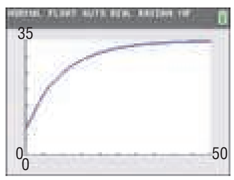
(c) 5730 yr

13. (a) 5:18 PM
(b)



(c) About 14.3 min **(d)** The temperature of the pizza approaches 70°F.

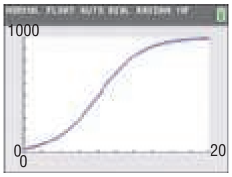
15. (a) 18.63°C; 25.07°C
(b)



17. 1.7 ppm; 7.17 days, or 172 h **19.** 0.26 M; 6.58 hr, or 395 min **21.** 26.6 days

23. (a) 1000 **(b)** 43.9% **(c)** 30 g

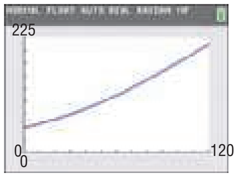
(d)



(e) 616.6 g **(f)** After 9.85 h **(g)** About 7.9 h

25. (a) $P(0) \approx 48$; In 1900, about 48 invasive species were present in the Great Lakes. **(b)** 1.7%

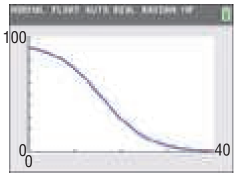
(c)



(d) About 176 **(e)** During 1999

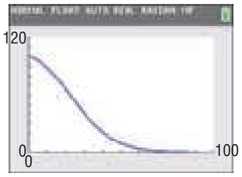
27. (a) In 1984, 91.8% of households did not own a personal computer.

(b)



(c) 70.6% **(d)** During 2011

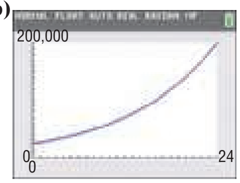
29. (a)



(b) 0.78, or 78%
(c) 50 people
(d) As n increases, the probability decreases.

31. (a) $P(t) = 25{,}000(2)^{t/8}$

(b)

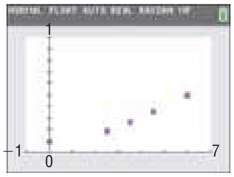


(c) 32,421 people **(d)** In about 13.42 yr
(e) $P(t) = 25{,}000e^{0.087t}$

33. $f(x) = -\frac{3}{2}x + 7$ **34.** Neither **35.** $2 \ln x + \frac{1}{2} \ln y - \ln z$ **36.** $2\sqrt[3]{5}$

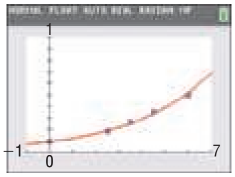
6.9 Assess Your Understanding *(page 500)*

1. (a)



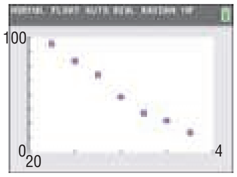
(b) $y = 0.0903(1.3384)^x$
(c) $N(t) = 0.0903e^{0.2915t}$

(d)



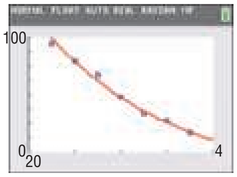
(e) 0.69
(f) After about 7.26 hr

3. (a)



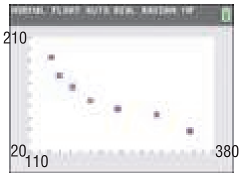
(b) $y = 118.7226(0.7013)^x$
(c) $A(t) = 118.7226e^{-0.3548t}$

(d)



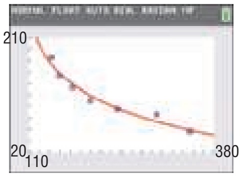
(e) 28.7%
(f) $k = -0.3548 = -35.48\%$ is the exponential growth rate. It represents the rate at which the percent of patients surviving advanced-stage breast cancer is decreasing.

5. (a)



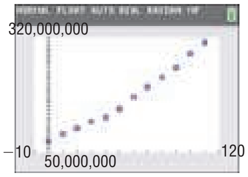
(b) $y = 330.0549 - 34.5008 \ln x$

(c)



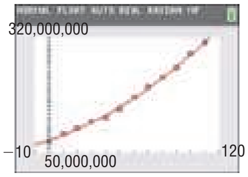
(d) 185 billion pounds
(e) Under by 5 billion pounds

7. (a)



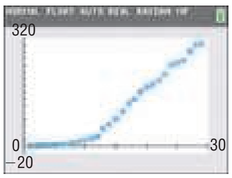
(b) $y = \dfrac{762{,}176{,}844.4}{1 + 8.7428e^{-0.0162x}}$

(c)



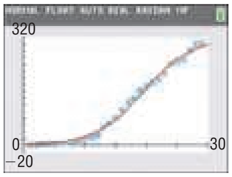
(d) 762,176,844
(e) Approximately 315,203,288 **(f)** 2023

9. (a)



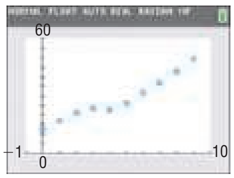
(b) $y = \dfrac{321.0384}{1 + 135.3081e^{-0.2516x}}$

(c)



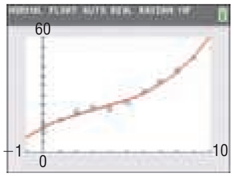
(d) 321.0 thousand cell sites
(e) 314.7 thousand cell sites

11. (a)



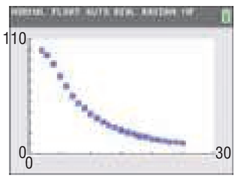
(b) Cubic
(c) $y = 0.0691x^3 - 0.6538x^2 + 4.4323x + 13.0352$

(d)



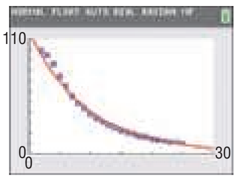
(e) About $74.6 billion

13. (a)



(b) Exponential
(c) $y = 115.5779(0.9012)^x$

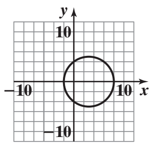
(d)



(e) 5.1%

14. $f(x) = \frac{1}{3}(x+3)(x+1)^2(x-2)$

15. $\frac{3\sqrt{2}}{2}$ **16.** $\sqrt{3}$ **17.**



Review Exercises *(page 506)*

1. (a) -4 **(b)** 1 **(c)** -6 **(d)** -6 **2. (a)** -26 **(b)** -241 **(c)** 16 **(d)** -1 **3. (a)** $\sqrt{11}$ **(b)** 1 **(c)** $\sqrt{\sqrt{6}+2}$ **(d)** 19

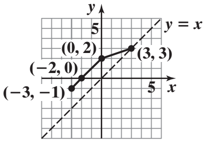
4. (a) e^4 **(b)** $3e^{-2}-2$ **(c)** e^{e^4} **(d)** -17

5. $(f\circ g)(x) = 1-3x$, all real numbers; $(g\circ f)(x) = 7-3x$, all real numbers; $(f\circ f)(x) = x$, all real numbers; $(g\circ g)(x) = 9x+4$, all real numbers

6. $(f\circ g)(x) = \sqrt{3+3x+3x^2}$, all real numbers; $(g\circ f)(x) = 1+\sqrt{3x}+3x$, $\{x \mid x \ge 0\}$;
$(f\circ f)(x) = \sqrt{3\sqrt{3x}}$, $\{x \mid x\ge 0\}$; $(g\circ g)(x) = 3+3x+4x^2+2x^3+x^4$, all real numbers

7. $(f\circ g)(x) = \dfrac{1+x}{1-x}$, $\{x \mid x\ne 0, x\ne 1\}$; $(g\circ f)(x) = \dfrac{x-1}{x+1}$, $\{x\mid x\ne -1, x\ne 1\}$; $(f\circ f)(x) = x$, $\{x\mid x\ne 1\}$; $(g\circ g)(x) = x$, $\{x\mid x\ne 0\}$

8. (a) one-to-one **(b)** $\{(2,1),(5,3),(8,5),(10,6)\}$ **9.**



10. $f^{-1}(x) = \dfrac{2x+3}{5x-2}$

$$f(f^{-1}(x)) = \frac{2\left(\frac{2x+3}{5x-2}\right)+3}{5\left(\frac{2x+3}{5x-2}\right)-2} = x$$

$$f^{-1}(f(x)) = \frac{2\left(\frac{2x+3}{5x-2}\right)+3}{5\left(\frac{2x+3}{5x-2}\right)-2} = x$$

Domain of f = range of f^{-1} = all real numbers except $\frac{2}{5}$

Range of f = domain of f^{-1} = all real numbers except $\frac{2}{5}$

11. $f^{-1}(x) = \dfrac{x+1}{x}$

$$f(f^{-1}(x)) = \frac{1}{\frac{x+1}{x}-1} = x$$

$$f^{-1}(f(x)) = \frac{\frac{1}{x-1}+1}{\frac{1}{x-1}} = x$$

Domain of f = range of f^{-1} = all real numbers except 1
Range of f = domain of f^{-1} = all real numbers except 0

12. $f^{-1}(x) = x^2+2, x\ge 0$
$f(f^{-1}(x)) = \sqrt{x^2+2-2} = |x| = x, x\ge 0$
$f^{-1}(f(x)) = (\sqrt{x-2})^2+2 = x$
Domain of f = range of f^{-1} = $[2,\infty)$
Range of f = domain of f^{-1} = $[0,\infty)$

13. $f^{-1}(x) = (x-1)^3$;
$f(f^{-1}(x)) = ((x-1)^3)^{1/3}+1 = x$
$f^{-1}(f(x)) = (x^{1/3}+1-1)^3 = x$
Domain of f = range of f^{-1} = $(-\infty,\infty)$
Range of f = domain of f^{-1} = $(-\infty,\infty)$

14. (a) 81 **(b)** 2 **(c)** $\frac{1}{9}$ **(d)** -3

15. $\log_5 z = 2$ **16.** $5^{13} = u$ **17.** $\left\{x \,\middle|\, x > \frac{2}{3}\right\}$; $\left(\frac{2}{3},\infty\right)$

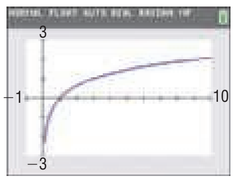
18. $\{x\mid x<1 \text{ or } x>2\}$; $(-\infty,1)\cup(2,\infty)$

19. -3 **20.** $\sqrt{2}$ **21.** 0.4

22. $\log_3 u + 2\log_3 v - \log_3 w$ **23.** $8\log_2 a + 2\log_2 b$

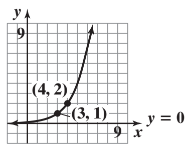
24. $2\log x + \frac{1}{2}\log(x^3+1)$ **25.** $2\ln(2x+3) - 2\ln(x-1) - 2\ln(x-2)$ **26.** $\frac{25}{4}\log_4 x$ **27.** $-2\ln(x+1)$ **28.** $\ln\left[\dfrac{16\sqrt{x^2+1}}{\sqrt{x(x-4)}}\right]$ **29.** 2.124

30.



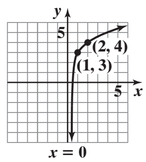
31. (a) Domain of f: $(-\infty,\infty)$

(b)



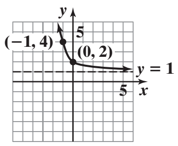
(c) Range of f: $(0,\infty)$
Horizontal asymptote: $y = 0$
(d) $f^{-1}(x) = 3+\log_2 x$
(e) Domain of f^{-1}: $(0,\infty)$
Range of f^{-1}: $(-\infty,\infty)$

(f)



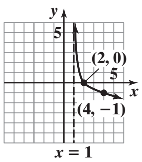
32. (a) Domain of f: $(-\infty,\infty)$

(b)



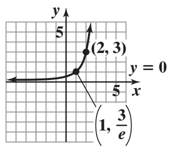
(c) Range of f: $(1,\infty)$
Horizontal asymptote: $y = 1$
(d) $f^{-1}(x) = -\log_3(x-1)$
(e) Domain of f^{-1}: $(1,\infty)$
Range of f^{-1}: $(-\infty,\infty)$

(f)



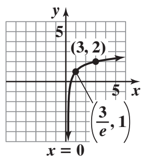
33. (a) Domain of f: $(-\infty,\infty)$

(b)



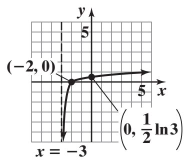
(c) Range of f: $(0,\infty)$
Horizontal asymptote: $y = 0$
(d) $f^{-1}(x) = 2+\ln\left(\dfrac{x}{3}\right)$
(e) Domain of f^{-1}: $(0,\infty)$
Range of f^{-1}: $(-\infty,\infty)$

(f)



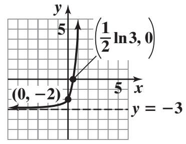
34. (a) Domain of f: $(-3,\infty)$

(b)



(c) Range of f: $(-\infty,\infty)$
Vertical asymptote: $x = -3$
(d) $f^{-1}(x) = e^{2x}-3$
(e) Domain of f^{-1}: $(-\infty,\infty)$
Range of f^{-1}: $(-3,\infty)$

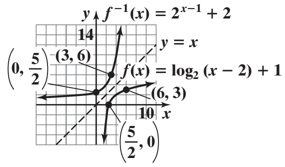
(f)



35. $\left\{-\frac{16}{9}\right\}$ **36.** $\left\{\frac{-1-\sqrt{3}}{2}, \frac{-1+\sqrt{3}}{2}\right\} \approx \{-1.366, 0.366\}$ **37.** $\left\{\frac{1}{4}\right\}$ **38.** $\left\{\frac{2\ln 3}{\ln 5 - \ln 3}\right\} \approx \{4.301\}$ **39.** $\{-2, 6\}$ **40.** $\{83\}$ **41.** $\left\{\frac{1}{2}, -3\right\}$

42. $\{1\}$ **43.** $\{-1\}$ **44.** $\{1 - \ln 5\} \approx \{-0.609\}$ **45.** $\left\{\log_3(-2+\sqrt{7})\right\} = \left\{\frac{\ln(-2+\sqrt{7})}{\ln 3}\right\} \approx \{-0.398\}$

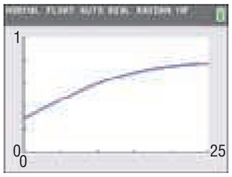
46. (a), (e)



(b) 3; $(6, 3)$
(c) 10; $(10, 4)$
(d) $\left\{x \middle| x > \frac{5}{2}\right\}$ or $\left(\frac{5}{2}, \infty\right)$
(e) $f^{-1}(x) = 2^{x-1} + 2$

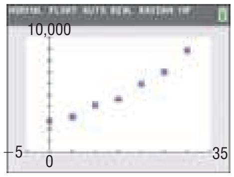
47. (a) 37.3 W **(b)** 6.9 dB **48. (a)** 11.77 **(b)** 9.56 in.
49. (a) 9.85 yr **(b)** 4.27 yr **50.** \$20,398.87; 4.04%; 17.5 yr
51. \$41,668.97 **52.** 24,765 yr ago **53.** 55.22 min, or 55 min, 13 sec
54. 7,615,278,125 **55.** 7.204 g; 0.519 g

56. (a) 0.3 **(b)** 0.8
(c)

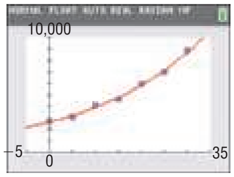


(d) In 2026

57. (a)

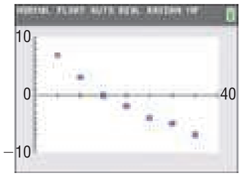


(b) $y = 2638.26(1.0407)^x$
(c) $A(t) = 2638.26e^{0.0399x}$
(d)

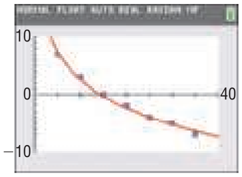


(e) 2021–22

58. (a)

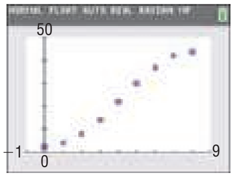


(b) $y = 18.921 - 7.096 \ln x$
(c)

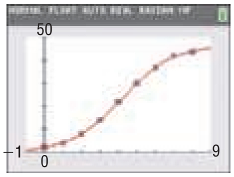


(d) Approximately −3°F

59. (a)



(b) $C = \dfrac{46.9292}{1 + 21.2733e^{-0.7306t}}$
(c)



(d) About 47 people; 50 people
(e) 2.4 days; during the tenth hour of day 3
(f) 9.5 days

Chapter Test *(page 509)*

1. (a) $f \circ g = \frac{2x+7}{2x+3}$; domain: $\left\{x \middle| x \neq -\frac{3}{2}\right\}$ **(b)** $(g \circ f)(-2) = 5$ **(c)** $(f \circ g)(-2) = -3$

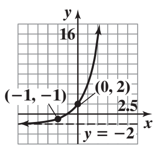
2. (a) The function is not one-to-one. **(b)** The function is one-to-one.

3. $f^{-1}(x) = \frac{2+5x}{3x}$; domain of $f = \left\{x \middle| x \neq \frac{5}{3}\right\}$, range of $f = \{y | y \neq 0\}$; domain of $f^{-1} = \{x | x \neq 0\}$; range of $f^{-1} = \left\{y \middle| y \neq \frac{5}{3}\right\}$

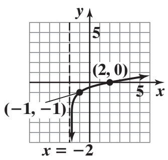
4. The point $(-5, 3)$ must be on the graph of f^{-1}. **5.** $\{5\}$ **6.** $\{4\}$ **7.** $\{625\}$ **8.** $e^3 + 2 \approx 22.086$ **9.** $\log 20 \approx 1.301$

10. $\log_3 21 = \frac{\ln 21}{\ln 3} \approx 2.771$ **11.** $\ln 133 \approx 4.890$

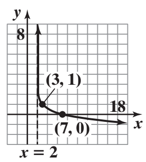
12. (a) Domain of f: $\{x | -\infty < x < \infty\}$ or $(-\infty, \infty)$
(b)



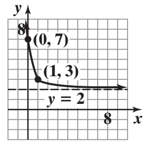
(c) Range of f: $\{y | y > -2\}$ or $(-2, \infty)$;
Horizontal asymptote: $y = -2$
(d) $f^{-1}(x) = \log_4(x+2) - 1$
(e) Domain of f^{-1}: $\{x | x > -2\}$ or $(-2, \infty)$
Range of f^{-1}: $\{y | -\infty < y < \infty\}$ or $(-\infty, \infty)$
(f)



13. (a) Domain of f: $\{x | x > 2\}$ or $(2, \infty)$
(b)



(c) Range of f: $\{y | -\infty < y < \infty\}$ or $(-\infty, \infty)$;
vertical asymptote: $x = 2$
(d) $f^{-1}(x) = 5^{1-x} + 2$
(e) Domain of f^{-1}: $\{x | -\infty < x < \infty\}$ or $(-\infty, \infty)$
Range of f^{-1}: $\{y | y > 2\}$ or $(2, \infty)$
(f)



14. $\{1\}$ **15.** $\{91\}$ **16.** $\{-\ln 2\} \approx \{-0.693\}$ **17.** $\left\{\frac{1-\sqrt{13}}{2}, \frac{1+\sqrt{13}}{2}\right\} \approx \{-1.303, 2.303\}$ **18.** $\left\{\frac{3\ln 7}{1 - \ln 7}\right\} \approx \{-6.172\}$

19. $\{2\sqrt{6}\} \approx \{4.899\}$ **20.** $2 + 3\log_2 x - \log_2(x-6) - \log_2(x+3)$ **21.** About 250.39 days **22. (a)** \$1033.82 **(b)** \$963.42 **(c)** 11.9 yr

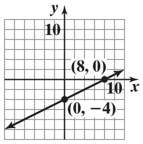
23. (a) About 83 dB **(b)** The pain threshold will be exceeded if 31,623 people shout at the same time.

Cumulative Review *(page 510)*

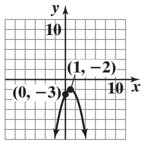
1. (a) Yes; no **(b)** Polynomial; the graph is smooth and continuous. **2. (a)** 10 **(b)** $2x^2 + 3x + 1$ **(c)** $2x^2 + 4xh + 2h^2 - 3x - 3h + 1$

3. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is on the graph. **4.** $\{-26\}$

5.

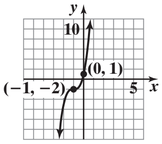


6. (a)



(b) All real numbers; $(-\infty, \infty)$ **7.** $f(x) = 2(x-4)^2 - 8 = 2x^2 - 16x + 24$

8. Exponential; $f(x) = 2 \cdot 3^x$ **9.**



10. (a) $f(g(x)) = \frac{4}{(x-3)^2} + 2$; domain: $\{x \mid x \neq 3\}$; 3

(b) $f(g(x)) = \log_2 x + 2$; domain: $\{x \mid x > 0\}$; $2 + \log_2 14$

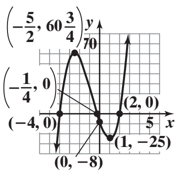
11. (a) Zeros: $-4, -\frac{1}{4}, 2$

(b) x-intercepts: $-4, -\frac{1}{4}, 2$; y-intercept: -8

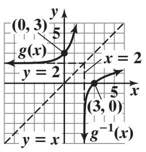
(c) Local maximum value of 60.75 occurs at $x = -2.5$.

Local minimum value of -25 occurs at $x = 1$.

(d)



12. (a), (c)



Domain g = range $g^{-1} = (-\infty, \infty)$

Range g = domain $g^{-1} = (2, \infty)$

(b) $g^{-1}(x) = \log_3(x - 2)$

13. $\left\{-\frac{3}{2}\right\}$ **14.** $\{2\}$

15. (a) $\{-1\}$ **(b)** $\{x \mid x > -1\}$ or $(-1, \infty)$

(c) $\{25\}$

16. (a)

(b) Logarithmic; $y = 49.293 - 10.563 \ln x$

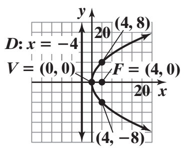
(c) Highest value of $|r|$

CHAPTER 7 Analytic Geometry

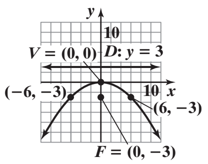
7.2 Assess Your Understanding *(page 522)*

6. parabola **7.** axis of symmetry **8.** latus rectum **9.** c **10.** $(3, 2)$ **11.** d **12.** c **13.** B **15.** E **17.** H **19.** C

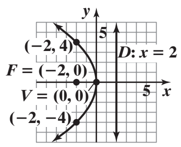
21. $y^2 = 16x$



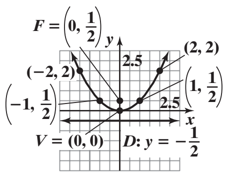
23. $x^2 = -12y$



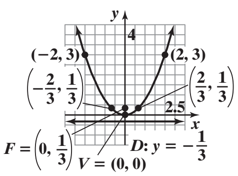
25. $y^2 = -8x$



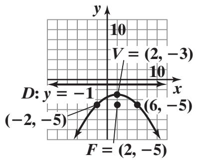
27. $x^2 = 2y$



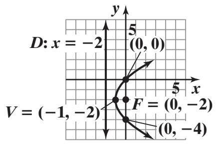
29. $x^2 = \frac{4}{3}y$



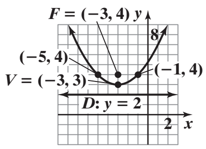
31. $(x-2)^2 = -8(y+3)$



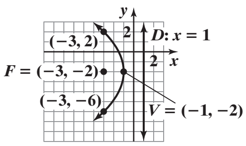
33. $(y+2)^2 = 4(x+1)$



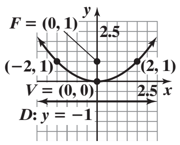
35. $(x+3)^2 = 4(y-3)$



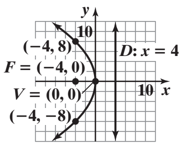
37. $(y+2)^2 = -8(x+1)$



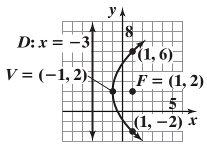
39. Vertex: $(0, 0)$; focus: $(0, 1)$; directrix: $y = -1$



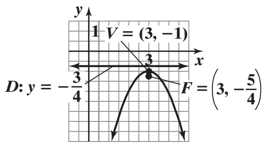
41. Vertex: $(0, 0)$; focus: $(-4, 0)$; directrix: $x = 4$



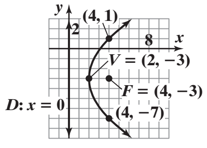
43. Vertex: $(-1, 2)$; focus: $(1, 2)$; directrix: $x = -3$



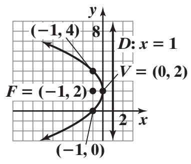
45. Vertex: $(3, -1)$; focus: $\left(3, -\frac{5}{4}\right)$; directrix: $y = -\frac{3}{4}$



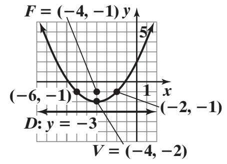
47. Vertex: $(2, -3)$; focus: $(4, -3)$; directrix: $x = 0$



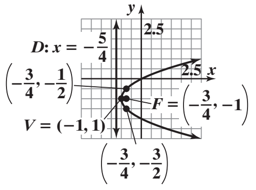
49. Vertex: $(0, 2)$; focus: $(-1, 2)$; directrix: $x = 1$



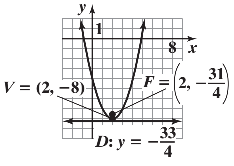
51. Vertex: $(-4, -2)$; focus: $(-4, -1)$; directrix: $y = -3$



53. Vertex: $(-1, -1)$; focus: $\left(-\frac{3}{4}, -1\right)$; directrix: $x = -\frac{5}{4}$



55. Vertex: $(2, -8)$; focus: $\left(2, -\frac{31}{4}\right)$; directrix: $y = -\frac{33}{4}$



57. $(y - 1)^2 = x$ **59.** $(y - 1)^2 = -(x - 2)$

61. $x^2 = 4(y - 1)$ **63.** $y^2 = \frac{1}{2}(x + 2)$

65. 1.5625 ft from the base of the dish, along the axis of symmetry

67. 1 in. from the vertex, along the axis of symmetry

69. 20 ft **71.** 0.78125 ft

73. 4.17 ft from the base, along the axis of symmetry

75. 24.31 ft, 18.75 ft, 7.64 ft

77. (a) $y = -\frac{2}{315}x^2 + 630$

(b) 567 ft: 119.7 ft; 478 ft: 267.3 ft; 308 ft: 479.4 ft **(c)** No

79. $Cy^2 + Dx = 0, C \neq 0, D \neq 0$

$$Cy^2 = -Dx$$
$$y^2 = -\frac{D}{C}x$$

This is the equation of a parabola with vertex at $(0, 0)$ and axis of symmetry the x-axis. The focus is $\left(-\frac{D}{4C}, 0\right)$; the directrix is the line $x = \frac{D}{4C}$. The parabola opens to the right if $-\frac{D}{C} > 0$ and to the left if $-\frac{D}{C} < 0$.

81. $Cy^2 + Dx + Ey + F = 0, C \neq 0$

$$Cy^2 + Ey = -Dx - F$$
$$y^2 + \frac{E}{C}y = -\frac{D}{C}x - \frac{F}{C}$$
$$\left(y + \frac{E}{2C}\right)^2 = -\frac{D}{C}x - \frac{F}{C} + \frac{E^2}{4C^2}$$
$$\left(y + \frac{E}{2C}\right)^2 = -\frac{D}{C}x + \frac{E^2 - 4CF}{4C^2}$$

(a) If $D \neq 0$, then the equation may be written as $\left(y + \frac{E}{2C}\right)^2 = -\frac{D}{C}\left(x - \frac{E^2 - 4CF}{4CD}\right)$.

This is the equation of a parabola with vertex at $\left(\frac{E^2 - 4CF}{4CD}, -\frac{E}{2C}\right)$ and axis of symmetry parallel to the x-axis.

(b)–(d) If $D = 0$, the graph of the equation contains no points if $E^2 - 4CF < 0$, is a single horizontal line if $E^2 - 4CF = 0$, and is two horizontal lines if $E^2 - 4CF > 0$.

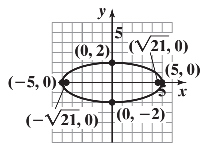
82. $(0, 2), (0, -2), (-36, 0)$; symmetric with respect to the x-axis **83.** $\{5\}$

84. $(2, 3)$ **85.** $2x + h + 2$

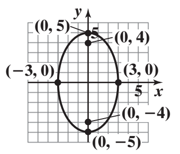
7.3 Assess Your Understanding *(page 533)*

7. ellipse **8.** b **9.** $(0, -5)$; $(0, 5)$ **10.** 5; 3; x **11.** $(-2, -3)$; $(6, -3)$ **12.** a **13.** C **15.** B

17. Vertices: $(-5, 0), (5, 0)$; foci: $(-\sqrt{21}, 0), (\sqrt{21}, 0)$

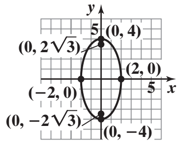


19. Vertices: $(0, -5), (0, 5)$; foci: $(0, -4), (0, 4)$



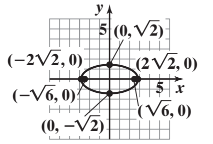
21. $\frac{x^2}{4} + \frac{y^2}{16} = 1$

Vertices: $(0, -4), (0, 4)$; foci: $(0, -2\sqrt{3}), (0, 2\sqrt{3})$



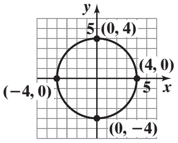
23. $\frac{x^2}{8} + \frac{y^2}{2} = 1$

Vertices: $(-2\sqrt{2}, 0), (2\sqrt{2}, 0)$; foci: $(-\sqrt{6}, 0), (\sqrt{6}, 0)$

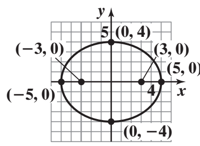


25. $\frac{x^2}{16} + \frac{y^2}{16} = 1$

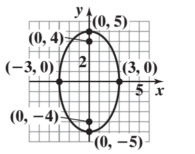
Vertices: $(-4, 0), (4, 0), (0, -4), (0, 4)$; focus: $(0, 0)$



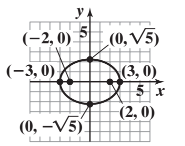
27. $\frac{x^2}{25} + \frac{y^2}{16} = 1$



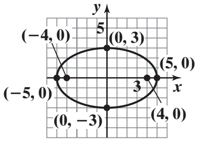
29. $\frac{x^2}{9} + \frac{y^2}{25} = 1$



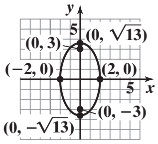
31. $\frac{x^2}{9} + \frac{y^2}{5} = 1$



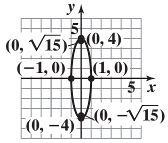
33. $\frac{x^2}{25} + \frac{y^2}{9} = 1$



35. $\frac{x^2}{4} + \frac{y^2}{13} = 1$



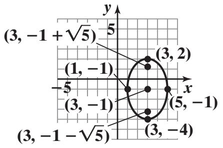
37. $x^2 + \frac{y^2}{16} = 1$



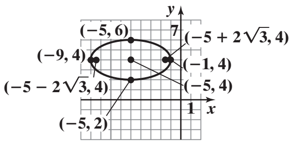
39. $\frac{(x+1)^2}{4} + (y-1)^2 = 1$

41. $(x-1)^2 + \frac{y^2}{4} = 1$

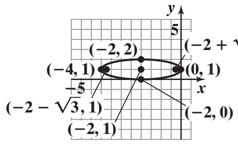
43. Center: $(3, -1)$; vertices: $(3, -4)$, $(3, 2)$; foci: $(3, -1-\sqrt{5})$, $(3, -1+\sqrt{5})$



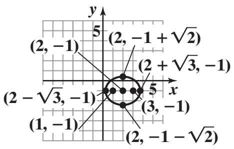
45. $\frac{(x+5)^2}{16} + \frac{(y-4)^2}{4} = 1$
Center: $(-5, 4)$; vertices: $(-9, 4)$, $(-1, 4)$; foci: $(-5-2\sqrt{3}, 4)$, $(-5+2\sqrt{3}, 4)$



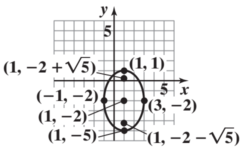
47. $\frac{(x+2)^2}{4} + (y-1)^2 = 1$
Center: $(-2, 1)$; vertices: $(-4, 1)$, $(0, 1)$; foci: $(-2-\sqrt{3}, 1)$, $(-2+\sqrt{3}, 1)$



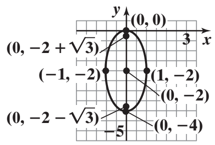
49. $\frac{(x-2)^2}{3} + \frac{(y+1)^2}{2} = 1$
Center: $(2, -1)$; vertices: $(2-\sqrt{3}, -1)$, $(2+\sqrt{3}, -1)$; foci: $(1, -1)$, $(3, -1)$



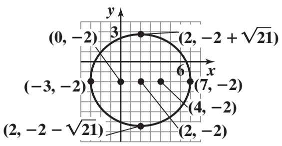
51. $\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1$
Center: $(1, -2)$; vertices: $(1, -5)$, $(1, 1)$; foci: $(1, -2-\sqrt{5})$, $(1, -2+\sqrt{5})$



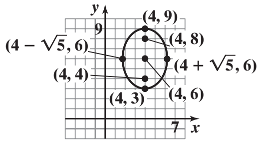
53. $x^2 + \frac{(y+2)^2}{4} = 1$
Center: $(0, -2)$; vertices: $(0, -4)$, $(0, 0)$; foci: $(0, -2-\sqrt{3})$, $(0, -2+\sqrt{3})$



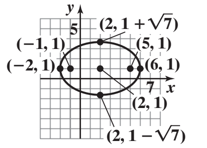
55. $\frac{(x-2)^2}{25} + \frac{(y+2)^2}{21} = 1$



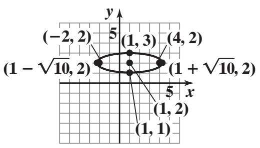
57. $\frac{(x-4)^2}{5} + \frac{(y-6)^2}{9} = 1$



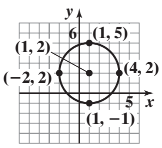
59. $\frac{(x-2)^2}{16} + \frac{(y-1)^2}{7} = 1$



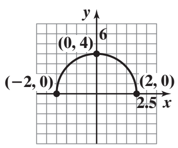
61. $\frac{(x-1)^2}{10} + (y-2)^2 = 1$



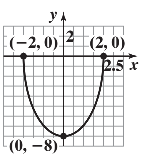
63. $\frac{(x-1)^2}{9} + \frac{(y-2)^2}{9} = 1$



65.



67.



69. $\frac{x^2}{100} + \frac{y^2}{36} = 1$ **71.** 43.3 ft **73.** 24.65 ft, 21.65 ft, 13.82 ft **75.** 30 ft **77.** The elliptical hole will have a major axis of length $2\sqrt{41}$ in. and a minor axis of length 8 in. **79.** 91.5 million mi; $\frac{x^2}{(93)^2} + \frac{y^2}{8646.75} = 1$

81. Perihelion: 460.6 million mi; mean distance: 483.8 million mi; $\frac{x^2}{(483.8)^2} + \frac{y^2}{233{,}524.2} = 1$

83. 35 million mi **85.** $5\sqrt{5} - 4$

87. $Ax^2 + Cy^2 + Dx + Ey + F = 0 \qquad A \neq 0, C \neq 0$

$$Ax^2 + Dx + Cy^2 + Ey = -F$$

$$A\left(x^2 + \frac{D}{A}x\right) + C\left(y^2 + \frac{E}{Cy}\right) = -F$$

$$A\left(x + \frac{D}{2A}\right)^2 + C\left(y + \frac{E}{2C}\right)^2 = -F + \frac{D^2}{4A} + \frac{E^2}{4C}$$

(a) If $\frac{D^2}{4A} + \frac{E^2}{4C} - F$ is of the same sign as A (and C), this is the equation of an ellipse with center at $\left(-\frac{D}{2A}, -\frac{E}{2C}\right)$.

(b) If $\frac{D^2}{4A} + \frac{E^2}{4C} - F$, the graph is the single point $\left(-\frac{D}{2A}, -\frac{E}{2C}\right)$.

(c) If $\frac{D^2}{4A} + \frac{E^2}{4C} - F$ is of the sign opposite that of A (and C), the graph contains no points, because in this case, the left side has the sign opposite that of the right side.

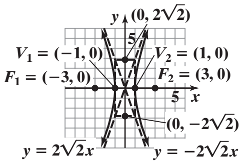
89. Zeros: $5 - 2\sqrt{3}, 5 + 2\sqrt{3}$; x-intercepts: $5 - 2\sqrt{3}, 5 + 2\sqrt{3}$ **90.** Domain: $\{x | x \neq 5\}$; Horizontal asymptote: $y = 2$; Vertical asymptote: $x = 5$

91. $-\frac{7}{5} \leq x \leq 3$ or $\left[-\frac{7}{5}, 3\right]$ **92.** $\left(\frac{5}{2}, 26\right)$, $(-3, -7)$

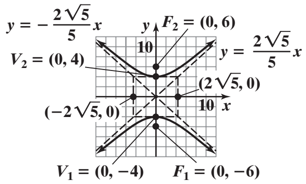
7.4 Assess Your Understanding *(page 546)*

7. hyperbola **8.** transverse axis **9.** b **10.** $(2, 4)$; $(2, -2)$ **11.** $(2, 6)$; $(2, -4)$ **12.** c **13.** 2; 3; x **14.** $y = -\frac{4}{9}x$; $y = \frac{4}{9}x$ **15.** B **17.** A

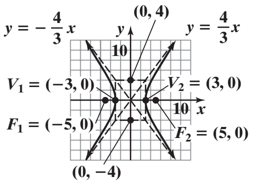
19. $x^2 - \frac{y^2}{8} = 1$



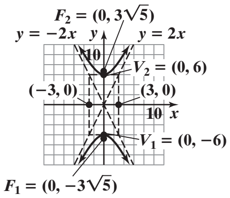
21. $\frac{y^2}{16} - \frac{x^2}{20} = 1$



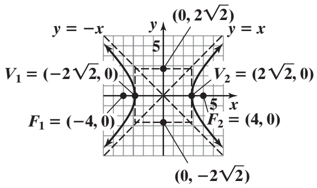
23. $\frac{x^2}{9} - \frac{y^2}{16} = 1$



25. $\frac{y^2}{36} - \frac{x^2}{9} = 1$

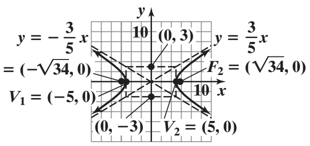


27. $\frac{x^2}{8} - \frac{y^2}{8} = 1$



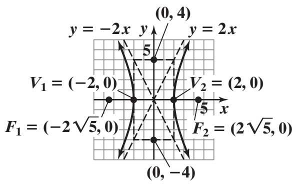
29. $\frac{x^2}{25} - \frac{y^2}{9} = 1$

Center: $(0, 0)$
Transverse axis: x-axis
Vertices: $(-5, 0)$, $(5, 0)$
Foci: $(-\sqrt{34}, 0)$, $(\sqrt{34}, 0)$
Asymptotes: $y = \pm\frac{3}{5}x$



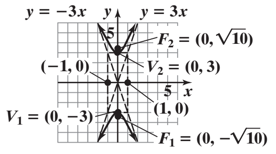
31. $\frac{x^2}{4} - \frac{y^2}{16} = 1$

Center: $(0, 0)$
Transverse axis: x-axis
Vertices: $(-2, 0)$, $(2, 0)$
Foci: $(-2\sqrt{5}, 0)$, $(2\sqrt{5}, 0)$
Asymptotes: $y = \pm 2x$



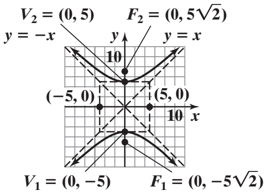
33. $\frac{y^2}{9} - x^2 = 1$

Center: $(0, 0)$
Transverse axis: y-axis
Vertices: $(0, -3)$, $(0, 3)$
Foci: $(0, -\sqrt{10})$, $(0, \sqrt{10})$
Asymptotes: $y = \pm 3x$



35. $\frac{y^2}{25} - \frac{x^2}{25} = 1$

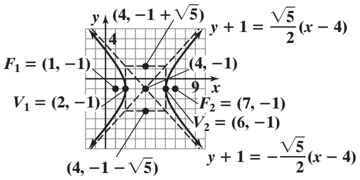
Center: $(0, 0)$
Transverse axis: y-axis
Vertices: $(0, -5)$, $(0, 5)$
Foci: $(0, -5\sqrt{2})$, $(0, 5\sqrt{2})$
Asymptotes: $y = \pm x$



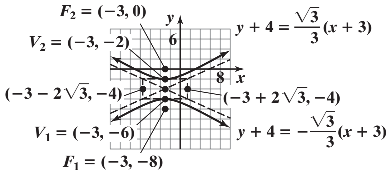
37. $x^2 - y^2 = 1$

39. $\frac{y^2}{36} - \frac{x^2}{9} = 1$

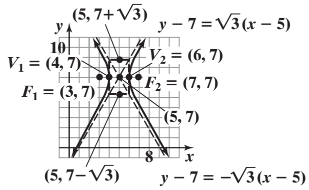
41. $\dfrac{(x-4)^2}{4} - \dfrac{(y+1)^2}{5} = 1$



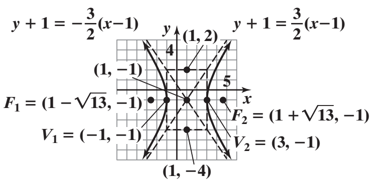
43. $\dfrac{(y+4)^2}{4} - \dfrac{(x+3)^2}{12} = 1$



45. $(x-5)^2 - \dfrac{(y-7)^2}{3} = 1$



47. $\dfrac{(x-1)^2}{4} - \dfrac{(y+1)^2}{9} = 1$



49. $\dfrac{(x-2)^2}{4} - \dfrac{(y+3)^2}{9} = 1$

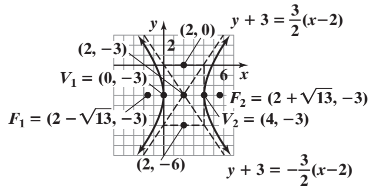
Center: $(2, -3)$

Transverse axis: parallel to x-axis

Vertices: $(0, -3)$, $(4, -3)$

Foci: $(2 - \sqrt{13}, -3)$, $(2 + \sqrt{13}, -3)$

Asymptotes: $y + 3 = \pm\dfrac{3}{2}(x - 2)$



51. $\dfrac{(y-2)^2}{4} - (x+2)^2 = 1$

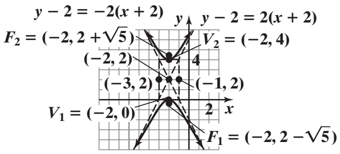
Center: $(-2, 2)$

Transverse axis: parallel to y-axis

Vertices: $(-2, 0)$, $(-2, 4)$

Foci: $(-2, 2 - \sqrt{5})$, $(-2, 2 + \sqrt{5})$

Asymptotes: $y - 2 = \pm 2(x + 2)$



53. $\dfrac{(x+1)^2}{4} - \dfrac{(y+2)^2}{4} = 1$

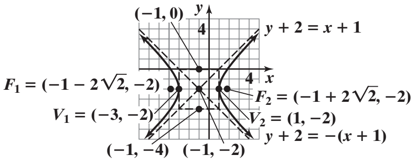
Center: $(-1, -2)$

Transverse axis: parallel to x-axis

Vertices: $(-3, -2)$, $(1, -2)$

Foci: $(-1 - 2\sqrt{2}, -2)$, $(-1 + 2\sqrt{2}, -2)$

Asymptotes: $y + 2 = \pm(x + 1)$



55. $(x-1)^2 - (y+1)^2 = 1$

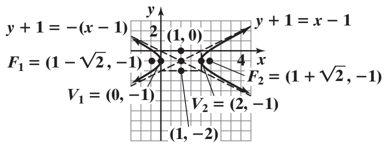
Center: $(1, -1)$

Transverse axis: parallel to x-axis

Vertices: $(0, -1)$, $(2, -1)$

Foci: $(1 - \sqrt{2}, -1)$, $(1 + \sqrt{2}, -1)$

Asymptotes: $y + 1 = \pm(x - 1)$



57. $\dfrac{(y-2)^2}{4} - (x+1)^2 = 1$

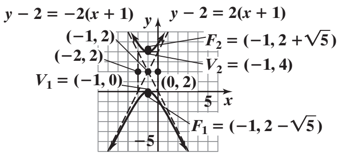
Center: $(-1, 2)$

Transverse axis: parallel to y-axis

Vertices: $(-1, 0)$, $(-1, 4)$

Foci: $(-1, 2 - \sqrt{5})$, $(-1, 2 + \sqrt{5})$

Asymptotes: $y - 2 = \pm 2(x + 1)$



59. $\dfrac{(x-3)^2}{4} - \dfrac{(y+2)^2}{16} = 1$

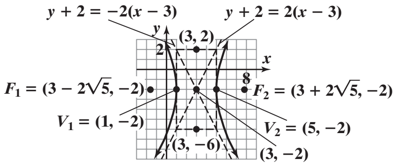
Center: $(3, -2)$

Transverse axis: parallel to x-axis

Vertices: $(1, -2)$, $(5, -2)$

Foci: $(3 - 2\sqrt{5}, -2)$, $(3 + 2\sqrt{5}, -2)$

Asymptotes: $y + 2 = \pm 2(x - 3)$



61. $\dfrac{(y-1)^2}{4} - (x+2)^2 = 1$

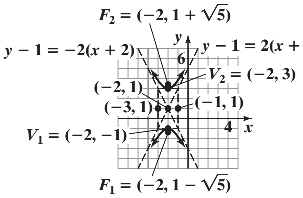
Center: $(-2, 1)$

Transverse axis: parallel to y-axis

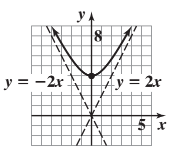
Vertices: $(-2, -1)$, $(-2, 3)$

Foci: $(-2, 1 - \sqrt{5})$, $(-2, 1 + \sqrt{5})$

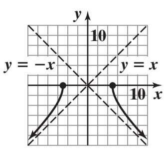
Asymptotes: $y - 1 = \pm 2(x + 2)$



63.



65.



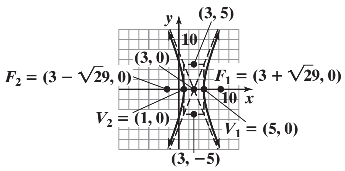
67. Center: $(3, 0)$

Transverse axis: parallel to x-axis

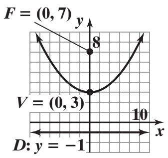
Vertices: $(1, 0)$, $(5, 0)$

Foci: $(3 - \sqrt{29}, 0)$, $(3 + \sqrt{29}, 0)$

Asymptotes: $y = \pm\dfrac{5}{2}(x - 3)$

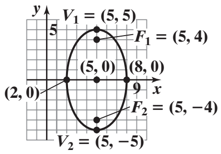


69. Vertex: $(0, 3)$; focus: $(0, 7)$; directrix: $y = -1$



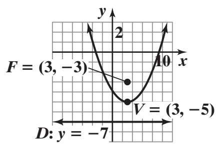
71. $\dfrac{(x-5)^2}{9} + \dfrac{y^2}{25} = 1$

Center: $(5, 0)$; vertices: $(5, 5)$, $(5, -5)$; foci: $(5, -4)$, $(5, 4)$



73. $(x-3)^2 = 8(y+5)$

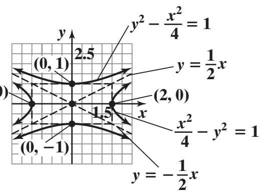
Vertex: $(3, -5)$; focus: $(3, -3)$; directrix: $y = -7$



75. The fireworks display is 50,138 ft north of the person at point A. **77.** The tower is 592.4 ft tall. **79. (a)** $y = \pm x$ **(b)** $\frac{x^2}{100} - \frac{y^2}{100} = 1, x \geq 0$
81. If the eccentricity is close to 1, the "opening" of the hyperbola is very small. As e increases, the opening gets bigger.

83. $\frac{x^2}{4} - y^2 = 1$; asymptotes: $y = \pm\frac{1}{2}x$

$y^2 - \frac{x^2}{4} = 1$; asymptotes: $y = \pm\frac{1}{2}x$



85. $Ax^2 + Cy^2 + F = 0$

$Ax^2 + Cy^2 = -F$

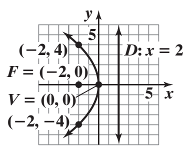
If A and C are opposite in sign and $F \neq 0$, this equation may be written as $\frac{x^2}{\left(-\frac{F}{A}\right)} + \frac{y^2}{\left(-\frac{F}{C}\right)} = 1$, where $-\frac{F}{A}$ and $-\frac{F}{C}$ are opposite in sign. This is the equation of a hyperbola with center $(0, 0)$. The transverse axis is the x-axis if $-\frac{F}{A} > 0$; the transverse axis is the y-axis if $-\frac{F}{A} < 0$.

87. $\left\{-\frac{5}{2}, \frac{3}{2}, \frac{5}{2}\right\}$ **88.** $f^{-1}(x) = \frac{6x+5}{x-1}$ **89.** $(-5, -4] \cup [4, 5)$ **90.** $\{6\}$

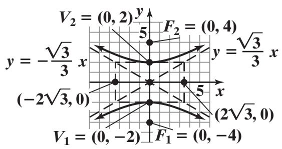
Review Exercises *(page 550)*

1. Parabola; vertex $(0, 0)$, focus $(-4, 0)$, directrix $x = 4$ **2.** Hyperbola; center $(0, 0)$, vertices $(5, 0)$ and $(-5, 0)$, foci $(\sqrt{26}, 0)$ and $(-\sqrt{26}, 0)$, asymptotes $y = \frac{1}{5}x$ and $y = -\frac{1}{5}x$ **3.** Ellipse; center $(0, 0)$, vertices $(0, 5)$ and $(0, -5)$, foci $(0, 3)$ and $(0, -3)$
4. $x^2 = -4(y - 1)$: Parabola; vertex $(0, 1)$, focus $(0, 0)$, directrix $y = 2$ **5.** $\frac{x^2}{2} - \frac{y^2}{8} = 1$: Hyperbola; center $(0, 0)$, vertices $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$, foci $(\sqrt{10}, 0)$ and $(-\sqrt{10}, 0)$, asymptotes $y = 2x$ and $y = -2x$ **6.** $(x - 2)^2 = 2(y + 2)$: Parabola; vertex $(2, -2)$, focus $\left(2, -\frac{3}{2}\right)$, directrix $y = -\frac{5}{2}$
7. $\frac{(y-2)^2}{4} - (x - 1)^2 = 1$: Hyperbola; center $(1, 2)$, vertices $(1, 4)$ and $(1, 0)$, foci $(1, 2 + \sqrt{5})$ and $(1, 2 - \sqrt{5})$, asymptotes $y - 2 = \pm 2(x - 1)$
8. $\frac{(x-2)^2}{9} + \frac{(y-1)^2}{4} = 1$: Ellipse; center $(2, 1)$, vertices $(5, 1)$ and $(-1, 1)$, foci $(2 + \sqrt{5}, 1)$ and $(2 - \sqrt{5}, 1)$
9. $(x - 2)^2 = -4(y + 1)$: Parabola; vertex $(2, -1)$, focus $(2, -2)$, directrix $y = 0$
10. $\frac{(x-1)^2}{4} + \frac{(y+1)^2}{9} = 1$: Ellipse; center $(1, -1)$, vertices $(1, 2)$ and $(1, -4)$, foci $(1, -1 + \sqrt{5})$ and $(1, -1 - \sqrt{5})$

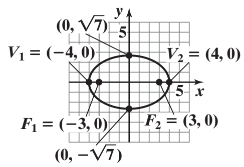
11. $y^2 = -8x$



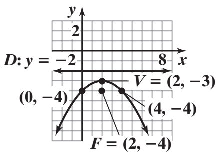
12. $\frac{y^2}{4} - \frac{x^2}{12} = 1$



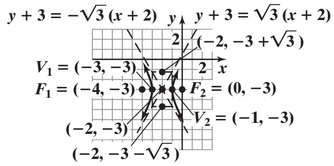
13. $\frac{x^2}{16} + \frac{y^2}{7} = 1$



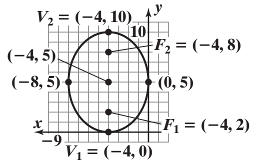
14. $(x - 2)^2 = -4(y + 3)$



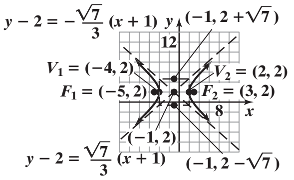
15. $(x + 2)^2 - \frac{(y+3)^2}{3} = 1$



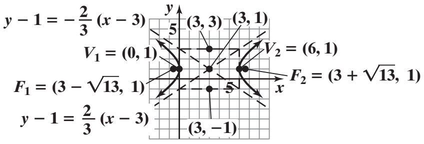
16. $\frac{(x+4)^2}{16} + \frac{(y-5)^2}{25} = 1$



17. $\frac{(x+1)^2}{9} - \frac{(y-2)^2}{7} = 1$



18. $\frac{(x-3)^2}{9} - \frac{(y-1)^2}{4} = 1$



19. $\frac{x^2}{5} - \frac{y^2}{4} = 1$ **20.** The ellipse $\frac{x^2}{16} + \frac{y^2}{7} = 1$ **21.** $\frac{1}{4}$ ft or 3 in. **22.** 19.72 ft, 18.86 ft, 14.91 ft **23.** 450 ft

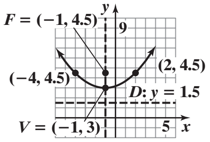
Chapter Test *(page 551)*

1. Hyperbola; center: $(-1, 0)$; vertices: $(-3, 0)$ and $(1, 0)$; foci: $(-1-\sqrt{13}, 0)$ and $(-1+\sqrt{13}, 0)$; asymptotes: $y = -\frac{3}{2}(x+1)$ and $y = \frac{3}{2}(x+1)$

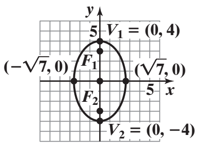
2. Parabola; vertex: $\left(1, -\frac{1}{2}\right)$; focus: $\left(1, \frac{3}{2}\right)$; directrix: $y = -\frac{5}{2}$

3. Ellipse; center: $(-1, 1)$; foci: $(-1-\sqrt{3}, 1)$ and $(-1+\sqrt{3}, 1)$; vertices: $(-4, 1)$ and $(2, 1)$

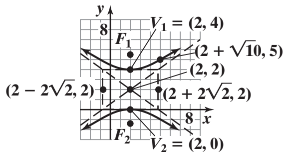
4. $(x+1)^2 = 6(y-3)$



5. $\frac{x^2}{7} + \frac{y^2}{16} = 1$



6. $\frac{(y-2)^2}{4} - \frac{(x-2)^2}{8} = 1$



7. The microphone should be located $\frac{2}{3}$ ft from the base of the reflector, along its axis of symmetry.

Cumulative Review *(page 551)*

1. $-6x + 5 - 3h$ **2.** $\left\{-5, -\frac{1}{3}, 2\right\}$ **3.** $\{x \mid -3 \le x \le 2\}$ or $[-3, 2]$ **4. (a)** Domain: $(-\infty, \infty)$; range: $(2, \infty)$ **(b)** $y = \log_3(x-2)$; domain: $(2, \infty)$; range: $(-\infty, \infty)$ **5. (a)** $\{18\}$ **(b)** $(2, 18]$

6. (a) $y = 2x - 2$ **(b)** $(x-2)^2 + y^2 = 4$ **(c)** $\frac{x^2}{9} + \frac{y^2}{4} = 1$ **(d)** $y = 2(x-1)^2$ **(e)** $y^2 - \frac{x^2}{3} = 1$ **(f)** $y = 4^x$

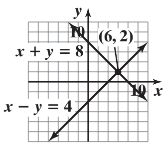
CHAPTER 8 Systems of Equations and Inequalities

8.1 Assess Your Understanding *(page 564)*

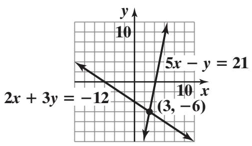
3. inconsistent **4.** consistent; independent **5.** $(3, -2)$ **6.** consistent; dependent **7.** b **8.** a **9.** $\begin{cases} 2(2) - (-1) = 5 \\ 5(2) + 2(-1) = 8 \end{cases}$

11. $\begin{cases} 3(2) - 4\left(\frac{1}{2}\right) = 4 \\ \frac{1}{2}(2) - 3\left(\frac{1}{2}\right) = -\frac{1}{2} \end{cases}$ **13.** $\begin{cases} 4 - 1 = 3 \\ \frac{1}{2}(4) + 1 = 3 \end{cases}$ **15.** $\begin{cases} 3(1) + 3(-1) + 2(2) = 4 \\ 1 - (-1) - 2 = 0 \\ 2(-1) - 3(2) = -8 \end{cases}$ **17.** $\begin{cases} 3(2) + 3(-2) + 2(2) = 4 \\ 2 - 3(-2) + 2 = 10 \\ 5(2) - 2(-2) - 3(2) = 8 \end{cases}$

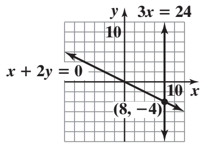
19. $x = 6, y = 2$; $(6, 2)$



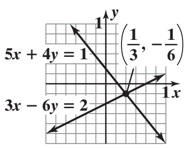
21. $x = 3, y = -6$; $(3, -6)$



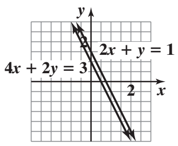
23. $x = 8, y = -4$; $(8, -4)$



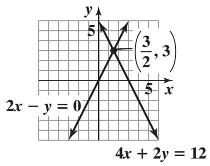
25. $x = \frac{1}{3}, y = -\frac{1}{6}$; $\left(\frac{1}{3}, -\frac{1}{6}\right)$



27. Inconsistent



29. $x = \frac{3}{2}, y = 3$; $\left(\frac{3}{2}, 3\right)$



31. $\{(x, y) \mid x = 4 - 2y, y \text{ is any real number}\}$, or $\left\{(x, y) \,\middle|\, y = \frac{4-x}{2}, x \text{ is any real number}\right\}$ **33.** $x = 1, y = 1$; $(1, 1)$ **35.** $x = \frac{3}{2}, y = 1$; $\left(\frac{3}{2}, 1\right)$

37. $x = 4, y = 3$; $(4, 3)$ **39.** $x = \frac{4}{3}, y = \frac{1}{5}$; $\left(\frac{4}{3}, \frac{1}{5}\right)$ **41.** $x = \frac{1}{5}, y = \frac{1}{3}$; $\left(\frac{1}{5}, \frac{1}{3}\right)$ **43.** $x = 8, y = 2, z = 0$; $(8, 2, 0)$

45. $x = 2, y = -1, z = 1$; $(2, -1, 1)$ **47.** Inconsistent **49.** $\{(x, y, z) \mid x = 5z - 2, y = 4z - 3; z \text{ is any real number}\}$ **51.** Inconsistent

53. $x = 1, y = 3, z = -2$; $(1, 3, -2)$ **55.** $x = -3, y = \frac{1}{2}, z = 1$; $\left(-3, \frac{1}{2}, 1\right)$ **57.** Length 30 ft; width 15 ft

59. 23 commercial launches and 69 noncommercial launches **61.** 22.5 lb **63.** Smartphone: \$325; tablet: \$640
65. Average wind speed 25 mph; average airspeed 175 mph **67.** 80 \$25 sets and 120 \$45 sets **69.** \$9.96

71. Mix 50 mg of first compound with 75 mg of second. **73.** $a = \frac{4}{3}, b = -\frac{5}{3}, c = 1$ **75.** $Y = 9000, r = 0.06$ **77.** $I_1 = \frac{10}{71}, I_2 = \frac{65}{71}, I_3 = \frac{55}{71}$

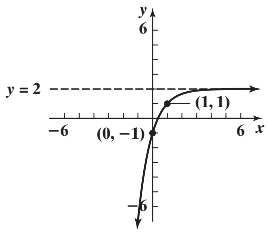
79. 100 orchestra, 210 main, and 190 balcony seats **81.** 1.5 chicken, 1 corn, 2 milk

83. If x = price of hamburgers, y = price of fries, and z = price of colas, then $x = 5.5 - z$, $y = \frac{41}{30} + \frac{1}{3}z$, $\$1.20 \le z \le \$1.80$. There is not sufficient information:

x	\$4.26	\$4.02	\$3.72
y	\$1.78	\$1.86	\$1.96
z	\$1.24	\$1.48	\$1.78

85. It will take Beth 30 hr, Bill 24 hr, and Edie 40 hr.

89.



90. (a) $2(2x-3)^3(x^3+5)(10x^3-9x^2+20)$ **(b)** $7(3x-5)^{-1/2}(x+3)^{-3/2}$

91. $\left[-\frac{2}{3}, 2\right]$ **92.** $f^{-1}(x) = \log_2(x-1)$

8.2 Assess Your Understanding *(page 580)*

1. matrix **2.** augmented **3.** third; fifth **4.** T **5.** b **6.** c **7.** $\left[\begin{array}{cc|c} 1 & -5 & 5 \\ 4 & 3 & 6 \end{array}\right]$ **9.** $\left[\begin{array}{cc|c} 2 & 3 & 6 \\ 4 & -6 & -2 \end{array}\right]$ **11.** $\left[\begin{array}{cc|c} 0.01 & -0.03 & 0.06 \\ 0.13 & 0.10 & 0.20 \end{array}\right]$

13. $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 3 & 3 & 0 & 5 \\ 1 & 1 & 2 & 2 \end{array}\right]$ **15.** $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 3 & -2 & 0 & 2 \\ 5 & 3 & -1 & 1 \end{array}\right]$ **17.** $\left[\begin{array}{ccc|c} 1 & -1 & -1 & 10 \\ 2 & 1 & 2 & -1 \\ -3 & 4 & 0 & 5 \\ 4 & -5 & 1 & 0 \end{array}\right]$ **19.** $\begin{cases} x - 3y = -2 & (1) \\ 2x - 5y = 5 & (2) \end{cases}$; $\left[\begin{array}{cc|c} 1 & -3 & -2 \\ 0 & 1 & 9 \end{array}\right]$

21. $\begin{cases} x - 3y + 4z = 3 & (1) \\ 3x - 5y + 6z = 6 & (2) \\ -5x + 3y + 4z = 6 & (3) \end{cases}$; $\left[\begin{array}{ccc|c} 1 & -3 & 4 & 3 \\ 0 & 4 & -6 & -3 \\ 0 & -12 & 24 & 21 \end{array}\right]$ **23.** $\begin{cases} x - 3y + 2z = -6 & (1) \\ 2x - 5y + 3z = -4 & (2) \\ -3x - 6y + 4z = 6 & (3) \end{cases}$; $\left[\begin{array}{ccc|c} 1 & -3 & 2 & -6 \\ 0 & 1 & -1 & 8 \\ 0 & -15 & 10 & -12 \end{array}\right]$

25. $\begin{cases} 5x - 3y + z = -2 & (1) \\ 2x - 5y + 6z = -2 & (2) \\ -4x + y + 4z = 6 & (3) \end{cases}$; $\left[\begin{array}{ccc|c} 1 & 7 & -11 & 2 \\ 2 & -5 & 6 & -2 \\ 0 & -9 & 16 & 2 \end{array}\right]$

27. $\begin{cases} x = 5 \\ y = -1 \end{cases}$
Consistent; $x = 5, y = -1$ or $(5, -1)$

29. $\begin{cases} x = 1 \\ y = 2 \\ 0 = 3 \end{cases}$
Inconsistent

31. $\begin{cases} x + 2z = -1 \\ y - 4z = -2 \\ 0 = 0 \end{cases}$
Consistent:
$\begin{cases} x = -1 - 2z \\ y = -2 + 4z \\ z \text{ is any real number or} \end{cases}$
$\{(x, y, z) \mid x = -1 - 2z, y = -2 + 4z, z \text{ is any real number}\}$

33. $\begin{cases} x_1 = 1 \\ x_2 + x_4 = 2 \\ x_3 + 2x_4 = 3 \end{cases}$
Consistent:
$\begin{cases} x_1 = 1, x_2 = 2 - x_4 \\ x_3 = 3 - 2x_4 \\ x_4 \text{ is any real number or} \end{cases}$
$\{(x_1, x_2, x_3, x_4) \mid x_1 = 1, x_2 = 2 - x_4, x_3 = 3 - 2x_4, x_4 \text{ is any real number}\}$

35. $\begin{cases} x_1 + 4x_4 = 2 \\ x_2 + x_3 + 3x_4 = 3 \\ 0 = 0 \end{cases}$
Consistent:
$\begin{cases} x_1 = 2 - 4x_4 \\ x_2 = 3 - x_3 - 3x_4 \\ x_3, x_4 \text{ are any real numbers or} \end{cases}$
$\{(x_1, x_2, x_3, x_4) \mid x_1 = 2 - 4x_4, x_2 = 3 - x_3 - 3x_4, x_3, x_4 \text{ are any real numbers}\}$

37. $\begin{cases} x_1 + x_4 = -2 \\ x_2 + 2x_4 = 2 \\ x_3 - x_4 = 0 \\ 0 = 0 \end{cases}$
Consistent:
$\begin{cases} x_1 = -2 - x_4 \\ x_2 = 2 - 2x_4 \\ x_3 = x_4 \\ x_4 \text{ is any real number or} \end{cases}$
$\{(x_1, x_2, x_3, x_4) \mid x_1 = -2 - x_4, x_2 = 2 - 2x_4, x_3 = x_4, x_4 \text{ is any real number}\}$

39. $x = 6, y = 2$; $(6, 2)$ **41.** $x = \frac{1}{2}, y = \frac{3}{4}$; $\left(\frac{1}{2}, \frac{3}{4}\right)$ **43.** $x = 4 - 2y$, y is any real number; $\{(x, y) \mid x = 4 - 2y, y \text{ is any real number}\}$

45. $x = \frac{3}{2}, y = 1$; $\left(\frac{3}{2}, 1\right)$ **47.** $x = \frac{4}{3}, y = \frac{1}{5}$; $\left(\frac{4}{3}, \frac{1}{5}\right)$ **49.** $x = 8, y = 2, z = 0$; $(8, 2, 0)$ **51.** $x = 2, y = -1, z = 1$; $(2, -1, 1)$ **53.** Inconsistent

55. $x = 5z - 2, y = 4z - 3$, where z is any real number; $\{(x, y, z) \mid x = 5z - 2, y = 4z - 3, z \text{ is any real number}\}$ **57.** Inconsistent

59. $x = 1, y = 3, z = -2$; $(1, 3, -2)$ **61.** $x = -3, y = \frac{1}{2}, z = 1$; $\left(-3, \frac{1}{2}, 1\right)$ **63.** $x = \frac{1}{3}, y = \frac{2}{3}, z = 1$; $\left(\frac{1}{3}, \frac{2}{3}, 1\right)$

65. $x = 1, y = 2, z = 0, w = 1$; $(1, 2, 0, 1)$ **67.** $y = 0, z = 1 - x$, x is any real number; $\{(x, y, z) \mid y = 0, z = 1 - x, x \text{ is any real number}\}$

69. $x = 2, y = z - 3$, z is any real number; $\{(x, y, z) \mid x = 2, y = z - 3, z \text{ is any real number}\}$ **71.** $x = \frac{13}{9}, y = \frac{7}{18}, z = \frac{19}{18}$; $\left(\frac{13}{9}, \frac{7}{18}, \frac{19}{18}\right)$

73. $x = \frac{7}{5} - \frac{3}{5}z - \frac{2}{5}w$, $y = -\frac{8}{5} + \frac{7}{5}z + \frac{13}{5}w$, where z and w are any real numbers; $\left\{(x, y, z, w) \,\middle|\, x = \frac{7}{5} - \frac{3}{5}z - \frac{2}{5}w, y = -\frac{8}{5} + \frac{7}{5}z + \frac{13}{5}w, z \text{ and } w \text{ are any real numbers}\right\}$ **75.** $y = -2x^2 + x + 3$ **77.** $f(x) = 3x^3 - 4x^2 + 5$ **79.** 1.5 salmon steak, 2 baked eggs, 1 acorn squash

81. \$4000 in Treasury bills, \$4000 in Treasury bonds, \$2000 in corporate bonds **83.** 8 Deltas, 5 Betas, 10 Sigmas **85.** $I_1 = \frac{44}{23}, I_2 = 2, I_3 = \frac{16}{23}, I_4 = \frac{28}{23}$

87. (a)

Amount Invested At		
7%	**9%**	**11%**
0	10,000	10,000
1000	8000	11,000
2000	6000	12,000
3000	4000	13,000
4000	2000	14,000
5000	0	15,000

(b)

Amount Invested At		
7%	**9%**	**11%**
12,500	12,500	0
14,500	8500	2000
16,500	4500	4000
18,750	0	6250

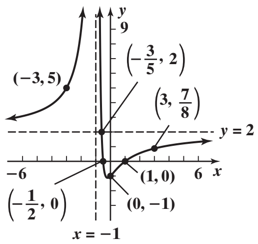
(c) All the money invested at 7% provides \$2100, more than what is required.

89.

First Supplement	Second Supplement	Third Supplement
50 mg	75 mg	0 mg
36 mg	76 mg	8 mg
22 mg	77 mg	16 mg
8 mg	78 mg	24 mg

94. $\{x \mid -1 < x < 6\}$, or $(-1, 6)$

95.



96. $\{x \mid x$ is any real nunber$\}$, or $(-\infty, \infty)$ **97.** $-5i, 5i, -2, 2$

8.3 Assess Your Understanding *(page 593)*

1. $ad - bc$ **2.** $\begin{vmatrix} 5 & 3 \\ -3 & -4 \end{vmatrix}$ **3.** F **4.** F **5.** F **6.** a **7.** 22 **9.** -2 **11.** 10 **13.** -26 **15.** $x = 6, y = 2; (6, 2)$ **17.** $x = 3, y = 2; (3, 2)$ **19.** $x = 8, y = -4; (8, -4)$ **21.** $x = 4, y = -2; (4, -2)$ **23.** Not applicable **25.** $x = \frac{1}{2}, y = \frac{3}{4}; \left(\frac{1}{2}, \frac{3}{4}\right)$ **27.** $x = \frac{1}{10}, y = \frac{2}{5}; \left(\frac{1}{10}, \frac{2}{5}\right)$ **29.** $x = \frac{3}{2}, y = 1; \left(\frac{3}{2}, 1\right)$ **31.** $x = \frac{4}{3}, y = \frac{1}{5}; \left(\frac{4}{3}, \frac{1}{5}\right)$ **33.** $x = 1, y = 3, z = -2; (1, 3, -2)$ **35.** $x = -3, y = \frac{1}{2}, z = 1; \left(-3, \frac{1}{2}, 1\right)$ **37.** Not applicable **39.** $x = 0, y = 0, z = 0; (0, 0, 0)$ **41.** Not applicable **43.** -4 **45.** 12 **47.** 8 **49.** 8 **51.** -5 **53.** $\frac{13}{11}$ **55.** 0 or -9

57.
$$\begin{aligned}
(y_1 - y_2)x - (x_1 - x_2)y + (x_1y_2 - x_2y_1) &= 0 \\
(y_1 - y_2)x + (x_2 - x_1)y &= x_2y_1 - x_1y_2 \\
(x_2 - x_1)y - (x_2 - x_1)y_1 &= (y_2 - y_1)x + x_2y_1 - x_1y_2 - (x_2 - x_1)y_1 \\
(x_2 - x_1)(y - y_1) &= (y_2 - y_1)x - (y_2 - y_1)x_1 \\
y - y_1 &= \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)
\end{aligned}$$

59. The triangle has an area of 5 square units. **61.** 50.5 square units **63.** $(x - 3)^2 + (y + 2)^2 = 25$

65. If $a = 0$, we have

$$\begin{aligned} by &= s \\ cx + dy &= t \end{aligned}$$

Thus, $y = \frac{s}{b}$ and

$$x = \frac{t - dy}{c} = \frac{tb - ds}{bc}$$

Using Cramer's Rule, we get

$$x = \frac{sd - tb}{-bc} = \frac{tb - sd}{bc}$$

$$y = \frac{-sc}{-bc} = \frac{s}{b}$$

If $b = 0$, we have

$$\begin{aligned} ax &= s \\ cx + dy &= t \end{aligned}$$

Since $D = ad \neq 0$, then $a \neq 0$ and $d \neq 0$.

Thus, $x = \frac{s}{a}$ and

$$y = \frac{t - cx}{d} = \frac{ta - cs}{ad}$$

Using Cramer's Rule, we get

$$x = \frac{sd}{ad} = \frac{s}{a}$$

$$y = \frac{ta - cs}{ad}$$

If $c = 0$, we have

$$\begin{aligned} ax + by &= s \\ dy &= t \end{aligned}$$

Since $D = ad \neq 0$, then $a \neq 0$ and $d \neq 0$.

Thus, $y = \frac{t}{d}$ and

$$x = \frac{s - by}{a} = \frac{sd - bt}{ad}$$

Using Cramer's Rule, we get

$$x = \frac{sd - bt}{ad}$$

$$y = \frac{at}{ad} = \frac{t}{d}$$

If $d = 0$, we have

$$\begin{aligned} ax + by &= s \\ cx &= t \end{aligned}$$

Since $D = -bc \neq 0$, then $b \neq 0$ and $c \neq 0$.

Thus, $x = \frac{t}{c}$ and

$$y = \frac{s - ax}{b} = \frac{sc - at}{bc}$$

Using Cramer's Rule, we get

$$x = \frac{-tb}{-bc} = \frac{t}{c}$$

$$y = \frac{at - sc}{-bc} = \frac{sc - at}{bc}$$

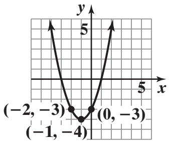
67.
$$\begin{aligned}
\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= -ka_{21}(a_{12}a_{33} - a_{32}a_{13}) + ka_{22}(a_{11}a_{33} - a_{31}a_{13}) - ka_{23}(a_{11}a_{32} - a_{31}a_{12}) \\
&= k[-a_{21}(a_{12}a_{33} - a_{32}a_{13}) + a_{22}(a_{11}a_{33} - a_{31}a_{13}) - a_{23}(a_{11}a_{32} - a_{31}a_{12})] = k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

69. $\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (a_{11}+ka_{21})(a_{22}a_{33}-a_{32}a_{23}) - (a_{12}+ka_{22})(a_{21}a_{33}-a_{31}a_{23}) + (a_{13}+ka_{23})(a_{21}a_{32}-a_{31}a_{22})$

$= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} + \cancel{ka_{21}a_{22}a_{33}} - \cancel{ka_{21}a_{32}a_{23}} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23}$
$\quad - \cancel{ka_{22}a_{21}a_{33}} + \cancel{ka_{22}a_{31}a_{23}} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} + \cancel{ka_{23}a_{21}a_{32}} - \cancel{ka_{23}a_{31}a_{22}}$
$= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}$
$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$
$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

70. $\dfrac{5-\sqrt{10}}{3}, \dfrac{5+\sqrt{10}}{3}$ **71.** $\pm\dfrac{1}{2}, \pm\dfrac{5}{2}, \pm 1, \pm 2, \pm 5, \pm 10$

72.



73. $x = \log_5 y$

Historical Problems *(page 608)*

1. (a) $2-5i \longleftrightarrow \begin{bmatrix} 2 & -5 \\ 5 & 2 \end{bmatrix}, 1+3i \longleftrightarrow \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$ **(b)** $\begin{bmatrix} 2 & -5 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ -1 & 17 \end{bmatrix}$ **(c)** $17+i$ **(d)** $17+i$

2. $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} a^2+b^2 & 0 \\ 0 & b^2+a^2 \end{bmatrix}$; the product is a real number.

3. (a) $x = k(ar+bs) + l(cr+ds) = r(ka+lc) + s(kb+ld)$
$y = m(ar+bs) + n(cr+ds) = r(ma+nc) + s(mb+nd)$ **(b)** $A = \begin{bmatrix} ka+lc & kb+ld \\ ma+nc & mb+nd \end{bmatrix}$

8.4 Assess Your Understanding *(page 608)*

1. square **2.** T **3.** F **4.** inverse **5.** T **6.** $A^{-1}B$ **7.** a **8.** d **9.** $\begin{bmatrix} 4 & 4 & -5 \\ -1 & 5 & 4 \end{bmatrix}$ **11.** $\begin{bmatrix} 0 & 12 & -20 \\ 4 & 8 & 24 \end{bmatrix}$ **13.** $\begin{bmatrix} -8 & 7 & -15 \\ 7 & 0 & 22 \end{bmatrix}$

15. $\begin{bmatrix} 28 & -9 \\ 4 & 23 \end{bmatrix}$ **17.** Not defined **19.** $\begin{bmatrix} 1 & 14 & -14 \\ 2 & 22 & -18 \\ 3 & 0 & 28 \end{bmatrix}$ **21.** $\begin{bmatrix} 15 & 21 & -16 \\ 22 & 34 & -22 \\ -11 & 7 & 22 \end{bmatrix}$ **23.** $\begin{bmatrix} 25 & -9 \\ 4 & 20 \end{bmatrix}$ **25.** $\begin{bmatrix} -13 & 7 & -12 \\ -18 & 10 & -14 \\ 17 & -7 & 34 \end{bmatrix}$ **27.** $\begin{bmatrix} -2 & 4 & 2 & 8 \\ 2 & 1 & 4 & 6 \end{bmatrix}$

29. $\begin{bmatrix} 5 & 14 \\ 9 & 16 \end{bmatrix}$ **31.** Not defined **33.** $\begin{bmatrix} 9 & 2 \\ 34 & 13 \\ 47 & 20 \end{bmatrix}$ **35.** $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ **37.** $\begin{bmatrix} 1 & -\frac{5}{2} \\ -1 & 3 \end{bmatrix}$ **39.** $\begin{bmatrix} 1 & -\frac{1}{a} \\ -1 & \frac{2}{a} \end{bmatrix}$ **41.** $\begin{bmatrix} 3 & -3 & 1 \\ -2 & 2 & -1 \\ -4 & 5 & -2 \end{bmatrix}$ **43.** $\begin{bmatrix} -\frac{5}{7} & \frac{1}{7} & \frac{3}{7} \\ \frac{9}{7} & \frac{1}{7} & -\frac{4}{7} \\ \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{bmatrix}$

45. $x=3, y=2; (3,2)$ **47.** $x=-5, y=10; (-5,10)$ **49.** $x=2, y=-1; (2,-1)$ **51.** $x=\frac{1}{2}, y=2; \left(\frac{1}{2}, 2\right)$ **53.** $x=-2, y=1; (-2,1)$

55. $x=\frac{2}{a}, y=\frac{3}{a}; \left(\frac{2}{a}, \frac{3}{a}\right)$ **57.** $x=-2, y=3, z=5; (-2,3,5)$ **59.** $x=\frac{1}{2}, y=-\frac{1}{2}, z=1; \left(\frac{1}{2}, -\frac{1}{2}, 1\right)$

61. $x=-\frac{34}{7}, y=\frac{85}{7}, z=\frac{12}{7}; \left(-\frac{34}{7}, \frac{85}{7}, \frac{12}{7}\right)$ **63.** $x=\frac{1}{3}, y=1, z=\frac{2}{3}; \left(\frac{1}{3}, 1, \frac{2}{3}\right)$ **65.** $\left[\begin{array}{cc|cc} 4 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{4} & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{2} & 1 \end{array}\right]$

67. $\left[\begin{array}{cc|cc} 15 & 3 & 1 & 0 \\ 10 & 2 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & \frac{1}{5} & \frac{1}{15} & 0 \\ 10 & 2 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & \frac{1}{5} & \frac{1}{15} & 0 \\ 0 & 0 & -\frac{2}{3} & 1 \end{array}\right]$

69. $\left[\begin{array}{ccc|ccc} -3 & 1 & -1 & 1 & 0 & 0 \\ 1 & -4 & -7 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 5 & 0 & 0 & 1 \\ 1 & -4 & -7 & 0 & 1 & 0 \\ -3 & 1 & -1 & 1 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 5 & 0 & 0 & 1 \\ 0 & -6 & -12 & 0 & 1 & -1 \\ 0 & 7 & 14 & 1 & 0 & 3 \end{array}\right]$

$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 5 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & -\frac{1}{6} & \frac{1}{6} \\ 0 & 1 & 2 & \frac{1}{7} & 0 & \frac{3}{7} \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 5 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & -\frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{7} & \frac{1}{6} & \frac{11}{42} \end{array}\right]$

71. $\begin{bmatrix} 0.01 & 0.05 & -0.01 \\ 0.01 & -0.02 & 0.01 \\ -0.02 & 0.01 & 0.03 \end{bmatrix}$ **73.** $\begin{bmatrix} 0.02 & -0.04 & -0.01 & 0.01 \\ -0.02 & 0.05 & 0.03 & -0.03 \\ 0.02 & 0.01 & -0.04 & 0.00 \\ -0.02 & 0.06 & 0.07 & 0.06 \end{bmatrix}$

75. $x = 4.57, y = -6.44, z = -24.07; (4.57, -6.44, -24.07)$ **77.** $x = -1.19, y = 2.46, z = 8.27; (-1.19, 2.46, 8.27)$

79. $x = -5, y = 7; (-5, 7)$ **81.** $x = -4, y = 2, z = \frac{5}{2}; \left(-4, 2, \frac{5}{2}\right)$

83. Inconsistent; $\varnothing$ **85.** $x = -\frac{1}{5}z + \frac{1}{5}, y = \frac{1}{5}z - \frac{6}{5}$, where z is any real number; $\left\{(x, y, z) \,\middle|\, x = -\frac{1}{5}z + \frac{1}{5}, y = \frac{1}{5}z - \frac{6}{5}, z \text{ is any real number}\right\}$

87. (a) $A = \begin{bmatrix} 6 & 9 \\ 3 & 12 \end{bmatrix}; B = \begin{bmatrix} 128.00 \\ 341.60 \end{bmatrix}$ **(b)** $AB = \begin{bmatrix} 3842.40 \\ 4483.20 \end{bmatrix}$; Nikki's total tuition is \$3842.40, and Joe's total tuition is \$4483.20.

89. (a) $\begin{bmatrix} 500 & 350 & 400 \\ 700 & 500 & 850 \end{bmatrix}; \begin{bmatrix} 500 & 700 \\ 350 & 500 \\ 400 & 850 \end{bmatrix}$ **(b)** $\begin{bmatrix} 15 \\ 8 \\ 3 \end{bmatrix}$ **(c)** $\begin{bmatrix} 11{,}500 \\ 17{,}050 \end{bmatrix}$ **(d)** $[0.10 \quad 0.05]$ **(e)** \$2002.50

91. (a) $K^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$ **(b)** $M = \begin{bmatrix} 13 & 1 & 20 \\ 8 & 9 & 19 \\ 6 & 21 & 14 \end{bmatrix}$ **(c)** Math is fun.

93. If $D = ad - bc \neq 0$, then $a \neq 0$ and $d \neq 0$, or $b \neq 0$ and $c \neq 0$. Assuming the former,

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{R_1 = \frac{1}{a}r_1} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{R_2 = -cr_1 + r_2} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{D}{a} & -\frac{c}{a} & 1 \end{array}\right] \xrightarrow{R_2 = \frac{a}{D}r_2} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{D} & \frac{a}{D} \end{array}\right] \xrightarrow{R_1 = -\frac{b}{a}r_2 + r_1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{D} & -\frac{b}{D} \\ 0 & 1 & -\frac{c}{D} & \frac{a}{D} \end{array}\right]$$

95. (a) $B_3 = A + A^2 + A^3 = \begin{bmatrix} 2 & 4 & 5 & 2 & 3 \\ 5 & 3 & 2 & 5 & 4 \\ 4 & 2 & 2 & 4 & 2 \\ 2 & 2 & 3 & 2 & 3 \\ 1 & 3 & 2 & 1 & 2 \end{bmatrix}$; Yes, all pages can reach every other page within 3 clicks. **(b)** Page 3

97. (a) $(3 - 2\sqrt{3}, 3\sqrt{3} + 2)$ **(b)** $R^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$; This is the rotation matrix needed to get the translated coordinates back to the original coordinates.

102. $x^6 - 4x^5 - 3x^4 + 18x^3$ **103.** $f(12) = 5$ **104.** $\{0, 3\}$ **105.** 2,476,000 units

8.5 Assess Your Understanding *(page 618)*

5. Proper **7.** Improper; $1 + \frac{9}{x^2 - 4}$ **9.** Improper; $5x + \frac{22x - 1}{x^2 - 4}$ **11.** Improper; $1 + \frac{-2(x - 6)}{(x + 4)(x - 3)}$ **13.** $\frac{-4}{x} + \frac{4}{x - 1}$ **15.** $\frac{1}{x} + \frac{-x}{x^2 + 1}$

17. $\frac{-1}{x - 1} + \frac{2}{x - 2}$ **19.** $\frac{\frac{1}{4}}{x + 1} + \frac{\frac{3}{4}}{x - 1} + \frac{\frac{1}{2}}{(x - 1)^2}$ **21.** $\frac{\frac{1}{12}}{x - 2} + \frac{-\frac{1}{12}(x + 4)}{x^2 + 2x + 4}$ **23.** $\frac{\frac{1}{4}}{x - 1} + \frac{\frac{1}{4}}{(x - 1)^2} + \frac{-\frac{1}{4}}{x + 1} + \frac{\frac{1}{4}}{(x + 1)^2}$

25. $\frac{-5}{x + 2} + \frac{5}{x + 1} + \frac{-4}{(x + 1)^2}$ **27.** $\frac{\frac{1}{4}}{x} + \frac{1}{x^2} + \frac{-\frac{1}{4}(x + 4)}{x^2 + 4}$ **29.** $\frac{\frac{2}{3}}{x + 1} + \frac{\frac{1}{3}(x + 1)}{x^2 + 2x + 4}$ **31.** $\frac{\frac{2}{7}}{3x - 2} + \frac{\frac{1}{7}}{2x + 1}$ **33.** $\frac{\frac{3}{4}}{x + 3} + \frac{\frac{1}{4}}{x - 1}$

35. $\frac{1}{x^2 + 4} + \frac{2x - 1}{(x^2 + 4)^2}$ **37.** $\frac{-1}{x} + \frac{2}{x - 3} + \frac{-1}{x + 1}$ **39.** $\frac{4}{x - 2} + \frac{-3}{x - 1} + \frac{-1}{(x - 1)^2}$ **41.** $\frac{x}{(x^2 + 16)^2} + \frac{-16x}{(x^2 + 16)^3}$

43. $\frac{-\frac{8}{7}}{2x + 1} + \frac{\frac{4}{7}}{x - 3}$ **45.** $\frac{-\frac{2}{9}}{x} + \frac{-\frac{1}{3}}{x^2} + \frac{\frac{1}{6}}{x - 3} + \frac{\frac{1}{18}}{x + 3}$ **47.** $x - 2 + \frac{10x - 11}{x^2 + 3x - 4}$; $\frac{\frac{51}{5}}{x + 4} + \frac{-\frac{1}{5}}{x - 1}$; $x - 2 + \frac{\frac{51}{5}}{x + 4} - \frac{\frac{1}{5}}{x - 1}$

49. $x - \frac{x}{x^2 + 1}$ **51.** $x^2 - 4x + 7 + \frac{-11x - 32}{x^2 + 4x + 4}$; $\frac{-11}{x + 2} + \frac{-10}{(x + 2)^2}$; $x^2 - 4x + 7 - \frac{11}{x + 2} - \frac{10}{(x + 2)^2}$

53. $x+1+\dfrac{2x^3+x^2-x+1}{x^4-2x^2+1}$; $\dfrac{1}{x+1}+\dfrac{\frac{1}{4}}{(x+1)^2}+\dfrac{1}{x-1}+\dfrac{\frac{3}{4}}{(x-1)^2}$; $x+1+\dfrac{1}{x+1}+\dfrac{\frac{1}{4}}{(x+1)^2}+\dfrac{1}{x-1}+\dfrac{\frac{3}{4}}{(x-1)^2}$

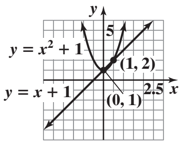
55. 3.85 years **56.** -2 **57.** Maximum, 1250 **58.** $(f\circ g)(x)=\dfrac{x-1}{x-2}$

Historical Problem *(page 626)*

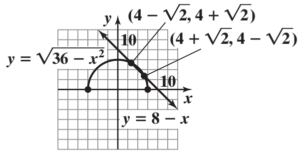
$x=6$ units, $y=8$ units

8.6 Assess Your Understanding *(page 626)*

5.



7.



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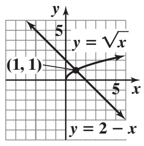

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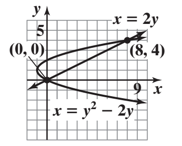

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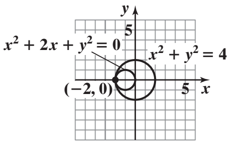

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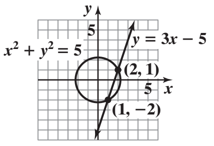
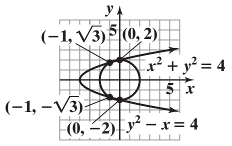
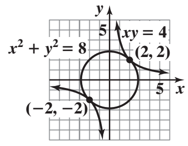


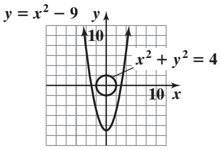
17.



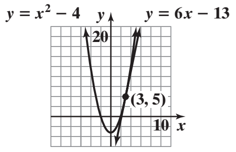
19.



21. No points of intersection



23.



25. $x=1, y=4; x=-1, y=-4; x=2\sqrt{2}, y=\sqrt{2}$; $x=-2\sqrt{2}, y=-\sqrt{2}$ or $(1,4), (-1,-4), (2\sqrt{2},\sqrt{2}), (-2\sqrt{2},-\sqrt{2})$

27. $x=0, y=1; x=-\dfrac{2}{3}, y=-\dfrac{1}{3}$ or $(0,1), \left(-\dfrac{2}{3},-\dfrac{1}{3}\right)$ **29.** $x=0, y=-1; x=\dfrac{5}{2}, y=-\dfrac{7}{2}$ or $(0,-1), \left(\dfrac{5}{2},-\dfrac{7}{2}\right)$

31. $x=2, y=\dfrac{1}{3}; x=\dfrac{1}{2}, y=\dfrac{4}{3}$ or $\left(2,\dfrac{1}{3}\right), \left(\dfrac{1}{2},\dfrac{4}{3}\right)$ **33.** $x=3, y=2; x=3, y=-2; x=-3, y=2; x=-3, y=-2$ or $(3,2), (3,-2), (-3,2), (-3,-2)$ **35.** $x=\dfrac{1}{2}, y=\dfrac{3}{2}; x=\dfrac{1}{2}, y=-\dfrac{3}{2}; x=-\dfrac{1}{2}, y=\dfrac{3}{2}; x=-\dfrac{1}{2}, y=-\dfrac{3}{2}$ or $\left(\dfrac{1}{2},\dfrac{3}{2}\right), \left(\dfrac{1}{2},-\dfrac{3}{2}\right), \left(-\dfrac{1}{2},\dfrac{3}{2}\right), \left(-\dfrac{1}{2},-\dfrac{3}{2}\right)$

37. $x=\sqrt{2}, y=2\sqrt{2}; x=-\sqrt{2}, y=-2\sqrt{2}$ or $(\sqrt{2},2\sqrt{2}), (-\sqrt{2},-2\sqrt{2})$ **39.** No real solution exists.

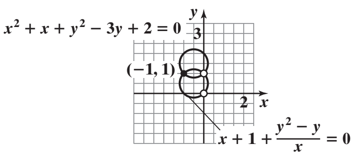
41. $x=\dfrac{8}{3}, y=\dfrac{2\sqrt{10}}{3}; x=-\dfrac{8}{3}, y=\dfrac{2\sqrt{10}}{3}; x=\dfrac{8}{3}, y=-\dfrac{2\sqrt{10}}{3}; x=-\dfrac{8}{3}, y=-\dfrac{2\sqrt{10}}{3}$ or $\left(\dfrac{8}{3},\dfrac{2\sqrt{10}}{3}\right), \left(-\dfrac{8}{3},\dfrac{2\sqrt{10}}{3}\right), \left(\dfrac{8}{3},-\dfrac{2\sqrt{10}}{3}\right), \left(-\dfrac{8}{3},-\dfrac{2\sqrt{10}}{3}\right)$

43. $x=1, y=\dfrac{1}{2}; x=-1, y=\dfrac{1}{2}; x=1, y=-\dfrac{1}{2}; x=-1, y=-\dfrac{1}{2}$ or $\left(1,\dfrac{1}{2}\right), \left(-1,\dfrac{1}{2}\right), \left(1,-\dfrac{1}{2}\right), \left(-1,-\dfrac{1}{2}\right)$ **45.** No real solution exists.

47. $x=\sqrt{3}, y=\sqrt{3}; x=-\sqrt{3}, y=-\sqrt{3}; x=2, y=1; x=-2, y=-1$ or $(\sqrt{3},\sqrt{3}), (-\sqrt{3},-\sqrt{3}), (2,1), (-2,-1)$

49. $x=0, y=-2; x=0, y=1; x=2, y=-1$ or $(0,-2), (0,1), (2,-1)$ **51.** $x=2, y=8$ or $(2,8)$ **53.** $x=81, y=3$ or $(81,3)$

55.

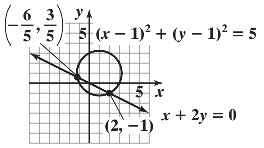


57. $x=0.48, y=0.62$ **59.** $x=-1.65, y=-0.89$

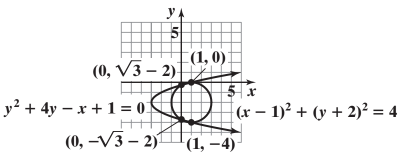
61. $x=0.58, y=1.86; x=1.81, y=1.05; x=0.58, y=-1.86; x=1.81, y=-1.05$

63. $x=2.35, y=0.85$

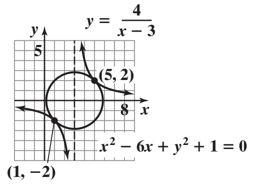
65.



67.



69.



71. 3 and 1; -1 and -3 **73.** 2 and 2; -2 and -2 **75.** $\dfrac{1}{2}$ and $\dfrac{1}{3}$ **77.** 5 **79.** 5 in. by 3 in. **81.** 2 cm and 4 cm **83.** Tortoise: 7 m/h, hare: $7\dfrac{1}{2}$ m/h

85. 12 cm by 18 cm **87.** $x=60$ ft; $y=30$ ft **89.** $l=\dfrac{P+\sqrt{P^2-16A}}{4}$; $w=\dfrac{P-\sqrt{P^2-16A}}{4}$ **91.** $y=4x-4$ **93.** $y=2x+1$

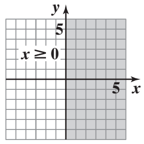
95. $y = -\frac{1}{3}x + \frac{7}{3}$ **97.** $y = 2x - 3$ **99.** $r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$; $r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ **101. (a)** 4.274 ft by 4.274 ft or 0.093 ft by 0.093 ft

102. $\left\{\frac{-3 - \sqrt{65}}{7}, \frac{-3 + \sqrt{65}}{7}\right\}$ **103.** $y = -\frac{2}{5}x - 3$ **104.** $y = \frac{1}{4}x + \frac{31}{4}$ **105.** 6.12%

8.7 Assess Your Understanding *(page 636)*

7. dashes; solid **8.** half-planes **9.** F **10.** unbounded

11.



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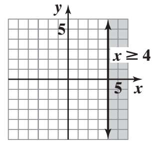

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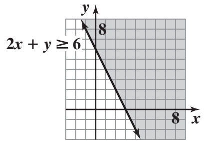

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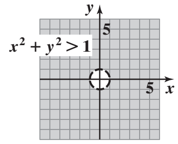

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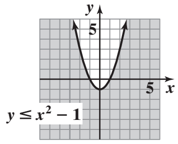

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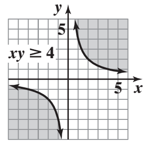

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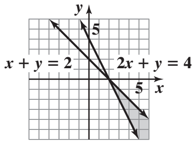

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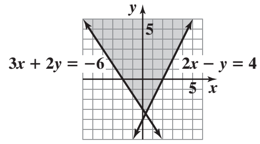

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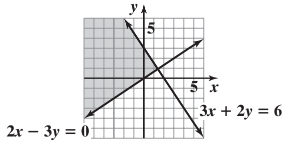
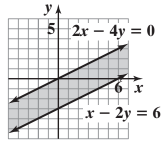
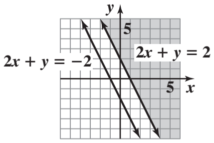


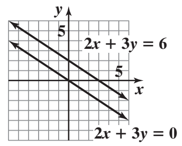
29.



31.



33. No solution



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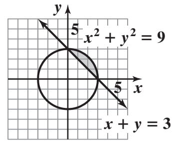

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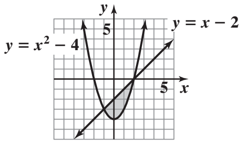

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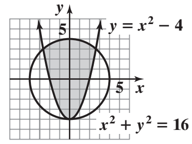

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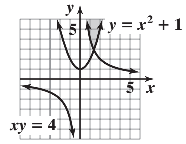
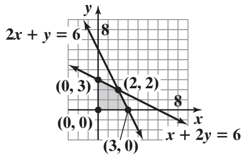
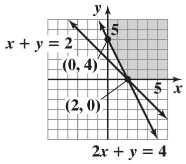


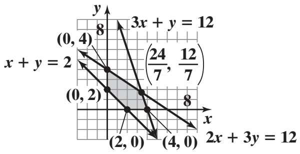
43. Bounded



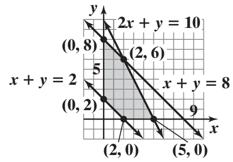
45. Unbounded



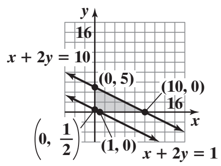
47. Bounded



49. Bounded



51. Bounded

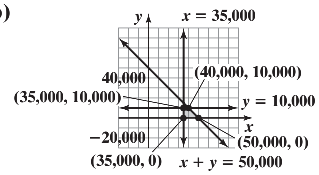


53. $\begin{cases} x \leq 4 \\ x + y \leq 6 \\ x \geq 0 \\ y \geq 0 \end{cases}$

55. $\begin{cases} x \leq 20 \\ y \geq 15 \\ x + y \leq 50 \\ x - y \leq 0 \\ x \geq 0 \end{cases}$

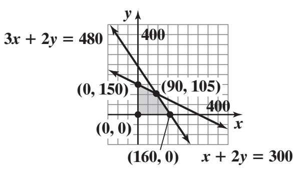
57. (a) $\begin{cases} x + y \leq 50{,}000 \\ x \geq 35{,}000 \\ y \leq 10{,}000 \\ y \geq 0 \end{cases}$

(b)



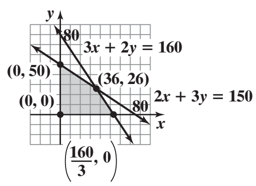
59. (a) $\begin{cases} x \geq 0 \\ y \geq 0 \\ x + 2y \leq 300 \\ 3x + 2y \leq 480 \end{cases}$

(b)



61. (a) $\begin{cases} 3x + 2y \leq 160 \\ 2x + 3y \leq 150 \\ x \geq 0 \\ y \geq 0 \end{cases}$

(b)



62. $\{-1 - 2i, -1 + 2i\}$ **63.** $d = \sqrt{74}$; midpoint: $\left(\frac{7}{2}, -\frac{11}{2}\right)$

64. $f(-1) = -5$; $f(2) = 28$ **65.** Vertical: $x = -3$; horizontal: $y = 5$

8.8 Assess Your Understanding *(page 644)*

1. objective function **2.** T **3.** Maximum value is 11; minimum value is 3. **5.** Maximum value is 65; minimum value is 4.
7. Maximum value is 67; minimum value is 20. **9.** The maximum value of z is 12, and it occurs at the point $(6, 0)$.
11. The minimum value of z is 4, and it occurs at the point $(2, 0)$. **13.** The maximum value of z is 20, and it occurs at the point $(0, 4)$.
15. The minimum value of z is 8, and it occurs at the point $(0, 2)$. **17.** The maximum value of z is 50, and it occurs at the point $(10, 0)$.
19. Produce 8 downhill and 24 cross-country; \$1760; \$1920 which is the profit when producing 16 downhill and 16 cross-country.
21. Rent 15 rectangular tables and 16 round tables for a minimum cost of \$1252. **23. (a)** \$10,000 in a junk bond and \$10,000 in Treasury bills **(b)** \$12,000 in a junk bond and \$8000 in Treasury bills **25.** 100 lb of ground beef should be mixed with 50 lb of pork.
27. Manufacture 10 racing skates and 15 figure skates. **29.** Order 2 metal samples and 4 plastic samples; \$34
31. (a) Configure with 10 first-class seats and 120 coach seats. **(b)** Configure with 15 first-class seats and 120 coach seats.

33. $\left\{-\frac{1}{32}, 1\right\}$ **34.** $\{-10\}$ **35.** 89.1 years **36.** $y = 3x + 7$

Review Exercises *(page 648)*

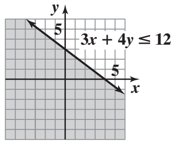
1. $x = 2, y = -1$ or $(2, -1)$ **2.** $x = 2, y = \frac{1}{2}$ or $\left(2, \frac{1}{2}\right)$ **3.** $x = 2, y = -1$ or $(2, -1)$ **4.** $x = \frac{11}{5}, y = -\frac{3}{5}$ or $\left(\frac{11}{5}, -\frac{3}{5}\right)$ **5.** Inconsistent
6. $x = 2, y = 3$ or $(2, 3)$ **7.** $y = -\frac{2}{5}x + 2$, where x is any real number, or $\left\{(x, y) \mid y = -\frac{2}{5}x + 2,\ x \text{ is any real number}\right\}$
8. $x = -1, y = 2, z = -3$ or $(-1, 2, -3)$
9. $x = \frac{7}{4}z + \frac{39}{4}, y = \frac{9}{8}z + \frac{69}{8}$, where z is any real number, or $\left\{(x, y, z) \middle| x = \frac{7}{4}z + \frac{39}{4}, y = \frac{9}{8}z + \frac{69}{8}, z \text{ is any real number}\right\}$ **10.** Inconsistent

11. $\begin{cases} 3x + 2y = 8 \\ x + 4y = -1 \end{cases}$ **12.** $\begin{cases} x + 2y + 5z = -2 \\ 5x \quad\quad - 3z = 8 \\ 2x - y \quad\quad = 0 \end{cases}$ **13.** $\begin{bmatrix} 4 & -4 \\ 3 & 9 \\ 4 & 4 \end{bmatrix}$ **14.** $\begin{bmatrix} 6 & 0 \\ 12 & 24 \\ -6 & 12 \end{bmatrix}$ **15.** $\begin{bmatrix} 4 & -3 & 0 \\ 12 & -2 & -8 \\ -2 & 5 & -4 \end{bmatrix}$ **16.** $\begin{bmatrix} 9 & -31 \\ -6 & -3 \end{bmatrix}$

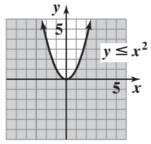
17. $\begin{bmatrix} \frac{1}{2} & -1 \\ -\frac{1}{6} & \frac{2}{3} \end{bmatrix}$ **18.** $\begin{bmatrix} -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \\ \frac{1}{7} & \frac{1}{7} & -\frac{2}{7} \\ \frac{3}{7} & -\frac{4}{7} & \frac{1}{7} \end{bmatrix}$ **19.** Singular **20.** $x = \frac{2}{5}, y = \frac{1}{10}$ or $\left(\frac{2}{5}, \frac{1}{10}\right)$ **21.** $x = 9, y = \frac{13}{3}, z = \frac{13}{3}$ or $\left(9, \frac{13}{3}, \frac{13}{3}\right)$

22. Inconsistent **23.** $x = -\frac{1}{2}, y = -\frac{2}{3}, z = -\frac{3}{4}$, or $\left(-\frac{1}{2}, -\frac{2}{3}, -\frac{3}{4}\right)$
24. $z = -1, x = y + 1$, where y is any real number, or $\{(x, y, z) \mid x = y + 1, z = -1, y \text{ is any real number}\}$
25. $x = 4, y = 2, z = 3, t = -2$ or $(4, 2, 3, -2)$ **26.** 5 **27.** 108 **28.** -100 **29.** $x = 2, y = -1$ or $(2, -1)$ **30.** $x = 2, y = 3$ or $(2, 3)$
31. $x = -1, y = 2, z = -3$ or $(-1, 2, -3)$ **32.** 16 **33.** -8 **34.** $\frac{-\frac{3}{2}}{x} + \frac{\frac{3}{2}}{x - 4}$ **35.** $\frac{-3}{x - 1} + \frac{3}{x} + \frac{4}{x^2}$ **36.** $\frac{-\frac{1}{10}}{x + 1} + \frac{\frac{1}{10}x + \frac{9}{10}}{x^2 + 9}$
37. $\frac{x}{x^2 + 4} + \frac{-4x}{(x^2 + 4)^2}$ **38.** $\frac{\frac{1}{2}}{x^2 + 1} + \frac{\frac{1}{4}}{x - 1} + \frac{-\frac{1}{4}}{x + 1}$ **39.** $x = -\frac{2}{5}, y = -\frac{11}{5}; x = -2, y = 1$ or $\left(-\frac{2}{5}, -\frac{11}{5}\right), (-2, 1)$
40. $x = 2\sqrt{2}, y = \sqrt{2}; x = -2\sqrt{2}, y = -\sqrt{2}$ or $(2\sqrt{2}, \sqrt{2}), (-2\sqrt{2}, -\sqrt{2})$
41. $x = 0, y = 0; x = -3, y = 3; x = 3, y = 3$ or $(0, 0), (-3, 3), (3, 3)$
42. $x = \sqrt{2}, y = -\sqrt{2}; x = -\sqrt{2}, y = \sqrt{2}; x = \frac{4}{3}\sqrt{2}, y = -\frac{2}{3}\sqrt{2}; x = -\frac{4}{3}\sqrt{2}, y = \frac{2}{3}\sqrt{2}$ or $(\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2}), \left(\frac{4}{3}\sqrt{2}, -\frac{2}{3}\sqrt{2}\right)$, $\left(-\frac{4}{3}\sqrt{2}, \frac{2}{3}\sqrt{2}\right)$ **43.** $x = 1, y = -1$ or $(1, -1)$

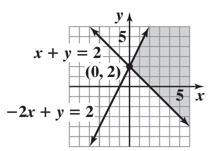
44.



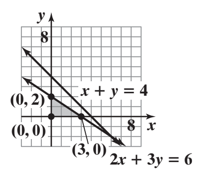
45.



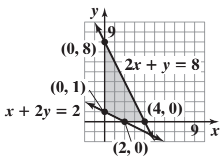
46. Unbounded



47. Bounded



48. Bounded



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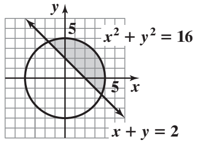

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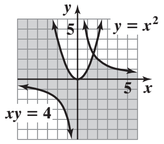
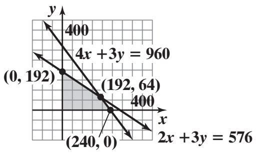


51. The maximum value is 32 when $x = 0$ and $y = 8$. **52.** The minimum value is 3 when $x = 1$ and $y = 0$. **53.** 10 **54.** A is any real number, $A \neq 10$. **55.** $y = -\frac{1}{3}x^2 - \frac{2}{3}x + 1$ **56.** Mix 70 lb of \$6.00 coffee and 30 lb of \$9.00 coffee. **57.** Buy 1 small, 5 medium, and 2 large.

58. (a) $\begin{cases} x \geq 0 \\ y \geq 0 \\ 4x + 3y \leq 960 \\ 2x + 3y \leq 576 \end{cases}$

(b)



59. Speedboat: 36.67 km/hr; Aguarico River: 3.33 km/hr
60. Bruce: 4 hr; Bryce: 2 hr; Marty: 8 hr
61. Produce 35 gasoline engines and 15 diesel engines; the factory is producing an excess of 15 gasoline engines and 0 diesel engines.

Chapter Test *(page 650)*

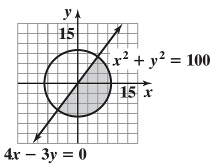
1. $x = 3, y = -1$ or $(3, -1)$ **2.** Inconsistent **3.** $x = -z + \frac{18}{7}, y = z - \frac{17}{7}$, where z is any real number, or $\left\{(x, y, z) \,\middle|\, x = -z + \frac{18}{7}, y = z - \frac{17}{7}, z \text{ is any real number}\right\}$ **4.** $x = \frac{1}{3}, y = -2, z = 0$ or $\left(\frac{1}{3}, -2, 0\right)$ **5.** $\left[\begin{array}{rrr|r} 4 & -5 & 1 & 0 \\ -2 & -1 & 0 & -25 \\ 1 & 5 & -5 & 10 \end{array}\right]$

6. $\begin{cases} 3x + 2y + 4z = -6 \\ 1x + 0y + 8z = 2 \\ -2x + 1y + 3z = -11 \end{cases}$ or $\begin{cases} 3x + 2y + 4z = -6 \\ x + 8z = 2 \\ -2x + y + 3z = -11 \end{cases}$ **7.** $\begin{bmatrix} 6 & 4 \\ 1 & -11 \\ 5 & 12 \end{bmatrix}$ **8.** $\begin{bmatrix} -11 & -19 \\ -3 & 5 \\ 6 & -22 \end{bmatrix}$ **9.** $\begin{bmatrix} 4 & 10 & 26 \\ 1 & -11 & 2 \\ -1 & 26 & 3 \end{bmatrix}$ **10.** $\begin{bmatrix} 16 & 17 \\ 3 & -10 \end{bmatrix}$

11. $\begin{bmatrix} 2 & -1 \\ -\frac{5}{2} & \frac{3}{2} \end{bmatrix}$ **12.** $\begin{bmatrix} 3 & 3 & -4 \\ -2 & -2 & 3 \\ -4 & -5 & 7 \end{bmatrix}$ **13.** $x = \frac{1}{2}, y = 3$ or $\left(\frac{1}{2}, 3\right)$ **14.** $x = -\frac{1}{4}y + 7$, where y is any real number, or $\left\{(x, y) \,\middle|\, x = -\frac{1}{4}y + 7, y \text{ is any real number}\right\}$ **15.** $x = 1, y = -2, z = 0$ or $(1, -2, 0)$ **16.** Inconsistent **17.** -29 **18.** -12

19. $x = -2, y = -5$ or $(-2, -5)$ **20.** $x = 1, y = -1, z = 4$ or $(1, -1, 4)$ **21.** $(1, -3)$ and $(1, 3)$ **22.** $(3, 4)$ and $(1, 2)$

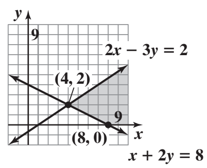
23.



24. $\frac{3}{x + 3} + \frac{-2}{(x + 3)^2}$

25. $\frac{-\frac{1}{3}}{x} + \frac{\frac{1}{3}x}{(x^2 + 3)} + \frac{5x}{(x^2 + 3)^2}$

26. Unbounded

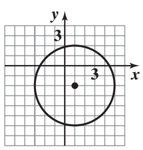


27. The maximum value of z is 64, and it occurs at the point $(0, 8)$. **28.** Flare jeans cost \$24.50, camisoles cost \$8.50, and T-shirts cost \$6.00.

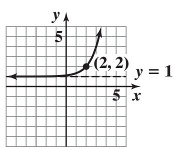
Cumulative Review *(page 651)*

1. $\left\{0, \frac{1}{2}\right\}$ **2.** $\{5\}$ **3.** $\left\{-1, -\frac{1}{2}, 3\right\}$ **4.** $\{-2\}$ **5.** $\left\{\frac{5}{2}\right\}$ **6.** $\left\{\frac{1}{\ln 3}\right\}$ **7.** Odd; symmetric with respect to the origin

8. Center: $(1, -2)$; radius $= 4$

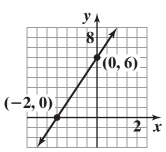


9. Domain: all real numbers
Range: $\{y \mid y > 1\}$
Horizontal asymptote: $y = 1$

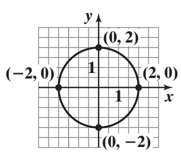


10. $f^{-1}(x) = \frac{5}{x} - 2$
Domain of f: $\{x \mid x \neq -2\}$
Range of f: $\{y \mid y \neq 0\}$
Domain of f^{-1}: $\{x \mid x \neq 0\}$
Range of f^{-1}: $\{y \mid y \neq -2\}$

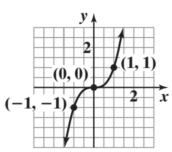
11. (a)



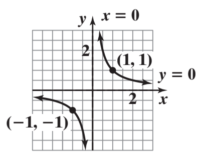
(b)



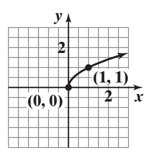
(c)



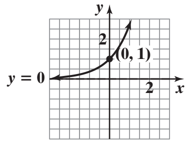
(d)



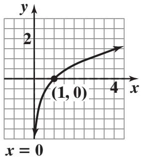
(e)



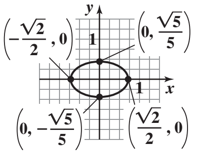
(f)



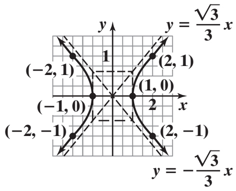
(g)



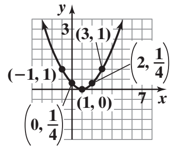
(h)



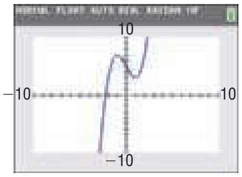
(i)



(j)



12. (a)



; Zero: −2.28

(b) Local maximum of 7 at $x = -1$; local minimum of 3 at $x = 1$
(c) $(-\infty, -1], [1, \infty)$

CHAPTER 9 Sequences; Induction; the Binomial Theorem

9.1 Assess Your Understanding *(page 663)*

5. sequence **6.** True **7.** $n(n-1)\cdot\cdot\cdot 3\cdot 2\cdot 1$ **8.** b **9.** summation **10.** b **11.** 3,628,800 **13.** 504 **15.** 1260 **17.** $s_1 = 1, s_2 = 2, s_3 = 3, s_4 = 4, s_5 = 5$

19. $a_1 = \frac{1}{3}, a_2 = \frac{1}{2}, a_3 = \frac{3}{5}, a_4 = \frac{2}{3}, a_5 = \frac{5}{7}$ **21.** $c_1 = 1, c_2 = -4, c_3 = 9, c_4 = -16, c_5 = 25$ **23.** $s_1 = \frac{1}{2}, s_2 = \frac{2}{5}, s_3 = \frac{2}{7}, s_4 = \frac{8}{41}, s_5 = \frac{8}{61}$

25. $t_1 = -\frac{1}{6}, t_2 = \frac{1}{12}, t_3 = -\frac{1}{20}, t_4 = \frac{1}{30}, t_5 = -\frac{1}{42}$ **27.** $b_1 = \frac{1}{e}, b_2 = \frac{2}{e^2}, b_3 = \frac{3}{e^3}, b_4 = \frac{4}{e^4}, b_5 = \frac{5}{e^5}$ **29.** $a_n = \frac{n}{n+1}$ **31.** $a_n = \frac{1}{2^{n-1}}$

33. $a_n = (-1)^{n+1}$ **35.** $a_n = (-1)^{n+1}n$ **37.** $a_1 = 2, a_2 = 5, a_3 = 8, a_4 = 11, a_5 = 14$ **39.** $a_1 = -2, a_2 = 0, a_3 = 3, a_4 = 7, a_5 = 12$

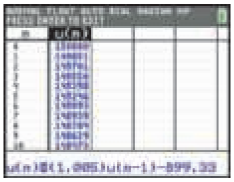
41. $a_1 = 5, a_2 = 10, a_3 = 20, a_4 = 40, a_5 = 80$ **43.** $a_1 = 3, a_2 = \frac{3}{2}, a_3 = \frac{1}{2}, a_4 = \frac{1}{8}, a_5 = \frac{1}{40}$ **45.** $a_1 = 1, a_2 = 2, a_3 = 2, a_4 = 4, a_5 = 8$

47. $a_1 = A, a_2 = A + d, a_3 = A + 2d, a_4 = A + 3d, a_5 = A + 4d$

49. $a_1 = \sqrt{2}, a_2 = \sqrt{2+\sqrt{2}}, a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}},$ $a_4 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}},$ $a_5 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}$ **51.** $3 + 4 + \cdots + (n+2)$ **53.** $\frac{1}{2} + 2 + \frac{9}{2} + \cdots + \frac{n^2}{2}$ **55.** $1 + \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{3^n}$ **57.** $\frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{3^n}$ **59.** $\ln 2 - \ln 3 + \ln 4 - \cdots + (-1)^n \ln n$ **61.** $\sum_{k=1}^{20} k$

63. $\sum_{k=1}^{13} \frac{k}{k+1}$ **65.** $\sum_{k=0}^{6} (-1)^k \left(\frac{1}{3^k}\right)$ **67.** $\sum_{k=1}^{n} \frac{3^k}{k}$ **69.** $\sum_{k=0}^{n} (a + kd)$ or $\sum_{k=1}^{n+1} [a + (k-1)d]$ **71.** 200 **73.** 820 **75.** 1110 **77.** 1560 **79.** 3570

81. 44,000 **83. (a)** 2162 **(b)** After 26 months **85. (a)** $A_0 = 1500, A_n = (1.0125)A_{n-1} + 750$ **(b)** After 99 quarters **(c)** \$213,073.11
87. (a) \$2930 **(b)** 14 payments have been made. **(c)** 36 payments; \$3584.62 **(d)** \$584.62
89. (a) $a_0 = 150{,}000, a_n = (1.005)a_{n-1} - 899.33$
(b) \$149,850.67 **(c)**



(d) After 58 payments or 4 years and 10 months later
(e) After 359 payments of \$899.33, plus last payment of \$895.10
(f) \$173,754.57

(g) (a) $a_0 = 150{,}000, a_n = (1.005)a_{n-1} - 999.33$
(b) \$149,750.67 **(c)**

(d) After 37 payments or 3 years and 1 month later
(e) After 278 payments of \$999.33, plus last payment of \$353.69(1.005) = \$355.46
(f) \$128,169.20

91. 21 pairs **93.** Fibonacci sequence **95. (a)** 3.630170833 **(b)** 3.669060828 **(c)** 3.669296668 **(d)** 12
97. (a) $a_1 = 0.4; a_2 = 0.7; a_3 = 1; a_4 = 1.6; a_5 = 2.8; a_6 = 5.2; a_7 = 10; a_8 = 19.6$
(b) Except for term 5, which has no match, Bode's formula provides excellent approximations for the mean distances of the planets from the sun.
(c) The mean distance of Ceres from the sun is approximated by $a_5 = 2.8$, and that of Uranus is $a_8 = 19.6$.
(d) $a_9 = 38.8; a_{10} = 77.2$
(e) Pluto's distance is approximated by a_9, but no term approximates Neptune's mean distance from the sun.
(f) According to Bode's Law, the mean orbital distance of Eris will be 154 AU from the sun.
99. $a_0 = 2; a_5 = 2.236067977; 2.236067977$ **101.** $a_0 = 4; a_5 = 4.582575695; 4.582575695$ **103.** 1, 3, 6, 10, 15, 21, 28

105. $u_n = 1 + 2 + 3 + \cdots + n = \sum_{k=1}^{n} k = \dfrac{n(n+1)}{2}$, and from Problem 104, $u_{n-1} = \dfrac{(n+1)(n+2)}{2}$.

Thus, $u_{n+1} + u_n = \dfrac{(n+1)(n+2)}{2} + \dfrac{n(n+1)}{2} = \dfrac{(n+1)[(n+2)+n]}{2} = (n+1)^2$.

108. \$2654.39 **109.** $[-4, -1] \cup [4, \infty)$ **110.** HA: $y = 2$; VA: $x = -2$ **111.** $(y-4)^2 = 16(x+3)$

9.2 Assess Your Understanding *(page 672)*

1. arithmetic **2.** F **3.** 17 **4.** T **5.** d **6.** c **7.** $s_n - s_{n-1} = (n+4) - [(n-1)+4] = n + 4 - (n+3) = n + 4 - n - 3 = 1$, a constant; $d = 1$; $s_1 = 5, s_2 = 6, s_3 = 7, s_4 = 8$

9. $a_n - a_{n-1} = (2n-5) - [2(n-1)-5] = 2n - 5 - (2n - 2 - 5) = 2n - 5 - (2n-7) = 2n - 5 - 2n + 7 = 2$, a constant; $d = 2$; $a_1 = -3, a_2 = -1, a_3 = 1, a_4 = 3$

11. $c_n - c_{n-1} = (6-2n) - [6 - 2(n-1)] = 6 - 2n - (6 - 2n + 2) = 6 - 2n - (8-2n) = 6 - 2n - 8 + 2n = -2$, a constant; $d = -2$; $c_1 = 4, c_2 = 2, c_3 = 0, c_4 = -2$

13. $t_n - t_{n-1} = \left(\frac{1}{2} - \frac{1}{3}n\right) - \left[\frac{1}{2} - \frac{1}{3}(n-1)\right] = \frac{1}{2} - \frac{1}{3}n - \left(\frac{1}{2} - \frac{1}{3}n + \frac{1}{3}\right) = \frac{1}{2} - \frac{1}{3}n - \left(\frac{5}{6} - \frac{1}{3}n\right) = \frac{1}{2} - \frac{1}{3}n - \frac{5}{6} + \frac{1}{3}n = -\frac{1}{3}$, a constant; $d = -\frac{1}{3}$; $t_1 = \frac{1}{6}, t_2 = -\frac{1}{6}, t_3 = -\frac{1}{2}, t_4 = -\frac{5}{6}$

15. $s_n - s_{n-1} = \ln 3^n - \ln 3^{n-1} = n \ln 3 - (n-1)\ln 3 = n \ln 3 - (n \ln 3 - \ln 3) = n \ln 3 - n \ln 3 + \ln 3 = \ln 3$, a constant; $d = \ln 3$; $s_1 = \ln 3, s_2 = 2\ln 3, s_3 = 3 \ln 3, s_4 = 4 \ln 3$

17. $a_n = 3n - 1$; $a_{51} = 152$ **19.** $a_n = 8 - 3n$; $a_{51} = -145$ **21.** $a_n = \frac{1}{2}(n-1)$; $a_{51} = 25$ **23.** $a_n = \sqrt{2}n$; $a_{51} = 51\sqrt{2}$ **25.** 200 **27.** -266 **29.** $\frac{83}{2}$

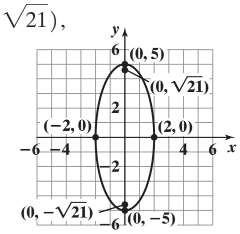
31. $a_1 = -13$; $d = 3$; $a_n = a_{n-1} + 3$; $a_n = -16 + 3n$ **33.** $a_1 = -53$; $d = 6$; $a_n = a_{n-1} + 6$; $a_n = -59 + 6n$

35. $a_1 = 28$; $d = -2$; $a_n = a_{n-1} - 2$; $a_n = 30 - 2n$ **37.** $a_1 = 25$; $d = -2$; $a_n = a_{n-1} - 2$; $a_n = 27 - 2n$ **39.** n^2 **41.** $\frac{n}{2}(9 + 5n)$ **43.** 1260 **45.** 324

47. 30,919 **49.** 10,036 **51.** 6080 **53.** -1925 **55.** 15,960 **57.** $-\frac{3}{2}$ **59.** 24 terms **61.** 1185 seats **63.** 210 beige and 190 blue

65. $\{T_n\} = \{-5.5n + 67\}$; $T_5 = 39.5°F$ **67.** The amphitheater has 1647 seats. **69.** 8 yr **72.** 16.42% **73.** Yes

74. Ellipse: Center: $(0, 0)$; Vertices: $(0, -5)$, $(0, 5)$; Foci: $(0, -\sqrt{21})$, $(0, \sqrt{21})$,



75. $\begin{bmatrix} \frac{1}{2} & 0 \\ \frac{3}{2} & -1 \end{bmatrix}$

Historical Problems *(page 680)*

1. $1\frac{2}{3}$ loaves, $10\frac{5}{6}$ loaves, 20 loaves, $29\frac{1}{6}$ loaves, $38\frac{1}{3}$ loaves **2. (a)** 1 person **(b)** 2401 kittens **(c)** 2800

9.3 Assess Your Understanding *(page 680)*

1. $a_n = a_1 \cdot r^{n-1}$ **2.** $-\frac{2}{3}$ **3.** c **4.** $\frac{a}{1-r}$ **5.** b **6.** T **7.** F **8.** T **9.** $r = 3$; $s_1 = 3, s_2 = 9, s_3 = 27, s_4 = 81$

11. $r = \frac{1}{2}$; $a_1 = -\frac{3}{2}, a_2 = -\frac{3}{4}, a_3 = -\frac{3}{8}, a_4 = -\frac{3}{16}$ **13.** $r = 2$; $c_1 = \frac{1}{4}, c_2 = \frac{1}{2}, c_3 = 1, c_4 = 2$ **15.** $r = 2^{1/3}$; $e_1 = 2^{1/3}, e_2 = 2^{2/3}, e_3 = 2, e_4 = 2^{4/3}$

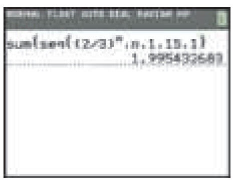
17. $r = \frac{3}{2}$; $t_1 = \frac{1}{2}, t_2 = \frac{3}{4}, t_3 = \frac{9}{8}, t_4 = \frac{27}{16}$ **19.** $a_5 = 162$; $a_n = 2 \cdot 3^{n-1}$ **21.** $a_5 = 5$; $a_n = 5 \cdot (-1)^{n-1}$ **23.** $a_5 = 0$; $a_n = 0$

25. $a_5 = 4\sqrt{2}$; $a_n = (\sqrt{2})^n$ **27.** $a_7 = \frac{1}{64}$ **29.** $a_9 = 1$ **31.** $a_8 = 0.00000004$ **33.** $a_n = 7 \cdot 2^{n-1}$ **35.** $a_n = -3 \cdot \left(-\frac{1}{3}\right)^{n-1} = \left(-\frac{1}{3}\right)^{n-2}$

37. $a_n = -(-3)^{n-1}$ **39.** $a_n = \frac{7}{15}(15)^{n-1} = 7 \cdot 15^{n-2}$ **41.** $-\frac{1}{4}(1 - 2^n)$ **43.** $2\left[1 - \left(\frac{2}{3}\right)^n\right]$ **45.** $1 - 2^n$

47.

49.



51.

53. Converges; $\frac{3}{2}$ **55.** Converges; 16 **57.** Converges; $\frac{8}{5}$ **59.** Diverges **61.** Converges; $\frac{20}{3}$ **63.** Diverges **65.** Converges; $\frac{18}{5}$ **67.** Converges; 6

69. Arithmetic; $d = 1$; 1375 **71.** Neither **73.** Arithmetic; $d = -\frac{2}{3}$; -700 **75.** Neither **77.** Geometric; $r = \frac{2}{3}$; $2\left[1 - \left(\frac{2}{3}\right)^{50}\right]$

79. Geometric; $r = -2$; $-\frac{1}{3}[1 - (-2)^{50}]$ **81.** Geometric; $r = 3^{1/2}$; $-\frac{\sqrt{3}}{2}(1 + \sqrt{3})(1 - 3^{25})$

83. -4 **85.** \$47,271.37 **87. (a)** 0.775 ft **(b)** 8th **(c)** 15.88 ft **(d)** 20 ft **89.** \$349,496.41 **91.** \$96,885.98 **93.** \$305.10 **95.** 1.845×10^{19} **97.** 10
99. \$72.67 per share **101.** December 20, 2015; \$9999.92 **103.** \$5633.36 **105.** Option B results in more money (\$524,287 versus \$500,500).

107. Total pay: \$41,943.03; pay on day 22: \$20,971.52 **112.** 2.121 **113.** $R = 134$ **114.** $\frac{x^2}{4} - \frac{y^2}{12} = 1$ **115.** 54

9.4 Assess Your Understanding *(page 687)*

1. (I) $n = 1$: $2(1) = 2$ and $1(1 + 1) = 2$
(II) If $2 + 4 + 6 + \cdots + 2k = k(k + 1)$, then $2 + 4 + 6 + \cdots + 2k + 2(k + 1) = (2 + 4 + 6 + \cdots + 2k) + 2(k + 1)$
$= k(k + 1) + 2(k + 1) = k^2 + 3k + 2 = (k + 1)(k + 2) = (k + 1)[(k + 1) + 1]$.

3. (I) $n = 1$: $1 + 2 = 3$ and $\frac{1}{2}(1)(1 + 5) = \frac{1}{2}(6) = 3$

(II) If $3 + 4 + 5 + \cdots + (k + 2) = \frac{1}{2}k(k + 5)$, then $3 + 4 + 5 + \cdots + (k + 2) + [(k + 1) + 2]$

$= [3 + 4 + 5 + \cdots + (k + 2)] + (k + 3) = \frac{1}{2}k(k + 5) + k + 3 = \frac{1}{2}(k^2 + 7k + 6) = \frac{1}{2}(k + 1)(k + 6) = \frac{1}{2}(k + 1)[(k + 1) + 5]$.

5. (I) $n = 1$: $3(1) - 1 = 2$ and $\frac{1}{2}(1)[3(1) + 1] = \frac{1}{2}(4) = 2$

(II) If $2 + 5 + 8 + \cdots + (3k - 1) = \frac{1}{2}k(3k + 1)$, then $2 + 5 + 8 + \cdots + (3k - 1) + [3(k + 1) - 1]$

$= [2 + 5 + 8 + \cdots + (3k - 1)] + (3k + 2) = \frac{1}{2}k(3k + 1) + (3k + 2) = \frac{1}{2}(3k^2 + 7k + 4) = \frac{1}{2}(k + 1)(3k + 4)$

$= \frac{1}{2}(k + 1)[3(k + 1) + 1]$.

7. (I) $n = 1$: $2^{1-1} = 1$ and $2^1 - 1 = 1$
(II) If $1 + 2 + 2^2 + \cdots + 2^{k-1} = 2^k - 1$, then $1 + 2 + 2^2 + \cdots + 2^{k-1} + 2^{(k+1)-1} = (1 + 2 + 2^2 + \cdots + 2^{k-1}) + 2^k$
$= 2^k - 1 + 2^k = 2(2^k) - 1 = 2^{k+1} - 1$.

9. (I) $n = 1$: $4^{1-1} = 1$ and $\frac{1}{3}(4^1 - 1) = \frac{1}{3}(3) = 1$

(II) If $1 + 4 + 4^2 + \cdots + 4^{k-1} = \frac{1}{3}(4^k - 1)$, then $1 + 4 + 4^2 + \cdots + 4^{k-1} + 4^{(k+1)-1} = (1 + 4 + 4^2 + \cdots + 4^{k-1}) + 4^k$

$= \frac{1}{3}(4^k - 1) + 4^k = \frac{1}{3}[4^k - 1 + 3(4^k)] = \frac{1}{3}[4(4^k) - 1] = \frac{1}{3}(4^{k+1} - 1)$.

11. (I) $n = 1$: $\frac{1}{1 \cdot 2} = \frac{1}{2}$ and $\frac{1}{1 + 1} = \frac{1}{2}$

(II) If $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{k(k + 1)} = \frac{k}{k + 1}$, then $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{k(k + 1)} + \frac{1}{(k + 1)[(k + 1) + 1]}$

$= \left[\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{k(k + 1)}\right] + \frac{1}{(k + 1)(k + 2)} = \frac{k}{k + 1} + \frac{1}{(k + 1)(k + 2)} = \frac{k(k + 2) + 1}{(k + 1)(k + 2)}$

$= \frac{k^2 + 2k + 1}{(k + 1)(k + 2)} = \frac{(k + 1)^2}{(k + 1)(k + 2)} = \frac{k + 1}{k + 2} = \frac{k + 1}{(k + 1) + 1}$.

13. (I) $n = 1$: $1^2 = 1$ and $\frac{1}{6} \cdot 1 \cdot 2 \cdot 3 = 1$

(II) If $1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{1}{6}k(k + 1)(2k + 1)$, then $1^2 + 2^2 + 3^2 + \cdots + k^2 + (k + 1)^2$

$= (1^2 + 2^2 + 3^2 + \cdots + k^2) + (k + 1)^2 = \frac{1}{6}k(k + 1)(2k + 1) + (k + 1)^2 = \frac{1}{6}(2k^3 + 9k^2 + 13k + 6)$

$= \frac{1}{6}(k + 1)(k + 2)(2k + 3) = \frac{1}{6}(k + 1)[(k + 1) + 1][2(k + 1) + 1]$.

15. (I) $n = 1$: $5 - 1 = 4$ and $\frac{1}{2}(1)(9 - 1) = \frac{1}{2} \cdot 8 = 4$

(II) If $4 + 3 + 2 + \cdots + (5 - k) = \frac{1}{2}k(9 - k)$, then $4 + 3 + 2 + \cdots + (5 - k) + [5 - (k + 1)]$

$= [4 + 3 + 2 + \cdots + (5 - k)] + 4 - k = \frac{1}{2}k(9 - k) + 4 - k = \frac{1}{2}(9k - k^2 + 8 - 2k) = \frac{1}{2}(-k^2 + 7k + 8)$

$= \frac{1}{2}(k + 1)(8 - k) = \frac{1}{2}(k + 1)[9 - (k + 1)]$.

17. (I) $n = 1: 1 \cdot (1 + 1) = 2$ and $\frac{1}{3} \cdot 1 \cdot 2 \cdot 3 = 2$

(II) If $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k(k+1) = \frac{1}{3}k(k+1)(k+2)$, then $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k(k+1)$
$+ (k+1)[(k+1)+1] = [1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k(k+1)] + (k+1)(k+2)$
$= \frac{1}{3}k(k+1)(k+2) + \frac{1}{3} \cdot 3(k+1)(k+2) = \frac{1}{3}(k+1)(k+2)(k+3) = \frac{1}{3}(k+1)[(k+1)+1][(k+1)+2].$

19. (I) $n = 1: 1^2 + 1 = 2$, which is divisible by 2.
(II) If $k^2 + k$ is divisible by 2, then $(k+1)^2 + (k+1) = k^2 + 2k + 1 + k + 1 = (k^2 + k) + 2k + 2$. Since $k^2 + k$ is divisible by 2 and $2k + 2$ is divisible by 2, $(k+1)^2 + (k+1)$ is divisible by 2.

21. (I) $n = 1: 1^2 - 1 + 2 = 2$, which is divisible by 2.
(II) If $k^2 - k + 2$ is divisible by 2, then $(k+1)^2 - (k+1) + 2 = k^2 + 2k + 1 - k - 1 + 2 = (k^2 - k + 2) + 2k$. Since $k^2 - k + 2$ is divisible by 2 and $2k$ is divisible by 2, $(k+1)^2 - (k+1) + 2$ is divisible by 2.

23. (I) $n = 1$: If $x > 1$, then $x^1 = x > 1$.
(II) Assume, for an arbitrary natural number k, that if $x > 1$ then $x^k > 1$. Multiply both sides of the inequality $x^k > 1$ by x. If $x > 1$, then $x^{k+1} > x > 1$.

25. (I) $n = 1$: $a - b$ is a factor of $a^1 - b^1 = a - b$.
(II) If $a - b$ is a factor of $a^k - b^k$, then $a^{k+1} - b^{k-1} = a(a^k - b^k) + b^k(a - b)$.
Since $a - b$ is a factor of $a^k - b^k$ and $a - b$ is a factor of $a - b$, then $a - b$ is a factor of $a^{k+1} - b^{k+1}$.

27. (a) $n = 1$: $(1 + a)^1 = 1 + a \geq 1 + 1 \cdot a$
(b) Assume that there is an integer k for which the inequality holds. So $(1+a)^k \geq 1 + ka$. We need to show that $(1+a)^{k+1} \geq 1 + (k+1)a$.
$(1+a)^{k+1} = (1+a)^k(1+a) \geq (1+ka)(1+a) = 1 + ka^2 + a + ka = 1 + (k+1)a + ka^2 \geq 1 + (k+1)a.$

29. If $2 + 4 + 6 + \cdots + 2k = k^2 + k + 2$, then $2 + 4 + 6 + \cdots + 2k + 2(k+1)$
$= (2 + 4 + 6 + \cdots + 2k) + 2k + 2 = k^2 + k + 2 + 2k + 2 = k^2 + 3k + 4 = (k^2 + 2k + 1) + (k+1) + 2$
$= (k+1)^2 + (k+1) + 2.$
But $2 \cdot 1 = 2$ and $1^2 + 1 + 2 = 4$. The fact is that $2 + 4 + 6 + \cdots + 2n = n^2 + n$, not $n^2 + n + 2$ (Problem 1).

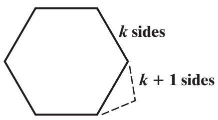
31. (I) $n = 1$: $[a + (1-1)d] = a$ and $1 \cdot a + d\frac{1 \cdot (1-1)}{2} = a$.

(II) If $a + (a+d) + (a+2d) + \cdots + [a + (k-1)d] = ka + d\frac{k(k-1)}{2}$, then

$$a + (a+d) + (a+2d) + \cdots + [a + (k-1)d] + [a + ((k+1)-1)d] = ka + d\frac{k(k-1)}{2} + a + kd$$

$$= (k+1)a + d\frac{k(k-1) + 2k}{2} = (k+1)a + d\frac{(k+1)(k)}{2} = (k+1)a + d\frac{(k+1)[(k+1)-1]}{2}.$$

33. (I) $n = 3$: The sum of the angles of a triangle is $(3-2) \cdot 180° = 180°$.
(II) Assume that for some $k \geq 3$, the sum of the angles of a convex polygon of k sides is $(k-2) \cdot 180°$.
A convex polygon of $k + 1$ sides consists of a convex polygon of k sides plus a triangle (see the illustration).
The sum of the angles is $(k-2) \cdot 180° + 180° = (k-1) \cdot 180° = [(k+1) - 2] \cdot 180°$.



35. $\{251\}$ **36.** $\begin{bmatrix} 0 & 4 & -5 \\ 1 & -5 & 3 \end{bmatrix}$ **37.** $x = \frac{1}{2}, y = -3; \left(\frac{1}{2}, -3\right)$ **38.** $\begin{bmatrix} 7 & -3 \\ -7 & 8 \end{bmatrix}$

9.5 Assess Your Understanding *(page 693)*

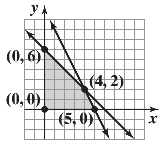
1. Pascal triangle **2.** 1; n **3.** F **4.** Binomial Theorem **5.** 10 **7.** 21 **9.** 50 **11.** 1 **13.** $\approx 1.8664 \times 10^{15}$ **15.** $\approx 1.4834 \times 10^{13}$
17. $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$ **19.** $x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64$ **21.** $81x^4 + 108x^3 + 54x^2 + 12x + 1$
23. $x^{10} + 5x^8y^2 + 10x^6y^4 + 10x^4y^6 + 5x^2y^8 + y^{10}$ **25.** $x^3 + 6\sqrt{2}x^{5/2} + 30x^2 + 40\sqrt{2}x^{3/2} + 60x + 24\sqrt{2}x^{1/2} + 8$
27. $a^5x^5 + 5a^4bx^4y + 10a^3b^2x^3y^2 + 10a^2b^3x^2y^3 + 5ab^4xy^4 + b^5y^5$ **29.** 17,010 **31.** $-101{,}376$ **33.** 41,472 **35.** $2835x^3$ **37.** $314{,}928x^7$ **39.** 495

41. 3360 **43.** 1.00501 **45.** $\binom{n}{n-1} = \frac{n!}{(n-1)!\,[n-(n-1)]!} = \frac{n!}{(n-1)!\,1!} = \frac{n \cdot (n-1)!}{(n-1)!} = n; \binom{n}{n} = \frac{n!}{n!\,(n-n)!} = \frac{n!}{n!\,0!} = \frac{n!}{n!} = 1$

47. $2^n = (1+1)^n = \binom{n}{0}1^n + \binom{n}{1}(1)^{n-1}(1) + \cdots + \binom{n}{n}1^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$ **49.** 1

51. $\left\{\frac{\ln 5}{\ln 6 - \ln 5}\right\} \approx \{8.827\}$ **52.** $g(x) = x - 4$ **53.** $x = 1, y = 3, z = -2; (1, 3, -2)$

54. Bounded



Review Exercises *(page 695)*

1. $a_1 = -\frac{4}{3}, a_2 = \frac{5}{4}, a_3 = -\frac{6}{5}, a_4 = \frac{7}{6}, a_5 = -\frac{8}{7}$ **2.** $c_1 = 2, c_2 = 1, c_3 = \frac{8}{9}, c_4 = 1, c_5 = \frac{32}{25}$ **3.** $a_1 = 3, a_2 = 2, a_3 = \frac{4}{3}, a_4 = \frac{8}{9}, a_5 = \frac{16}{27}$

4. $a_1 = 2, a_2 = 0, a_3 = 2, a_4 = 0, a_5 = 2$ **5.** $6 + 10 + 14 + 18 = 48$ **6.** $\sum_{k=1}^{13} (-1)^{k+1}\frac{1}{k}$ **7.** Arithmetic; $d = 1$; $S_n = \frac{n}{2}(n + 11)$ **8.** Neither

9. Geometric; $r = 8$; $S_n = \frac{8}{7}(8^n - 1)$ **10.** Arithmetic; $d = 4$; $S_n = 2n(n - 1)$ **11.** Geometric; $r = \frac{1}{2}$; $S_n = 6\left[1 - \left(\frac{1}{2}\right)^n\right]$ **12.** Neither

13. 9515 **14.** –1320 **15.** $\frac{1093}{2187} \approx 0.49977$ **16.** 682 **17.** 35 **18.** $\frac{1}{10^{10}}$ **19.** $9\sqrt{2}$ **20.** $\{a_n\} = \{5n - 4\}$ **21.** $\{a_n\} = \{n - 10\}$ **22.** Converges; $\frac{9}{2}$
23. Converges; $\frac{4}{3}$ **24.** Diverges **25.** Converges; 8

26. **(I)** $n = 1: 3 \cdot 1 = 3$ and $\frac{3 \cdot 1}{2}(1 + 1) = 3$

(II) If $3 + 6 + 9 + \cdots + 3k = \frac{3k}{2}(k + 1)$, then $3 + 6 + 9 + \cdots + 3k + 3(k + 1) = (3 + 6 + 9 + \cdots + 3k) + (3k + 3)$

$= \frac{3k}{2}(k + 1) + (3k + 3) = \frac{3k^2}{2} + \frac{3k}{2} + \frac{6k}{2} + \frac{6}{2} = \frac{3}{2}(k^2 + 3k + 2) = \frac{3}{2}(k + 1)(k + 2) = \frac{3(k + 1)}{2}[(k + 1) + 1].$

27. **(I)** $n = 1: 2 \cdot 3^{1-1} = 2$ and $3^1 - 1 = 2$
(II) If $2 + 6 + 18 + \cdots + 2 \cdot 3^{k-1} = 3^k - 1$, then $2 + 6 + 18 + \cdots + 2 \cdot 3^{k-1} + 2 \cdot 3^{(k+1)-1} = (2 + 6 + 18 + \cdots + 2 \cdot 3^{k-1}) + 2 \cdot 3^k$
$= 3^k - 1 + 2 \cdot 3^k = 3 \cdot 3^k - 1 = 3^{k+1} - 1.$

28. **(I)** $n = 1$: $(3 \cdot 1 - 2)^2 = 1$ and $\frac{1}{2} \cdot 1 \cdot [6(1)^2 - 3(1) - 1] = 1$

(II) If $1^2 + 4^2 + 7^2 + \cdots + (3k - 2)^2 = \frac{1}{2}k(6k^2 - 3k - 1)$, then $1^2 + 4^2 + 7^2 + \cdots + (3k - 2)^2 + [3(k + 1) - 2]^2$

$= [1^2 + 4^2 + 7^2 + \cdots + (3k - 2)^2] + (3k + 1)^2 = \frac{1}{2}k(6k^2 - 3k - 1) + (3k + 1)^2 = \frac{1}{2}(6k^3 - 3k^2 - k) + (9k^2 + 6k + 1)$

$= \frac{1}{2}(6k^3 + 15k^2 + 11k + 2) = \frac{1}{2}(k + 1)(6k^2 + 9k + 2) = \frac{1}{2}(k + 1)[6(k + 1)^2 - 3(k + 1) - 1].$

29. 10 **30.** $x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$ **31.** $81x^4 - 432x^3 + 864x^2 - 768x + 256$ **32.** 144 **33.** 84

34. **(a)** 8 bricks **(b)** 1100 bricks **35.** 360 **36.** **(a)** $20\left(\frac{3}{4}\right)^3 = \frac{135}{16}$ ft **(b)** $20\left(\frac{3}{4}\right)^n$ ft **(c)** 13 times **(d)** 140 ft **37.** \$151,873.77 **38.** \$58,492.93

Chapter Test *(page 697)*

1. $0, \frac{3}{10}, \frac{8}{11}, \frac{5}{4}, \frac{24}{13}$ **2.** 4, 14, 44, 134, 404 **3.** $2 - \frac{3}{4} + \frac{4}{9} = \frac{61}{36}$ **4.** $-\frac{1}{3} - \frac{14}{9} - \frac{73}{27} - \frac{308}{81} = -\frac{680}{81}$ **5.** $\sum_{k=1}^{10} (-1)^k\left(\frac{k + 1}{k + 4}\right)$ **6.** Neither

7. Geometric; $r = 4$; $S_n = \frac{2}{3}(1 - 4^n)$ **8.** Arithmetic: $d = -8$; $S_n = n(2 - 4n)$ **9.** Arithmetic; $d = -\frac{1}{2}$; $S_n = \frac{n}{4}(27 - n)$

10. Geometric; $r = \frac{2}{5}$; $S_n = \frac{125}{3}\left[1 - \left(\frac{2}{5}\right)^n\right]$ **11.** Neither **12.** Converges; $\frac{1024}{5}$ **13.** $243m^5 + 810m^4 + 1080m^3 + 720m^2 + 240m + 32$

14. First we show that the statement holds for $n = 1$. $\left(1 + \frac{1}{1}\right) = 1 + 1 = 2$. The equality is true for $n = 1$, so Condition I holds. Next we assume that $\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\cdots\left(1 + \frac{1}{n}\right) = n + 1$ is true for some k, and we determine whether the formula then holds for $k + 1$. We assume that $\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\cdots\left(1 + \frac{1}{k}\right) = k + 1$. Now we need to show that $\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\cdots\left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{k + 1}\right)$ $= (k + 1) + 1 = k + 2$. We do this as follows:

$\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\cdots\left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{k + 1}\right) = \left[\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\cdots\left(1 + \frac{1}{k}\right)\right]\left(1 + \frac{1}{k + 1}\right)$

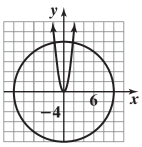
$= (k + 1)\left(1 + \frac{1}{k + 1}\right)$(induction assumption) $= (k + 1) \cdot 1 + (k + 1) \cdot \frac{1}{k + 1} = k + 1 + 1 = k + 2$

Condition II also holds. Thus, the formula holds true for all natural numbers.

15. After 10 years, the Durango will be worth \$6103.11. **16.** The weightlifter will have lifted a total of 8000 pounds after 5 sets.

Cumulative Review *(page 697)*

1. $\{-3, 3, -3i, 3i\}$ **2. (a)**



(b) $\left\{\left(\sqrt{\dfrac{-1+\sqrt{3601}}{18}}, \dfrac{-1+\sqrt{3601}}{6}\right), \left(-\sqrt{\dfrac{-1+\sqrt{3601}}{18}}, \dfrac{-1+\sqrt{3601}}{6}\right)\right\}$

(c) The circle and the parabola intersect at

$$\left(\sqrt{\frac{-1+\sqrt{3601}}{18}}, \frac{-1+\sqrt{3601}}{6}\right), \left(-\sqrt{\frac{-1+\sqrt{3601}}{18}}, \frac{-1+\sqrt{3601}}{6}\right).$$

3. $\left\{\ln\left(\dfrac{5}{2}\right)\right\}$ **4.** $y = 5x - 10$ **5.** $(x+1)^2 + (y-2)^2 = 25$ **6. (a)** 5 **(b)** 13 **(c)** $\dfrac{6x+3}{2x-1}$ **(d)** $\left\{x \middle| x \neq \dfrac{1}{2}\right\}$ **(e)** $\dfrac{7x-2}{x-2}$ **(f)** $\{x | x \neq 2\}$

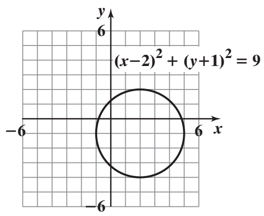
(g) $g^{-1}(x) = \dfrac{1}{2}(x-1)$; all reals **(h)** $f^{-1}(x) = \dfrac{2x}{x-3}$; $\{x | x \neq 3\}$ **7.** $\dfrac{x^2}{7} + \dfrac{y^2}{16} = 1$ **8.** $(x+1)^2 = 4(y-2)$

CHAPTER 10 Counting and Probability

10.1 Assess Your Understanding *(page 704)*

5. subset; $\subseteq$ **6.** finite **7.** $n(A) + n(B) - n(A \cap B)$ **8.** T **9.** $\varnothing, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}, \{a, b, c, d\}$ **11.** 25 **13.** 40 **15.** 25 **17.** 37 **19.** 18 **21.** 5 **23.** 15 different arrangements **25.** 9000 numbers **27.** 175; 125 **29. (a)** 15 **(b)** 15 **(c)** 15 **(d)** 25 **(e)** 40 **31. (a)** 13.8 million **(b)** 79.5 million **33.** 480 portfolios

36.



37. $(0, -11)$ **38.** 2, 5, −2 **39.** $\left\{\dfrac{1}{18}\right\}$

10.2 Assess Your Understanding *(page 711)*

3. permutation **4.** combination **5.** $\dfrac{n!}{(n-r)!}$ **6.** $\dfrac{n!}{(n-r)!r!}$ **7.** 30 **9.** 24 **11.** 1 **13.** 1680 **15.** 28 **17.** 35 **19.** 1 **21.** 10,400,600

23. $\{abc, abd, abe, acb, acd, ace, adb, adc, ade, aeb, aec, aed, bac, bad, bae, bca, bcd, bce, bda, bdc, bde, bea, bec, bed, cab, cad, cae, cba, cbd, cbe, cda, cdb, cde, cea, ceb, ced, dab, dac, dae, dba, dbc, dbe, dca, dcb, dce, dea, deb, dec, eab, eac, ead, eba, ebc, ebd, eca, ecb, ecd, eda, edb, edc\}$; 60

25. {123, 124, 132, 134, 142, 143, 213, 214, 231, 234, 241, 243, 312, 314, 321, 324, 341, 342, 412, 413, 421, 423, 431, 432}; 24

27. $\{abc, abd, abe, acd, ace, ade, bcd, bce, bde, cde\}$; 10 **29.** {123, 124, 134, 234}; 4 **31.** 16 **33.** 8 **35.** 24 **37.** 60 **39.** 18,278 **41.** 35 **43.** 1024 **45.** 120 **47.** 132,860 **49.** 336 **51.** 90,720 **53. (a)** 63 **(b)** 35 **(c)** 1 **55.** 1.157×10^{76} **57.** 362,880 **59.** 660 **61.** 15

63. (a) 125,000; 117,600 **(b)** A better name for a *combination* lock would be a *permutation* lock because the order of the numbers matters.

67. Horizontal asymptote: $y = 0$; vertical asymptote: $x = 4$ **68.** $(g \circ f)(x) = 4x^2 - 2x - 2$ **69.** $\{x | 3 < x \leq 8\}$ or $(3, 8]$ **70.** $a_5 = 80$

Historical Problem *(page 721)*

1. (a) $\{AAAA, AAAB, AABA, AABB, ABAA, ABAB, ABBA, ABBB, BAAA, BAAB, BABA, BABB, BBAA, BBAB, BBBA, BBBB\}$

(b) $P(A \text{ wins}) = \dfrac{C(4,2) + C(4,3) + C(4,4)}{2^4} = \dfrac{6+4+1}{16} = \dfrac{11}{16}$; $P(B \text{ wins}) = \dfrac{C(4,3) + C(4,4)}{2^4} = \dfrac{4+1}{16} = \dfrac{5}{16}$

10.3 Assess Your Understanding *(page 721)*

1. equally likely **2.** complement **3.** F **4.** T **5.** 0, 0.01, 0.35, 1 **7.** Probability model **9.** Not a probability model

11. $S = \{\text{HH, HT, TH, TT}\}$; $P(\text{HH}) = \dfrac{1}{4}$, $P(\text{HT}) = \dfrac{1}{4}$, $P(\text{TH}) = \dfrac{1}{4}$, $P(\text{TT}) = \dfrac{1}{4}$

13. $S = \{$HH1, HH2, HH3, HH4, HH5, HH6, HT1, HT2, HT3, HT4, HT5, HT6, TH1, TH2, TH3, TH4, TH5, TH6, TT1, TT2, TT3, TT4, TT5, TT6$\}$; each outcome has the probability of $\dfrac{1}{24}$.

15. $S = \{$HHH, HHT, HTH, HTT, THH, THT, TTH, TTT$\}$; each outcome has the probability of $\dfrac{1}{8}$.

17. $S = \{$1 Yellow, 1 Red, 1 Green, 2 Yellow, 2 Red, 2 Green, 3 Yellow, 3 Red, 3 Green, 4 Yellow, 4 Red, 4 Green$\}$; each outcome has the probability of $\dfrac{1}{12}$; thus, $P(2 \text{ Red}) + P(4 \text{ Red}) = \dfrac{1}{12} + \dfrac{1}{12} = \dfrac{1}{6}$.

19. $S=$ {1 Yellow Forward, 1 Yellow Backward, 1 Red Forward, 1 Red Backward, 1 Green Forward, 1 Green Backward, 2 Yellow Forward, 2 Yellow Backward, 2 Red Forward, 2 Red Backward, 2 Green Forward, 2 Green Backward, 3 Yellow Forward, 3 Yellow Backward, 3 Red Forward, 3 Red Backward, 3 Green Forward, 3 Green Backward, 4 Yellow Forward, 4 Yellow Backward, 4 Red Forward, 4 Red Backward, 4 Green Forward, 4 Green Backward}; each outcome has the probability of $\frac{1}{24}$; thus, $P(1 \text{ Red Backward}) + P(1 \text{ Green Backward}) = \frac{1}{24}+\frac{1}{24}=\frac{1}{12}$.
21. $S=$ {11 Red, 11 Yellow, 11 Green, 12 Red, 12 Yellow, 12 Green, 13 Red, 13 Yellow, 13 Green, 14 Red, 14 Yellow, 14 Green, 21 Red, 21 Yellow, 21 Green, 22 Red, 22 Yellow, 22 Green, 23 Red, 23 Yellow, 23 Green, 24 Red, 24 Yellow, 24 Green, 31 Red, 31 Yellow, 31 Green, 32 Red, 32 Yellow, 32 Green, 33 Red, 33 Yellow, 33 Green, 34 Red, 34 Yellow, 34 Green, 41 Red, 41 Yellow, 41 Green, 42 Red, 42 Yellow, 42 Green, 43 Red, 43 Yellow, 43 Green, 44 Red, 44 Yellow, 44 Green}; each outcome has the probability of $\frac{1}{48}$; thus, $E=$ {22 Red, 22 Green, 24 Red, 24 Green}; $P(E)=\frac{n(E)}{n(S)}=\frac{4}{48}=\frac{1}{12}$.

23. A, B, C, F **25.** B **27.** $P(H)=\frac{4}{5}$; $P(T)=\frac{1}{5}$ **29.** $P(1)=P(3)=P(5)=\frac{2}{9}$; $P(2)=P(4)=P(6)=\frac{1}{9}$ **31.** $\frac{3}{10}$ **33.** $\frac{1}{2}$ **35.** $\frac{1}{6}$ **37.** $\frac{1}{8}$ **39.** $\frac{1}{4}$

41. $\frac{1}{6}$ **43.** $\frac{1}{18}$ **45.** 0.55 **47.** 0.70 **49.** 0.30 **51.** 0.858 **53.** 0.56 **55.** 0.936 **57.** $\frac{17}{20}$ **59.** $\frac{11}{20}$ **61.** $\frac{1}{2}$ **63.** $\frac{17}{50}$ **65.** $\frac{12}{25}$

67. (a) 0.71 **(b)** 0.91 **(c)** 0.94 **(d)** 0.20 **(e)** 0.43 **(f)** 0.09 **(g)** 0.85 **(h)** 0.57 **69. (a)** $\frac{25}{33}$ **(b)** $\frac{25}{33}$ **71.** 0.167

73. $\frac{1}{292{,}201{,}338} \approx 0.00000000342$ **74.** 2; left; 3; down **75.** $2y\sqrt[3]{3x^2y^2}$ **76.** $\{22\}$ **77.** $(2, -3, -1)$

Review Exercises *(page 725)*

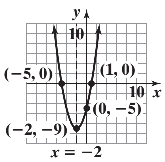
1. $\varnothing$, {Dave}, {Joanne}, {Erica}, {Dave, Joanne}, {Dave, Erica}, {Joanne, Erica}, {Dave, Joanne, Erica} **2.** 17 **3.** 24 **4.** 29 **5.** 34 **6.** 7 **7.** 45 **8.** 25 **9.** 7 **10.** 336 **11.** 56 **12.** 60 **13.** 128 **14.** 3024 **15.** 1680 **16.** 91 **17.** 1,600,000 **18.** 216,000
19. 256 (allowing numbers with initial zeros, such as 011) **20.** 12,600 **21. (a)** 381,024 **(b)** 1260
22. (a) $8.634628387 \times 10^{45}$ **(b)** 0.6531 **(c)** 0.3469 **23. (a)** 0.062 **(b)** 0.938 **24.** $\frac{4}{9}$ **25.** 0.2; 0.26 **26. (a)** 0.68 **(b)** 0.58 **(c)** 0.32

Chapter Test *(page 726)*

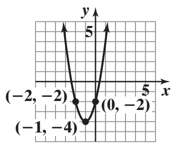
1. 22 **2.** 3 **3.** 8 **4.** 45 **5.** 5040 **6.** 151,200 **7.** 462 **8.** There are 54,264 ways to choose 6 different colors from the 21 available colors.
9. There are 840 distinct arrangements of the letters in the word REDEEMED. **10.** There are 56 different exacta bets for an 8-horse race.
11. There are 155,480,000 possible license plates using the new format. **12. (a)** 0.95 **(b)** 0.30 **13. (a)** 0.25 **(b)** 0.55 **14.** 0.19 **15.** 0.000033069
16. $P(\text{exactly 2 fours}) = \frac{625}{3888} \approx 0.1608$

Cumulative Review *(page 727)*

1. $\left\{\frac{1}{3}-\frac{\sqrt{2}}{3}i, \frac{1}{3}+\frac{\sqrt{2}}{3}i\right\}$ **2.**

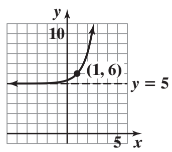


3.



4. $\{x \mid 3.99 \le x \le 4.01\}$ or $[3.99, 4.01]$

5. $\left\{-\frac{1}{2}+\frac{\sqrt{7}}{2}i, -\frac{1}{2}-\frac{\sqrt{7}}{2}i, -\frac{1}{5}, 3\right\}$ **6.**



Domain: all real numbers
Range: $\{y \mid y > 5\}$
Horizontal asymptote: $y = 5$

7. 2 **8.** $\left\{\frac{8}{3}\right\}$ **9.** $x=2, y=-5, z=3$ **10.** 125; 700

Credits

Photos

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Text

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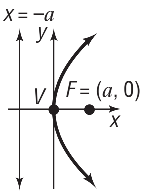
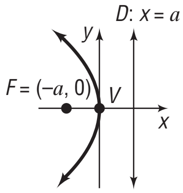
Subject Index

CONICS

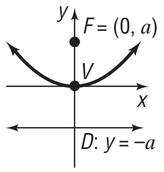
Parabola



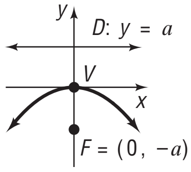
$y^2 = 4ax$



$y^2 = -4ax$

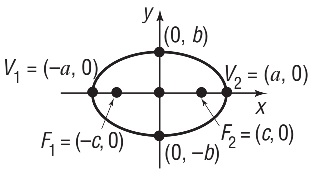


$x^2 = 4ay$

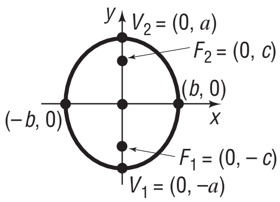


$x^2 = -4ay$

Ellipse

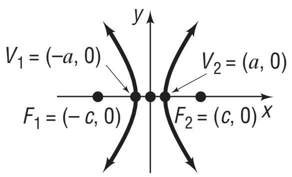


$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b, \quad c^2 = a^2 - b^2$$



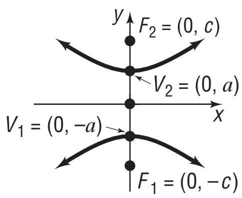
$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \quad a > b, \quad c^2 = a^2 - b^2$$

Hyperbola



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad c^2 = a^2 + b^2$$

Asymptotes: $y = \dfrac{b}{a}x, \quad y = -\dfrac{b}{a}x$



$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1, \quad c^2 = a^2 + b^2$$

Asymptotes: $y = \dfrac{a}{b}x, \quad y = -\dfrac{a}{b}x$

PROPERTIES OF LOGARITHMS

$\log_a(MN) = \log_a M + \log_a N$

$\log_a\left(\dfrac{M}{N}\right) = \log_a M - \log_a N$

$\log_a M^r = r\log_a M$

$\log_a M = \dfrac{\log M}{\log a} = \dfrac{\ln M}{\ln a}$

$a^r = e^{r \ln a}$

PERMUTATIONS/COMBINATIONS

$0! = 1 \qquad 1! = 1$

$n! = n(n-1)\cdot \ldots \cdot (3)(2)(1)$

$P(n, r) = \dfrac{n!}{(n-r)!}$

$C(n, r) = \dbinom{n}{r} = \dfrac{n!}{(n-r)!\,r!}$

BINOMIAL THEOREM

$$(a+b)^n = a^n + \binom{n}{1}ba^{n-1} + \binom{n}{2}b^2a^{n-2} + \cdots + \binom{n}{n-1}b^{n-1}a + b^n$$

ARITHMETIC SEQUENCE

$$a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + [a_1 + (n-1)d] = \frac{n}{2}[2a_1 + (n-1)d] = \frac{n}{2}[a_1 + a_n]$$

GEOMETRIC SEQUENCE

$$a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-1} = a_1 \cdot \frac{1 - r^n}{1 - r}$$

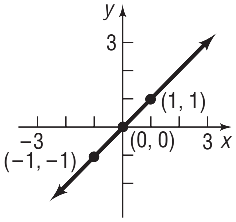
GEOMETRIC SERIES

If $|r| < 1$, $a_1 + a_1r + a_1r^2 + \cdots = \displaystyle\sum_{k=1}^{\infty} a_1r^{k-1} = \frac{a_1}{1-r}$

LIBRARY OF FUNCTIONS

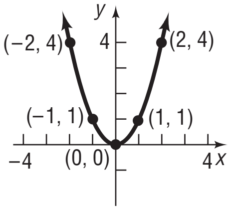
Identity Function

$f(x) = x$



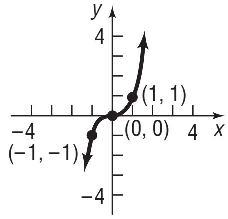
Square Function

$f(x) = x^2$



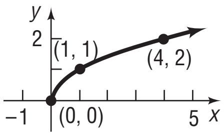
Cube Function

$f(x) = x^3$



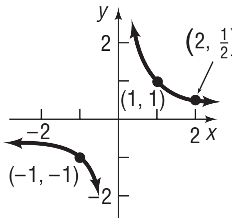
Square Root Function

$f(x) = \sqrt{x}$



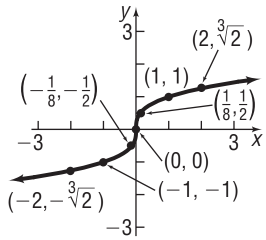
Reciprocal Function

$f(x) = \dfrac{1}{x}$



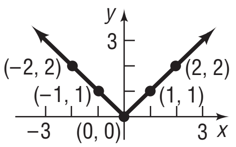
Cube Root Function

$f(x) = \sqrt[3]{x}$



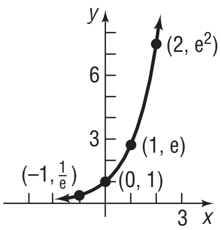
Absolute Value Function

$f(x) = |x|$



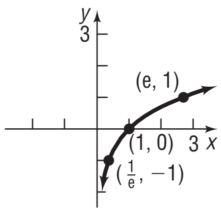
Exponential Function

$f(x) = e^x$



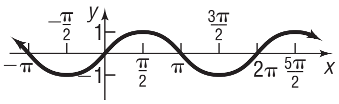
Natural Logarithm Function

$f(x) = \ln x$



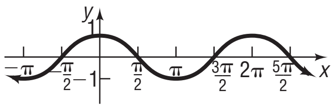
Sine Function

$f(x) = \sin x$



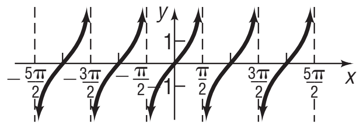
Cosine Function

$f(x) = \cos x$



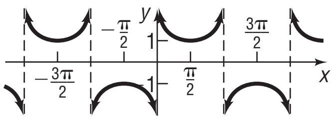
Tangent Function

$f(x) = \tan x$



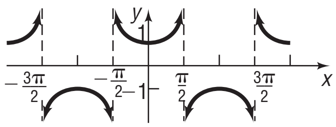
Cosecant Function

$f(x) = \csc x$



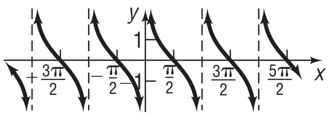
Secant Function

$f(x) = \sec x$



Cotangent Function

$f(x) = \cot x$



FORMULAS/EQUATIONS

Distance Formula

If $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, the distance from P_1 to P_2 is

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Standard Equation of a Circle

The standard equation of a circle of radius r with center at (h, k) is

$$(x - h)^2 + (y - k)^2 = r^2$$

Slope Formula

The slope m of the line containing the points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{if } x_1 \neq x_2$$

$$m \text{ is undefined} \qquad \text{if } x_1 = x_2$$

Point–Slope Equation of a Line

The equation of a line with slope m containing the point (x_1, y_1) is

$$y - y_1 = m(x - x_1)$$

Slope–Intercept Equation of a Line

The equation of a line with slope m and y-intercept b is

$$y = mx + b$$

Quadratic Formula

The solutions of the equation $ax^2 + bx + c = 0$, $a \neq 0$, are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 - 4ac > 0$, there are two unequal real solutions.
If $b^2 - 4ac = 0$, there is a repeated real solution.
If $b^2 - 4ac < 0$, there are two complex solutions that are not real.

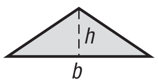
GEOMETRY FORMULAS

Circle

r = Radius, A = Area, C = Circumference

$$A = \pi r^2 \qquad C = 2\pi r$$

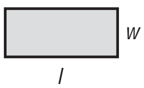
Triangle



b = Base, h = Altitude (Height), A = area

$$A = \tfrac{1}{2}bh$$

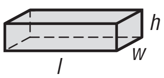
Rectangle



l = Length, w = Width, A = area, P = perimeter

$$A = lw \qquad P = 2l + 2w$$

Rectangular Box (closed)



l = Length, w = Width, h = Height, V = Volume, S = Surface area

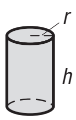
$$V = lwh \qquad S = 2lw + 2lh + 2wh$$

Sphere

r = Radius, V = Volume, S = Surface area

$$V = \tfrac{4}{3}\pi r^3 \qquad S = 4\pi r^2$$

Right Circular Cylinder (closed)



r = Radius, h = Height, V = Volume, S = Surface area

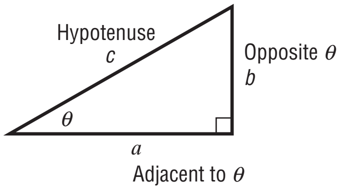
$$V = \pi r^2 h \qquad S = 2\pi r^2 + 2\pi rh$$

TRIGONOMETRIC FUNCTIONS

Of an Acute Angle

$$\sin\theta = \frac{b}{c} = \frac{\text{Opposite}}{\text{Hypotenuse}} \qquad \cos\theta = \frac{a}{c} = \frac{\text{Adjacent}}{\text{Hypotenuse}} \qquad \tan\theta = \frac{b}{a} = \frac{\text{Opposite}}{\text{Adjacent}}$$

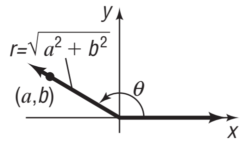
$$\csc\theta = \frac{c}{b} = \frac{\text{Hypotenuse}}{\text{Opposite}} \qquad \sec\theta = \frac{c}{a} = \frac{\text{Hypotenuse}}{\text{Adjacent}} \qquad \cot\theta = \frac{a}{b} = \frac{\text{Adjacent}}{\text{Opposite}}$$



Of a General Angle

$$\sin\theta = \frac{b}{r} \qquad \cos\theta = \frac{a}{r} \qquad \tan\theta = \frac{b}{a} \quad a \neq 0$$

$$\csc\theta = \frac{r}{b} \quad b \neq 0 \qquad \sec\theta = \frac{r}{a} \quad a \neq 0 \qquad \cot\theta = \frac{a}{b} \quad b \neq 0$$



TRIGONOMETRIC IDENTITIES

Fundamental Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$\cot^2\theta + 1 = \csc^2\theta$$

Half-Angle Formulas

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$
$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$
$$\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}$$

Double-Angle Formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$\cos(2\theta) = 2\cos^2\theta - 1$$
$$\cos(2\theta) = 1 - 2\sin^2\theta$$
$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

Even-Odd Identities

$$\sin(-\theta) = -\sin\theta \qquad \csc(-\theta) = -\csc\theta$$
$$\cos(-\theta) = \cos\theta \qquad \sec(-\theta) = \sec\theta$$
$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta$$

Product-to-Sum Formulas

$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$
$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$
$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$

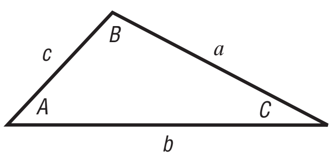
Sum and Difference Formulas

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$
$$\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

Sum-to-Product Formulas

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$
$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

SOLVING TRIANGLES



Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$